

A Appendix for Section 2

Proof of Proposition 1.

Condition (i):

Without loss of generality, write the preferences of the representative consumer as

$$U^*(c_1, \dots, c_N, L_1, \dots, L_F, L^*) = D(\mathbf{c})L^*v((1 - L)/L^*).$$

Define $v^*(L, L^*) = L^*v((1 - L)/L^*)$ and note that v^* is constant returns to scale. Hence, U^* is a product of two homothetic functions, it is itself homothetic. Since preferences are ordinal, we can assume, without loss of generality, that U^* is constant-returns-to-scale, and has associated with it an ideal price index $P_{u^*}(\mathbf{p}, \mathbf{w}, w^*)$ where \mathbf{p} is the price of goods, \mathbf{w} is the price of factors, and w^* is the price of L^* (in fixed supply).

By the first welfare theorem, and Hulten's theorem, we have

$$\frac{d \log U^*}{d \log A_i} = \frac{p_i y_i}{P_{u^*} U^*} \equiv \lambda_i^*,$$

for all i . Hence, by Young's theorem, we have

$$\frac{d \lambda_i^*}{d \log A_j} = \frac{d \lambda_j^*}{d \log A_i}.$$

Denote the Domar weight of i by

$$\lambda_i = \frac{p_i y_i}{P_c C},$$

and note that $\lambda_i = \lambda_i^*(P_{u^*} U^* / P_c C)$. Then we have

$$\begin{aligned} \frac{d \lambda_i^*}{d \log A_j} &= \frac{\lambda_i^*}{\lambda_i} \frac{d \lambda_i}{d \log A_j} + \frac{d \log(P_c C / P_u U)}{d \log A_j}, \\ &= \frac{P_c C}{P_u U} \frac{d \lambda_i}{d \log A_j} + 0. \end{aligned}$$

where the second line follows from the fact that the elasticity of substitution between \mathcal{D} and v^* is 1. Similarly,

$$\frac{d \lambda_j^*}{d \log A_i} = \frac{\lambda_j^*}{\lambda_j} \frac{d \lambda_j}{d \log A_i} + 0.$$

Hence,

$$\frac{\lambda_j^*}{\lambda_j} \frac{d \lambda_j}{d \log A_i} = \frac{\lambda_i^*}{\lambda_i} \frac{d \lambda_i}{d \log A_j},$$

or

$$\frac{d \lambda_j}{d \log A_i} = \frac{d \lambda_i}{d \log A_j}.$$

Condition (ii):

This can be derived as an immediate consequence of Proposition 6. It can also be proved from first-principles by observing that

$$d \log Y / d \log A_i = d \log TFP / d \log A_i + d \log L / d \log A_i = \lambda_i + d \log L / d \log A_i,$$

where the second equation follows from Hulten's theorem, and the fact that labor's share of income is always one. The symmetry of partial derivatives then implies

$$\frac{d}{d \log A_j} (\lambda_i + d \log L / d \log A_i) = \frac{d}{d \log A_i} (\lambda_j + d \log L / d \log A_j).$$

As long as L is a continuously differentiable function of \mathbf{A} , this implies

$$\frac{d \lambda_i}{d \log A_j} = \frac{d \lambda_j}{d \log A_i}.$$

□

B Appendix for Section 3

Input Shares, Input Expenditures, and Input Quantities

Using Propositions 3 and 4 as well as Corollary 1, it is easy to derive the elasticities of input shares, expenditures and quantities of the different producers to the different productivities. These results can actually be derived by relabeling the network to treat the sales of good l to producer i as going through a new fictitious producer specific to i and l .

Corollary 2. (*Input Shares, Input Expenditures, and Input Quantities*) *The elasticities of input expenditures and input quantities of the different producers to the different productivities are given by*

$$\frac{d \log \lambda_{il}}{d \log A_k} = \frac{d \log \lambda_i}{d \log A_k} + \frac{\lambda_i}{\lambda_{il}} (\theta_i - 1) \text{Cov}_{\Omega(i)}(\Psi^{(k)}, I_{(l)}) - \frac{\lambda_i}{\lambda_{il}} (\theta_i - 1) \text{Cov}_{\Omega(i)}\left(\sum_g \Psi^{(g)} \frac{d \log \Lambda_g}{d \log A_k}, I_{(l)}\right), \quad (29)$$

$$\frac{d \log p_l x_{il}}{d \log A_k} = \frac{d \log \lambda_{il}}{d \log A_k} + \frac{d \log Y}{d \log A_k}, \quad (30)$$

$$\frac{d \log x_{il}}{d \log A_k} = \frac{d \log p_l x_{il}}{d \log A_k} - \frac{d \log p_l}{d \log A_k}, \quad (31)$$

where $d \log \lambda_i / d \log A_k$, $d \log Y / d \log A_k$, and $d \log p_l / d \log A_k$ are given in Propositions 3 and 4, and $I_{(l)}$ is the l th column of the identity matrix. These formulas can be applied to factors to characterize the elasticities of the expenditures on factors and factor quantities of the different producers by treating factors as producers of non-reproducible goods using $l = f$ and replacing p_l by w_f , y_i by L_f , and λ_{il} by Λ_{if} .

These results can actually easily be derived by relabeling the network to treat the sales of good l to producer i as going through a new fictitious producer specific to i and l .

B.1 Example: Intermediate Labor Reallocation

Imagine a very simple example where labor comes in two forms: producer-specific quasi-fixed labor F_i which cannot be reallocated across producers, and general labor M which can be reallocated. Assume that these two forms of labor enter into production according to a Cobb-Douglas aggregate with shares $1 - \beta$ for fixed labor and β for general labor. We then get

$$\begin{aligned} \frac{d \log Y}{d \log A_k} &= \lambda_k \\ \frac{d \log \lambda_i}{d \log A_k} &= \frac{(\theta - 1)}{1 + (\theta - 1)(1 - \beta)} [\delta_{ik} - \lambda_k], \quad \frac{d \log \Lambda_M}{d \log A_k} = 0, \\ \frac{d \log p_i}{d \log A_k} &= -\delta_{ik} \left[1 - \frac{(1 - \beta)(\theta - 1)}{1 + (\theta - 1)(1 - \beta)} \right] + \lambda_k \frac{1}{1 + (\theta - 1)(1 - \beta)}, \quad \frac{d \log w_M}{d \log A_k} = \lambda_k, \\ \frac{d \log p_i y_i}{d \log A_k} &= \lambda_k + \frac{(\theta - 1)}{1 + (\theta - 1)(1 - \beta)} [\delta_{ik} - \lambda_k], \quad \frac{d \log w_M M}{d \log A_k} = \lambda_k, \\ \frac{d \log y_i}{d \log A_k} &= \delta_{ik} \left[1 + \frac{\beta(\theta - 1)}{1 + (\theta - 1)(1 - \beta)} \right] - \lambda_k \frac{(\theta - 1)\beta}{1 + (\theta - 1)(1 - \beta)}, \\ &\quad \frac{d \log M_i}{d \log A_k} = \frac{(\theta - 1)}{1 + (\theta - 1)(1 - \beta)} (\delta_{ik} - \lambda_k). \end{aligned}$$

B.2 Example: Roundabout Economy

In the roundabout economy, the gross output y_1 of the single producer is split between its use in final demand c_1 and its use as an intermediate input x_1 . Aggregate output is $Y = c_1$. We

have

$$\begin{aligned}\frac{d \log Y}{d \log A_1} &= \lambda_1, \\ \frac{d \log \lambda_1}{d \log A_1} &= (\theta - 1)(1 - \lambda_1), \quad \frac{d \log \Lambda_L}{d \log A_1} = 0, \\ \frac{d \log p_1}{d \log A_1} &= 0, \quad \frac{d \log w_L}{d \log A_1} = \lambda_1, \\ \frac{d \log p_1 y_1}{d \log A_1} &= \lambda_1 + (\theta - 1)(1 - \lambda_1), \quad \frac{d \log w_L L_1}{d \log A_1} = \lambda_1, \\ \frac{d \log y_1}{d \log A_1} &= \lambda_1 + (\theta - 1)(1 - \lambda_1), \quad \frac{d \log L_1}{d \log A_1} = 0.\end{aligned}$$

In this economy, the impact of a positive productivity shock $d \log A_1 > 0$ on output is $\lambda_1 d \log A_1 > d \log A_1$ in accordance with Hulten's theorem. The price of producer 1 stays unchanged. When $\theta > 1$ its share increases and its gross output increases by more than by more than $\lambda_1 d \log A_1$. These patterns are reversed when $\theta < 1$.

