QUANTUM SCHUBERT POLYNOMIALS AND QUANTUM COHOMOLOGY

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In this expository note we will follow the 1997 paper of Fomin, Gelfand, and Postnikov, exploring quantum Schubert polynomials and their relationship with the quantum cohomology of the flag manifold. We will see that quantum Schubert polynomials correspond to Schubert classes of the quantum cohomology.

Contents

1. Brief Recollections

In this section we briefly describe the some recollections to set the stage.

Before we begin, some quick notes about notation: the original paper of FGP uses e_i^k to denote the elementary symmetric polynomial of degree i in k variables; we will instead adopt[1] $e_i[k]$. Similarly FGP use E_i^k to denote the quantum elementary symmetric polynomial; we will interchangeably use $e_i^{\mathrm{q}}[k]$ (or $e_i^q[k]$), which is more suggestive. We may also write $\mathbb{Z}[q_{[n-1]}]$ for $\mathbb{Z}[q_1, \cdots, q_{n-1}]$ and $\mathbb{Z}[x_{[n]}]$ for $\mathbb{Z}[x_1, \cdots, x_n]$, or also the shorter $\mathbb{Z}[q]$ and $\mathbb{Z}[x]$. We will use the notation $\pi = [\pi(1), \cdots, \pi(n)]$ to mark where π sends $1, \cdots, n$. We will also use[2] $\pi_\circ$ to denote the permutation of maximal length, i.e. $[n, n-1, \cdots, 1]$. By convention we take $e_0[k] = 1$ and $e_i[k] = 0$ for $i > k$; similarly we take $e_i^q[k] = 0$ for $i > k$.

Recall that we are concerned with the $\binom{n}{2}$-dimensional flag manifold Fl_n, whose points are complete flags of the space $\mathbb{C}^n$:

$$\mathrm{Fl}_n \coloneqq \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n\},$$

where the strict inclusions imply $\dim V_i = i$. We had broken this into

$$\mathrm{Fl}_n = \bigsqcup_{\pi \in S_n} \Omega_\pi,$$

where for a fixed reference flag E we defined

$$\Omega_\pi \coloneqq \{W_\bullet : \dim(W_i \cap E_j) = r_{ij}(\pi)\},$$

[1] I suppose it stylistically makes more sense to write $e_i(x_1, \cdots, x_k) = e_i(x_{[k]})$, but this is a bit too much, and in any case, that the variables involved are the x's is understood and suppressed.

[2] Perhaps it is more suggestive to write $\pi_{\binom{n}{2}}$ since there is only one permutation of maximal length $\binom{n}{2}$, but again this is a bit much.

where

$$r_{ij}(\pi) = |\{k : 1 \leq k \leq i,\ \pi(k) \leq j\}| = \sum_{\substack{\mu \leq i \\ \nu \leq j}} P(\pi)_{\mu\nu}.$$

Recall also the construction

$$\Omega^*_\pi \coloneqq \{W_\bullet : \dim(W_i \cap F_j) = r_{ij}(\pi^*)\},$$

where

$$\pi^* \coloneqq \pi_\circ \pi, \quad \text{i.e. } \pi^*(i) = n + 1 - \pi(i).$$

Recall we had also taken the closure of these to form the "Schubert variety"

$$X_\pi \coloneqq \widehat{\Omega}_\pi$$

and the "dual Schubert variety"

$$X^*_\pi \coloneqq \widehat{\Omega}^*_\pi;$$

these enjoy the description

$$X_\pi = \{W_\bullet : \dim(W_i \cap E_j) \geq r_{ij}(\pi)\}$$

and

$$X^*_\pi = \{W_\bullet : \dim(W_i \cap F_j) \geq r_{ij}(\pi^*)\},$$

as well as

$$X_\pi = \bigcup_{\tau \leq \pi} \Omega_\tau$$

and

$$X^*_\pi = \bigcup_{\tau \geq \pi} \Omega^*_\tau,$$

where the inequalities refer to the strong Bruhat order[3] on S_n. Recall that

$$\dim X_\pi = \dim \Omega_\pi = \ell(\pi) = \operatorname{codim} \Omega^*_\pi = \operatorname{codim} X^*_\pi,$$

so that we may consider the "Schubert class"

$$\sigma_\pi \coloneqq [X^*_\pi] \in H^{2\ell(\pi)}(\mathrm{Fl}_n : \mathbb{Z}),$$

where by $[X^*_\pi]$ in cohomology we mean the Poincare dual of the homology fundamental class of the submanifold. Recall that a consequence of the cell decomposition of Fl_n was that

$$H^\bullet(\mathrm{Fl}_n : \mathbb{Z}) \cong \bigoplus_{\pi \in S_n} \mathbb{Z}\sigma_\pi.$$

Recall that these satisfied the "Monk's rule",

Theorem (classical Monk)**.** For s_{ab} a transposition of a and b, $j \in [n-1]$, and $\pi \in S_n$,

$$\sigma_\pi \cdot \sigma_i = \sum_{\substack{a,b: 1 \leq a \leq i < b \leq n \\ \ell(\pi s_{ab}) = \ell(\pi)+1}} \sigma_{\pi s_{ab}}$$

□

Recall that the cohomology of Fl_n is

[3] Recall one characterization of this is $\tau \leq \pi \iff X_\tau \subseteq X_\pi \iff r_{ij}(\tau) \geq r_{ij}(\pi)$; the other was some/every reduced word for π containing as a subword a reduced word for τ.

