COMPUTING THE RESOLUTION COEFFICIENTS IN A CONJECTURED COMBINATORIAL RESOLUTION CATEGORIFYING THE JACOBI-TRUDI DETERMINANT FORMULA

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1. Summary

The goal is to categorify the Jacobi-Trudi determinant formula as a BGG-esque resolution. For 3-part partitions of n given by $(\lambda_1, \lambda_2, \lambda_3) \vdash n$, we present a conjecture for a combinatorial model of a resolution for the irreducible representations of the symmetric group S_n associated to the partition λ. This resolution consists of permutation modules (which can be thought of as free functors applied to species). When the maps in this resolution are expanded as linear combinations of certain natural maps between these permutation modules, we have calculated what the coefficients ought to be (i.e. necessary conditions for exactness). This is not an explicit formula, but a generating function for it; in particular one could write a program to find what they are. We conjecture that this resolution is exact (i.e. that these conditions are sufficient).

1.1. Background and Motivation.

Background and Motivation. It is well-known that the irreducible representations Π_λ of the symmetric group S_n are labeled by partitions λ of n, and that the characters are the Schur functions s_λ, which are described by the Jacobi-Trudi determinant formula as $s_\lambda = \det(h_{\lambda_i+j-i})_{ij}$. Here h_k is the complete homogeneous symmetric polynomial, which is also the character of the permutation representation E_k of S_k. So it is true on the level of characters that the irreducible representations of S_n are given by some alternating sum of (induction-) products of permutation representations, and it is natural to ask whether one can "categorify" this by giving a resolution of Π_λ by permutation representations whose alternating sum of characters gives the Jacobi-Trudi determinant formula.

In 1988 (and in 1992 in a follow-up paper by Akin), Zelevinsky [1] and Akin [2] independently described how to obtain a resolution of the irreducible representations of the symmetric groups which categorifies Jacobi-Trudi. They do so by taking the BGG resolution of an irreducible representation of the general linear group, applying a special functor (tensor with a fixed representation, quotient by the nilpotent subalgebra $\mathfrak{n}^-$, and take the weight space at a fixed weight), and then appealing to Schur–Weyl duality. Akin's second paper [3] moreover described how one could theoretically describe the differential maps in this resolution in terms of Shapovalov elements, but the maps are not very explicit. In 2011 Boltje and Hartmann [4] gave a conjectured resolution which is different than that of Akin and Zelevinsky, and this conjecture was proved by Santana and Yudin [5] in 2012 with the help of the bar resolution and the Schur functor.

In summer of 2019 at PRISE (a Harvard research program for its students), I thought about this problem through the lens of combinatorial species, which is a theory of calculus on functors (called *species*) between the core of the category of finite sets and the category of sets. By composing a species with the free functor from the category of sets to that of vector spaces, one obtains a representation of the symmetric groups; under this scheme, the permutation representations are obtained by the "set species". By categorifying the Jacobi-Trudi determinant formula, one is able to build irreducible representations (and therefore all representations) of the symmetric group from formal alternating sums of permutation representations in a meaningful way.

I derived a conjecture for a combinatorial resolution of the irreducible representations of S_n over characteristic zero corresponding to a partition of length 3 (lower length cases are much easier). The objects in this resolution are the (induction products of) permutation representations prescribed by the Jacobi-Trudi determinant – namely, letting E_λ be the induction product $\mathrm{Ind}_{S_{\lambda_1}\times\cdots\times S_{\lambda_k}}^{S_n}(E_{\lambda_1}\otimes\cdots\otimes E_{\lambda_k})$, the i-th term of the resolution is $\bigoplus_{\ell(w)=i} E_{w\circ\lambda}$ (where $w\circ\lambda$ denotes the affine action of the Weyl group on weights; note the similarity to the BGG resolution, where the terms are instead direct sums of Verma modules) – but the differential maps are more complicated and are obtained from a combinatorial model. Roughly, the combinatorial model places $E_{w\circ\lambda}$ (where $w \in S_3$) in a hexagonal shape (the shape of the affine Weyl action orbit on the weights), fills in the interior with other appropriate auxiliary permutation representations (labeled by weights in the convex hull of the affine Weyl action orbit), and has certain natural maps between two adjacent permutation representations. (In general, for partitions with a higher number of parts, this shape would be a permutohedron.) The differential maps of the resolution are then linear combinations of compositions along appropriate paths of these natural maps. I was able to use combinatorial species to calculate the coefficients in this linear combination as coefficients of a generating function involving Catalan-like series, and with these coefficients the sequence forms a complex; however, exactness proved to be difficult, and I was unable to prove that this sequence was exact. I believe that this proposed resolution is secretly the same as that of Akin and Zelevinsky (but introduces new combinatorial structure) and different from the one proposed by Boltje and Hartmann.

1.2. **A short explanation of why this question and some other questions.** Some further notes: this project grew out of a fascination I have for combinatorial species, which are defined as functors $\mathcal{F}\colon \operatorname{Cor}(\mathsf{fntSet}) \longrightarrow \mathsf{Set}$ from the groupoid of finite sets to the category of sets, i.e. set-valued, or discrete, representations (of the symmetric groups). In particular, by applying the free functor from the category of sets to the category of vector spaces to a species, one obtains an actual representation.

Like for representations, there is also a notion of "irreducible species" (called "molecular species" in the literature[1]) Irreducible species enjoy a beautifully simple classification theorem (though perhaps on some level this is expected, since species is really just group actions on sets): irreducible species concentrated on degree n are in bijection with conjugacy classes of subgroups of S_n, with the bijection given by

$$H \subseteq S_n \longleftrightarrow X^n/H.$$

I was then curious as to why this difference in simplicity exists between species and representations, as well as how one might obtain irreducible representations from irreducible species. This "resolution" that I propose here is one way to do so, for the terms in this resolution (and the Jacobi-Trudi determinant formula) are indeed free functors applied to irreducible species (indeed, $E_\mu = X^{|\mu|}/S_{\mu_1} \times \cdots \times S_{\mu_k}$).

However, I had found other examples too of obtaining irreps from irreducible species, for example the sign representation

$$\Pi_{(1,\cdots,1)} = \operatorname{Free}(E_n^{\pm} - E_n).$$

Unfortunately I do not know how to generalize these.