C Appendix for Section 4

For the example in Figure 2, applying Proposition 5, we get

$$\begin{aligned}\lambda_i \frac{d \log \lambda_i}{d \log A_j} &= - \sum_{c \in \mathcal{C}} (\sigma - 1) \chi_c b_{cj} b_{ci} \\ &- \frac{1}{\Lambda_H \Lambda_L} \frac{\left[\sum_{c \in \mathcal{C}} (\sigma - 1) \chi_c b_{ci} [\omega_{iH} - E_{b_c}(\omega_{(H)})] \right] \left[\sum_{c \in \mathcal{C}} (\sigma - 1) \chi_c b_{cj} [\omega_{jH} - E_{b_c}(\omega_{(H)})] \right]}{1 + \frac{1}{\Lambda_H \Lambda_L} \sum_{c \in \mathcal{C}} (\sigma - 1) \chi_c \text{Var}_{b_c}(\omega_{(H)}) - \sum_k \omega_{kH} (b_{1k} - b_{2k})} \\ &+ \frac{(b_{1i} - b_{2i}) \left[\sum_{c \in \mathcal{C}} (\sigma - 1) \chi_c b_{cj} [\omega_{jH} - E_{b_c}(\omega_{(H)})] \right]}{1 + \frac{1}{\Lambda_H \Lambda_L} \sum_{c \in \mathcal{C}} (\sigma - 1) \chi_c \text{Var}_{b_c}(\omega_{(H)}) - \sum_k \omega_{kH} (b_{1k} - b_{2k})}.\end{aligned}$$

The first two terms on the right-hand side are symmetric for the impact of a shock to j on i and for a shock to i on j , but the third term is not: it is zero for the impact of a shock to j on i if $b_{1i} = b_{2i}$ but it is nonzero for the impact of a shock to i on j if $b_{1j} \neq b_{2j}$ and if sector j 's exposure to high-skilled labor is different from the average $\omega_{jH} \neq E_{b_c}(\omega_{(H)})$.

D Appendix for Section 5

Input Shares, Input Expenditures, and Input Quantities

Using Propositions 6 and 7, it is easy to derive the elasticities of input shares, expenditures and quantities of the different producers to the different productivities. As in the case of inelastic factor supplies, these results can actually easily be derived by relabeling the network to treat the sales of good l to producer i as going through a new fictitious producer specific to i and l . In fact, equations (29), (30), and (31) in Corollary 2 still apply. The only difference is that now $d \log \lambda_i / d \log A_k$, $d \log Y / d \log A_k$, and $d \log p_l / d \log A_k$ must be taken from Propositions 6 and 7 instead of Propositions 3 and 4.

Aggregate Technology Shocks in the Brock-Mirman RBC Model

We provide an example which illustrates how to apply our framework inter-temporally and dynamically. We consider a simple case of the Real Business Cycle (RBC) model, and show how capital accumulation, which propagates productivity shocks and generates positive comovement in quantities, can be seen as a particular case of positive comovement driven by intermediate goods.

We consider the Brock-Mirman parametrization of the neoclassical growth model which can be solved in closed form. There is a representative agent with log-balanced-growth per-period preferences $\log c_t - \nu(1 + \zeta_L^{-1})L_t^{(1+\zeta_L^{-1})}$, and discount factor $\beta < 1$. There is an aggregate per-period production function $y_t = A_t K_t^\alpha L_t^{1-\alpha}$ with initial capital stock K_0 . Capital fully depreciates in every period.

The solution is well known. The equilibrium conditions for this economy are

$$y_t = A_t K_t^\alpha L_t^{1-\alpha}, \quad y_t = c_t + K_{t+1},$$

$$\frac{c_{t+1}}{c_t} = \beta \alpha \frac{y_{t+1}}{K_{t+1}}, \quad \nu L_t^{\zeta_L^{-1}} c_t = (1 - \alpha) \frac{y_t}{L_t}.$$

It is easily verified that the solution features constant labor

$$L_t = L, \quad c_t = c y_t, \quad K_{t+1} = (1 - c) y_t$$

with $L = [(1 - \alpha) / (\nu c)]^{\zeta_L / (1 + \zeta_L)}$ and $c = 1 - \beta \alpha$.

We imagine that the economy is at a steady state with $A_t = 1$ for all t . At $t = 0$, a one-time unanticipated shock hits the economy in the form of a new path for productivity. We know that output solves the difference equation $y_{t+1} = A_{t+1} (\beta \alpha y_t)^\alpha L^{1-\alpha}$. Hence we get positive comovement through intermediate inputs via capital accumulation.

We can capture these properties using our formalism. We index goods by time $t = 0, \dots, \infty$

and we treat capital as an intermediate input. The factors are labor L_t in the different periods and the initial capital stock $K_0 = K$. The production function of the good produced in period t is a Cobb-Douglas aggregate of labor and of the good produced in period $t - 1$.

The input-output matrix is given by $\Omega_{t,L_t} = 1 - \alpha$ and $\Omega_{(t+1)t} = \alpha$ for all $t \geq 0$, $\Omega_{0K_0} = \alpha$, and all the other entries are 0. We have $\Lambda_{K_0} = (1 - \beta)\alpha / (1 - \beta\alpha)$, $\Lambda_{L_t} = (1 - \beta)\beta^t(1 - \alpha) / (1 - \beta\alpha)$, $\lambda_t = \beta^t(1 - \beta) / (1 - \beta\alpha)$, and $\varrho = 1$. Applying our formulas is straightforward since all the elasticities of substitution are unitary: because of log utility over consumption and because of the assumption of a Cobb-Douglas per-period production function with full depreciation. We get

$$\begin{aligned} \frac{d \log Y}{d \log A_t} &= \lambda_t, \\ \frac{d \log \lambda_s}{d \log A_t} &= 0, \quad \frac{d \log \Lambda_{L_s}}{d \log A_t} = 0, \quad \frac{d \log \Lambda_{K_0}}{d \log A_t} = 0, \\ \frac{d \log p_s}{d \log A_t} &= -\alpha^{s-t} 1_{\{t \leq s\}} + \lambda_t, \quad \frac{d \log w_{L_s}}{d \log A_t} = \lambda_t, \quad \frac{d \log w_{K_0}}{d \log A_t} = \lambda_t, \\ \frac{d \log(p_s y_s)}{d \log A_t} &= \lambda_t, \quad \frac{d \log(w_{L_s} L_s)}{d \log A_t} = \lambda_t, \quad \frac{d \log(w_{K_0} K_0)}{d \log A_t} = \lambda_t, \\ \frac{d \log y_s}{d \log A_t} &= \alpha^{s-t} 1_{\{t \leq s\}}, \quad \frac{d \log L_s}{d \log A_t} = 0, \quad \frac{d \log K_0}{d \log A_t} = 0. \end{aligned}$$

We confirm the property of positive comovement through intermediate inputs via capital accumulation that we noted in the closed-form solution. This approach also provides a new and useful intuition for why the Brock-Mirman case is so tractable. Indeed, to capture capital as an intermediate good when there is less than full depreciation, we need to model the production function of producer t as a Cobb-Douglas aggregate of labor and a CES aggregate of all the goods in period $s \leq t - 1$ with an infinite elasticity of substitution.²⁵ Because some elasticities of substitution are not unitary (since they are infinite), the solution is more complex, even to the first order, and requires solving an infinite-dimensional linear system of equations, i.e. a set of difference equations.²⁶ The Brock-Mirman case trivializes these complications because all the elasticities of substitution are unitary.