Theorem.

$$H^\bullet(\mathrm{Fl}_n : \mathbb{Z}) \cong \mathbb{Z}[x_1, \cdots, x_n]/I_n$$

via

$$\sigma_\pi \longleftrightarrow \mathfrak{S}_\pi/I_n.$$

□

where

$$I_n \coloneqq \langle e_1[n], \cdots, e_n[n]\rangle.$$

We had constructed the so-called "Schubert polynomials" to construct this isomorphism. Recall this went thusly: for S_n acting on $\mathbb{Z}[x_1, \cdots, x_n]$ via $\pi \cdot f(x) = f(x_{\pi(1)}, \cdots, x_{\pi(n)})$, we defined a "divided difference"

$$\partial_{ij} f \coloneqq \frac{f - s_{ij} f}{x_i - x_j}$$

which sort of "symmetrizes" f in the i-th and j-th variables. We used the shorthand $\partial_i \coloneqq \partial_{i,i+1}$. Recall that these satisfied the "nilCoxeter relations", as well as some others:

Proposition (nilCoxeter relations et al)**.**

$$\partial_i^2 = 0,$$
$$\partial_i \partial_j = \partial_j \partial_i \qquad \forall\, |i - j| > 1,$$
$$\partial_{i+1}\partial_i\partial_{i+1} = \partial_i\partial_{i+1}\partial_i,$$

and

$$\partial_\pi \partial_\tau = \begin{cases} \partial_{\pi\tau} & \ell(\pi\tau) = \ell(\pi) + \ell(\tau) \\ 0 & \text{else} \end{cases},$$

and

$$\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g),$$

and, for a linear form (homogeneous of degree 1) $L = \sum \lambda_i x_i$,

$$\partial_\pi(fL) = \partial_\pi(f)\pi(L) + \sum_{i<j : \ell(\pi s_{ij}) = \ell(\pi) - 1} (\lambda_i - \lambda_j)\partial_{\pi s_{ij}} f.$$

□

Compare this to

Proposition (Coxeter relations)**.** For $s_i \in S_n$ simple transpositions,

$$s_i^2 = \mathrm{id},$$
$$s_i s_j = s_j s_i \qquad \forall\, |i - j| > 1,$$
$$s_{i+1}s_i s_{i+1} = s_i s_{i+1} s_i.$$

□

Note that in particular ∂_i commutes with multiplication by any function symmetric in x_i and x_{i+1}. We then used these to construct the Schubert polynomials:

$$\boxed{\mathfrak{S}_\pi \coloneqq \partial_{\pi^{*,-1}}(x^\delta) = \partial_{\pi^{-1}\pi_\circ}(x_1^{n-1} x_2^{n-2} \cdots x_n^0)},$$

where we denote $\delta = (n-1, n-2, \cdots, 1, 0)$. Recall that these satisfied

Proposition. The Schubert polynomials are also uniquely characterized by

$$\begin{aligned}\mathfrak{S}_{\pi_\circ} &= x^\delta,\\ \mathfrak{S}_{\pi s_i} &= \partial_i \mathfrak{S}_\pi \qquad \forall\, \ell(\pi s_i) = \ell(\pi) - 1.\end{aligned}$$

They also enjoy

$$\partial_\tau \mathfrak{S}_\pi = \begin{cases} \mathfrak{S}_{\pi\tau^{-1}} & \ell(\pi\tau^{-1}) = \ell(\pi) - \ell(\tau) \\ 0 & \text{else} \end{cases}$$

and a variant of Monk's

$$\mathfrak{S}_\pi \mathfrak{S}_{s_i} = \sum_{\substack{a,b:\, 1\le a\le i<b \\ \ell(\pi s_{ab}) = \ell(\pi)+1}} \mathfrak{S}_{\pi s_{ab}}.$$

□

The punchline was that the Schubert classes σ_π mapping to $\mathfrak{S}_\pi / I_n$ exhibited the isomorphism $H^\bullet(\mathrm{Fl}_n) \cong \mathbb{Z}[x_1, \cdots, x_n]/I_n$.

2. Quantum Cohomology of the Flag Manifold

In this section we describe the goal of quantum Schubert polynomials.

We can consider the "(small) quantum cohomology"

Definition. Let the *(small) quantum cohomology* be

$$\mathrm{Q}H^\bullet(\mathrm{Fl}_n : \mathbb{Z}) \overset{\mathsf{Mod}_\mathbb{Z}}{\coloneqq} H^\bullet(\mathrm{Fl}_n : \mathbb{Z}) \otimes_\mathbb{Z} \mathbb{Z}[q_1, \cdots, q_{n-1}]$$

as $\mathbb{Z}$-modules; the multiplication structure linear over $\mathbb{Z}[q_1, \cdots, q_{n-1}]$ is given by

$$\sigma_\pi \cdot \sigma_\tau = \sum_{\varpi \in S_n} \sum_d \langle \sigma_\pi, \sigma_\tau, \sigma_\varpi \rangle_d q_1^{d_1} \cdots q_{n-1}^{d_{n-1}} \sigma_{\pi_\circ \varpi}.$$