Some questions present themselves. What of representations of other groups/algebras (for example classical Lie groups/algebras)? How would classical phenomena such as Schur-Weyl present themselves in this context? What of species over finite fields? Could species be generalized to other groups in a meaningful way which presents rich theory? If there are other ways (other than the Jacobi-Trudi determinant, which we explored a little in this note) to go from irreducible species to irreducible representations, is there a way to classify these constructions?

2. Setup

2.1. **The Objects.** Let Π_λ be the irreducible representation of S_n associated to the partition $\lambda \vdash n$. Let λ be a 3-part partition. From the Jacobi-Trudi determinant formula, we know that the Schur function can be expressed in terms of the complete homogeneous functions:

$$s_\lambda = \det \begin{pmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} \\ h_{\lambda_3-2} & h_{\lambda_3-1} & h_{\lambda_3} \end{pmatrix}.$$

Since characters determine representations, this means

$$\Pi_\lambda = \det \begin{pmatrix} E_{\lambda_1} & E_{\lambda_1+1} & E_{\lambda_1+2} \\ E_{\lambda_2-1} & E_{\lambda_2} & E_{\lambda_2+1} \\ E_{\lambda_3-2} & E_{\lambda_3-1} & E_{\lambda_3} \end{pmatrix}, \tag{$*$}$$

[1]As I understand it, this terminology is borrowed from chemistry.

where by E_k we mean an abuse of notation: E_k is usually the set species (i.e. a set representation), outputting the set $[k]$ on input $[k]$ and the empty set on everything else. The E_k here, being a vector space representation, is the free functor applied to the set species above and in particular a representation of S_k. As there is only one element in the species, it is clear that applying the free functor gives us the trivial representation.

This determinant equation for Π_λ is a priori only true as an equality of virtual representations; a resolution will, so to speak, breathe life into it.

Let us make the shorthand

$$E_{a,b,c} \coloneqq E_a E_b E_c,$$

the product of species (or rather the free functor applied to it). For example, the vector space $E_{1,2}$ is the free functor applied to the set $\left\{ \binom{1}{2,3}, \binom{2}{1,3}, \binom{3}{1,2} \right\}$. Note that, as representations, this is the induction product of three trivial representations:

$$E_{a,b,c} = \operatorname{Ind}_{S_a \times S_b \times S_c}^{S_{a+b+c}} (\operatorname{triv}_a \otimes \operatorname{triv}_b \otimes \operatorname{triv}_c).$$

2.2. **The Resolution.** The resolution we propose is

$$\begin{array}{c} 0 \\ \downarrow \\ E_{\lambda_3-2,\lambda_2,\lambda_1+2} \\ \downarrow \partial_3 \\ E_{\lambda_3-2,\lambda_1+1,\lambda_2+1} \oplus E_{\lambda_2-1,\lambda_3-1,\lambda_1+2} \\ \downarrow \partial_2 \\ E_{\lambda_1,\lambda_3-1,\lambda_2+1} \oplus E_{\lambda_2-1,\lambda_1+1,\lambda_3} \\ \downarrow \partial_1 \\ E_{\lambda_1,\lambda_2,\lambda_3} \\ \downarrow \partial_0 \\ \Pi_{\lambda_1,\lambda_2,\lambda_3} \\ \downarrow \\ 0 \end{array}$$

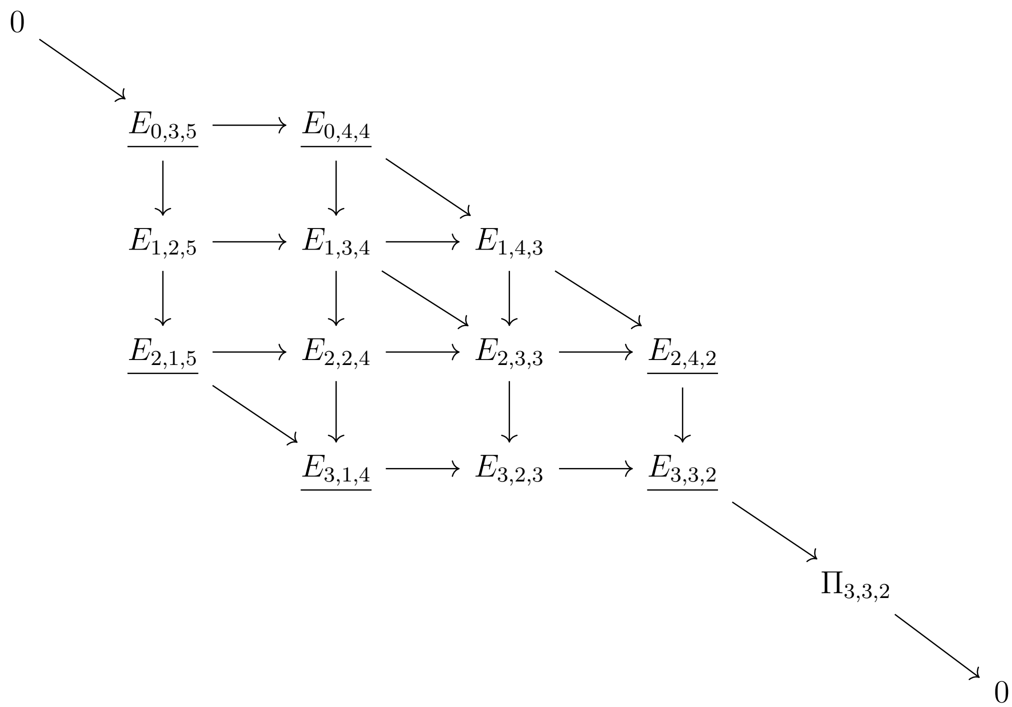
Note that the terms are precisely those in the Jacobi-Trudi determinant. In other words, the resolution looks like

$$0 \longrightarrow \bigoplus_{\ell(w)=3} E_{w\circ\lambda} \xrightarrow{\partial_3} \bigoplus_{\ell(w)=2} E_{w\circ\lambda} \xrightarrow{\partial_2} \bigoplus_{\ell(w)=1} E_{w\circ\lambda} \xrightarrow{\partial_1} E_\lambda \xrightarrow{\partial_0} \Pi_\lambda \longrightarrow 0$$

where the sums are over $w \in S_3$ and $w \circ \lambda$ denotes the affine action of the Weyl group S_3.