Analysis of the Steady-State of the Ramsey Model with Capital

We find

$$\frac{d \log Y}{d \log A_k} = \frac{\lambda_k}{\Lambda_L} - \frac{d \log \Lambda_L}{d \log A_k},$$

²⁵The input-output matrix is given by $\Omega_{t,L_t} = 1 - \alpha$, $\Omega_{t,K_0} = \alpha(\beta v)^t$, $\Omega_{t,t-s} = \alpha(\beta v)^{t-1-s} / (1 - \beta v)$ for all $s \leq t - 1$, and all the other entries given by 0.

²⁶Moreover, the inter-temporal elasticity of substitution could also be non-unitary, adding further complications.

$$\begin{aligned} \frac{d \log \lambda_i}{d \log A_k} &= (\theta - 1)(\delta_{ik} - \lambda_k) + (\theta - 1) \frac{\lambda_k}{\Lambda_L} (\omega_{iK} - \Lambda_K), \\ \frac{d \log \Lambda_K}{d \log A_k} &= - \frac{\Lambda_L}{\Lambda_K} \frac{d \log \Lambda_L}{d \log A_k}, \\ \frac{d \log \Lambda_L}{d \log A_k} &= \frac{\lambda_k - (\theta - 1)(\omega_{kK} - \Lambda_K) - (\theta - 1) \frac{Var_\lambda(\omega_{(L)})}{\Lambda_L} - (\theta_{KL} - 1) \sum_j \frac{\lambda_j}{\Lambda_L} \omega_{jK} \omega_{jL}}{\Lambda_L \left(1 - (\theta_{KL} - 1) \sum_j \frac{\Lambda_K}{\Lambda_L} \frac{\lambda_j}{\Lambda_L} \omega_{jL} \omega_{jK} \right)}, \\ \frac{d \log p_i}{d \log A_k} &= -\delta_{ik} + \omega_{iL} \frac{\lambda_k}{\Lambda_L}, \quad \frac{d \log r_K}{d \log A_k} = 0, \quad \frac{d \log w_L}{d \log A_k} = \frac{\lambda_k}{\Lambda_L}, \end{aligned}$$

$$\begin{aligned} \frac{d \log p_i y_i}{d \log A_k} &= (\theta - 1)(\delta_{ik} - \lambda_k) + (\theta - 1) \frac{\lambda_k}{\Lambda_L} (\omega_{iK} - \Lambda_K) + \frac{\lambda_k}{\Lambda_L} - \frac{d \log \Lambda_L}{d \log A_k}, \\ \frac{d \log r_K K}{d \log A_k} &= \frac{\lambda_k}{\Lambda_L} - \frac{1}{\Lambda_K} \frac{d \log \Lambda_L}{d \log A_k}, \quad \frac{d \log w_L L}{d \log A_k} = \frac{\lambda_k}{\Lambda_L}, \end{aligned}$$

$$\begin{aligned} \frac{d \log y_i}{d \log A_k} &= \delta_{ik} + (\theta - 1)(\delta_{ik} - \lambda_k) + (\theta - 1) \frac{\lambda_k}{\Lambda_L} (\omega_{iK} - \Lambda_K) + \omega_{iK} \frac{\lambda_k}{\Lambda_L} - \frac{d \log \Lambda_L}{d \log A_k}, \\ \frac{d \log K}{d \log A_k} &= \frac{\lambda_k}{\Lambda_L} - \frac{1}{\Lambda_K} \frac{d \log \Lambda_L}{d \log A_k}, \quad \frac{d \log L}{d \log A_k} = 0. \end{aligned}$$

Consider the response of the economy to a positive productivity shock $d \log A_k$ to producer k . The labor share stays constant if the elasticity of substitution across producers is $\theta = 1$ or if the factor intensity of all producers are the same, and if the elasticities of substitution between capital and labor of all producers is $\theta_{KL} = 1$.

Consider first the case where $\theta = \theta_{KL} = 1$. The supply of capital increases by $\lambda_k / \Lambda_L d \log A_k$, and this creates the following differences with the case where factor supplies are inelastic and factor intensities are homogenous analyzed in Section 3.3. Output increases by $\lambda_k / \Lambda_L d \log A_k = \lambda_k d \log A_k + \Lambda_K d \log K > \lambda_k d \log A_k$. The real wage increases by $\lambda_k / \Lambda_L d \log A_k$. This increase in the real wage is reflected into the prices of the different producers according to their labor intensities via $\omega_{iL} \lambda_k / \Lambda_L d \log A_k$. The sales of the different producers and the total factor payments increase by $\lambda_k / \Lambda_L d \log A_k$. The output of the different producers increases in proportion to their capital intensity $\omega_{iK} \lambda_k / \Lambda_L$.

Now consider the case where $\theta_{KL} = 1$ but producers are substitutes with $\theta > 1$ (the effects below are reversed when producers are complements with $\theta < 1$). If producer k is sufficiently more capital intensive than average so that $\omega_{kK} > \Lambda_K$, then the labor share decreases, which has the following effects, over and above (1) the effects identified when $\theta = \theta_j = 1$ and (2) the effects arising when factor supplies are inelastic and factor intensities are homogenous, a case isomorphic to the one-inelastic-factor case analyzed in Section 3.3. It magnifies the effect on

aggregate output since producer k relies more on capital and hence expands more. It magnifies the increase in the capital stock. It does not change the real wage because the average increase in capital intensity exactly offsets the increase in the capital stock. It increases the sales of all producers, but more so for producers that are more capital intensive than average. It increases the output of all producers, but more so for producers that are more capital intensive than average.

Finally, consider what happens when capital and labor are complements in the production function of producers so that $\theta_{KL} < 1$ (the effects are reversed when capital and labor are substitutes in production with $\theta_{KL} > 1$). It is important to note that the θ_{KL} matter only to the extent that it modifies the change $d \log \Lambda_L$ in how the labor share responds to the shock $d \log A_k$: because the shock increases the real wage and because $\theta_{KL} < 1$, the labor share increases for all producers, and this effect mitigates the decrease in the labor share. All the other expressions remain unchanged as a function of the change in the labor share. For example, the increase in aggregate output is mitigated, etc.

Overall, a rich pattern of comovement emerges. Like above, this model introduces a force for positive comovement via elastic capital supply. It can generate positive comovement not only in output and sales, but also in capital.

E Appendix for Section 6

E.1 Comparative Statics Results: Shocks to Markups/Wedges

In inefficient economies, it is also interesting to characterize the propagation of shocks to markups/wedges

Aggregate Output and Shares

The following proposition, taken from Baqaee and Farhi (2017b), provides a joint characterization of the elasticities of aggregate output and factor shares to the different markup/wedge shocks.