Here the "quantum Gromov-Witten invariants" $\langle \sigma_\pi, \sigma_\tau, \sigma_\varpi \rangle_d$ are defined as thus: For an algebraic map

$$f \colon \mathbb{P}^1 \longrightarrow \mathrm{Fl}_n,$$

we say f has "multidegree $d = (d_1, \cdots, d_{n-1})$" if

$$f_*[\mathbb{P}^1] = \sum_i d_i [X_{s_i}],$$

where $[\mathbb{P}^1] \in H_2(\mathbb{P}^1)$ and $[X_{s_i}] \in H_2(\mathrm{Fl}_n)$ are the homology fundamental classes. The "moduli space" is

$$\mathcal{M}_d(\mathbb{P}^1, \mathrm{Fl}_n) \coloneqq \{f \text{ algebraic of multidegree } d\},$$

which turns out to be a smooth algebraic variety of complex dimension

$$\binom{n}{2} + 2\sum_{i=1}^{n-1} d_i.$$

For a subvariety $X \subset \mathrm{Fl}_n$ and a point $t \in \mathbb{P}^1$ then define

$$X(t) \coloneqq \{f \in \mathcal{M}_d(\mathbb{P}^1, \mathrm{Fl}_n) : f(t) \in X\};$$

it turns out $\operatorname{codim}_{\mathcal{M}_d(\mathbb{P}^1,\mathrm{Fl}_n)} X(t) = \operatorname{codim}_{\mathrm{Fl}_n} X$. Then for $\pi_1, \cdots, \pi_m \in S_n$ the "Gromov-Witten invariant of genus 0" is defined by

$$\langle \sigma_{\pi_1}, \cdots, \sigma_{\pi_m} \rangle_d := \begin{cases} \left|\bigcap (g_i X_{\pi_i^*})(t_i)\right| & \sum \ell(\pi_i) = \binom{n}{2} + 2\sum_i d_i \\ 0 & \text{else} \end{cases},$$

where $g_1, \cdots, g_m \in \mathrm{GL}_n$ are generic elements and $t_1, \cdots, t_m \in \mathbb{P}^1$ are distinct points. It turns out the sizes of these $\bigcap (g_i X_{\pi_i^*})(t_i)$ are finite for $\sum \ell(\pi_i) = \binom{n}{2} + 2\sum_i d_i$ and independent of choice of t and g.

By definition this quantum multiplication is commutative; it turns out it is also associative. By setting $q_1 = \cdots = q_{n-1} = 0$ we recover classical multiplication, since the only surviving terms are $d = 0$, in which case an algebraic map of multidegree zero is necessarily constant; then we are looking for the number of constant maps $f : \mathbb{P}^1 \to \mathrm{Fl}_n$ (which is equivalent to a choice of a flag/point inside Fl_n) which send points t_i to inside varieties $X_{\pi_i^*}$, i.e. the classical intersection number. Moreover, by definition

$$\langle \sigma_\pi, \sigma_\tau, \sigma_\varpi \rangle_d = 0$$

unless $\ell(\pi) + \ell(\tau) + \ell(\varpi) = \binom{n}{2} + 2\sum_i d_i$, i.e. $\ell(\pi) + \ell(\tau) = \ell(\varpi^*) + 2\sum_i d_i = \ell(\pi_\circ \varpi) + 2\sum_i d_i$, so that by setting $\deg \sigma_\pi = \ell(\pi)$ and $\deg q = 2$ we have

$$\deg(\sigma_\pi \cdot \sigma_\tau) = \deg(q^d \sigma_{\pi_\circ \varpi}) = \ell(\varpi^*) + 2\sum d_i$$

and so quantum multiplication respects grading.

Letting

Definition. The *quantum elementary symmetric polynomial* is

$$e_i^q[k] := [\lambda^i] \det \left(1 + \lambda \begin{pmatrix} x_1 & q_1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & q_{n-1} \\ & & -1 & x_n \end{pmatrix} \right),$$

where $[\lambda^i]$ denotes the coefficient in front of λ^i. The notation $E_i[k]$ may also be adopted.

We may then define a $\mathbb{Z}[q]$-linear ring homomorphism

$$\begin{aligned} \mathbb{Z}[q_{[n-1]}][x_{[n]}] &\longrightarrow \mathrm{Q}H^\bullet(\mathrm{Fl}_n : \mathbb{Z}) \\ x_1 + \cdots + x_i &\longmapsto \sigma_{s_i}, \end{aligned}$$

where we may denote $\mathbb{Z}[q][x]$ for the cumbersome $\mathbb{Z}[q_{[n-1]}][x_{[n]}]$ and suppress the n. Then it turns out (and FGP cites)

Theorem. The kernel of this map is

$$I_n^q := \langle e_1^q[n], \cdots, e_n^q[n] \rangle,$$

and there is an induced ring isomorphism

$$\mathbb{Z}[q_{[n-1]}][x_{[n]}]/I_n^q \longrightarrow \mathrm{Q}H^\bullet(\mathrm{Fl}_n : \mathbb{Z}).$$

□

The punchline is this: quantum Schubert polynomials will be the polynomials corresponding to the quantum Schubert classes under this isomorphism.

3. Classical Polynomials

Recall the classical elementary symmetric polynomials in k variables:

$$e_i[k] := \sum_{1 \le \mu_1 < \cdots < \mu_i \le k} x_{\mu_1} \cdots x_{\mu_i}.$$

By convention $e_0[k] = 1$ and $e_i[k] = 0$ for $i > k$ and $i < 0$. These satisfy some properties:

Proposition. The $e_i[k]$ satisfy

$$\begin{aligned} e_i[k] &= e_i[k-1] + x_k e_{i-1}[k-1], \\ (e_i[k+1] - e_i[k])e_{j-1}[k] &= (e_j[k+1] - e_j[k])e_{i-1}[k], \\ e_i[k]e_j[k] &= e_i[k+1]e_j[k] + \sum_{\mu \ge 1} e_{i-\mu}[k+1]e_{j+\mu}[k] - \sum_{\mu \ge 1} e_{i-\mu}[k]e_{j+\mu}[k+1]. \end{aligned}$$

Moreover, in relations to divided differences,

$$\begin{aligned} \partial_{\neq k} e_i[k] &= 0, \\ [\partial_{\neq k}, e_i[k]] &= 0, \\ \partial_k e_i[k] &= e_{i-1}[k-1]. \end{aligned}$$

□

The proofs of these are just direct computation and so we skip them here. The first two relations with divided differences are immediate from definition, and the last follows from the pseudo-Leibniz rule for $\partial_i(fg)$ and $e_i[k] = e_i[k-1] + x_k e_{i-1}[k-1]$, which is also obvious, and from which easily $(e_i[k+1] - e_i[k])e_{j-1}[k] = (e_j[k+1] - e_j[k])e_{i-1}[k]$ follows. The last equation of the first half follows from induction on i and a computation.

