A novel solution to the probleme des menages, feat. Catalan

by **AstrapiGnosis**, Jul 17, 2017, 11:16 PM

We present a novel solution to the problème des ménages using the theory of species, which brings to light some of the relationships between the Catalan numbers and the ménages problem ^ .

The problème des ménages asks for the number of seatings of n couples around a circular table so that in addition to an alternating male-female pattern, no man sits next to his wife. A similar, linear version of the question asks for the number of seatings in a straight line. We attack the latter question first.

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Linear Case:

We firstly seat the n females, then seat the males; the problem then becomes counting the number of permutations σ of $[n]$ so that $\sigma(i) \neq i, i+1$ for $1 \leq i \leq n-1$ and $\sigma(n) \neq n$ (we will call this "the good condition"). Multiplying this by $n!$ for the number of ways there were to sit the females to begin with will give us our answer.

Setup Outline: Since the good condition in this problem is fundamentally dependent on the labels, we recast this to an unlabeled problem. We represent a permutation on $[n]$ as follows: Consider a line of dots. The first dot will correspond to 1, the second one to 2, etc., but it is important to note the distinction between this correspondence and labels - the line of dots is unlabeled. For $i < j$, we represent $\sigma(i) = j$ as an arrow going from the i-th dot (henceforth referred to as simply i) to the j-th dot above the line of dots, and for $j < i$ we represent $\sigma(i) = j$ as an arrow from i to j going below the line of dots (we represent $\sigma(i) = i$ as simply i going to itself). In other words, arrows pointing to the right arc above the line of dots while arrows pointing to the left go below it. Naturally every dot has an indegree and an outdegree of 1 (since it's a permutation we're supposed to be representing here). In this scheme, we seek the number of configurations such that no dot has an arrow going to itself and so that no dot has an arrow arching above the line to the dot to the immediate right.

Terminology: We define some terminology. Given an horizontal line of dots, we create a partial order on the dots so that "dot d_1 is not to the right of dot d_2" $\Leftrightarrow$ $d_1 \leq d_2$, with equality if d_1 and d_2 are the same dot. Recall that we can represent any permutation as a set of oriented cycles; when we impose any permutation onto this line of dots, it will still maintain this property. Let us call these cycles on the line of dots "*loops*"; note that, after recasting onto the line of dots, these cycles-turned-loops may be "*tangled*", i.e. their arrows may cross each other. For any given dot d, if i is the dot whose arrow points towards d, then we call i "*d's input*"; similarly, if o is the dot to whom d points, we call o "*d's output*". Consider a set of loops, each with a left-most dot l_i - the loop to whom $\min\{l_i\}$ belongs is called the "*left-most loop*". Similarly, consider a set of loops, each with a right-most dot r_i - the loop to whom $\max\{r_i\}$ belongs is called the "*right-most loop*". We will call a dot with an arrow pointing to itself a "*fixed point*", a dot with an arrow pointing directly to the dot to the immediate right a "*bridge*"; we will call "$\sigma(i) \neq i, i+1$ for $1 \leq i \leq n-1$ and $\sigma(n) \neq n$", or equivalently the lack of fixed points and bridges, the "*good condition*"

General Motivation: We attempt to hide/collapse all the violations of the good condition in any permutation by combining them into a single dot via composition.

Species: General notation conventions dictate that type generating series (a.k.a. unlabeled generating series) ought to have a tilda over them so as to distinguish them from exponential generating series; however, for the sake of convenience, and since we don't really utilize exponential generating series here, we will simply refer to the type generating series of a species F as $F(x)$.
We let Per be the typical species of permutations (i.e. a set of oriented cycles) with cycle index series $Z_{\mathrm{Per}}(x_1, x_2, x_3, \cdots) = \dfrac{1}{(1-x_1)(1-x_2)(1-x_3)\cdots}$ and LinOrd be the typical species of linear orderings
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with cycle index series $Z_{\text{LinOrd}}(x_1, x_2, x_3, \cdots) = \frac{1}{1 - x_1}$.
The species Tan, loosely speaking, will refer to the species of representations of permutations as loops on a line of dots. More rigorously, we define the species Tan (standing for Tangles, since the loops on the line of dots seem reminiscent of tangled yarn) to be the species of digraphs on a linear order of vertices such that each vertex has an indegree and outdegree of 1. In other words, Tan is a Cartesian product between Per and LinOrd, or $\text{Tan} = \text{Per} \times \text{LinOrd}$. Note that the type generating series of Tan is $\text{Tan}(x) = \sum_{n=0}^{\infty} n!x^n$.
We define the species LinMen (standing for Linear Ménages) as the subspecies of Tan with the additional property that there are no fixed points and no bridges. Note that the number of unlabeled LinMen structures on $[n]$ is the answer we seek; in other words, we seek $[x^n]\text{LinMen}(x)$.