2.3. **The Combinatorial Diagram.** There's also a combinatorial picture for the resolution above here. Perhaps an example will illustrate this best.

2.3.1. *An Example.* Let $\lambda = (3,3,2) \vdash 8$ and consider the diagram (combinatorial rule will be described next paragraph)



then one can see that the underlined terms are precisely the terms in the resolution we desire, where we direct sum $E_{0,4,4}$ and $E_{2,1,5}$ together and direct sum $E_{2,4,2}$ and $E_{3,1,4}$ together to get the resolution. Note the structure here of two isosceles triangles of possibly different "lengths" i.e. number of segments.

Roughly, the rule here is to start on the top left with $(\lambda_3 - 2, \lambda_2, \lambda_1 + 2)$ and move either to the east or the south (or southeast when appropriate), shifting a number from the third slot to the second slot on every move to the east and shifting a number from the second slot to the first slot on every move south (a move southeast will directly shift a number of the third slot to the first slot). Do this until all six desired members of the resolution are hit, and take the convex hull.

2.3.2. *In General.* Let $[\pi(1)\pi(2)\pi(3)]$ denote the permutation $(1,2,3) \mapsto (\pi(1), \pi(2), \pi(3))$. In general, we build the combinatorial diagram as follows. We place the results of the affine Weyl actions on the weight[2] λ in the plane[3] in $\mathfrak{h}(\mathfrak{sl}_3)^*$ and add it to the diagram:

[2]Considered as a weight of $\mathfrak{sl}_3$, technically λ is considered the same up to multiples of $(1,1,1)$; but we can figure out what the original partition is given the requirement $\lambda \vdash n$.

[3]Perhaps it is more customary to place the dominant weight in the top right, but here it is in the bottom right.

$[321] \circ \lambda$	$[312] \circ \lambda$		
$[231] \circ \lambda$		$[213] \circ \lambda$	
	$[132] \circ \lambda$	$[123] \circ \lambda = \lambda$	

This is the starting point for our diagram.

Recall that the positive roots for $\mathfrak{sl}_3$ are $\alpha_1 = \phi_1 - \phi_2 = (1, -1, 0)$, $\alpha_2 = \phi_2 - \phi_3 = (0, 1, -1)$, $\alpha_3 = \phi_1 - \phi_3 = (1, 0, -1) = \alpha_1 + \alpha_2$, where ϕ_i outputs the i-th diagonal term of a matrix in $\mathfrak{sl}_3$.

Then, for every $\ell(u) = \ell(w) - 1$, consider all ordered[4] tuples $(i_j)_j$, where $i_j \in \{1, 2, 3\}$, such that

$$u \circ \lambda = w \circ \lambda + \sum_j \alpha_{i_j}.$$

For each such tuple, add the weights (ranging k from 0 to the length of the tuple)

$$w \circ \lambda + \sum_{j \leq k} \alpha_{i_j}$$

to the diagram. Pictorially, we place $w \circ \lambda + \sum_{j \leq k+1} \alpha_{i_j}$ to the south of $w \circ \lambda + \sum_{j \leq k} \alpha_{i_j}$ if $\alpha_{i_{k+1}} = \alpha_1$; east if $\alpha_{i_{k+1}} = \alpha_2$; and southeast if $\alpha_{i_{k+1}} = \alpha_3$.

Lastly, add an arrow between $w \circ \lambda + \sum_{j \leq k} \alpha_{i_j}$ and $w \circ \lambda + \sum_{j \leq k+1} \alpha_{i_j}$ (which, again, points south if $i_{k+1} = 1$; east if $i_{k+1} = 2$; and southeast if $i_{k+1} = 3$).

This completes our combinatorial diagram.

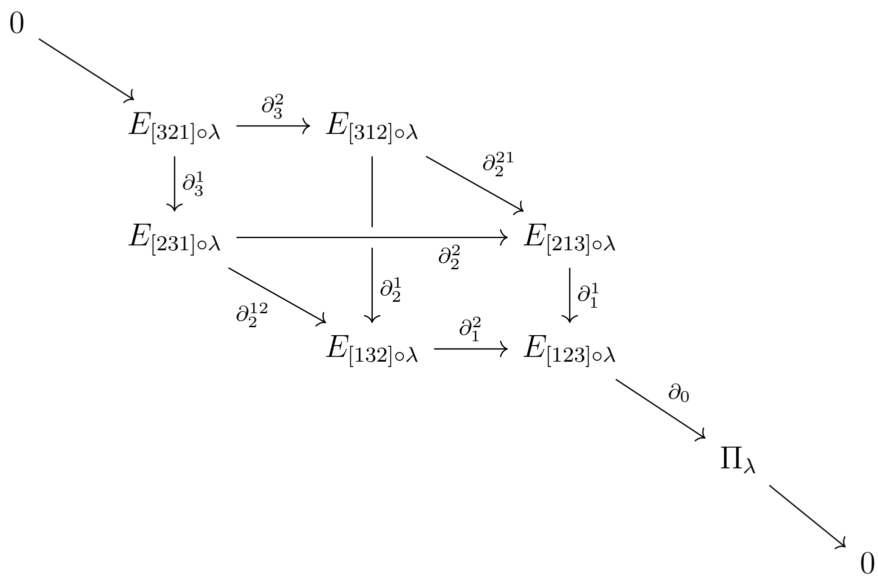
Then, replace each weight μ with $E_{\mu'}$, where $\mu' \equiv \mu$ in $\mathfrak{h}(\mathfrak{sl}_3)^*$ but we choose $\mu' \vdash n$. The arrows become maps which we describe in the next section[5].

Let us draw a general picture. We omit the non-underlined auxiliary terms. Then the resolution looks like[6]

[4]It is important that it is ordered; we consider $(1, 2, 1, 2)$ to be different from $(1, 1, 2, 2)$, even though $w \circ \lambda + \alpha_1 + \alpha_2 + \alpha_1 + \alpha_2 = w \circ \lambda + \alpha_1 + \alpha_1 + \alpha_2 + \alpha_2$.

[5]One way to think of this is as a functor from a diagram of weights to representations.

[6]The scheme for labelling is this: the subscript denotes the index of the map in the resolution, and the super script has 1 for down and 2 for right.



