SPECTRAL SEQUENCES AND AN APPLICATION TO THE SYMMETRY OF TOR

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In this expository note we will quickly develop spectral sequences and use them to prove the symmetry of the Tor functor.

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It seems most common in the literature for spectral sequences to begin on page 1. I suppose it doesn't really matter, but it seems as if by starting at page 0 many of the indices in the setup are greatly simplified. I therefore adopt this approach and hope that I do not mess up any indices in transitioning.

We begin with some groundwork on spectral sequences; in the interest of length we will not prove any results about spectral sequences. We will however use them to prove the symmetry of the Tor functor, a property I used in M231a but which we did not prove.

1. Bare Basics: Bi-things

Recall that the notion of a graded module:

Definition. A *graded R-module* is an indexed family

$$M = (M_\mu)_{\mu\in\mathbb{Z}}$$

of R-modules.

A *graded map of degree a* between graded modules M and N is

$$f\colon M \longrightarrow N$$

a family of maps

$$f = \left(f_\mu\colon M_\mu \longrightarrow N_{\mu+a}\right)_{\mu\in\mathbb{Z}}$$

where we write $\deg f = a$.

These objects still admit index-wise notions of submodules, quotient modules, kernels, and images (and therefore exactness). Note as a minor detail that there would need to be some index correction sometimes; for example, $A \xrightarrow{f} B \xrightarrow{g} C$ is exact when $\operatorname{Img} f_{\mu-\deg f} = \operatorname{Ker} g_\mu$.

A (homological, i.e. decreasing index) chain complex can then be phrased as a graded module equipped with a map d such that $\deg \mathrm{d} = -1$ and $\mathrm{d}^2 = 0$. Recall that given such information we may then construct homology.

Similarly one can define a bigraded module:

Definition. A *bigraded R-module* is an indexed family

$$M = (M_{\mu,\nu})_{\mu,\nu\in\mathbb{Z}}$$

of R-modules.

A *graded map of degree* (a, b) between bigraded modules M and N is

$$f\colon M \longrightarrow N$$

a family of maps

$$f = \big(f_{\mu,\nu}\colon M_{\mu,\nu} \longrightarrow N_{\mu+a,\nu+b}\big)_{\mu,\nu\in\mathbb{Z}}$$

where we write $\deg f = (a, b)$.

Just as in the singly-graded case, this admits index-wise notions of submodules[1] and quotient modules and kernels and images and exactness.

There is also an analogous notion of a chain complex, in the sense that here we may also take homology:

Definition. A *differential bigraded module* is (M, d) for a bigraded module M and a bigraded map (the *differential*) $\mathrm{d}\colon M \longrightarrow M$ with

$$\mathrm{d}^2 = 0.$$

For $\deg \mathrm{d} = (a, b)$, we may then construct homology in much the same manner:

$$H_{\mu,\nu}(M, \mathrm{d}) := {\mathrm{Ker}\, \mathrm{d}_{\mu,\nu}}\big/{\mathrm{Img}\, \mathrm{d}_{\mu-a,\nu-b}}.$$

2. Exact Couples

There is the notion of an exact couple:

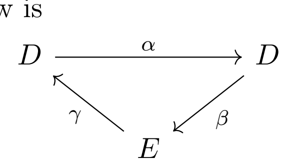
Definition. An *exact couple* is $(D, E, \alpha, \beta, \gamma)$ where D, E are bigraded modules and α, β, γ are bigraded maps between them fitting in a diagram

$$D \xrightarrow{\alpha} D \xrightarrow{\beta} E \xrightarrow{\gamma} D \xrightarrow{\alpha} D$$

which is exact, i.e.

$$\begin{aligned} \mathrm{Ker}\, \alpha &= \mathrm{Img}\, \gamma, \\ \mathrm{Ker}\, \beta &= \mathrm{Img}\, \alpha, \\ \mathrm{Ker}\, \gamma &= \mathrm{Img}\, \beta. \end{aligned}$$

The diagram that people usually draw is



[1]E.g. $N \subseteq M$ if $N_{\mu\nu} \subseteq M_{\mu\nu}$ for every index.