We may also define

Definition. A *standard elementary monomial* is, for $\vec{i} = (i_1, \cdots, i_m)$ with $i_k \le k$,

$$e_{\vec{i}} = e_{i_1, \cdots, i_m} := e_{i_1}[1] \cdots e_{i_m}[m].$$

By convention if some $i_k > k$ we can set $e_{\vec{i}} = 0$. This can be thought of as a product of $e_i[k]$ without repetition of k.

Recall that the elementary symmetric polynomials formed a basis in the ring of symmetric polynomials. We will prove a result in a similar spirit:

Theorem (straightening)**.** The standard elementary monomials form a $\mathbb{Z}$-basis in $\mathbb{Z}[x_1, \cdots, x_\infty]$ the ring of polynomials in infinite variables:

$$\mathbb{Z}[x_1, \cdots, x_\infty] = \mathbb{Z}\{e_{\vec{i}}\}_{0 \le i_k \le k}.$$

□

Proof. First let us see that $\{e_{\vec{i}}\}_{0 \le i_k \le k}$ spans all polynomials. To do this note that

$$x_i = e_1[i] - e_1[i-1];$$

use this to expand f into a linear combination of products of $e_i[k]$, and apply the following straightening algorithm:

If there is a monomial in f which is not standard (i.e. contains a repeated k, or a "bracket index collision"), we can take the minimal k for which it contains $e_i[k]e_j[k]$, so that

$$f = e_i[k]e_j[k] \prod e_{\text{blah}}[\text{blah}] + \sum \prod e_{\text{blah}}[\text{blah}].$$

Then use

$$e_i[k]e_j[k] = e_i[k+1]e_j[k] + \sum_{\mu\geq 1} e_{i-\mu}[k+1]e_{j+\mu}[k] - \sum_{\mu\geq 1} e_{i-\mu}[k]e_{j+\mu}[k+1]$$

to replace $e_i[k]e_j[k]$ with the right-hand side. By minimality of k doing this will only shift collisions of the bracket indices upwards (e.g. k to $k+1$). We can try to repeat this until there are no bracket index collisions left; indeed, this process must terminate since it preserves the number of $e_i[k]$ in each monomial, and (if we order the monomials so that the bracket index increases from left to right) in moving from k to $k+1$ we shift possible collisions to the right; but each monomial of f can only have finitely many factors. Hence this process, which terminates, will turn f into a linear combination of standard elementary monomials.

Now let us check linear independence. AFSOC the standard elementary monomials were linearly dependent, with some

$$\sum \prod e_{\text{blah}}[\text{blah}] = 0$$

nontrivial vanishing linear combination of minimal degree. Let k be the minimal bracket index appearing in this sum, i.e.

$$e_i[k] \prod e_{\text{blah}}[\text{blah}] + \sum \prod e_{\text{blah}}[\text{blah}] = 0;$$

then by our cited properties of e we have

$$\partial_k \left(\sum_{\text{without } k} \prod e_{\text{blah}}[\text{blah}] \right) = 0$$

since we can commute ∂_k with $e_{\text{blah}}[\text{blah}]$ until there is only one left, whereupon $\partial_k e_{\text{blah}}[\text{blah}] = 0$. For those monomials *with* k, we have

$$\partial_k(e_i[k]e_j[k+1]\cdots) = \partial_k(e_i[k])e_j[k+1]\cdots = e_{i-1}[k-1]e_j[k+1]\cdots \neq 0,$$

where again ∂_k commutes with everyone after $e_i[k]$ (there is no one before it by minimality). Note well that this drops the degree by 1. Hence by applying ∂_k to this vanishing sum we obtain another nontrivial vanishing sum of smaller degree, contradicting minimality of degree.

Hence $e_{\vec{i}}$ forms a basis over $\mathbb{Z}$ of all polynomials over $\mathbb{Z}$, as claimed. ■

Recall

Theorem. Each of the following forms a $\mathbb{Z}$-basis:

$$\begin{aligned} \mathbb{Z}[x_1, \cdots, x_n]/I_n &= \mathbb{Z}\{x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}\}_{a_i \leq n-i}, \\ &= \mathbb{Z}\{e_{i_1,\cdots,i_{n-1}}\}, \\ &= \mathbb{Z}\{\mathfrak{S}_\pi\}_{\pi\in S_n}, \end{aligned}$$

where by each polynomial we mean their cosets under quotient by I_n. Moreover each of these three families span the same module L_n complementary to the ideal I_n. In the last case, $\{\mathfrak{S}_\pi\}_{\pi\in S_n}$ moreover forms a basis of L_n.
□

The last of these three was proved in the notes for M269, and the second follows from our above proof. We skip the proof of the first.

4. Quantum Polynomials

Recall earlier we had defined

$$\left(\begin{array}{l}\textbf{Definition. } \text{The } \textit{quantum elementary symmetric polynomial} \text{ is} \\ \qquad e_i^q[k] := [\lambda^i] \det\left(1 + \lambda \begin{pmatrix} x_1 & q_1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & q_{n-1} \\ & & -1 & x_n \end{pmatrix}\right), \\ \text{where } [\lambda^i] \text{ denotes the coefficient in front of } \lambda^i. \text{ The notation } E_i[k] \text{ may also be adopted.}\end{array}\right.$$

where by convention $e_i^q[k] = 0$ for $i > k$ or $i < 0$. It is clear that by taking $q = 0$ we obtain the classical $e_i[k]$.