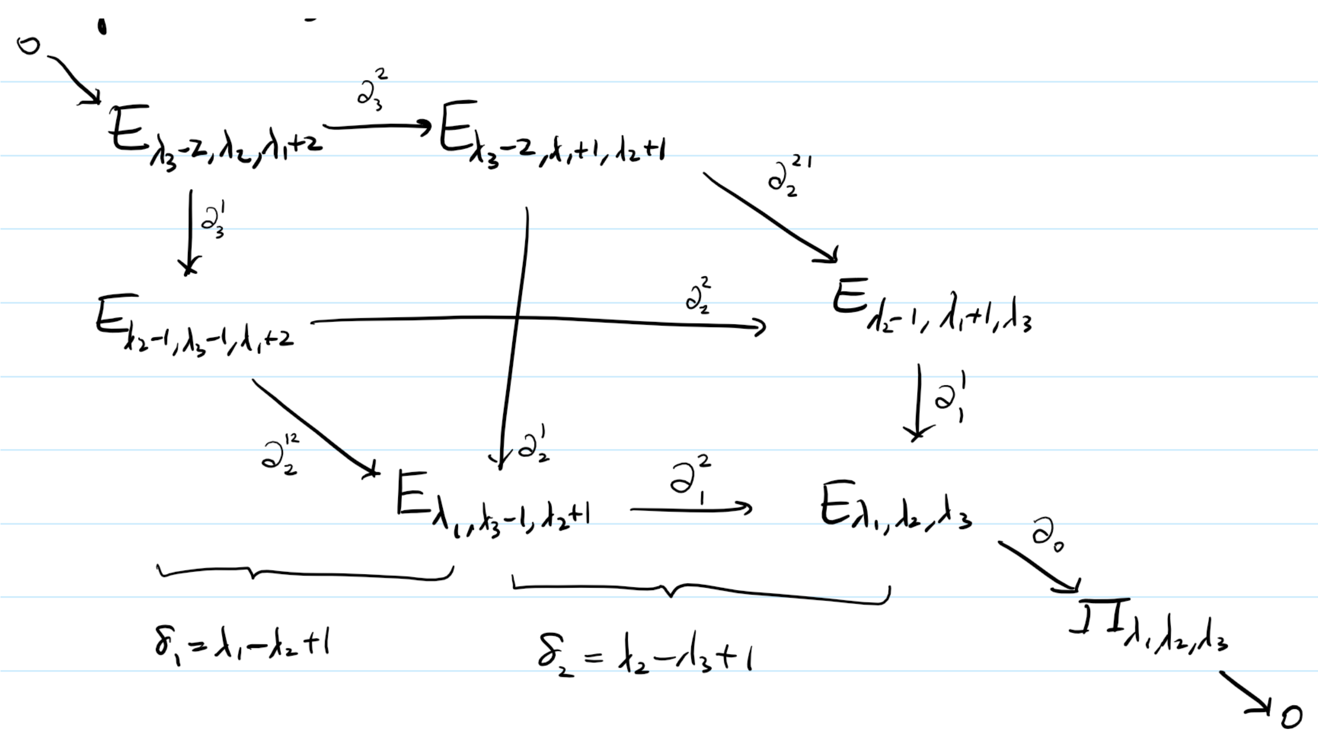
Here the number of segments in the left half of the hexagon is given by

$$\delta_1 := \lambda_1 - \lambda_2 + 1, \tag{$*$}$$

while the number of segments in the right half is given by

$$\delta_2 := \lambda_2 - \lambda_3 + 1. \tag{$*$}$$

The segments involved in this diagram are either going right, down, or diagonally right-down. Here's a handdrawn picture:



As an abuse of notation, we may also denote any of the ∂'s to be the set of paths going only right and/or down and/or diagonally right-down along segments between them. For example, we may denote ∂_3^2 to be the set of (necessarily horizontal) paths between $E_{[321]}$ and $E_{[312]}$ (of which there is only one), and ∂_2^{21} to be the set of paths along segments between $E_{[312]}$ and $E_{[213]}$ (this would be a Catalan number except diagonal paths are allowed). Similarly, $\partial_2^{21} \circ \partial_3^2$ may also denote the set of paths going from $E_{[321]}$ to $E_{[312]}$ to $E_{[213]}$. We apologize in advance for this notation.

It remains now to describe the differential maps $\partial\colon E_{w\circ\lambda} \longrightarrow E_{u\circ\lambda}$.

3. The Maps

Now that we have described the objects of this resolution and the shape of the maps between them, we must describe the actual maps.

To each segment in the diagram is associated a natural symmetrizing map which is a map of representations. For example, the map associated to

$$E_{0,3,5} \longrightarrow E_{0,4,4}$$

sends

$$\varphi_{0,3,5\to 0,4,4}\colon \begin{pmatrix} 1,2,3 \\ 4,5,6,7,8 \end{pmatrix} \longmapsto \begin{pmatrix} 1,2,3,4 \\ 5,6,7,8 \end{pmatrix} + \begin{pmatrix} 1,2,3,5 \\ 4,6,7,8 \end{pmatrix} + \begin{pmatrix} 1,2,3,6 \\ 4,5,7,8 \end{pmatrix} + \begin{pmatrix} 1,2,3,7 \\ 4,5,6,8 \end{pmatrix} + \begin{pmatrix} 1,2,3,8 \\ 4,5,6,7 \end{pmatrix}.$$

Let shift_k be an operation on tuples of numbers shifting one from the k-th spot to the $k-1$-th spot. That is,

$$\text{shift}_k\colon (a_1, \cdots, a_{k-1}, a_k, \cdots, a_m) \longmapsto (a_1, \cdots, a_{k-1}+1, a_k-1, \cdots, a_m).$$

Then the maps between E_μ and $E_{\text{shift}_k^m(\mu)}$ are given by

$$\begin{aligned} \partial_{\mu\to\text{shift}_k^m(\mu)}\colon E_\mu &\longrightarrow E_{\text{shift}_k^m(\mu)} \\ v &\longmapsto \frac{1}{m!} \bigcirc_{\mu\to\text{shift}_k^m(\mu)} \varphi(v), \end{aligned} \qquad (*)$$

where $\bigcirc_{\mu\to\text{shift}_k^m(\mu)}\varphi$ is the composition over all segments in $\mu \to \text{shift}_k^m(\mu)$ of the natural symmetrizing maps associated to each segment.