Given an exact couple (which we will denote with (D^0, E^0)), one can obtain others:

Proposition. Construct the *first derived couple* $(D^1, E^1, \alpha^1, \beta^1, \gamma^1)$ by

$$\begin{aligned} D^1 &:= \operatorname{Img}\alpha^0 \subseteq D^0, \\ E^1 &:= H_{\bullet\bullet}(E^0, \mathrm{d}^0), \\ \mathrm{d}^0 &:= \beta^0\gamma^0 \colon E^0 \longrightarrow E^0, \\ \alpha^1 &:= \alpha^0|_{D^1}, \\ \beta^1 &:= [\beta^0(\alpha^0)^{\circ-1}\square], \\ \gamma^1 &:= \gamma^0, \end{aligned}$$

where $(\alpha^0)^{\circ-1} = \alpha^{0,\circ-1}$ refers to the compositional inverse of α^0 in

$$\beta^1(y) = [\beta^0(\alpha^0)^{\circ-1}(y)] \quad \text{for } y \in D^1$$

and

$$\gamma^1[z] = \gamma^0(z) \quad \text{for } [z] \in E^1.$$

Note well that $D^1 \subseteq D^0$, so the indexing we are taking here is $D^1_{\mu,\nu} := \operatorname{Img}\alpha_{\mu-a_1,\nu-a_2}$. Note also that

$$\begin{aligned} \deg\alpha^1 &= \deg\alpha^0 = (a_1, a_2), \\ \deg\beta^1 &= \deg\beta^0 - \deg\alpha^0 = (b_1 - a_1, b_2 - a_2), \\ \deg\gamma^1 &= \deg\gamma^0. \end{aligned}$$

The content of the claim is that this is still an exact couple.

We can then iterate this construction to obtain $(D^k, E^k, \alpha^k, \beta^k, \gamma^k)$ the *k-th derived couple*, defined recursively as the derived couple of the $(k-1)$-th derived couple, which would be (we drop the zero in $\alpha^0 = \alpha$ and similarly for other maps)

$$\begin{aligned} D^k &= \operatorname{Img}\alpha^{\circ k} \subseteq D^0 \\ (D^k_{\mu,\nu} &= \operatorname{Img}\alpha_{\mu-a_1,\nu-a_2}\alpha_{\mu-2a_1,\nu-2a_2}\cdots\alpha_{\mu-ka_1,\nu-ka_2}), \\ E^k &= H_{\bullet\bullet}(E^{k-1}, d^{k-1}), \\ d^k &= \beta\alpha^{\circ-k}\gamma, \\ \alpha^k &= \alpha|_{D^k}, \\ \beta^k &= \beta\alpha^{\circ-k}, \\ \gamma^k &= \gamma, \end{aligned}$$

with degrees

$$\begin{aligned} \deg\alpha^k &= \deg\alpha, \\ \deg\beta^k &= \deg\beta - k\deg\alpha, \\ \deg\gamma^k &= \deg\gamma. \end{aligned}$$

□

(wow this took up a whole page)

3. Filtrations

Recall that a filtration of a module M is a family of submodules

$$\cdots \subseteq M^{\mu-1} \subseteq M^{\mu} \subseteq M^{\mu+1} \subseteq \cdots .$$

Recall that given such a filtration we could consider

$$\operatorname{Gr}^{\mu} M \coloneqq M^{\mu}/M^{\mu-1}$$

and

$$\operatorname{Gr} M \coloneqq \bigoplus_{\mu} \operatorname{Gr}^{\mu} M.$$

Recall that given a complex C, we may consider filtrations:

Definition. A *(increasing) filtration* of a complex (C, d) is a family of graded submodules

$$(F^{\mu}C)_{\mu\in\mathbb{Z}}$$

with

$$\cdots \subseteq F^{\mu-1}C \subseteq F^{\mu}C \subseteq F^{\mu+1}C \subseteq \cdots$$

which is compatible with d:

$$\mathrm{d}\colon (F^{\mu}C)_n \longrightarrow (F^{\mu}C)_{n-1}.$$

A filtration is moreover said to be *bounded* if, for each n, there are a, b depending on n such that

$$F^{a}C_n = 0,$$
$$F^{b}C_n = C_n.$$

We may shorten $(F^{\mu}C)_n$ to $F^{\mu}C_n$. We might also write C^{μ} for $F^{\mu}C$, and ${C^{\mu}}_n$ for $F^{\mu}C_n$, in line with the philosophy that these things should be considered as filtrations of complexes rather than complexes of filtrations.