We can try to expand the definition of the determinant

$$\det \begin{pmatrix} 1+\lambda x_1 & \lambda q_1 & & \\ -\lambda & \ddots & \ddots & \\ & \ddots & \ddots & \lambda q_{n-1} \\ & & -\lambda & 1+\lambda x_n \end{pmatrix},$$

as an alternating sum, where we would find that, due to all the zeros, the only nonvanishing terms are for the identity permutation and simple transpositions. For the simple transpositions, the signs are moreover negated since each simple transposition would activate $-\lambda$, whose sign cancels out $\operatorname{sgn}(s_i) = -1$. Here's an example for $n = 4$:

$$\begin{aligned} \det \begin{pmatrix} 1+\lambda x_1 & \lambda q_1 & & \\ -\lambda & 1+\lambda x_2 & \lambda q_2 & \\ & -\lambda & 1+\lambda x_3 & \lambda q_3 \\ & & -\lambda & 1+\lambda x_4 \end{pmatrix} &= (1+\lambda x_1)(1+\lambda x_2)(1+\lambda x_3)(1+\lambda x_4) \\ &\quad + \lambda\lambda q_1(1+\lambda x_3)(1+\lambda x_4) \\ &\quad + (1+\lambda x_1)\lambda\lambda q_2(1+\lambda x_4) \\ &\quad + (1+\lambda x_1)(1+\lambda x_2)\lambda\lambda q_3 \\ &\quad + \lambda\lambda q_1 \lambda\lambda q_3, \end{aligned}$$

from which the following description[4] of $e_i^q[k]$ becomes clear:

$$e_i^q[k] = \sum_{(\mu,\mu+1),(\nu) \text{ disjoint cover } i \text{ pts}} q_{\vec{\mu}} x_{\vec{\nu}},$$

where we think of q_i as covering $(i, i+1)$ and x_i as covering (i). For example,

$$\begin{aligned} e_2^q[3] &= x_1x_2 + x_1x_3 + x_2x_3 + q_1 + q_2, \\ e_3^q[3] &= x_1x_2x_3 + q_1x_3 + q_2x_1. \end{aligned}$$

From this description we immediately see that $e_i^q[k]$ is a homogeneous polynomial of degree i if we set $\deg x = 1$ and $\deg q = 2$. From the description the first of the following properties is also transparent.

[4]Note well that the simple transpositions in the determinant cannot collide.

Proposition. The quantum elementary symmetric polynomials enjoy

$$e^q_i[k] = e^q_i[k-1] + x_k e^q_{i-1}[k-1] + q_{k-1} e^q_{i-2}[k-2],$$

and for $i, j \leq k$,

$$e^q_i[k]e^q_j[k+1] + e^q_{i+1}[k]e^q_j[k] + q_k e^q_{i-1}[k-1]e^q_j[k] = e^q_j[k]e^q_i[k+1] + e^q_{j+1}[k]e^q_i[k] + q_k e^q_{j-1}[k-1]e^q_i[k].$$

Since the divided difference operators do not see the q variables, we still have

$$\begin{aligned}\partial_{\neq k} e^q_i[k] &= 0,\\ [\partial_{\neq k}, e^q_i[k]] &= 0,\\ \partial_k e^q_i[k] &= e^q_{i-1}[k-1].\end{aligned}$$

□

The second property follows from the first by subtracting

$$e^q_j[k](e^q_{i+1}[k+1] - e^q_{i+1}[k]) = e^q_j[k](x_{k+1}e^q_i[k] + q_k e^q_{i-1}[k-1])$$

from

$$e^q_i[k](e^q_{j+1}[k+1] - e^q_{j+1}[k]) = e^q_i[k](x_{k+1}e^q_j[k] + q_k e^q_{j-1}[k-1]).$$

Note well that these specialized at $q = 0$ gives the classical properties we cited earlier.

Similarly to the classical case, we may define

Definition. The *quantum standard elementary monomials* are

$$e^q_{\vec{i}} = e^q_{i_1,\cdots,i_m} := e^q_{i_1}[1] \cdots e^q_{i_m}[m].$$

The notation $E_{\vec{i}}$ may also be used.

Since standard elementary monomials form a basis of $\mathbb{Z}[x]$, we may uniquely expand

$$\mathfrak{S}_\pi = \sum c_{i_1,\cdots,i_{n-1}} e_{i_1,\cdots,i_{n-1}};$$

let us then define

Definition. The *quantum Schubert polynomials* are

$$\mathfrak{S}^q_\pi := \sum c_{i_1,\cdots,i_{n-1}} e^q_{i_1,\cdots,i_{n-1}}.$$

By using these properties of e^q and a straightening argument in the spirit of the classical case, one may obtain

Theorem. Each of the following forms a $\mathbb{Z}[q_{[n-1]}]$-basis:

$$\begin{aligned}\mathbb{Z}[q_1, \cdots, q_{n-1}][x_1, \cdots, x_n]/I^q_n &= \mathbb{Z}[q_{[n-1]}]\{x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}\}_{a_i \leq n-i},\\ &= \mathbb{Z}[q_{[n-1]}]\{e^q_{i_1,\cdots,i_{n-1}}\},\\ &= \mathbb{Z}[q_{[n-1]}]\{\mathfrak{S}^q_\pi\}_{\pi \in S_n},\end{aligned}$$

where by each polynomial we mean their cosets under quotient by I^q_n. Moreover each of these three families span the same module L^q_n complementary to the ideal I^q_n. In the second and last last cases, $\{e^q_{i_1,\cdots,i_{n-1}}\}$ and $\{\mathfrak{S}^q_\pi\}_{\pi \in S_n}$ moreover form bases of L^q_n.