Consider the subspecies Cat ^ of Tan such that (1). no arrow crosses another (in other words the loops are not tangled) and (2). in any loop, the right-most dot points directly to the left-most dot.

Claim: Letting X be the species of dots, Cat satisfies

$$\text{Cat} = X \cdot \text{Cat} \cdot \text{Cat} + 1.$$

Proof: First we give construction. Given two Cats (by this we mean two structures which are members of the Cat species) (it is important to note that these Cats may be empty structures, which is good) and an X so that the X is in between the Cats like so (we will name them Cat_1 and Cat_2 to differentiate between the one on the left and the one on the right, but they are both members of the same species):

Cat_1 $\quad X \quad$ Cat_2

we may connect X to the right-most dot of the left-most loop of Cat_1 as follows: we know by definition that the right-most dot r points directly to the left-most dot l, so we delete the arrow between r and l, draw an arrow pointing from r to X, and draw an arrow pointing from X to l. This operation gives rise to another structure, which is also a Cat because: (1). it is easy to see that the new structure is still not tangled (the new arrows will reach over any other loops instead of tangling with them) and (2). since X is now the new right-most dot of the left-most loop and since we drew an arrow pointing from X to l, the right-most dot still points directly to the left-most dot; all other loops of Cat_1 are clearly left unchanged by this operation and thus still maintain this right-left connection. So this operation will take two Cats and an X and give us a new Cat.
Next we give deconstruction. Given a Cat, we can take the right-most dot r (with input i and output l, which is the left-most dot by definition) of the left-most loop, delete the arrow going from i to r, delete the arrow going from r to l, and draw an arrow going from i to l (which will maintain the right-left connection condition). Again, this process clearly does not introduce tangles. r is now our singleton X from before, and we have broken down a Cat into two smaller Cats and a singleton. So the construction operation is unique, and thus $\text{Cat} \approx X \cdot \text{Cat} \cdot \text{Cat}$; we need the $+1$ to account for the empty structure. ■

Remark: Note that this means $Z_{\text{Cat}}(x_1, x_2, x_3, \cdots) = \frac{1 - \sqrt{1 - 4x_1}}{2x_1}$ and, of course, $\text{Cat}(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$.

Another humorous remark: note that Cat is almost literally the embodiment of violating the good condition. We can almost think of a Cat as a set of bubbles of violations.

Now, given any permutation on the dots, we can collapse all the violations of the good condition into one single dot as follows:

Claim (First Main Result):

$$\big(\mathrm{LinMen} \circ (\mathrm{Cat} - 1)\big) \cdot \mathrm{Cat} = \mathrm{Tan},$$

or

$$\boxed{\big(\mathrm{LinMen} \circ (\mathrm{Cat} - 1)\big) \cdot \mathrm{Cat} = \mathrm{Per} \times \mathrm{LinOrd}},$$

where $\times$ refers to the Cartesian product.

Proof: First we give construction. We compose LinMen with $\mathrm{Cat} - 1$ (-1 because we want to avoid the empty structure, as compositions aren't well-defined for empty structures) as follows: given a dot d with input i and output o in the underlying set of LinMen, we delete the arrows into and out of d, inflate d into a $\mathrm{Cat} - 1$, delete the arrow going from the right-most dot of the right-most loop of $\mathrm{Cat} - 1$ to the left-most dot of the same loop, draw an arrow going from i to the left-most dot of the right-most loop of $\mathrm{Cat} - 1$, and draw an arrow going from the right-most dot of the right-most loop of $\mathrm{Cat} - 1$ to o (note that this maintains that each dot still has an indegree and outdegree of 1). Now that we have $\mathrm{LinMen} \circ (\mathrm{Cat} - 1)$, we stick on a $\cdot\mathrm{Cat}$ (possible empty!) at the end because this construction for composition does not account for a separate, non-tangled Cat at the very right. Since each dot still has an indegree and outdegree of 1, the resulting structure is a Tan.