Given a filtration, we may obtain an exact couple by looking at the long exact sequence of a pair. Namely, by considering

$$0 \longrightarrow F^{\mu-1}C \xrightarrow{\iota^{\mu-1}} F^{\mu}C \xrightarrow{\pi^{\mu}} F^{\mu}C/F^{\mu-1}C \longrightarrow 0$$

we obtain a long exact sequence

$$\cdots \xrightarrow{\quad\alpha\quad} H_{n+1}(F^{\mu}C) \xrightarrow{\beta} H_{n+1}(F^{\mu}C/F^{\mu-1}C) \xrightarrow{\gamma}$$
$$H_n(F^{\mu-1}C) \xrightarrow{\alpha=\iota_*^{\mu-1}} H_n(F^{\mu}C) \xrightarrow{\beta=\pi_*^{\mu}} H_n(F^{\mu}C/F^{\mu-1}C) \xrightarrow{\gamma=\delta}$$
$$H_{n-1}(F^{\mu-1}C) \xrightarrow{\quad\alpha\quad} \cdots$$

Proposition. Given a filtration of a complex, we may obtain an exact couple with

$$\begin{aligned} D_{\mu,\nu} &= H_{\mu+\nu}(F^\mu C), \\ E_{\mu,\nu} &= H_{\mu+\nu}(F^\mu C / F^{\mu-1} C), \\ \alpha &= \iota_*^{\mu-1}, \\ \beta &= \pi_*^\mu, \\ \gamma &= \delta, \end{aligned}$$

so that the k-th derived exact couple is

$$\begin{aligned} D^k_{\mu,\nu} &= \operatorname{Img}\left(H_{\mu+\nu}(F^{\mu-k} C) \overset{(\iota^{\mu-1} \cdots \iota^{\mu-k})_*}{\longrightarrow} H_{\mu+\nu}(F^\mu C) \right), \\ E^k_{\mu,\nu} &= \operatorname{Ker} \mathrm{d}^{k-1}_{\mu,\nu} \big/ \operatorname{Img} \mathrm{d}^{k-1}_{\mu+k,\nu-(k-1)}, \\ \deg \alpha &= (1,-1), \\ \deg \beta &= (-k,k), \\ \deg \gamma &= (-1,0), \\ \deg \mathrm{d}^k &= (-k-1,k). \end{aligned}$$

□

Just so there's a picture, here is what this looks like at $k = 0$:

$$\begin{array}{l} \cdots \xrightarrow{\alpha} D_{\mu,\nu+1} \xrightarrow{\beta} E_{\mu,\nu+1} \xrightarrow{\gamma} \\ D_{\mu-1,\nu+1} \xrightarrow{\alpha} D_{\mu,\nu} \xrightarrow{\beta} E_{\mu,\nu} \xrightarrow{\gamma=\delta} \\ D_{\mu-1,\nu} \xrightarrow{\alpha} \cdots \end{array}$$

4. Spectral Sequences

Let us define what a spectral sequence is. Again you should be warned that people tend to start at page 1; because I like to be special I will start at page 0[2].

Definition. A *spectral sequence* is a sequence

$$(E^k, \mathrm{d}^k)_{k \geq 0}$$

of differential bigraded modules such that

$$E^k = H_{\bullet\bullet}(E^{k-1}, \mathrm{d}^{k-1}).$$

Then, as we saw above, every filtration of a complex produces an exact couple whose derived exact couples give a spectral sequence.

If the filtration is moreover bounded (recall this means that for each n, eventually $F^{\text{small}} C_n = 0$ and $F^{\text{big}} C_n = C_n$), then we may say something much better:

[2]Possibly I'll learn sometime later down the line that there's actually a generating function associated to these things and for that reason you should want to start at page 1, but whatever.