Proposition 12. *(Aggregate Output and Shares in Inefficient Economies) In inefficient economies with markups/wedges, the elasticities of aggregate output to the different markup/wedges are given by*

$$\frac{d \log Y}{d \log \mu_k} = -\tilde{\lambda}_k - \sum_g \tilde{\Lambda}_g \frac{d \log \Lambda_g}{d \log \mu_k}, \quad (32)$$

where the elasticities of the factor shares to the different productivities are given by the solution of the

following system of equations

$$\frac{d \log \Lambda_f}{d \log \mu_k} = -\lambda_k \frac{\Psi_{kf}}{\Lambda_f} - \sum_j (\theta_j - 1) \frac{\mu_j^{-1} \lambda_j}{\Lambda_f} \text{Cov}_{\tilde{\Omega}(i)} \left(\tilde{\Psi}_{(k)} - \sum_g \tilde{\Psi}_{(g)} \frac{d \log \Lambda_g}{d \log \mu_k}, \Psi_{(f)} \right). \quad (33)$$

The elasticities of the shares of the other producers to the different productivities are given by

$$\frac{d \log \lambda_i}{d \log \mu_k} = \delta_{ik} - \lambda_k \frac{\Psi_{ki}}{\lambda_i} - \sum_j (\theta_j - 1) \frac{\mu_j^{-1} \lambda_j}{\lambda_i} \text{Cov}_{\tilde{\Omega}(i)} \left(\tilde{\Psi}_{(k)} - \sum_g \tilde{\Psi}_{(g)} \frac{d \log \Lambda_g}{d \log \mu_k}, \Psi_{(i)} \right). \quad (34)$$

The intuition for these results is that positive markup shocks, just like negative productivity shocks, increase the prices of the corresponding producer. These effects propagate in similar ways and are encapsulated in the second terms on the right-hand side of the share propagation equations (33) and (34). But compared to productivity shocks, positive markup shocks also have another effect by reducing the expenditure on the inputs used by the affected producer and hence their sales shares. These effects are captured by the first terms on the right-hand side of these share propagation equations.

The profit share moves for two reasons following a positive markup shock. First, there is a mechanical increase in the profit share even in the absence of any reallocation of resources. Second, there is a movement in the profit share because resources are reallocated towards or away from the more distorted parts of the economy, which reflect changes in allocative efficiency. To isolate the effects of changes in allocative efficiency on output, we need to net out the first effect. This is what the first term on the right-hand side of the output aggregation equation (32) does. Another way to think about it is that a positive markup shock acts like a negative productivity shock, with the difference that it also releases resources, which can ultimately be expressed as released factors with a positive effect on output measured by $\sum_f \tilde{\Lambda}_f \lambda_k \Psi_{kf} / \Lambda_f$.

Prices

We now characterize the elasticities of prices to the different markups/wedges.

Proposition 13. (Prices) *In inefficient economies with markups/wedges, the elasticities of the sales, prices, and output quantities of the different producers to the different markups/wedges are given by*

$$\frac{d \log w_f}{d \log \mu_k} = \frac{d \log \Lambda_f}{d \log \mu_k} + \frac{d \log Y}{d \log \mu_k}, \quad (35)$$

$$\frac{d \log p_i}{d \log \mu_k} = \tilde{\Psi}_{ik} + \sum_g \tilde{\Psi}_{ig} \frac{d \log w_f}{d \log \mu_k}, \quad (36)$$

where $d \log \Lambda_f / d \log A_k$ and $d \log Y / d \log A_k$ are given in Proposition 12.

The intuition is very similar to that for Proposition 9.

Sales, and Quantities

Propositions 13 and 13 can be used to characterize the elasticities of the sales and output quantities of the different producers to the different markups/wedges, along the same lines as in Corollary 1. In fact, equations (8) and (9) in Corollary 1 still apply. The only difference is that now derivatives with respect to A_k must be replaced by derivatives with respect to μ_k . In addition $d \log \lambda_i / d \log \mu_k$, $d \log Y / d \log \mu_k$, and $d \log p_i / d \log \mu_k$ must be taken from Propositions 12 and 13 instead of Propositions 3 and 4.

Input Shares, Input Expenditures, and Input Quantities

Using Propositions 8 and 9, it is easy to derive the elasticities of input shares, expenditures and quantities of the different producers to the different productivities. As in Corollary 2, these results can actually easily be derived by relabeling the network to treat the sales of good l to producer i as going through a new fictitious producer specific to i and l . In fact, equations (29), (30), and (31) in Corollary 2 still apply. There are only two differences. First, equation (29) must be replaced by

$$\begin{aligned} \frac{d \log \lambda_{il}}{d \log A_k} &= \frac{d \log \lambda_i}{d \log A_k} \\ &+ \frac{\mu_i^{-1} \lambda_i}{\lambda_{il}} (\theta_i - 1) \text{Cov}_{\tilde{\Omega}^{(i)}}(\tilde{\Psi}_{(k)}, I_{(l)}) - \frac{\mu_i^{-1} \lambda_i}{\lambda_{il}} (\theta_i - 1) \text{Cov}_{\tilde{\Omega}^{(i)}}\left(\sum_g \tilde{\Psi}_{(g)} \frac{d \log \Lambda_g}{d \log A_k}, I_{(l)}\right). \end{aligned}$$

Second, now $d \log \lambda_i / d \log A_k$, $d \log Y / d \log A_k$, and $d \log p_l / d \log A_k$ must be taken from Propositions 8 and 9 instead of Propositions 3 and 4.

Similarly, we can derive elasticities with respect to the different markups/wedges. As above, these results can actually easily be derived by relabeling the network to treat the sales of good l to producer i as going through a new fictitious producer specific to i and l . The difference is that we must now replace derivatives with respect to A_k with derivatives with respect to μ_k . In addition $d \log \lambda_i / d \log \mu_k$, $d \log Y / d \log \mu_k$, and $d \log p_i / d \log \mu_k$ must be taken from Propositions 12 and 13 instead of Propositions 3 and 4.

E.2 Elastic Factor Supplies

To streamline the exposition we use the following differential notation convention. For a variable X , we write

$$d \log X = \sum_k \frac{d \log X}{d \log A_k} d \log A_k + \sum_k \frac{d \log X}{d \log \mu_k} d \log \mu_k$$

and we interpret an equality of the type $d \log X = d \log Y$ to mean that for all k , we have

$$\frac{d \log X}{d \log A_k} = \frac{d \log Y}{d \log A_k} \quad \text{and} \quad \frac{d \log X}{d \log \mu_k} = \frac{d \log Y}{d \log \mu_k}.$$

Aggregate Output and Shares

As is standard by now, we start with the elasticities of aggregate output and factor shares. The following proposition is taken from Baqaee and Farhi (2017b).