Next we give deconstruction ^ . We define a continuous violation of the good condition to be any continuous sequence of bridges interspersed only by Cat structures (recall that a fixed point is also a Cat structure). Given a Tan, we take each continuous violation with left-most dot l_i and right-most dot r_i, notice that by connecting l_i and r_i we arrive at a larger Cat structure, delete the arrow going from the input i_i ^ of l_i to l_i, collapse the entirety of the continuous violation into r_i, and draw an arrow going from i_i to r_i. If there are no bridges to preamble one of the Cat structures in Tan, we tag that Cat structure onto a continuous violation to the immediate right; that is, we will collapse that Cat together with the continuous violation to the right. If there are no continuous violations to the immediate right, we take the dot to the immediate right, think of it as a Cat structure on one dot, and collapse those two Cat structures together. If there is a disconnected Cat structure to the right of everything else, we simply leave it alone. In this way, we have hidden all violations (with the possible exception of the Cat to the right), and the resulting structure is thus a LinMen structure to the left of a Cat structure. So the construction operation is unique, as desired. ■

Now that we have the species result, we may translate to algebra.

Algebra: Again, these are type generating series we are dealing with here, but we drop the tildas for ease of notation.

Recall the definition of the cycle index series:

$$Z_F(x_1, x_2, x_3, \cdots) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in S_n} |\mathrm{Fix}(F[\sigma])| {x_1}^{\sigma_1} {x_2}^{\sigma_2} {x_3}^{\sigma_3} \cdots$$

where $|\mathrm{Fix}(F[\sigma])|$ means the number of F-structures that are left unchanged (fixed) by relabeling according to σ and σ_i refers to the number of cycles of length i in the cycle representation of σ. Consider the species LinMen (recall that LinMen is defined as a species of digraphs on a linear order of vertices such that each vertex has an indegree and outdegree of 1 which in addition has no fixed points and no bridges); when labeled, it amounts to a simple attachment of superficial labels to the unlabeled graph, which has a pre-ordained structure set upon it by the linear order (note that this means the number of labeled LinMen structures on $[n]$ is simply $n!$ times the number of unlabeled LinMen structures). Because there is already a linear order in place, the labels don't really affect the fundamental structure of the graph, so when we non-trivially permute these labels the labeled graph is never fixed. As a result, for any non-identity $\sigma \in S_n$, or in other words for any $\sigma \in S_n$ for which $\sigma_1 \neq n$, $|\mathrm{Fix}(\mathrm{LinMen}[\sigma])| = 0$. Thus the only non-zero terms in Z_{LinMen} will depend only on x_1, as all other variables are raised to the 0-th power; this means that $Z_{\mathrm{LinMen}}(x_1, x_2, x_3, \cdots) = Z_{\mathrm{LinMen}}(x_1, 0, 0, \cdots) = \mathrm{LinMen}(x)$. We could also have arrived at this conclusion by recalling that $\mathrm{LinMen} \in \mathrm{Tan} = \mathrm{Per} \times \mathrm{LinOrd}$; since Z_{LinOrd}

depends only on x_1, even though Z_{Per} depends on all the x_is, the Hadamard product of the two power series will eliminate all terms of Z_{Per} which have $x_2, x_3, \cdots$s in them, since

$$\left(\sum_{\vec{n}} a_{\vec{n}} \frac{\vec{x}^{\vec{n}}}{1^{n_1} n_1! 2^{n_2} n_2! 3^{n_3} n_3! \cdots}\right) \times \left(\sum_{\vec{n}} b_{\vec{n}} \frac{\vec{x}^{\vec{n}}}{1^{n_1} n_1! 2^{n_2} n_2! 3^{n_3} n_3! \cdots}\right) = \sum_{\vec{n}} a_{\vec{n}} b_{\vec{n}} \frac{\vec{x}^{\vec{n}}}{1^{n_1} n_1! 2^{n_2} n_2! 3^{n_3} n_3! \cdots};$$

and since LinMen is strictly non-virtual, if Z_{Tan} depends only on x_1, then so does Z_{LinMen}.