Proposition. If $E = (E^k, \mathrm{d}^k)_k$ is a spectral sequence arising from a bounded filtration of a complex, then for each (μ, ν) there exists a bound b depending on μ, ν beyond which $E_{\mu,\nu}$ stabilizes:

$$E^{\geq b}_{\mu,\nu} = E^b_{\mu,\nu}.$$

In this case, let us define

$$E^{\infty}_{\mu,\nu} := E^b_{\mu,\nu}.$$

□

The punchline is that, in this case, we are able to say something about the homology of the original complex we started with. Unfortunately we are only able to do this for the Gr of the homology, but in the case of vector spaces we can then add them all up for the real thing. I guess if we are not in vector-space-land then things get trickier.

Definition. Given a filtration on C, define a filtration on $H_\bullet(C)$ via the following process:

Let $\iota^{\mu,\infty}$ be the inclusion map

$$\iota^{\mu,\infty} \colon F^\mu C \hookrightarrow C,$$

with

$$\iota^{\mu,\infty}_* \colon H_\bullet(F^\mu C) \longrightarrow H_\bullet(C);$$

then define

$$F^\mu H_n(C) := \operatorname{Img} \iota^{\mu,\infty}_*.$$

This forms a filtration of $H_n(C)$.

Thankfully, if $F^\bullet C$ is bounded, it turns out so is $F^\bullet H(C)$:

Lemma. If $F^\bullet C$ is a bounded filtration of C, then $F^\bullet H(C)$ is also bounded with the same bound. That is, if for each n we have $a = a(n)$ and $b = b(n)$ with

$$F^a C_n = 0,$$
$$F^b C_n = C_n,$$

then

$$F^a H_n(C) = 0,$$
$$F^b H_n(C) = H_n(C).$$

□

The punchline is

Theorem. For $F^\bullet C$ a bounded filtration with spectral sequence E, we have

$$E^\infty_{\mu,\nu} \cong \operatorname{Gr}^\mu H_{\mu+\nu}(C) = F^\mu H_{\mu+\nu}(C) \big/ F^{\mu-1} H_{\mu+\nu}(C).$$

□

5. Bicomplexes

We will apply the above to the case of bicomplexes, which are defined as

Definition. A *bicomplex* is $(M, \mathrm{d}^{\leftarrow}, \mathrm{d}^{\downarrow})$ a bigraded module equipped with bigraded maps

$$\deg \mathrm{d}^{\leftarrow} = (-1, 0),$$
$$\deg \mathrm{d}^{\downarrow} = (0, -1)$$

such that

$$\mathrm{d}^{\leftarrow}\mathrm{d}^{\leftarrow} = 0,$$
$$\mathrm{d}^{\downarrow}\mathrm{d}^{\downarrow} = 0,$$
$$\mathrm{d}^{\leftarrow}\mathrm{d}^{\downarrow} + \mathrm{d}^{\downarrow}\mathrm{d}^{\leftarrow} = 0,$$

where the last identity written out in indices would look like

$$\mathrm{d}^{\leftarrow}_{\mu,\nu-1}\mathrm{d}^{\downarrow}_{\mu,\nu} + \mathrm{d}^{\downarrow}_{\mu-1,\nu}\mathrm{d}^{\leftarrow}_{\mu,\nu} = 0.$$

Given such a bicomplex, one may construct

Definition. The *total complex* $\mathrm{Tot}(M)$ of a bicomplex is a complex with n-th term defined as

$$\mathrm{Tot}(M)_n := \bigoplus_{\mu+\nu=n} M_{\mu,\nu}$$

with differential

$$\mathrm{d}_n := \sum_{\mu+\nu=n} \mathrm{d}^{\downarrow}_{\mu,\nu} + \mathrm{d}^{\leftarrow}_{\mu,\nu} \colon \mathrm{Tot}(M)_n \longrightarrow \mathrm{Tot}(M)_{n-1}.$$

The fact that $\mathrm{Tot}(M)$ so equipped with d forms a complex follows from the anticommutativity condition $\mathrm{d}^{\leftarrow}\mathrm{d}^{\downarrow} + \mathrm{d}^{\downarrow}\mathrm{d}^{\leftarrow} = 0$.