□

Proof. The second of these statements can be proved in a similar way to the classical case: we can apply a straightening algorithm based on

$$e^q_i[k]e^q_j[k+1] + e^q_{i+1}[k]e^q_j[k] + q_k e^q_{i-1}[k-1]e^q_j[k] = e^q_j[k]e^q_i[k+1] + e^q_{j+1}[k]e^q_i[k] + q_k e^q_{j-1}[k-1]e^q_i[k],$$

which we can relabel to be

$$e^q_i[k]e^q_j[k] = e^q_j[k]e^q_{i-1}[k+1]+e^q_{j+1}[k]e^q_{i-1}[k]+q_k e^q_{j-1}[k-1]e^q_{i-1}[k]-e^q_{i-1}[k]e^q_j[k+1]-q_k e^q_{i-2}[k-1]e^q_j[k];$$

note well that the bracket index collision term goes from being $e^q_i[k]e^q_j[k]$ to $e^q_{j+1}[k]e^q_{i-1}[k]$, so that by repeatedly applying this identity to the bracket collision term eventually the $e^q_{i-1}[k]$ becomes $e^q_{i+j}[k]e^q_0[k] = e^q_{i+j}[k]$, eliminating the bracket index collision. Similarly linear independence can be proved using the divided difference operators.

The third statement follows from the classical case and the first two statements. ■

Since $e^q_i[k]$ is homogeneous of degree i, we have $e^q_{\vec{i}}$ is homogeneous of degree $\sum \vec{i}$. Since the classical Schubert polynomial $\mathfrak{S}_\pi = \sum c_{\vec{i}} e_{\vec{i}}$ is homogeneous of degree $\ell(\pi)$, it follows that the quantum Schubert polynomial $\mathfrak{S}^q_\pi = \sum c_{\vec{i}} e^q_{\vec{i}}$ is also homogeneous of degree $\ell(\pi)$, where $\deg x = 1$ and $\deg q = 2$. Since setting $q = 0$ reduces e^q to e, it follows that setting $q = 0$ reduces $\mathfrak{S}^q_\pi$ to $\mathfrak{S}_\pi$.

We will lastly cite one fact about the quantum Schubert polynomials, which is proved in sections 5 and 6 of FGP. For $F \in \mathbb{Z}[q][x]/I^q_n$ consider

$$[\mathfrak{S}^q_{\pi_\circ}](F)$$

the coefficient in front of $\mathfrak{S}^q_{\pi_\circ} = e^q_{1,\cdots,n-1} = e^q_1[1]\cdots e^q_{n-1}[n-1] = x_1 \cdot x_1x_2 \cdot x_1x_2x_3 \cdot \cdots \cdot x_1 \cdots x_{n-1} = x_1^{n-1} \cdots x_{n-1}^1 = x^\delta$ in the expansion of F in the basis of quantum Schubert polynomials. This coefficient satisfies a certain orthogonality relation:

Theorem (orthogonality)**.** For $\pi, \tau \in S_n$,

$$[\mathfrak{S}^q_{\pi_\circ}]\big(\mathfrak{S}^q_\pi \mathfrak{S}^q_\tau\big) = \begin{cases} 1 & \tau = \pi^* \\ 0 & \text{else} \end{cases},$$

where recall $\pi^* = \pi_\circ \pi$.
□

We skip this proof for the sake of brevity.

5. Quantum Cohomology and Quantum Polynomials

Throughout this section we work module I^q_n. Let Q_π be the quantum polynomials in $\mathbb{Z}[q][x]/I^q_n$ corresponding to the Schubert classes under the cited isomorphism

$$\mathrm{Q}H^\bullet(\mathrm{Fl}_n : \mathbb{Z}) \cong \mathbb{Z}[q][x]/I^q_n.$$

As Schubert classes form a basis of cohomology, Q_π will form a basis of $\mathbb{Z}[q][x]/I^q_n$. We will show that

$$Q_\pi = \mathfrak{S}^q_\pi.$$

To this end, recall that quantum multiplication respects grading; hence Q_π must be homogeneous of degree $\ell(\pi)$, where $\deg x = 1$ and $\deg q = 2$.

Since setting $q = 0$ turns quantum cohomology into regular homology, it will send quantum Schubert classes to classical Schubert classes, and we should want setting $q = 0$ to turn Q_π into $\mathfrak{S}_\pi$.

Since the Gromov-Witten invariants are nonnegative integers by definition, we require the structure constants of $\mathbb{Z}[q][x]/I_n^q$ with respect to the basis $\{Q_\pi\}$ (meaning what happens when we expand a product of two basis elements in the basis) to be nonnegative integers.

Let $\mathbb{Z}_{0+}$ denote the nonnegative integers, and let $\mathbb{Z}_{0+}[q]$ denote the set of polynomials in q whose coefficients are in $\mathbb{Z}_{0+}$. Denote

$$\mathcal{Q}_{0+} = \mathbb{Z}_{0+}[q]\{Q_\pi\}_{\pi \in S_n}$$

the set of linear combinations of Q_π with coefficients in $\mathbb{Z}_{0+}[q]$; by the nonnegativity of the structure constants, we have $\mathcal{Q}_{0+}$ is a semiring.