Remark: note that, since the only terms of Z_{Per} which matter during a Hadamard product are $\frac{1}{1-x_1}$, and since $Z_{\mathrm{LinOrd}} = \frac{1}{1-x_1}$,

$$Z_{\mathrm{Tan}} = \left(\frac{1}{1-x_1}\right) \times \left(\frac{1}{1-x_1}\right) = \left(\sum_{i=0}^{\infty} i! \frac{{x_1}^i}{i!}\right) \times \left(\sum_{i=0}^{\infty} i! \frac{{x_1}^i}{i!}\right) = \sum_{n=0}^{\infty} n!^2 \frac{{x_1}^n}{n!},$$

which means $\mathrm{Tan}(x) = \sum_{n=0}^{\infty} n! x^n$, as expected.

Recall that, in the unlabeled scheme, substitution works as follows:

$$(F \circ G)(x) = Z_F\Big(G(x), G(x^2), G(x^3), \cdots\Big),$$

meaning that

$$\Big(\mathrm{LinMen} \circ (\mathrm{Cat}-1)\Big) = Z_{\mathrm{LinMen}}\Big(\mathrm{Cat}(x)-1, \mathrm{Cat}(x^2)-1, \mathrm{Cat}(x^3)-1, \cdots\Big) = Z_{\mathrm{LinMen}}\Big(\mathrm{Cat}(x)-1\Big) = \mathrm{LinMen}(\mathrm{Cat}(x)-1).$$

Now we can finally use the species result from earlier. We know that $\Big(\mathrm{LinMen} \circ (\mathrm{Cat}-1)\Big) \cdot \mathrm{Cat} = \mathrm{Tan}$, which means

$$\mathrm{LinMen}(\mathrm{Cat}(x)-1) \cdot \mathrm{Cat}(x) = \mathrm{Tan}(x).$$

Letting $\mathrm{LinMen}(x) = \sum_{i=0}^{\infty} m_i x^i$ and making the substitution $y = \frac{1-\sqrt{1-4x}}{2x} - 1 \iff x = \frac{y}{(1+y)^2}$, we get:

$$\begin{aligned}
\mathrm{LinMen}(y) \cdot (y+1) &= \mathrm{Tan}\left(\frac{y}{(1+y)^2}\right) \\
\mathrm{LinMen}(y) &= \frac{1}{1+y} \mathrm{Tan}\left(\frac{y}{(1+y)^2}\right) \\
&= \frac{1}{y} \frac{y}{1+y} \mathrm{Tan}\left(\frac{\left(\frac{y}{1+y}\right)^2}{y}\right) \\
&= \frac{1}{y}\left(y - y^2 + y^3 - y^4 + \cdots\right) \sum_{i=0}^{\infty} i! \frac{\left(y - y^2 + y^3 - y^4 + \cdots\right)^{2i}}{y^i} \\
&= \sum_{i=0}^{\infty} i! \frac{\left(y - y^2 + y^3 - y^4 + \cdots\right)^{2i+1}}{y^{i+1}}
\end{aligned}$$

For a given term, $i!\frac{\left(y-y^2+y^3-y^4+\cdots\right)^{2i+1}}{y^{i+1}}$, note that

$$\begin{aligned}
[y^n]i!\frac{\left(y-y^2+y^3-y^4+\cdots\right)^{2i+1}}{y^{i+1}} &= [y^{n+i+1}]i!\left(y-y^2+y^3-y^4+\cdots\right)^{2i+1} \\
&= (-1)^{n+i+1}i![y^{n+i+1}]\left(-y-y^2-y^3-y^4-\cdots\right)^{2i+1} \\
&= (-1)^{n+i}i![y^{n+i+1}]\left(+y+y^2+y^3+y^4+\cdots\right)^{2i+1} \\
&= (-1)^{n+i}i!\binom{(n+i+1)-(2i+1)+(2i+1)-1}{(2i+1)-1} \\
&= (-1)^{n+i}i!\binom{n+i}{2i}
\end{aligned}$$

since $[y^{n+i+1}]\left(y+y^2+y^3+y^4+\cdots\right)^{2i+1}$ is equivalent to decompositions (ordered partitions) of $n+i+1$ into $2i+1$ positive parts. This means that

$$\begin{aligned}
[y^n]\mathrm{LinMen}(y) = m_n &= [y^n]\sum_{i=0}^{\infty} i!\frac{\left(y-y^2+y^3-y^4+\cdots\right)^{2i+1}}{y^{i+1}} \\
&= \sum_{i=0}^{\infty}[y^n]i!\frac{\left(y-y^2+y^3-y^4+\cdots\right)^{2i+1}}{y^{i+1}} \\
&= \sum_{i=0}^{\infty}(-1)^{n+i}i!\binom{n+i}{2i} \\
&= \sum_{i=0}^{n}(-1)^{n+i}i!\binom{n+i}{2i} \\
&\Longrightarrow_{\text{substitute } n-i=k} \sum_{k=0}^{n}(-1)^k(n-k)!\binom{2n-k}{2n-2k}
\end{aligned}$$

$$[y^n]\mathrm{LinMen}(y) = |\mathrm{LinMen}_n|/\sim = \boxed{\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}(n-k)!},$$

as we could have verified using the principle of inclusion and exclusion in, like, ten seconds.