There are two types of filtrations one may endow this bicomplex with: they are

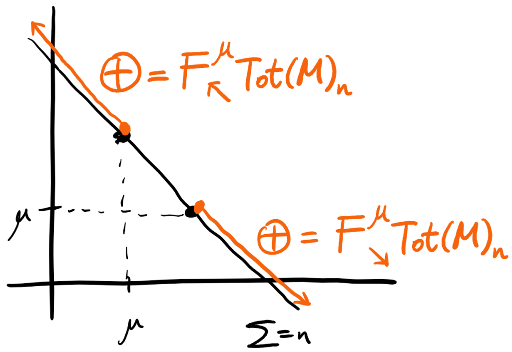
$$\begin{aligned} F^{\mu}_{\nwarrow} \mathrm{Tot}(M)_n &= \bigoplus_{i\le\mu} M_{i,n-i} \\ &= \cdots \oplus M_{\mu-2,\nu+2} \oplus M_{\mu-1,\nu+1} \oplus M_{\mu,\nu} \end{aligned}$$

and

$$\begin{aligned} F^{\mu}_{\searrow} \mathrm{Tot}(M)_n &= \bigoplus_{i\le\mu} M_{n-i,i} \\ &= M_{\nu,\mu} \oplus M_{\nu+1,\mu-1} \oplus M_{\nu+2,\mu-2} \oplus \cdots . \end{aligned}$$

Pictorially here is what is going on[3]:

[3] I think in the literature these are referred to as ${}^{\mathrm{I}}F^{\mu}\,\mathrm{Tot}(M)_n$ and ${}^{\mathrm{II}}F^{\mu}\,\mathrm{Tot}(M)_n$ respectively, but I'm not too fond of this notation.



That these are filtrations, i.e. that $\mathrm{d}\colon F^\mu \operatorname{Tot}(M)_n \longrightarrow F^\mu \operatorname{Tot}(M)_{n-1}$, requires checking but is easily done.

As an aside, given a bicomplex, we may also consider its transpose, i.e. considering $(M^\dagger, \mathrm{d}^{\dagger,\leftarrow}, \mathrm{d}^{\dagger,\downarrow})$ with

$$\begin{aligned} M^\dagger_{\mu,\nu} &= M_{\nu,\mu}, \\ \mathrm{d}^{\dagger,\leftarrow}_{\mu,\nu} &= \mathrm{d}^{\downarrow}_{\nu,\mu}, \\ \mathrm{d}^{\dagger,\downarrow}_{\mu,\nu} &= \mathrm{d}^{\leftarrow}_{\nu,\mu}. \end{aligned}$$

Then it is straightforward to see that

$$\operatorname{Tot}(M) = \operatorname{Tot}(M^\dagger)$$

and

$$F^\mu_\nwarrow \operatorname{Tot}(M)_n = F^\mu_\searrow \operatorname{Tot}(M)_n.$$

A situation which commonly arises is that of a *first quadrant bicomplex*, i.e. one for which

$$M_{\mu,\nu} = 0 \qquad \text{for } \mu < 0 \text{ or } \nu < 0.$$

In this case it is readily apparent (since they get cut off by the axes) that

$$F^\bullet_\nwarrow \operatorname{Tot}(M),\ F^\bullet_\searrow \operatorname{Tot}(M)$$

are both bounded filtrations, in which case the punchline theorem holds, so that

$$\begin{aligned} E^{\nwarrow\infty}_{\mu,\nu} &\cong \mathrm{Gr}^\mu_\nwarrow H_n(\operatorname{Tot} M), \\ E^{\searrow\infty}_{\mu,\nu} &\cong \mathrm{Gr}^\mu_\searrow H_n(\operatorname{Tot} M). \end{aligned}$$

But $H_n(\operatorname{Tot} M)$ is only one way in which one might take homology. Another idea is this: since we are given a bicomplex, we may consider each column separately. Each column (say at $x = \mu$) $M_{\mu,\bullet}$ is a complex, so that we may take its homology:

$$H(M_{\mu,\bullet}).$$

In other words, we are taking the double-indexed homology of the differential bigraded module with respect to $\mathrm{d}^\downarrow$:

$$H^\downarrow_{\mu,\nu} := H_\nu(M_{\mu,\bullet}) = H_{\mu,\nu}(M, \mathrm{d}^\downarrow).$$