In particular, this means that for example the map between $E_{[321]}$ and $E_{[312]}$ is given by

$$\boxed{\partial_3^2 = \frac{1}{\delta_1!} \bigcirc_{\gamma\in\partial_3^2} \varphi}, \qquad (*)$$

where here in $\bigcirc_{\gamma\in\partial_3^2}\varphi$ the ∂_3^2 is the set of the singular path γ between $E_{[321]}$ and $E_{[312]}$ and the $\bigcirc\varphi$ means the composition of all natural maps associated to each segment in the path.

On the other hand, the diagonal paths are given by for example

$$\boxed{\partial_2^{21} = -\sum_{\gamma\in\partial_2^{21}} a_{\text{diag}\,\gamma} \bigcirc_\gamma \varphi}, \qquad (*)$$

where $\text{diag}\,\gamma$ denotes the number of diagonal "shortcuts" taken in the path γ. The coefficients $a_{\text{diag}\,\gamma}$ here are yet to be determined, and we will describe them next.

4. Prelude: A Quick Bit of Counting

Before we can describe the coefficients a above, let us first define some auxiliary objects.

The vector space $E_{[321]} = E_{\lambda_3-2,\lambda_2,\lambda_1+2}$ has a natural basis given by the elements of the set species $E_{\lambda_3-2}E_{\lambda_2}E_{\lambda_1+2}$. For example, the vector space $E_{1,2}$ has a natural basis $\left\{ \binom{1}{2,3}, \binom{2}{1,3}, \binom{3}{1,2} \right\}$. For a path $\gamma \in \partial_2^{21} \circ \partial_3^2$, the map

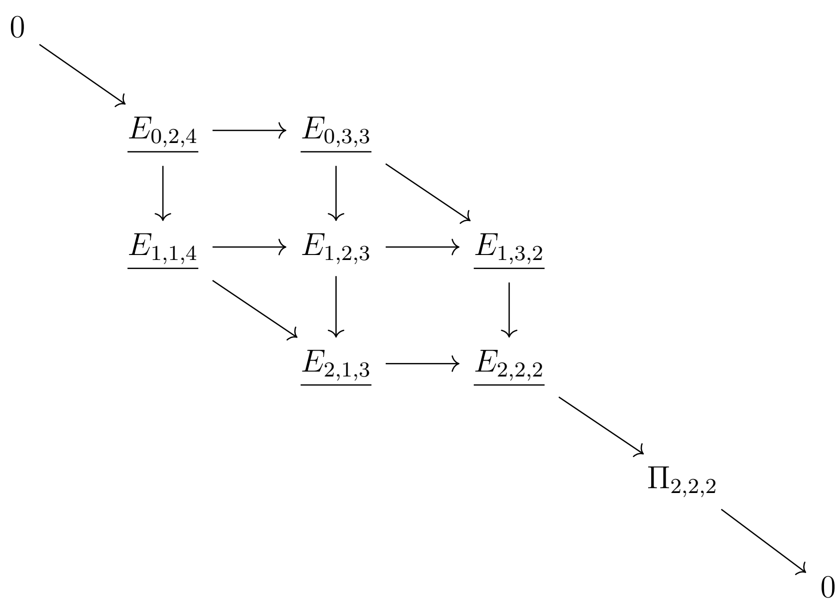
$$\frac{1}{\delta_1!} \bigcirc_{\gamma} \varphi$$

then sends each vector in this natural basis to a linear combination of similar such natural basis vectors. We can then count the number of natural basis vectors in the image with precisely i numbers moving from the third slot directly to the first slot. Let this number be aptly called

$$\# \operatorname{vec}_{\# \operatorname{imp}=i}(\gamma),$$

where a number moving from the third slot directly to the first slot is aptly called an "impurity", shortened here to imp.

Perhaps an example will illustrate things better.



Let X denote moving right, Y denote moving down, and T denote moving diagonally. Let $\gamma \in \partial_2^{21} \circ \partial_2^2$ be given by XT (read left to right); then

$$\frac{1}{\delta_1!} \bigcirc_{\gamma} \varphi \colon \begin{pmatrix} \\ 1,2 \\ 3,4,5,6 \end{pmatrix} \longmapsto \begin{pmatrix} 4 \\ 1,2,3 \\ 5,6 \end{pmatrix} + \begin{pmatrix} 5 \\ 1,2,3 \\ 4,6 \end{pmatrix} + \begin{pmatrix} 6 \\ 1,2,3 \\ 4,5 \end{pmatrix} + \begin{pmatrix} 3 \\ 1,2,4 \\ 5,6 \end{pmatrix} + \begin{pmatrix} 5 \\ 1,2,4 \\ 3,6 \end{pmatrix} + \begin{pmatrix} 6 \\ 1,2,4 \\ 3,5 \end{pmatrix}$$
$$+ \begin{pmatrix} 3 \\ 1,2,5 \\ 4,6 \end{pmatrix} + \begin{pmatrix} 4 \\ 1,2,5 \\ 3,6 \end{pmatrix} + \begin{pmatrix} 6 \\ 1,2,5 \\ 3,4 \end{pmatrix} + \begin{pmatrix} 3 \\ 1,2,6 \\ 4,5 \end{pmatrix} + \begin{pmatrix} 4 \\ 1,2,6 \\ 3,5 \end{pmatrix} + \begin{pmatrix} 5 \\ 1,2,6 \\ 3,4 \end{pmatrix},$$

so that we may see

$$\# \operatorname{vec}_{\# \operatorname{imp}=0}(\gamma) = 0,$$
$$\# \operatorname{vec}_{\# \operatorname{imp}=1}(\gamma) = 12.$$

Inspection will reveal that the reason for $\# \operatorname{vec}_{\# \operatorname{imp}=0}(\gamma) = 0$ is because we moved diagonally; moving diagonally is by definition going directly from the third slot to the first slot, and so of course we have impurities.