Lastly, we will cite a result from a paper of Ciocan-Fontanine which states that, for a cycle $\pi = s_{k-i+1} \cdots s_k \in S_n$, we have

$$Q_{s_{k-i+1}\cdots s_k} = e_i^q[k]$$

is the quantum elementary symmetric polynomial; in particular this implies that every $e_i^q[k] \in \mathcal{Q}_{0+}$, which is all we will need. This is an analogue of the classical fact that

$$\mathfrak{S}_{s_{k-i+1}\cdots s_k} = e_i[k].$$

In summary,

Lemma. The Q_π will satisfy:

- Q_π is homogeneous of degree $\ell(\pi)$, where $\deg x = 1$ and $\deg q = 2$;
- Setting $q = 0$ turns Q_π into $\mathfrak{S}_\pi$;
- The structure constants of $\mathbb{Z}[q][x]/I_n^q$ with respect to the basis $\{Q_\pi\}$ are nonnegative integers; this implies that
$$[\mathfrak{S}^q_{\pi_\circ}]\big(Q_{\pi_1} \cdots Q_{\pi_m}\big) = [Q_{\pi_\circ}]\big(Q_{\pi_1} \cdots Q_{\pi_m}\big) \in \mathbb{Z}_{0+}[q];$$
- Each quantum elementary symmetric polynomial belongs to the semiring $\mathcal{Q}_{0+}$, i.e. $e_i^q[k] \in \mathcal{Q}_{0+}$; combined with the previous fact this means all quantum standard elementary polynomials have $e^q_{\vec{i}} \in \mathcal{Q}_{0+}$.

□

The implication of the third bullet point above is true since, by degree considerations (the first two properties; we will also see this in the proof below), the transition matrix between the bases $\{Q_\pi\}$ and $\{\mathfrak{S}^q_\pi\}$ is unipotent triangular (with respect to, say, an order based on increasing $\ell(\pi)$), so to find the coefficient corresponding to the longest permutation $\pi_\circ$ it suffices to expand in either basis.

We now prove that

Theorem. $\mathfrak{S}^q_\pi$ are the corresponding elements to σ_π under the isomorphism

$$\mathrm{Q}H^\bullet(\mathrm{Fl}_n : \mathbb{Z}) \cong \mathbb{Z}[q_{[n-1]}][x_{[n]}]/I_n^q.$$

□

i.e. that $Q_\pi = \mathfrak{S}^q_\pi$.

Proof. Let us focus on a length $l \le \binom{n}{2}$. By the classical bases of $\mathbb{Z}[x]/I_n$, the polynomials $\{\mathfrak{S}_\pi\}_{\ell(\pi)=l}$ are related to $\{e_{i_1,\cdots,i_{n-1}}\}_{\sum i = l}$ by an invertible linear transformation.

Moreover, when we write $e_{\vec{i}}$ in the basis of $\mathfrak{S}_\pi$, the coefficients are nonnegative integers since each $e_{\vec{i}}$ is a product of the Schubert polynomials $e_i[k]$, and the structure constants of the classical cases (being intersection numbers) are nonnegative integers.

Since the linear transformation relating $\{\mathfrak{S}_\pi\}_{\ell(\pi)=l}$ and $\{e_{i_1,\cdots,i_{n-1}}\}_{\sum i=l}$ is invertible, each $\mathfrak{S}_\pi$ must appear in the expansion of at least one $e_{\vec{i}}$. Hence

$$\sum_{\sum i=l} e_{i_1,\cdots,i_{n-1}} = \sum_{\ell(\pi)=l} c_\pi \mathfrak{S}_\pi$$

for positive constants

$$c_\pi > 0.$$

Quantum-ifying, this becomes

$$\sum_{\sum i=l} e^q_{i_1,\cdots,i_{n-1}} = \sum_{\ell(\pi)=l} c_\pi \mathfrak{S}^q_\pi \in \mathcal{Q}_{0+},$$

where we have used the fourth bullet in the lemma earlier to say that $\sum e^q_{\vec{i}} \in \mathcal{Q}_{0+}$. Hence

$$\sum_{\ell(\pi)=l} c_\pi \mathfrak{S}^q_\pi \in \mathcal{Q}_{0+}.$$

As the Q_π form a basis, we can try to expand $\mathfrak{S}^q_\pi$ in terms of the Q. By degree considerations we can only have (since if there isn't enough degree you can always tack on more q's, but if there's too much then there's nothing you can do)

$$\mathfrak{S}^q_\pi = \sum_{\ell(\tau)\le\ell(\pi)} a_\tau Q_\tau$$

for $a_\tau \in \mathbb{Z}[q]$ (so that for $\ell(\tau) < \ell(\pi)$, a_τ must have q terms for the degrees to match); moreover by taking $q = 0$ we see that $a_\pi = 1$ necessarily (since $Q_\tau|_{q=0} = \mathfrak{S}_\pi$). Then

$$\mathfrak{S}^q_\pi = Q_\pi + \sum_{\ell(\tau)<\ell(\pi)} a_\tau Q_\tau.$$

A similar expression in the other direction also holds.