After doing so, we may lay these homologies down in a row (say the *nu*-th row) with

$$\cdots \longleftarrow H_\nu(M_{\mu-1,\bullet}) \xleftarrow{\mathrm{d}^\leftarrow_{\mu,\nu}} H_\nu(M_{\mu,\bullet}) \xleftarrow{\mathrm{d}^\leftarrow_{\mu+1,\nu}} H_\nu(M_{\mu+1,\bullet}) \longleftarrow \cdots$$

where the arrows are for example

$$\begin{aligned} H_\nu(M_{\mu,\bullet}) &\longrightarrow H_\nu(M_{\mu-1,\bullet}) \\ [z] &\longmapsto [\mathrm{d}^{\leftarrow}_{\mu,\nu}(z)] \end{aligned}$$

which we will also denote by $\mathrm{d}^{\leftarrow}_{\mu,\nu}$ (it is a straightforward check that this is well-defined). Hence we can take homology yet again; we will denote this second homology by

$$H^{\leftarrow}_\mu H^{\downarrow}_\nu(M) \coloneqq H_\mu(H^{\downarrow}_{\bullet,\nu}(M)).$$

Analogously, we may consider each row and consider its homology

$$H^{\leftarrow}_{\mu,\nu} \coloneqq H_\nu(M_{\bullet,\mu}) = H_{\mu,\nu}(M, \mathrm{d}^{\leftarrow});$$

we can arrange these in the μ-th column as

$$\begin{array}{c} \vdots \\ \downarrow \\ H_\mu(M_{\bullet,\nu+1}) \\ \big\downarrow \mathrm{d}^{\downarrow}_{\mu,\nu+1} \\ H_\mu(M_{\bullet,\nu}) \\ \big\downarrow \mathrm{d}^{\downarrow}_{\mu,\nu} \\ H_\mu(M_{\bullet,\nu-1}) \\ \big\downarrow \mathrm{d}^{\downarrow}_{\mu,\nu-1} \\ \vdots \end{array}$$

whereupon we may again take the homology to obtain

$$H^{\downarrow}_\mu H^{\leftarrow}_\nu(M) \coloneqq H_\mu(H^{\leftarrow}_{\nu,\bullet}(M)).$$

The punchline is then that

Theorem. For M a bicomplex contained in the first quadrant, we have

$$E^{\nwarrow\infty}_{\mu,\nu} \cong \mathrm{Gr}^{\mu}_{\nwarrow} H_n(\mathrm{Tot}\, M) = F^{\mu}_{\nwarrow} H_n(\mathrm{Tot}\, M) \big/ F^{\mu-1}_{\nwarrow} H_n(\mathrm{Tot}\, M),$$

$$E^{\searrow\infty}_{\mu,\nu} \cong \mathrm{Gr}^{\mu}_{\searrow} H_n(\mathrm{Tot}\, M) = F^{\mu}_{\searrow} H_n(\mathrm{Tot}\, M) \big/ F^{\mu-1}_{\searrow} H_n(\mathrm{Tot}\, M).$$

Moreover, the first two pages are

$$E^{\nwarrow 0}_{\mu,\nu} = H^{\downarrow}_{\mu,\nu}(M),$$
$$E^{\nwarrow 1}_{\mu,\nu} = H^{\leftarrow}_{\mu} H^{\downarrow}_{\nu}(M)$$

and

$$E^{\searrow 0}_{\mu,\nu} = H^{\leftarrow}_{\mu,\nu}(M),$$
$$E^{\searrow 1}_{\mu,\nu} = H^{\downarrow}_{\mu} H^{\leftarrow}_{\nu}(M).$$

In the case that E^1 lies entirely on either the horizontal or the vertical axis, we actually have

$$E^1 \cong E^\infty.$$

If it is the horizontal axis, then

$$E^1_{n,0} \cong H_n(\mathrm{Tot}\, M);$$

if it is the vertical axis, then

$$E^1_{0,n} \cong H_n(\mathrm{Tot}\, M).$$

□

6. Application to Tor

Finally, we shall use all this machinery to prove the symmetry of the Tor functor. Recall that the Tor functor came from the following process: to compute $\mathrm{Tor}^R_n(M, N)$ for M, N R-modules, we take a projective resolution of the first module M:

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

and consider its deleted form

$$\cdots \longrightarrow P_1 \longrightarrow P_0;$$

then apply the tensor product $\otimes_R N$ (which is covariant)

$$\cdots \longrightarrow P_1 \otimes N \longrightarrow P_0 \otimes N \longrightarrow 0$$

which is a complex; then define

$$\mathrm{Tor}^R_n(M, N) \coloneqq H_n(P_\bullet \otimes_R N).$$

As the tensor product is symmetric, one might ask whether it matters if the first variable is resolved or if the second variable is. To answer this we give the following:

Fact. The Tor functor is symmetric.