Proposition 14. (*Aggregate Output and Factor Shares*) *The elasticities of aggregate output and factor shares to the different productivities and markups/wedges are given by the following system of linear equations:*

$$d \log Y = \varrho \left[\sum_k \tilde{\lambda}_k d \log A_k - \sum_k \tilde{\lambda}_k d \log \mu_k - \sum_f \tilde{\Lambda}_f \frac{1}{1 + \zeta_f} d \log \Lambda_f \right], \quad (37)$$

$$\begin{aligned} d \log \Lambda_f = & - \sum_k \lambda_k \frac{\Psi_{kf}}{\Lambda_f} d \log \mu_k \\ & + \sum_j (\theta_j - 1) \frac{\mu_j^{-1} \lambda_j}{\Lambda_f} Cov_{\tilde{\Omega}^{(j)}} \left(\sum_k \tilde{\Psi}_{(k)} d \log A_k - \sum_k \tilde{\Psi}_{(k)} d \log \mu_k, \Psi_{(f)} \right) \\ & - \sum_j (\theta_j - 1) \frac{\mu_j^{-1} \lambda_j}{\Lambda_f} Cov_{\tilde{\Omega}^{(j)}} \left(\sum_g \tilde{\Psi}_{(g)} \frac{1}{1 + \zeta_g} d \log \Lambda_g + \sum_g \tilde{\Psi}_{(g)} \frac{\gamma_g - \zeta_g}{1 + \zeta_g} d \log Y, \Psi_{(f)} \right), \quad (38) \end{aligned}$$

where $\varrho = 1 / (\sum_f \tilde{\Lambda}_f \frac{1 + \gamma_f}{1 + \zeta_f})$ as above. Similarly the elasticities of the different sales shares to the different productivities and markups/wedges are given by

$$\begin{aligned} d \log \lambda_i = & \sum_k (\delta_{ik} - \lambda_k \frac{\Psi_{ki}}{\Lambda_i}) d \log \mu_k \\ & + \sum_j (\theta_j - 1) \frac{\mu_j^{-1} \lambda_j}{\lambda_i} Cov_{\tilde{\Omega}^{(j)}} \left(\sum_k \tilde{\Psi}_{(k)} d \log A_k - \sum_k \tilde{\Psi}_{(k)} d \log \mu_k, \Psi_{(i)} \right) \\ & - \sum_j (\theta_j - 1) \frac{\mu_j^{-1} \lambda_j}{\lambda_i} Cov_{\tilde{\Omega}^{(j)}} \left(\sum_g \tilde{\Psi}_{(g)} \frac{1}{1 + \zeta_g} d \log \Lambda_g + \sum_g \tilde{\Psi}_{(g)} \frac{\gamma_g - \zeta_g}{1 + \zeta_g} d \log Y, \Psi_{(i)} \right). \quad (39) \end{aligned}$$

These equations naturally generalize their counterparts for efficient economies with elastic factor supplies and for inefficient economies with inelastic factor supplies. The intuition combines those given for these cases. For example, the intuition for equation (37) is as follows. With inelastic factors, a decline in factor income shares, *ceteris paribus*, *increases* output since it

represents a reduction in the misallocation of resources and an increase in aggregate TFP. With elastic factor supply, the output effect is dampened by the presence of $1/(1 + \zeta_f) < 1$. This is due to the fact that a reduction in factor income shares, while increasing aggregate TFP, reduces factor payments and factor supplies, which in turn reduces output. Hence, when factors are elastic, increases in allocative efficiency from assigning more resources to more monopolistic producers are counteracted by reductions in factor supplies due to the associated suppression of factor demand.²⁷

Prices

We continue with the elasticities of prices to the different productivities and markups/wedges.

Proposition 15. (*Prices*) *The elasticities of prices to the different productivities and markups/wedges are given by:*

$$d \log w_f = \frac{1}{1 + \zeta_f} d \log \Lambda_f + \frac{1 + \gamma_f}{1 + \zeta_f} d \log Y, \quad (40)$$

$$d \log p_i = - \sum_k \tilde{\Psi}_{ik} d \log A_k + \sum_k \tilde{\Psi}_{ik} d \log \mu_k + \sum_g \tilde{\Psi}_{ig} d \log w_g. \quad (41)$$

Sales and Quantities

Propositions 14 and 15 can be used to characterize the elasticities of the sales and output quantities of the different producers to the different productivities and markups/wedges exactly as in Sections 6.1 and E.1. The only difference is that now the elasticities of aggregate output, factor shares, and sales shares are given by Proposition 14 and the elasticities of prices are given by Proposition 15.

Input Shares, Input Expenditures, and Input Quantities

Propositions 14 and 15 can be used to characterize the elasticities of input shares, expenditures and quantities to the different productivities and markups/wedges exactly as in Sections 6.1 and E.1. The only difference is that now the elasticities of aggregate output, factor shares, and sales shares are given by Proposition 14 and the elasticities of prices are given by Proposition

²⁷In the limit where factor supplies become infinitely elastic, the influence of the allocative efficiency effects disappear from output, since more factors can always be marshaled on the margin at the same real price. To see this, consider the case with a single factor called labor, and factor supply function $G_L(w/P_y, Y) = (w/P_y Y)^\nu$, which can be derived from a standard labor-leisure choice model. In this case, $\gamma_L = \zeta_L = \nu$, and so equation (37) implies that $d \log Y = \sum_k \tilde{\lambda}_k (d \log A_k - d \log \mu_k) - 1/(1 + \nu) \sum_f \tilde{\Lambda}_f d \log \Lambda_f$. When labor supply becomes infinitely elastic $\nu \rightarrow \infty$, this simplifies to $d \log Y = \sum_k \tilde{\lambda}_k (d \log A_k - d \log \mu_k)$, so that changes in allocative efficiency have no effect on output, even though they affect aggregate TFP.

15. As above, these results can actually easily be derived by relabeling the network to treat the sales of good l to producer i as going through a new fictitious producer specific to i and l .

E.3 Examples for economies with wedges

E.3.1 Incidence of Capital Taxation

We consider the a horizontal economy where producers have different productivities and different markups/wedges. Each producer produces from capital and labor. Labor is homogeneous and its total supply is inelastic. Capital comes in two forms, taxable capital and non-taxable capital. Their depreciation rates are δ_{K_T} and $\delta_{K_{NT}}$ and both of them are in infinitely elastic supply at user costs $\rho + \delta_{K_T}$ and $\rho + \delta_{K_{NT}}$.

Each producer produces from a capital aggregate and labor according to

$$\frac{y_i}{\bar{y}_i} = \frac{A_i}{\bar{A}_i} \left(\omega_{iK_i} \left(\frac{K_i}{\bar{K}_i} \right)^{\frac{\theta_{KL}-1}{\theta_{KL}}} + \omega_{iL} \left(\frac{L_i}{\bar{L}_i} \right)^{\frac{\theta_{KL}-1}{\theta_{KL}}} \right)^{\frac{\theta_{KL}}{\theta_{KL}-1}},$$

with $\omega_{iK_i} + \omega_{iL} = 1$, where

$$\frac{K_i}{\bar{K}_i} = \left(\omega_{K_iK_T} \left(\frac{K_{iT}}{\bar{K}_{iT}} \right)^{\frac{\theta_K-1}{\theta_K}} + \omega_{K_iK_{NT}} \left(\frac{K_{iNT}}{\bar{K}_{iNT}} \right)^{\frac{\theta_K-1}{\theta_K}} \right)^{\frac{\theta_K}{\theta_K-1}},$$

with $\omega_{K_iK_T} + \omega_{K_iK_{NT}} = 1$. We define $\psi_{iK_T} = \omega_{iK_i}\omega_{K_iK_T}$, $\psi_{iK_{NT}} = \omega_{iK_i}\omega_{K_iK_{NT}}$.