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Cyclic Case (the typical problème des ménages):

Again, we reduce the problem to counting permutations on $[n]$ such that $\sigma(i)\not\equiv i, i+1 \mod n \ \ \forall i$.

We proceed in essentially the same manner as before, with a few modifications: we define the "*egregious condition*" to be "$\sigma(i)\not\equiv i, i+1 \mod n$", a "*ditch*" to be a connection from the right-most dot to the left-most dot of a loop, and a "*graveyard*" to be a connection from the global right-most dot to the global left-most dot.

Define the species Bad to be a subspecies of LinMen (from above) which furthermore has an arrow pointing from its right-most dot to its left-most dot (note that this is not just within one loop, but globally across the entire structure), and define the species CycMen to be a subspecies of LinMen which furthermore does not have a graveyard.
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Clearly $\mathrm{Bad} + \mathrm{CycMen} = \mathrm{Men}$, so CycMen will follow if we can find Bad. Define the species $\mathrm{Tan}_{n\to 1}$ to be the subspecies of Tan which furthermore has an arrow pointing from its right-most dot to its left-most dot. Since $\mathrm{Tan}_{n\to 1}$ is equivalently the species of permutations for which $\sigma(n) = 1$, we see that its type generating series is

$$\mathrm{Tan}_{n\to 1}(x) = \sum_{n=1}^{\infty}(n-1)!x^n.$$

Claim (Second Main Result):

$$\mathrm{Bad} \circ (\mathrm{Cat} - 1) = \mathrm{Cat} \cdot \mathrm{Tan}_{n\to 1},$$

or

$$\boxed{\big(\mathrm{LinMen} - \mathrm{CycMen}\big) \circ (\mathrm{Cat} - 1) = \mathrm{Cat} \cdot \Big(\big((\mathrm{Per}^{\bullet} + \mathrm{Per}) \times \mathrm{LinOrd}\big) \cdot X \cdot X + X\Big)},$$

where $^{\bullet}$ refers to the pointing operation, X is again the species of dots, and $\times$ is again the Cartesian product.

Proof: First we give construction. We compose Bad with $\mathrm{Cat} - 1$ just like we did LinMen with $\mathrm{Cat} - 1$ before ^ (put tersely, connect to the left-most dot of the right-most loop). This will result in a $\mathrm{Tan}_{n\to 1}$ structure except in the case where the left-most dot of Bad was used to compose with a disconnected $\mathrm{Cat} - 1$ structure, which would result in a smaller $\mathrm{Cat} - 1$ hanging over the $\mathrm{Tan}_{n\to 1}$ since we connected only to the right-most loop of the original $\mathrm{Cat} - 1$. Thus this operation will result in a Cat structure (possible empty) to the left of a $\mathrm{Tan}_{n\to 1}$ structure.

Next we give deconstruction. Given a Cat structure to the left of a $\mathrm{Tan}_{n\to 1}$ structure, we collapse violations of the good condition (not the egregious, but the good) exactly as before; most importantly, we collapse the Cat structure to the left of $\mathrm{Tan}_{n\to 1}$ as before, by collapsing it together with the left-most continuous violation of $\mathrm{Tan}_{n\to 1}$ (possibly a singleton). Since this is reversible, as before, our construction is unique, as desired.