□

Proof. All tensor products will be taken over R, which we drop for brevity.

Take P, Q to be projective resolutions of M, N respectively:

$$\cdots \longrightarrow P_2 \xrightarrow{\mathrm{d}^P_2} P_1 \xrightarrow{\mathrm{d}^P_1} P_0 \longrightarrow M \longrightarrow 0$$

and

$$\cdots \longrightarrow Q_2 \xrightarrow{\mathrm{d}^Q_2} Q_1 \xrightarrow{\mathrm{d}^Q_1} Q_0 \longrightarrow N \longrightarrow 0.$$

Consider then the bicomplex formed as follows: at the (μ, ν)-th place let us place down $P_\mu \otimes Q_\nu$, so

$$M_{\mu,\nu} \coloneqq P_\mu \otimes Q_\nu,$$

and let

$$\begin{aligned} \mathrm{d}^{\leftarrow}_{\mu,\nu} &\coloneqq \mathrm{d}^P_\mu \otimes 1, \\ \mathrm{d}^{\downarrow}_{\mu,\nu} &\coloneqq (-1)^\mu 1 \otimes \mathrm{d}^Q_\nu. \end{aligned}$$

Pictorially, here is what this bicomplex looks like:

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \\ \cdots & \longleftarrow & P_{\mu-1} \otimes Q_\nu & \xleftarrow{\mathrm{d}^P_\mu \otimes 1} & P_\mu \otimes Q_\nu & \longleftarrow & \cdots \\ & & \Big\downarrow{\scriptstyle (-1)^{\mu-1} 1 \otimes \mathrm{d}^Q_\nu} & & \Big\downarrow{\scriptstyle (-1)^{\mu} 1 \otimes \mathrm{d}^Q_\nu} & & \\ \cdots & \longleftarrow & P_{\mu-1} \otimes Q_{\nu-1} & \xleftarrow{\mathrm{d}^P_\mu \otimes 1} & P_\mu \otimes Q_{\nu-1} & \longleftarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \end{array}$$

It is clear that the rows and columns formed this way are exact; this is moreover a bicomplex, i.e. each square moreover anticommutes, precisely due to the $(-1)^\mu$ sign; this is evident from looking at the diagram. Since the resolutions are labelled by nonnegative integers, this bicomplex is a first quadrant complex. Moreover note well that its total complex is the traditional one-dimensional tensor product of two chain complexes:

$$\mathrm{Tot}(P \otimes Q) = P_\bullet \otimes Q_\bullet$$

with the correct differential $\mathrm{d} = \mathrm{d}^P \otimes 1 + (-1)^\mu 1 \otimes \mathrm{d}^Q$.

Let us compute some iterated homologies. Fixing a column $x = \mu$, let us consider its homology; it must vanish in degree above zero since $Q_\bullet$ was exact, and tensoring with P_μ (which is projective, implying it is flat) preserves exactness, so

$$H_{>0}(P_\mu \otimes Q_\bullet) = 0.$$

At the zeroth degree it is

$$H_0(P_\mu \otimes Q_\bullet) = \mathrm{Coker}(P_\mu \otimes Q_1 \to P_\mu \otimes Q_0);$$

as the sequence

$$\cdots \longrightarrow Q_2 \xrightarrow{\mathrm{d}^Q_2} Q_1 \xrightarrow{\mathrm{d}^Q_1} Q_0 \longrightarrow N \longrightarrow 0$$

is exact, so is its tensor with the flat module P_μ

$$\cdots \longrightarrow P_\mu \otimes Q_2 \xrightarrow{1 \otimes \mathrm{d}^Q_2} P_\mu \otimes Q_1 \xrightarrow{1 \otimes \mathrm{d}^Q_1} P_\mu \otimes Q_0 \longrightarrow P_\mu \otimes N \longrightarrow 0,$$