Each producer charges a markup μ_i , and we denote the the average markup by $\mu_{\mathcal{I}} = (\sum \lambda_i \mu_i^{-1})^{-1}$. In addition, each producer is taxed on its revenues, net of the wage bill, from which it can deduct the non-taxable capital bill and the depreciation rate on both capitals so that the revenues generated are

$$\tau[p_i y_i - w_L L_i - \delta_{K_T} K_{iT} - (\rho + \delta_{K_{NT}}) K_{iNT}]$$

and the pure profits left to the firm are

$$(1 - \tau)[p_i y_i - w_L L_i - (\frac{\rho}{1 - \tau} + \delta_{K_T}) K_{iT} - (\rho + \delta_{K_{NT}}) K_{iNT}].$$

This introduces a wedge $\mu_{K_T} \equiv [\rho/(1 - \tau) + \delta_{K_T}]/(\rho + \delta_{K_T})$ between the effective user cost $\rho/(1 - \tau) + \delta_{K_T}$ of taxable capital for firms and that $\rho + \delta_{K_T}$ for suppliers of this capital. This example provides a concrete illustration of the wedge/markup isomorphism which have been relying on.

In this economy, we have $\tilde{\lambda}_i = \lambda_i$ with $\sum_i \lambda_i = 1$, $\tilde{\Lambda}_L = \sum_i \lambda_i \omega_{iL}$, $\Lambda_L = \sum_i \mu_i^{-1} \lambda_i \omega_{iL}$, $\tilde{\Lambda}_{K_T} = \sum_i \lambda_i \psi_{iK_T}$, $\Lambda_{K_T} = \sum_i \mu_i^{-1} \lambda_i \psi_{iK_T}$, $\tilde{\Lambda}_{K_{NT}} = \sum_i \lambda_i \psi_{iK_{NT}}$, $\Lambda_{K_{NT}} = \sum_i \mu_i^{-1} \lambda_i \psi_{iK_{NT}}$, and $\varrho = 1/\tilde{\Lambda}_L$.

Total tax revenues as a share of aggregate output can be computed to be

$$\frac{R}{Y} = \tau \left((1 - \mu_{\mathcal{I}}^{-1}) + \frac{\frac{\rho}{1-\tau}}{\frac{\rho}{1-\tau} + \delta_{K_T}} \mu_{K_T} \Lambda_{K_T} \right).$$

As is apparent from this expression, corporate taxation taxes pure monopoly profits in addition to the taxable part (depreciation is deductible) of the effective rental return of taxable capital for firms.

We shall only concern ourselves with the incidence of a change in the corporate tax $d\tau$ on aggregate output $d \log Y$ and the aggregate labor share $d \Lambda_L$. This can be computed by performing comparative statics with respect to

$$d \log \mu_{K_T} = \frac{\frac{\rho}{1-\tau}}{\frac{\rho}{1-\tau} + \delta_{K_T}} \frac{d\tau}{1-\tau}.$$

We choose to focus our attention on the effects of changes in the corporate tax on the wage bill $d(w_L L)$ as a fraction of aggregate output Y , as well as a fraction of the static revenue change dR^{static} (for a fixed tax base) and of the dynamic revenue change dR^{dynamic} (taking into account the change in the endogenous change in the tax base). We only report some selected results. We refer the reader to Appendix E for detailed derivations and broader characterizations: for example we also derive the changes in aggregate output, factor shares, capital stock, sales and quantities of the different producers, etc.

We start by characterizing the change in the wage bill as a share of aggregate output:

$$\frac{d(w_L L)}{Y} = -\tilde{\Lambda}_{K_T} \mu_{\mathcal{I}}^{-1} \left(1 + \text{Corr}_{\lambda}(\mu^{-1}, \omega_{(L)}) \right) \frac{\frac{\rho}{1-\tau}}{\frac{\rho}{1-\tau} + \delta_{K_T}} \frac{d\tau}{1-\tau}.$$

There are several interesting things to note. First, $d(w_L L)/Y$ is proportional to the tax change $d\tau$, to the share of the user cost of taxable capital which is taxable $[\rho/(1-\tau)]/[\rho/(1-\tau) + \delta_{K_T}]$, and to the cost-based share of taxable capital $\tilde{\Lambda}_{K_T}$. Second, it is proportional $\mu_{\mathcal{I}}^{-1}(1 + \text{Corr}_{\lambda}(\mu^{-1}, \omega_{(L)}))$. This constant is simply the ratio $\Lambda_L/\tilde{\Lambda}_L$ of the revenue-based to the cost-based labor share because the the cost of labor is a fraction of the marginal product of labor: it depends negatively on the average markup $\mu_{\mathcal{I}}$ and on the correlation between the labor intensity and the markup of the different producers. Third, it does not depend on any elasticity of substitution θ , θ_{KL} , or $\theta_{K_T K_{NT}}$. This is a consequence of our joint assumptions of infinitely elastic capital supplies and inelastic labor supply. This last observation implies in particular that potential changes in allocative efficiency resulting from changes in corporate taxes are not reflected in the real wage or the wage bill. They show up instead in changes in capital, capital revenues, and profits.

We then turn to the characterization of the change in the wage bill as a fraction of the static and dynamic change in tax revenues. For brevity, we call these the static and dynamic ratios. We have

$$\frac{d(w_L L)}{d^{\text{static}} R} = -\frac{1}{1-\tau} \frac{1}{1 + (1 - \mu_I^{-1}) \frac{1}{\mu_{K_T} \Lambda_{K_T}} \frac{\frac{\rho}{1-\tau} + \delta_{K_T}}{\frac{\rho}{1-\tau}}} \frac{1 + \text{Corr}_\lambda(\mu_i^{-1}, \omega_{iL})}{1 + \text{Corr}_\lambda(\mu_i^{-1}, \psi_{iK_T})}.$$

The general expression for the change in the wage bill as a fraction of the dynamic change in tax revenues $d(w_L L)/dR^{\text{dynamic}}$ is more involved and can be found in the appendix. To build intuition, it is helpful to consider the following special cases.

1. We start with the case where all producers have the same production functions up to producer-specific productivity shocks, there are no markups, and all the capital stock is taxed. We get

$$\begin{aligned} \frac{d(w_L L)}{d^{\text{static}} R} &= -\frac{1}{1-\tau'} \\ \frac{d(w_L L)}{d^{\text{dynamic}} R} &= -\frac{\Lambda_L}{\Lambda_L - \tau \frac{\frac{\rho}{1-\tau}}{\frac{\rho}{1-\tau} + \delta_{K_T}} \theta_{KL}}. \end{aligned}$$

In this case, both the static and the dynamic ratios are greater than one. The static ratio only depends on the tax rate. The dynamic ratio also depends on the labor share, the elasticity of substitution between capital and labor, and the importance of depreciation in the user cost. Basically, the wage bears the whole burden of corporate taxation. Reductions in the capital tax lead to a reduction in tax revenues and in the deadweight loss of taxation, both of which are reflected in increases in the real wage.