Let

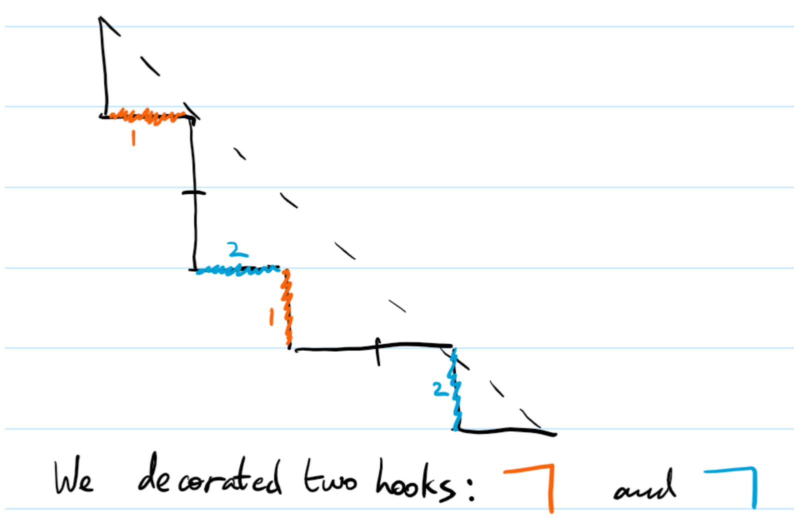
$$c_{i,d} \coloneqq \sum_{\substack{\operatorname{diag}\gamma = d \\ \gamma \in \partial_2^{21} \circ \partial_3^2}} \# \operatorname{vec}_{\# \operatorname{imp}=i}(\gamma) \tag{$*$}$$

for $d \leq i$.

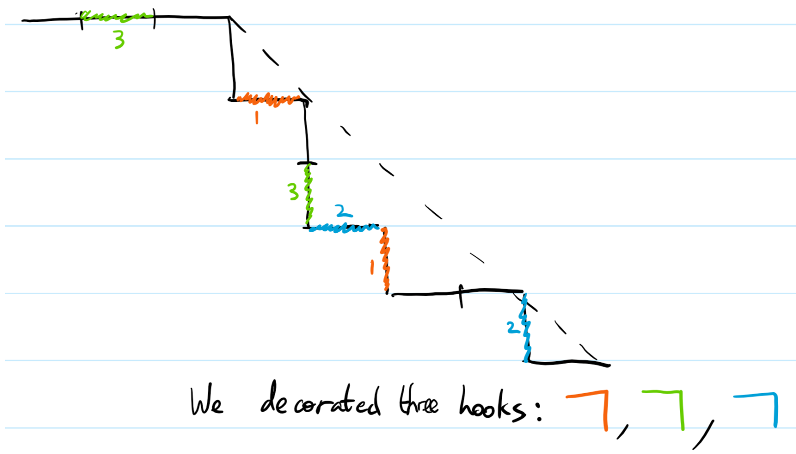
For a path $\gamma \in \partial_2^{21} \circ \partial_3^2$, we can choose to decorate certain segments. Let us decorate it in such a way that we decorate an arbitrary number of pairs of segments (one right and one down) such that the down segment always comes after the right segment in a pair. Let

$$\operatorname{hook} \gamma$$

denote the number of such pairs we decorate. Let such a pair be called a hook. Here's a drawn example for $\delta_1 = 0$:



Here's a drawn example for $\delta_1 = 3$:



Then some combinatorics[7] gives

$$c_{i,d} = \binom{\lambda_1+2}{\delta_1}\binom{\lambda_2}{\underbrace{1,\cdots,1}_{\delta_2-i},\lambda_2-\delta_2+i}\binom{\lambda_2+1}{\underbrace{1,\cdots,1}_{\delta_2},\lambda_2-\delta_2+1}\cdot\#\{\gamma\in\partial_2^{21}\circ\partial_3^2 : \operatorname{diag}\gamma = d, \operatorname{hook}\gamma = i-d\}, \tag{$*$}$$

where

$$\begin{aligned}
&\#\{\gamma\in\partial_2^{21}\circ\partial_3^2 : \operatorname{diag}\gamma = d, \operatorname{hook}\gamma = i-d\}_{\delta_1,\delta_2}\\
&= \binom{2(\delta_2-d)+d}{d}\#\{\gamma\in\partial_2^{21}\circ\partial_3^2 : \operatorname{diag}\gamma = 0, \operatorname{hook}\gamma = i-d\}_{\delta_1,\delta_2-d}\\
&= \binom{2\delta_2-d}{d}\#\{\gamma\in\partial_2^{21}\circ\partial_3^2 : \operatorname{diag}\gamma = 0, \operatorname{hook}\gamma = i-d\}_{\delta_1,\delta_2-d}\\
&= \binom{2\delta_2-d}{d}\operatorname{hcat}^{\urcorner:i-d}_{\delta_1,\delta_2-d},
\end{aligned}$$

where we have created the shorthand

$$\operatorname{hcat}^{\urcorner:h}_{\delta_1,\delta_2} := \#\{\gamma\in\partial_2^{21}\circ\partial_3^2 : \operatorname{diag}\gamma = 0, \operatorname{hook}\gamma = h\}_{\delta_1,\delta_2}. \tag{$*$}$$

Note for example that $\operatorname{hcat}^{\urcorner:0}_{\delta_1,\delta_2} = \operatorname{cat}_{\delta_2}$.

[7] Each hook is supposed to be us taking a number from the third slot and marking it for impurity i.e. moving to the first slot later, which is why we require the down move to be after the right move. Indeed, for every path γ with the prescribed number of diagonals and hooks, to count the number of vectors with i impurities, we have $\binom{\lambda_1+2}{\delta_1}$ from traversing along ∂_3^2, $\binom{\lambda_2}{\underbrace{1,\cdots,1}_{\delta_2-i},\lambda_2-\delta_2+i}$ to choose in what order should the original members of the second slot move to the first slot, and $\binom{\lambda_2+1}{\underbrace{1,\cdots,1}_{\delta_2},\lambda_2-\delta_2+1}$ to choose the order in which the required δ_2 numbers leave the last slot. Note that in this scheme we do not worry about who should the impurities be and in what order, and this is because the path γ already has decorated hooks; we simply prescribe in what orders the numbers should leave, and whichever number should get assigned to a hook will naturally become an impurity.

In conclusion,

$$c_{i,d} = \sum_{\substack{\operatorname{diag}\gamma = d \\ \gamma \in \partial_2^{21} \circ \partial_3^2}} \binom{2\delta_2 - d}{d} \operatorname{hcat}_{\delta_1, \delta_2 - d}^{\urcorner : i-d} . \qquad (*)$$

The name hcat is supposed to be suggestive: h here stands for horizontal extension. In words, the number $\operatorname{hcat}_{\delta_1,\delta_2}^{\urcorner:h}$ counts the number of Dyck paths on $2\delta_2$ segments with an extended horizontal beginning of length δ_1 such that there are h hooks.