Before we conclude the proof, we should probably explain why

$$\mathrm{Tan}_{n\to 1} = \big((\mathrm{Per}^{\bullet} + \mathrm{Per}) \times \mathrm{LinOrd}\big) \cdot X \cdot X + X.$$

To see this, note that given an underlying set U we can first choose two vertices to be the first and last dots, hence the $\cdot X \cdot X$ on the right; then, to find where in the Per these two vertices will go, we can either distinguish a vertex in Per for the two vertices to go after, or not distinguish anything in Per at all and simply let the two vertices form a cycle of their own, hence the $\mathrm{Per}^{\bullet} + \mathrm{Per}$; lastly, we take the Cartesian product of $\mathrm{Per}^{\bullet} + \mathrm{Per}$ with LinOrd because we impose a linear order on all the dots except the two vertices, which are the first and last dots, who must of course be the first and last dots, hence the $\times\mathrm{LinOrd}$, which $\cdot X \cdot X$ is outside of; at the very end of it all we need to account for the edge case in which the first dot is the last dot, hence the $+X$ at the very end. ■

Remark:

$$\begin{aligned}
Z_{\mathrm{Tan}_{n\to 1}} &= Z_{(\mathrm{Per}^\bullet+\mathrm{Per})\times\mathrm{LinOrd}}\cdot x_1\cdot x_1 + x_1\\
&= {x_1}^2\left(Z_{\mathrm{Per}^\bullet+\mathrm{Per}}\times Z_{\mathrm{LinOrd}}\right)+x_1\\
&= {x_1}^2\left(\left(x_1\frac{\partial}{\partial x_1}Z_{\mathrm{Per}}+Z_{\mathrm{Per}}\right)\times Z_{\mathrm{LinOrd}}\right)+x_1\\
&= {x_1}^2\left(\left(x_1\frac{\partial}{\partial x_1}\left(\frac{1}{1-x_1}+\mathrm{error}\right)+\left(\frac{1}{1-x_1}+\mathrm{error}\right)\right)\times\frac{1}{1-x_1}\right)+x_1\\
&= {x_1}^2\left(\left(\frac{x_1}{(1-x_1)^2}+\frac{1}{1-x_1}\right)\times\frac{1}{1-x_1}\right)+x_1\\
&= {x_1}^2\left(\left(\sum_{i=0}^\infty i\cdot i!\frac{{x_1}^i}{i!}+\sum_{i=0}^\infty i!\frac{{x_1}^i}{i!}\right)\times\frac{1}{1-x_1}\right)+x_1\\
&= {x_1}^2\left(\left(\sum_{i=0}^\infty (i+1)i!\frac{{x_1}^i}{i!}\right)\times\left(\sum_{i=0}^\infty i!\frac{{x_1}^i}{i!}\right)\right)+x_1\\
&= {x_1}^2\left(\sum_{i=0}^\infty (i+1)i!^2\frac{{x_1}^i}{i!}\right)+x_1\\
&= \sum_{i=0}^\infty (i+1)!i!\frac{{x_1}^{i+2}}{i!}+x_1\\
&= \sum_{n=1}^\infty (n-1)!n!\frac{{x_1}^n}{n!},
\end{aligned}$$

where the error terms die off because we are taking a Hadamard product with a series which is dependent only on x_1. This cycle index series makes sense, as it implies $\mathrm{Tan}_{n\to 1}(x)=\sum_{n=1}^\infty (n-1)!x^n$.

Algebra: Again, we drop the tildas.

Since the cycle index series of LinMen was a series only in x_1, and since Bad and CycMen are strictly non-virtual species which sum up to LinMen, the cycle index series of Bad must also be a series in only x_1, which again allows for exceedingly pleasant compositions. After the same argument as before for why composition is nice, we arrive at

$$\mathrm{Bad}(\mathrm{Cat}(x)-1)=\mathrm{Cat}(x)\cdot\mathrm{Tan}_{n\to 1}(x).$$

Letting $\mathrm{Bad}(x)=\sum_{i=0}^\infty b_i x^i$ and making the substitution $y=\frac{1-\sqrt{1-4x}}{2x}-1\iff x=\frac{y}{(1+y)^2}$, we get:

$$\begin{aligned}
\mathrm{Bad}(y) &= (y+1)\mathrm{Tan}_{n\to 1}\left(\frac{y}{(1+y)^2}\right)\\
&= (y+1)\mathrm{Tan}_{n\to 1}\left(\frac{\left(\frac{y}{1+y}\right)^2}{y}\right)\\
&= (y+1)\sum_{i=1}^\infty (i-1)!\frac{\left(y-y^2+y^3-y^4+\cdots\right)^{2i}}{y^i}
\end{aligned}$$