2. Next we relax the assumption that there are no markups. All producers have the same production functions up to producer-specific productivity shocks, charge the same markup, and all the capital stock is taxed. We get

$$\begin{aligned} \frac{d(w_L L)}{d^{\text{static}} R} &= -\frac{1}{1-\tau} \frac{1}{1 + \frac{\frac{\rho}{1-\tau} + \delta_{K_T}}{\frac{\rho}{1-\tau}} \frac{\mu_I - 1}{\tilde{\Lambda}_{K_T}}}, \\ \frac{d(w_L L)}{d^{\text{dynamic}} R} &= -\frac{\tilde{\Lambda}_L}{\tilde{\Lambda}_L - \tau \frac{\frac{\rho}{1-\tau}}{\frac{\rho}{1-\tau} + \delta_{K_T}} \theta_{KL} + (\mu_I - 1) \left((1 - \tau) \frac{\tilde{\Lambda}_L}{\tilde{\Lambda}_{K_T}} \frac{\frac{\rho}{1-\tau} + \delta_{K_T}}{\frac{\rho}{1-\tau}} - \tau \theta_{KL} \right)}. \end{aligned}$$

The impact of the presence of markups is intuitive. The corporate tax also taxes monopoly profits. Reductions in the corporate tax rate lead to increases in net-of-tax monopoly profits which in turn mitigates the increase in the wage bill. This reduces the static ratio, and can even reduce it below one. A similar force is at play to lower the dynamic ratio, but

there is also a countervailing force: the increase in output resulting from the reduction in the corporate tax rate leads to a dynamic mitigation of the loss in revenues via an increase in profits, and this contributes to increasing the dynamic ratio.²⁸

3. We then modify the previous example to look at a case where some of the capital stock is not taxed. All producers have the same production functions up to producer-specific productivity shocks, charge the same markup, and all the capital stock is taxed. We get

$$\frac{d(w_L L)}{d^{\text{static}} R} = -\frac{1}{1-\tau} \frac{1}{1 + \frac{\frac{\rho}{1-\tau} + \delta_{K_T}}{\frac{\rho}{1-\tau}} \frac{\mu_{\mathcal{I}} - 1}{\tilde{\Lambda}_{K_T}}},$$

$$\frac{d(w_L L)}{d^{\text{dynamic}} R} = -\tilde{\Lambda}_L / \left[1 - \tau \frac{\frac{\rho}{1-\tau}}{\frac{\rho}{1-\tau} + \delta_{K_T}} \frac{\tilde{\Lambda}_{K_T}}{1 - \tilde{\Lambda}_L} \theta_{KL} \right. \\ \left. - \tau \frac{\tilde{\Lambda}_{K_{NT}} \tilde{\Lambda}_L}{1 - \tilde{\Lambda}_L} \frac{\frac{\rho}{1-\tau}}{\frac{\rho}{1-\tau} + \delta_{K_T}} \theta_{K_T K_{NT}} \right. \\ \left. + (\mu_{\mathcal{I}} - 1) \left((1 - \tau) \frac{\frac{\rho}{1-\tau} + \delta_{K_T}}{\frac{\rho}{1-\tau}} \frac{\tilde{\Lambda}_L}{\tilde{\Lambda}_{K_T}} - \tau \theta_{KL} - \tau \frac{\tilde{\Lambda}_{K_{NT}} \tilde{\Lambda}_L}{1 - \tilde{\Lambda}_L} \frac{\frac{\rho}{1-\tau}}{\frac{\rho}{1-\tau} + \delta_{K_T}} (\theta_{K_T K_{NT}} - 1) \right) \right].$$

For a given value of the cost-based taxable capital share, the expression for the static ratio is unchanged since the fact that only part of the capital is taxed affects the change in the wage bill and the static change in revenue in the same proportional way. The modification to the dynamic ratio is more complex and now involves the elasticity of substitution $\theta_{K_T K_{NT}}$ between taxable and non-taxable capital. For example, if there are no markups, then a reduction in the corporate tax leads to a substitution towards capital and away from non-taxable capital. This mitigates the dynamic loss of tax revenues in proportion to the elasticity of substitution $\theta_{K_T K_{NT}}$ between taxable and non-taxable capital.

4. Finally, we look at the case where producers have different production functions and charge different markups. For simplicity, we assume that all the capital stock is taxable. We get

$$\frac{d(w_L L)}{d^{\text{static}} R} = -\frac{1}{1-\tau} \frac{1}{1 + \frac{\frac{\rho}{1-\tau} + \delta_{K_T}}{\frac{\rho}{1-\tau}} \frac{1 - \mu_{\mathcal{I}}^{-1}}{\mu_{K_T} \tilde{\Lambda}_{K_T}}} \frac{1 + \text{Corr}_{\lambda}(\mu^{-1}, \omega_{(L)})}{1 + \text{Corr}_{\lambda}(\mu^{-1}, 1 - \omega_{(L)})}.$$

The expression for $d(w_L L) / d^{\text{dynamic}} R$ is more complicated and can be found in Appendix E. The presence of heterogeneity in factor intensities and in markups introduces new composi-

²⁸The term responsible for the reduction in the dynamic ratio is $(\mu_{\mathcal{I}} - 1)(1 - \tau) \frac{\frac{\rho}{1-\tau} + \delta_{K_T}}{\frac{\rho}{1-\tau}} \frac{\tilde{\Lambda}_L}{\tilde{\Lambda}_{K_T}}$ and the countervailing term is $-(\mu_{\mathcal{I}} - 1)\tau\theta_{KL}$.

tional effects in the static ratio. It also introduces new substitution effects in the dynamic ratio, where the elasticity of substitution between producers θ now plays a role. For brevity, we focus on the static ratio. If producers that charge higher markups are more capital intensive, this increases the labor share, and hence also the absolute increase in the real wage following a corporate tax cut, thereby increasing the static ratio. It also reduces the tax base by lowering the capital share, and hence the static revenue loss following a corporate tax cut, thereby further increasing the static ratio.

Corporate Taxation Example Continued

In this section, we derive the formulas behind the results in the corporate taxation example in Section E.3.1.

Tax revenues are given by

$$\begin{aligned}\frac{R}{Y} &= \tau \sum_i \lambda_i \left(1 - \mu_i^{-1} \omega_{iL} - \frac{\delta_{K_T}}{\frac{\rho}{1-\tau} + \delta_{K_T}} \mu_i^{-1} \psi_{iK_T} - \mu_i^{-1} \psi_{iK_{NT}} \right), \\ &= \tau \sum_i \lambda_i (1 - \mu_i^{-1}) + \frac{\frac{\rho\tau}{1-\tau}}{\frac{\rho}{1-\tau} + \delta_{K_T}} \sum_i \lambda_i \mu_i^{-1} \psi_{iK_T}, \\ &= \tau \left((1 - \mu_I^{-1}) + \frac{\frac{\rho}{1-\tau}}{\frac{\rho}{1-\tau} + \delta_{K_T}} \mu_{K_T} \Lambda_{K_T} \right).\end{aligned}$$

We have

$$\frac{d \log Y}{d \log \mu_{K_T}} = -\frac{\tilde{\Lambda}_{K_T}}{\tilde{\Lambda}_L} - \frac{d \log \Lambda_L}{d \log \mu_{K_T}},$$

$$\begin{aligned}\frac{d \log \Lambda_L}{d \log \mu_{K_T}} &= (\theta - 1) \frac{\tilde{\Lambda}_{K_T}}{\tilde{\Lambda}_L} \frac{Var_\lambda(\omega_{(L)})}{\Lambda_L} - (\theta - 1) \frac{Cov_\lambda(\psi_{(K_T)}, \omega_{(L)})}{\Lambda_L} \\ &\quad + \frac{\tilde{\Lambda}_{K_T}}{\tilde{\Lambda}_L} (\theta_{KL} - 1) \sum_j \frac{\mu_j^{-1} \lambda_j}{\Lambda_L} \omega_{jL} (1 - \omega_{jL}) + (\theta_{KL} - 1) \sum_j \frac{\mu_j^{-1} \lambda_j}{\Lambda_L} \omega_{jL} \psi_{jK_T},\end{aligned}$$