Then we can expand all of $\sum_{\ell(\pi)=l} c_\pi \mathfrak{S}^q_\pi \in \mathcal{Q}_{0+}$ in this basis,

$$\sum_{\ell(\pi)=l} c_\pi \mathfrak{S}^q_\pi = \sum_{\ell(\pi)=l} c_\pi Q_\pi + \sum_{\ell(\tau)<l} \mathrm{coeff}_\tau Q_\tau \in \mathcal{Q}_{0+},$$

which implies

$$\sum_{\ell(\tau)<l} \mathrm{coeff}_\tau Q_\tau \in \mathcal{Q}_{0+},$$

which implies

$$\sum_{\ell(\pi)=l} c_\pi \mathfrak{S}^q_\pi - \sum_{\ell(\pi)=l} c_\pi Q_\pi = \sum_{\ell(\pi)=l} c_\pi (\mathfrak{S}^q_\pi - Q_\pi) \in \mathcal{Q}_{0+}.$$

Now consider $\vec{j} = (j_1, \cdots, j_{n-1})$ with $\sum j > \binom{n}{2} - l$; since by the last bullet of the lemma $e^q_{\vec{j}} \in \mathcal{Q}_{0+}$, we can expand $e^q_{\vec{j}} = \sum_\tau \mathrm{nonneg}_\tau Q_\tau$ for nonnegative coefficients $\mathrm{nonneg}_\tau \in \mathbb{Z}_{0+}[q]$. If we multiply this by Q_π, by the third bullet point we would have

$$[\mathfrak{S}^q_{\pi_\circ}]\left(Q_\pi \sum_\tau \mathrm{nonneg}_\tau Q_\tau\right) = \sum_\tau \mathrm{nonneg}_\tau [Q_{\pi_\circ}](Q_\pi Q_\tau) \in \mathbb{Z}_{0+}[q],$$

i.e.

$$[\mathfrak{S}^q_{\pi_\circ}]\left(e^q_{\vec{j}} Q_\pi\right) \in \mathbb{Z}_{0+}[q]. \qquad (*)$$

Indeed, if we multiplied $e^q_{\vec{j}} = \sum_\tau \mathrm{nonneg}_\tau Q_\tau$ by any element of $\mathcal{Q}_{0+}$, we would obtain the same result; in particular apply this to $\sum_{\ell(\pi)=l} c_\pi(\mathfrak{S}^q_\pi - Q_\pi) \in \mathcal{Q}_{0+}$ to get

$$[\mathfrak{S}^q_{\pi_\circ}]\left(e^q_{\vec{j}} \sum_{\ell(\pi)=l} c_\pi(\mathfrak{S}^q_\pi - Q_\pi)\right) \in \mathbb{Z}_{0+}[q].$$

This can be rewritten as

$$\begin{aligned}
\mathbb{Z}_{0+}[q] \ni [\mathfrak{S}^q_{\pi_\circ}]&\left(e^q_{\vec{j}} \sum_{\ell(\pi)=l} c_\pi(\mathfrak{S}^q_\pi - Q_\pi)\right) \\
&= \sum_{\ell(\pi)=l} c_\pi[\mathfrak{S}^q_{\pi_\circ}]\Big(e^q_{\vec{j}}\mathfrak{S}^q_\pi\Big) - \sum_{\ell(\pi)=l} c_\pi[\mathfrak{S}^q_{\pi_\circ}]\Big(e^q_{\vec{j}}Q_\pi\Big) \\
&= -\sum_{\ell(\pi)=l} c_\pi[\mathfrak{S}^q_{\pi_\circ}]\Big(e^q_{\vec{j}}Q_\pi\Big),
\end{aligned}$$

where

$$[\mathfrak{S}^q_{\pi_\circ}]\Big(e^q_{\vec{j}}\mathfrak{S}^q_\pi\Big) = 0$$

uses the orthogonality of $\mathfrak{S}$ quoted at the end of last section (expand $e^q_{\vec{j}}$ in terms of the quantum Schubert polynomials; by homogeneity all subsequent terms have degree i.e. length of labelling permutation equal to $\sum j$), as well as the fact that $\sum j > \ell(\pi^*) = \binom{n}{2} - l$ (so that the condition for nonvanishing can't hold). Then

$$-\sum_{\ell(\pi)=l} c_\pi[\mathfrak{S}^q_{\pi_\circ}]\Big(e^q_{\vec{j}}Q_\pi\Big) \in \mathbb{Z}_{0+}[q]. \tag{$*$}$$

Comparing the two starred equations, and recalling that $c_\pi > 0$ are positive, we conclude that it must be the case that

$$[\mathfrak{S}^q_{\pi_\circ}]\Big(e^q_{\vec{j}}Q_\pi\Big) = 0$$

for any l, any π with $\ell(\pi) = l$, and $\vec{j}$ with $\sum j > \ell(\pi^*)$. Since any $\mathfrak{S}^q_{\tau^*}$ for $\ell(\tau) < \ell(\pi)$ can be written as a linear combination of $e^q_{\vec{j}}$ for $\sum j = \ell(\tau^*) = \binom{n}{2} - \ell(\tau) > \binom{n}{2} - \ell(\pi) = \ell(\pi^*)$, this implies

$$[\mathfrak{S}^q_{\pi_\circ}](\mathfrak{S}^q_{\tau^*}Q_\pi) = 0 \qquad \forall\ \ell(\tau) < \ell(\pi); \tag{$*$}$$

expanding

$$Q_\pi = \mathfrak{S}^q_\pi + \sum_{\ell(\tau)<\ell(\pi)} b_\tau \mathfrak{S}^q_\tau,$$

we obtain

$$[\mathfrak{S}^q_{\pi_\circ}](\mathfrak{S}^q_{\tau^*}\mathfrak{S}^q_\pi) + \sum_{\ell(\varpi)<\ell(\pi)} b_\varpi[\mathfrak{S}^q_{\pi_\circ}](\mathfrak{S}^q_{\tau^*}\mathfrak{S}^q_\varpi) = 0 \qquad \forall\ \ell(\tau) < \ell(\pi),$$

which by applying orthogonality as we run across $\tau = \varpi$ with length less than π implies

$$b_\varpi = 0,$$

so that

$$Q_\pi = \mathfrak{S}^q_\pi,$$

as claimed. ■

This completes the proof.

6. References

S. Fomin, S. Gelfand, and A. Postnikov, *Quantum Schubert Polynomials*. 1997.