5. The Coefficients a

We now have the pieces necessary to describe a.

For the sequence we have described to be exact, it is necessary that

$$a_0 c_{0,0} = \binom{\lambda_2}{\delta_2}\binom{\lambda_1 + 2}{\delta_1 + \delta_2},$$

$$\sum_{d=0}^{i} a_d c_{i,d} = 0 \qquad 1 \leq i \leq \delta_2. \qquad (*)$$

This is obtained by formally setting every natural basis vector to 1 and requiring $\partial\partial = 0$, i.e. by counting the number of vectors with precisely i impurities (noting that $\partial_2^2 \circ \partial_3^1$ has no impurities).

Solving this system of linear equations gives

$$a_0 = \frac{\delta_1!}{\delta_2!(\delta_1 + \delta_2)!\operatorname{cat}_{\delta_2}},$$

$$a_i = -\frac{\sum_{d=0}^{i-1} \binom{2\delta_2 - d}{d} \operatorname{hcat}_{\delta_1,\delta_2-d}^{\urcorner:i-d} a_d}{\binom{2\delta_2 - i}{i}\operatorname{cat}_{\delta_2 - i}} \qquad 1 \leq i \leq \delta_2. \qquad (*)$$

By induction, there exist κ_i numbers such that

$$a_i = a_0 \frac{\kappa_i}{\binom{2\delta_2 - i}{i}} = \frac{\delta_1!}{\delta_2!(\delta_1+\delta_2)!\operatorname{cat}_{\delta_2}} \frac{\kappa_i}{\binom{2\delta_2 - i}{i}}, \qquad (*)$$

where

$$\kappa_0 = 1.$$

By using the inductive formula above for a_i, we get an inductive formula for κ_i given by

$$\kappa_i = -\frac{\sum_{d=0}^{i-1} \operatorname{hcat}_{\delta_1,\delta_2-d}^{\urcorner:i-d} \kappa_d}{\operatorname{cat}_{\delta_2 - i}} \qquad 1 \leq i \leq \delta_2. \qquad (*)$$

Written in matrix form, this is

$$[\kappa_{\delta_2 - i}]_{i=0}^{\delta_2 - 1} = -[\operatorname{hcat}_{\delta_1, j}^{\urcorner:j-i}]_{i,j=0}^{\delta_2-1}{}^{-1} [\operatorname{hcat}_{\delta_1,\delta_2}^{\urcorner:\delta_2 - i}]_{i=0}^{\delta_2-1}. \qquad (*)$$

Induction gives

$$\kappa_i = \sum_{\mu_1=0}^{i-1} \frac{-\operatorname{hcat}_{\delta_1,\delta_2-\mu_1}^{\urcorner:i-\mu_1}}{\operatorname{cat}_{\delta_2 - i}} \cdot \sum_{\mu_2=0}^{\mu_1-1} \frac{-\operatorname{hcat}_{\delta_1,\delta_2-\mu_2}^{\urcorner:\mu_1-\mu_2}}{\operatorname{cat}_{\delta_2-\mu_1}} \cdot \sum_{\mu_3=0}^{\mu_2-1} \frac{-\operatorname{hcat}_{\delta_1,\delta_2-\mu_3}^{\urcorner:\mu_2-\mu_3}}{\operatorname{cat}_{\delta_2-\mu_2}} \cdots. \qquad (*)$$

Hence

$$\boxed{a_i = \frac{\delta_1!}{\delta_2!(\delta_1+\delta_2)!\,\mathrm{cat}_{\delta_2}\binom{2\delta_2-i}{i}} \sum_{\mu_1=0}^{i-1} \frac{-\,\mathrm{hcat}^{\urcorner:i-\mu_1}_{\delta_1,\delta_2-\mu_1}}{\mathrm{cat}_{\delta_2-i}} \cdot \sum_{\mu_2=0}^{\mu_1-1} \frac{-\,\mathrm{hcat}^{\urcorner:\mu_1-\mu_2}_{\delta_1,\delta_2-\mu_2}}{\mathrm{cat}_{\delta_2-\mu_1}} \cdot \sum_{\mu_3=0}^{\mu_2-1} \frac{-\,\mathrm{hcat}^{\urcorner:\mu_2-\mu_3}_{\delta_1,\delta_2-\mu_3}}{\mathrm{cat}_{\delta_2-\mu_2}} \cdots}. \tag{$*$}$$

This completely characterizes the a.

As an example,

$$a_0 = \frac{\delta_1!}{\delta_2!(\delta_1+\delta_2)!\,\mathrm{cat}_{\delta_2}}$$

and[8]

$$a_1 = -\frac{\delta_1!}{\delta_2!(\delta_1+\delta_2)!\,\mathrm{cat}_{\delta_2}}\left(\frac{2\delta_1\delta_2}{\delta_2+1} + \delta_2 + 1 - \frac{4^{\delta_2}}{\binom{2\delta_2}{\delta_2}}\right).$$

The complexity in the expression for a_1 and the generating functions to come later makes us suspect that no closed form is available. But first let us explain how we got this formula for a_1 in the first place, and how one could conceivably compute all the coefficients (if one really wanted to...).

6. The Punchline: Generating Functions

Let X denote moving right and undecorated, Y denote moving down and undecorated, and T denote both decorated Xs and decorated Ys. Consider the species

$$\mathrm{Cat}^{\urcorner:k}(X,Y,T)$$

of Catalan paths with precisely k hooks and

$$\mathrm{hCat}^{\urcorner:k}(X,Y,T)$$

the species of Catalan paths with horizontal extension and precisely k hooks. For example, we would have

$$\mathrm{Cat}^{\urcorner:k}(X,Y,T) = \sum_{n,h=0}^{\infty} \mathrm{hcat}^{\urcorner:h}_{0,n+h}\, X^n Y^n T^{2h}.$$