so that the cokernel we desire is given by

$$H_0(P_\mu \otimes Q_\bullet) = P_\mu \otimes N.$$

Hence we have the zeroth page with respect to the first filtration (that this is the zeroth page is the content of a theorem which we cite from before)

$$E^{\nwarrow 0}_{\mu,\nu} = H^{\downarrow}_{\mu,\nu}(M) = \begin{cases} P_\mu \otimes N & \nu = 0 \\ 0 & \nu > 0 \end{cases}.$$

Taking the homology of this again, we obtain (again that this is equal to the first page is a theorem from before which we utilize)

$$E^{\nwarrow 1}_{\mu,\nu} = H^{\leftarrow}_{\mu} H^{\downarrow}_{\nu}(M) = \begin{cases} H_\mu(P_\bullet \otimes N) & \nu = 0 \\ 0 & \nu > 0 \end{cases},$$

i.e.

$$E^{\nwarrow 1} \text{ collapses onto the horizontal axis, where it is } H_n(P_\bullet \otimes N). \tag{$*$}$$

On the other hand, we can consider taking the horizontal homology first, and then taking homology vertically; i.e. we can try to compute the pages for the second filtration. To this end, fixing a row $y = \nu$, the homology of

$$\cdots \longrightarrow P_{\mu+1} \otimes Q_\nu \longrightarrow P_\mu \otimes Q_\nu \longrightarrow P_{\mu-1} \otimes Q_\nu \longrightarrow \cdots$$

again vanishes in positive degree, again since Q_ν is projective which implies it is flat. At degree zero it is

$$H_0(P_\bullet \otimes Q_\nu) = \operatorname{Coker}(P_1 \otimes Q_\nu \longrightarrow P_0 \otimes Q_\nu),$$

which we can compute since

$$\cdots \longrightarrow P_2 \xrightarrow{\mathrm{d}^P_2} P_1 \xrightarrow{\mathrm{d}^P_1} P_0 \longrightarrow M \longrightarrow 0$$

being exact implies

$$\cdots \longrightarrow P_2 \otimes Q_\nu \xrightarrow{\mathrm{d}^P_2 \otimes 1} P_1 \otimes Q_\nu \xrightarrow{\mathrm{d}^P_1 \otimes 1} P_0 \otimes Q_\nu \longrightarrow M \otimes Q_\nu \longrightarrow 0$$

is exact, so that

$$H_0(P_\bullet \otimes Q_\nu) = M \otimes Q_\nu.$$

This gives

$$E^{\searrow 0}_{\mu,\nu} = H^{\leftarrow}_{\mu,\nu}(M) = \begin{cases} M \otimes Q_\mu & \nu = 0 \\ 0 & \nu > 0 \end{cases},$$

which taking homology again gives

$$E^{\searrow 1}_{\mu,\nu} = H^{\downarrow}_{\mu} H^{\leftarrow}_{\nu}(M) = \begin{cases} H_\mu(M \otimes Q_\bullet) & \nu = 0 \\ 0 & \nu > 0 \end{cases},$$

i.e.

$$E^{\searrow 1} \text{ collapses onto the horizontal axis, where it is } H_n(M \otimes Q_\bullet). \tag{$*$}$$

But recall that the first page collapsing onto an axis implies that that axis is precisely $H_n(\operatorname{Tot} M)$, which does not depend on which filtration we take. Hence we have

$$H_n(P_\bullet \otimes N) \cong E^{\nwarrow 1}_{n,0} \cong H_n(\operatorname{Tot} M) \cong E^{\searrow 1}_{n,0} \cong H_n(M \otimes Q_\bullet), \tag{$**$}$$

so that

$$H_n(P_\bullet \otimes N) \cong H_n(M \otimes Q_\bullet)$$

and it does not matter which variable we resolve, as desired. Note incidentally that these two quantities are also equal to $H_n(\operatorname{Tot} M)$, where $\operatorname{Tot} M$ is the complex obtained by resolving *both* variables at once and tensoring the complexes together; this is sort of a neat characterization which fell out as a side-product. ■

7. References

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