For a given term, $(i-1)!\frac{\left(y-y^2+y^3-y^4+\cdots\right)^{2i}}{y^i}$, note that

$$\begin{aligned}[y^n](i-1)!\frac{\left(y-y^2+y^3-y^4+\cdots\right)^{2i}}{y^i} &= [y^{n+i}](i-1)!\left(y-y^2+y^3-y^4+\cdots\right)^{2i} \\ &= (-1)^{n+i}(i-1)![y^{n+i}]\left(-y-y^2-y^3-y^4-\cdots\right)^{2i} \\ &= (-1)^{n+i}(i-1)![y^{n+i}]\left(y+y^2+y^3+y^4+\cdots\right)^{2i} \\ &= (-1)^{n+i}(i-1)!\binom{(n+i)-(2i)+(2i)-1}{(2i)-1} \\ &= (-1)^{n+i}(i-1)!\binom{n+i-1}{2i-1}\end{aligned}$$

With the same ordered partition reasoning as before for the algebra. This means that

$$\begin{aligned}[y^n]\mathrm{Bad}(y) = b_n &= [y^n](y+1)\sum_{i=1}^{\infty}(i-1)!\frac{\left(y-y^2+y^3-y^4+\cdots\right)^{2i}}{y^i} \\ &= \sum_{i=1}^{\infty}[y^{n-1}](i-1)!\frac{\left(y-y^2+y^3-y^4+\cdots\right)^{2i}}{y^i} \\ &\quad+\sum_{i=1}^{\infty}[y^{n}](i-1)!\frac{\left(y-y^2+y^3-y^4+\cdots\right)^{2i}}{y^i} \\ &= \sum_{i=1}^{\infty}(-1)^{n+i-1}(i-1)!\binom{n+i-2}{2i-1} \\ &\quad+\sum_{i=1}^{\infty}(-1)^{n+i}(i-1)!\binom{n+i-1}{2i-1} \\ &= \sum_{i=1}^{n-1}(-1)^{n+i-1}(i-1)!\binom{n+i-2}{2i-1} \\ &\quad+\sum_{i=1}^{n}(-1)^{n+i}(i-1)!\binom{n+i-1}{2i-1} \\ &\Longrightarrow_{\text{substitute } n+1-i=k} \sum_{k=2}^{n}(-1)^k(n-k)!\binom{2n-k-1}{2n-2k+1} \\ &\quad+\sum_{k=1}^{n}(-1)^{k+1}(n-k)!\binom{2n-k}{2n-2k+1} \\ &= (n-1)!\binom{2n-1}{0}+\sum_{k=2}^{n}(-1)^k(n-k)!\left(\binom{2n-k-1}{k-2}-\binom{2n-k}{k-1}\right) \\ b_n &= \sum_{k=1}^{n}(-1)^{k+1}\binom{2n-k-1}{k-1}(n-k)!,\end{aligned}$$

where the last step followed from Pascal's. We then find, since $\mathrm{CycMen}+\mathrm{Bad}=\mathrm{LinMen}$,

$$
\begin{aligned}
[x^n]\mathrm{CycMen}(x) = |\mathrm{CycMen}_n|/\sim &= m_n - b_n \\
&= \sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}(n-k)! \\
&\quad -\sum_{k=1}^{n}(-1)^{k+1}\binom{2n-k-1}{k-1}(n-k)! \\
&= \binom{2n-0}{0}(n-0)! + \sum_{k=1}^{n}(-1)^k\left(\binom{2n-k-1}{k-1}+\binom{2n-k}{k}\right)(n-k)! \\
&= \binom{2n-0}{0}(n-0)! + \sum_{k=1}^{n}(-1)^k\left(\frac{k}{2n-k}\binom{2n-k}{k}+\binom{2n-k}{k}\right)(n-k)!
\end{aligned}
$$

$$
[x^n]\mathrm{CycMen}(x) = |\mathrm{CycMen}_n|/\sim = \boxed{\sum_{k=0}^{n}(-1)^k\frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!},
$$

as desired.

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Some closing remarks: These solutions, which are to the best of my knowledge novel, took me a ridiculously long time to find. Since the ménages problem had been so trivial by PIE, I had been hoping for an equally trivial, if not more so, species solution. I must say I was somewhat disappointed when the best species solution I found took so much thinking. At least this solution does explicitly expose some of the relationships between the Catalan numbers and the ménages numbers. However, I have not lost hope that something out there in the theory of species can trivialize this problem, or at least lend some motivation to some of the very ad-hoc combinatorial thinking here; when I find it, I will post it here.

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