Then their generating series satisfy

$$\boxed{\mathrm{Cat}^{\urcorner:k}(X,Y,T) = \frac{\sum_{i=1}^{k-1} XY\,\mathrm{Cat}^{\urcorner:i}\,\mathrm{Cat}^{\urcorner:k-i} + \sum_{j=1}^{k}\sum_{i=0}^{k-j} j!(T\partial_Y)^{\circ j}(XY\,\mathrm{Cat}^{\urcorner:i})(T\partial_X)^{\circ j}(\mathrm{Cat}^{\urcorner:k-j-i})}{1-2XY\,\mathrm{Cat}}} \tag{$*$}$$

and

$$\boxed{\mathrm{hCat}^{\urcorner:k}(X,Y,T) = \frac{\sum_{i=1}^{k} XY\,\mathrm{Cat}^{\urcorner:i}\,\mathrm{hCat}^{\urcorner:k-i} + \sum_{j=1}^{k}\sum_{i=0}^{k-j} j!(T\partial_Y)^{\circ j}(XY\,\mathrm{Cat}^{\urcorner:i})(T\partial_X)^{\circ j}(\mathrm{hCat}^{\urcorner:k-j-i})}{1-XY\,\mathrm{Cat}}}. \tag{$*$}$$

[8]Remark: I have found that a_1 here is associated to `https://oeis.org/search?q=1%2C8%2C47%2C244&sort=&language=english&go=Search`, which referenced the papers `https://www.combinatorics.org/ojs/index.php/eljc/article/view/v19i1p62` and `https://arxiv.org/pdf/1601.07988.pdf`. I am not sure what the combinatorial ramifications of this are.

Of course here

$$\mathrm{Cat}^{\urcorner:0}(X,Y,T) = \mathrm{Cat}(X,Y) = \frac{1-\sqrt{1-4XY}}{2XY}.$$

Notice that by computing the generating functions for every value of k between 0 and what we need, the above two equations allow us to find all the hcats we need, since the recursion for $\mathrm{Cat}^{\urcorner:k}$ never calls k itself. For example, a_1 above was computed in this way.

The above starred expressions for the generating series are obtained in a similar way to the recursion for the Catalan generating function. In short, every path we are concerned with can be broken off at its last contact point with the main diagonal, and we can do casework based on the number of hooks broken apart by this contact point.

To some degree (assuming one is willing to sit down and compute some generating functions), we have finished our description of our complex. The our claim is that

Conjecture. *The complex*

$$0 \longrightarrow E_{\lambda_3-2,\lambda_2,\lambda_1+2} \xrightarrow{\partial_3} E_{\lambda_3-2,\lambda_1+1,\lambda_2+1} \oplus E_{\lambda_2-1,\lambda_3-1,\lambda_1+2} \longrightarrow$$
$$\xrightarrow{\partial_2} E_{\lambda_1,\lambda_3-1,\lambda_2+1} \oplus E_{\lambda_2-1,\lambda_1+1,\lambda_3} \xrightarrow{\partial_1} E_{\lambda_1,\lambda_2,\lambda_3} \xrightarrow{\partial_0} \Pi_{\lambda_1,\lambda_2,\lambda_3} \longrightarrow 0$$

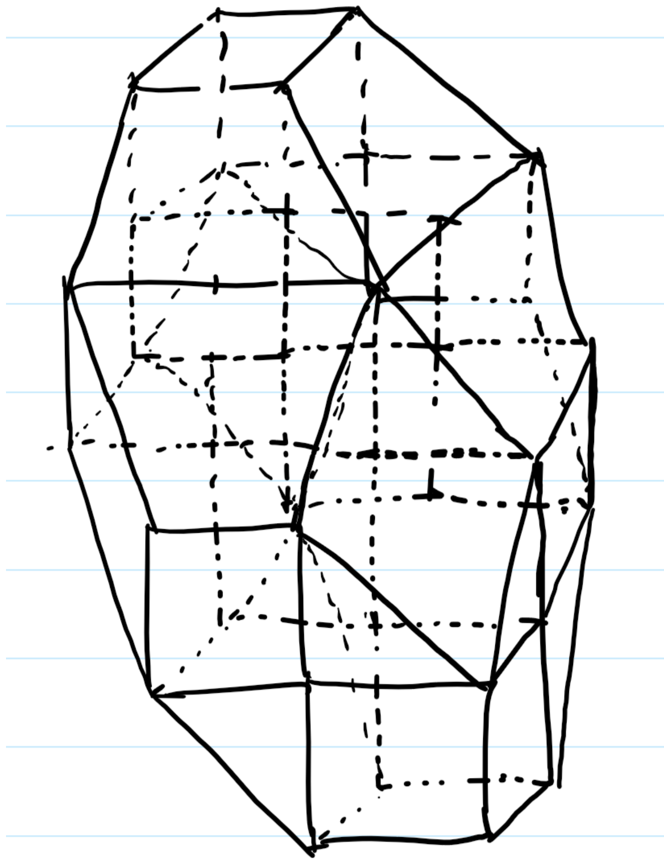
or more compactly

$$0 \longrightarrow \bigoplus_{\ell(w)=3} E_{w\circ\lambda} \xrightarrow{\partial_3} \bigoplus_{\ell(w)=2} E_{w\circ\lambda} \xrightarrow{\partial_2} \bigoplus_{\ell(w)=1} E_{w\circ\lambda} \xrightarrow{\partial_1} E_\lambda \xrightarrow{\partial_0} \Pi_\lambda \longrightarrow 0$$

is exact, where the differential maps are as prescribed in our work above.

7. Towards Higher Dimensions

We attempted to see what this resolution we propose would look like in three dimensions (i.e. for 4-part partitions), and we see that instead of a simple diamond/crystal shape, the shape of the resolution looks like



We conjecture that the choice of signs associated to each map in the resolution to make sure that every diamond has a plus and minus sign cancelling out (i.e. necessary for exactness) is given by

$$(-1)^{\text{dim of the convex hull of the vertices involved}}.$$

We have not explored this very much, but it seems to hold for the partition $(1,1,1,1) \vdash 4$. If our idea is correct, the coefficients in this case would roughly be given by some higher-dimensional modification of Catalan numbers.

This shape (or at least something very similar to it) is named the "permutohedron". When grading this permutohedron by the number of inversions, the permutohedron for S_k can be formally written as repeated tensor products $(0,1) \otimes (0,1,2) \otimes \cdots \otimes (0,1,\cdots,k-1)$. This appear to be a formal consequence of the combinatorial identity

$$\sum_{\sigma \in S_n} q^{l(\sigma)} = 1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1});$$

perhaps this is useful somehow. Also perhaps this can be categorified via the permutohedron somehow if we pick the right spaces to put on the vertices.

8. References

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