$$\begin{aligned}\frac{d \log \Lambda_{K_T}}{d \log \mu_{K_T}} &= -1 - (\theta - 1) \frac{Var_\lambda(\psi_{(K_T)})}{\mu_{K_T} \Lambda_{K_T}} - \frac{\tilde{\Lambda}_{K_T}}{\tilde{\Lambda}_L} (\theta - 1) \frac{Cov_\lambda(\psi_{(K_T)} + \psi_{(K_{NT})}, \psi_{(K_T)})}{\mu_{K_T} \Lambda_{K_T}} \\ &\quad - (\theta_{KL} - 1) \sum_j \frac{\mu_j^{-1} \lambda_j}{\mu_{K_T} \Lambda_{K_T}} \omega_{K_j K_T}^2 \omega_{jL} (1 - \omega_{jL}) \\ &\quad - \frac{\tilde{\Lambda}_{K_T}}{\tilde{\Lambda}_L} (\theta_{KL} - 1) \sum_j \frac{\mu_j^{-1} \lambda_j}{\mu_{K_T} \Lambda_{K_T}} \omega_{K_j K_T} (\omega_{K_j K_T} + \omega_{K_j K_{NT}}) \omega_{jL} (1 - \omega_{jL})\end{aligned}$$

$$- (\theta_{K_T K_{NT}} - 1) \sum_j \frac{\lambda_j (1 - \omega_{jL})}{\mu_{K_T} \Lambda_{K_T}} \omega_{K_j K_T} (1 - \omega_{K_j K_T}),$$

$$\frac{d \log \Lambda_{K_{NT}}}{d \log \mu_{K_T}} =,$$

$$\frac{d \log \lambda_i}{d \log \mu_{K_T}} = (\theta - 1) \psi_{iK_T} \left(\frac{\tilde{\Lambda}_{K_T}}{\tilde{\Lambda}_L} \frac{\omega_{iL}}{\psi_{iK_T}} - 1 \right),$$

$$\frac{d \log w_L}{d \log \mu_{K_T}} = \frac{d \log w_L L}{d \log \mu_{K_T}} = - \frac{\tilde{\Lambda}_{K_T}}{\tilde{\Lambda}_L}.$$

We also have

$$\frac{d(w_L L)}{Y} = - \frac{\Lambda_L}{\tilde{\Lambda}_L} \frac{\tilde{\Lambda}_{K_T}}{\mu_{K_T} \Lambda_{K_T}} \mu_{K_T} \Lambda_{K_T} d \log \mu_{K_T},$$

$$d \log \mu_{K_T} = \frac{\frac{\rho}{1-\tau}}{\frac{\rho}{1-\tau} + \delta_{K_T}} \frac{d \tau}{1-\tau}.$$

$$\begin{aligned} \frac{d^{\text{static}} R}{Y} &= \left((1 - \mu_{\mathcal{I}}^{-1}) + \frac{\frac{\rho}{1-\tau}}{\frac{\rho}{1-\tau} + \delta_{K_T}} \mu_{K_T} \Lambda_{K_T} \right) d \tau, \\ &= \left((1 - \mu_{\mathcal{I}}^{-1}) \frac{1}{\mu_{K_T} \Lambda_{K_T}} \frac{\frac{\rho}{1-\tau} + \delta_{K_T}}{\frac{\rho}{1-\tau}} + 1 \right) (1 - \tau) \mu_{K_T} \Lambda_{K_T} d \log \mu_{K_T}, \end{aligned}$$

$$\begin{aligned} \frac{d^{\text{dynamic}} R}{Y} &= \frac{d^{\text{static}} R}{Y} + \sum_i \tau \lambda_i (1 - \mu_i^{-1}) d \log \lambda_i \\ &+ \tau \frac{\frac{\rho}{1-\tau}}{\frac{\rho}{1-\tau} + \delta_{K_T}} \mu_{K_T} \Lambda_{K_T} \left(d \log \Lambda_{K_T} + \frac{\frac{\rho}{1-\tau} + \delta_{K_T}}{\frac{\rho}{1-\tau}} d \log \mu_{K_T} \right) - \frac{R}{Y} \left(\frac{\tilde{\Lambda}_{K_T}}{\tilde{\Lambda}_L} d \log \mu_{K_T} + d \log \Lambda_L \right). \end{aligned}$$

E.3.2 Fiscal Multipliers – the Quesnaysian-Keynesian Cross

The derivation is very similar to that in Baqaee (2015). We focus on the steady-state equilibrium featuring zero inflation, full employment, and no government spending after the first period and constant government taxes starting in period $t + 1$. In the first period t , the intertemporal budget constraint of the saver and the Euler equation pin down his consumption in period t

$$\begin{aligned} p_t c_{s,t} &= \frac{p_{t+1} c_{s,t+1}}{\rho_s (1 + i_t)} = \\ &= \frac{\left(w_{i,t+1} \bar{l}_{i,t+1} \Phi_{s,i} + r_{t+1} \bar{K}_{t+1} \Phi_{s,K} \right) (1 - \chi) + (1 - \rho) [D_t (1 + i_t) + B_t (1 + i_t) - G_t (1 - \chi)]}{\rho_s (1 + i_t)}, \end{aligned}$$

where χ is the share of taxes falling on borrowers, and variables with overlines are the endowments of factors in the full employment steady-state. The budget constraint for the borrower pins down his consumption in period t

$$p_t c_{b,t} = \left(\sum_i w_{i,t} l_{i,t} \Phi_{b,i} + r_t K_t \Phi_{b,K} \right) + \frac{D_t}{1+i_t} - D_{t-1},$$

where a change in the debt limit can set off a delveraging shock, similar to Eggertsson and Krugman (2012).

Equating supply and demand requires that

$$c_{s,t} = \frac{\sum_i w_{i,t} l_{i,t} + r_t K_t}{p_t} - c_{b,t} - \frac{G_t}{p_t}.$$

Substituting this into the Euler equation for the saver and imposing the zero lower bound on nominal interest rates gives

$$\begin{aligned} \sum_i w_{i,t} l_{i,t} + r_t k_t - \sum_i w_{i,t} l_{i,t} \Phi_{b,i} - r_t k_t \Phi_{b,K} + D_t - D_{t-1} - G \\ = \frac{1}{\rho} \left(w_{i,t+1} \bar{l}_i \Phi_{s,i} + r_{t+1} \bar{k} \Phi_{s,K} + (1 - \rho_s) (D_t + B_t - G(1 - \chi)) \right). \end{aligned}$$

Call this the AD curve. We know that

$$r_t k_t = \frac{(1 - b' \Psi_{(L)})}{b' \Psi_{(L)}} w_t l_t + \frac{(b - \delta)' \Psi_{(L)}}{b' \Psi_{(L)}} G_t,$$

where $w_t l_t = \sum_i w_{i,t} l_{i,t}$. Furthermore, we also know that

$$\lambda' = b' \Psi (1 - \mathcal{B}) + \delta' \Psi \mathcal{B},$$

or more specifically,

$$\frac{w_{i,t} l_{i,t}}{GDP} = \lambda_i^c \alpha_i (1 - \mathcal{B}) + \lambda_i^g \alpha_i \mathcal{B},$$

where λ^c is private consumption exposure to each good: $b' \Psi$, and λ^g is government consumption exposure to each good: $\delta' \Psi$. Hence, we can write

$$w_{i,t} l_{i,t} = w_t l_t = \lambda_i \alpha_i (w_t l_t + r_t k_t) = (\lambda_i^c (1 - \mathcal{B}) + \lambda_i^g \mathcal{B}) \alpha_i (w_t l_t + r_t k_t)$$

where we use the fact that, without loss of generality and due to stickiness of wages in the first

period, we can assume $w_{i,t} = w_t$. Combining this with the expression for $r_t k_t$ we get

$$w_t l_{i,t} = (\lambda_i^c (1 - \mathcal{B}) + \lambda_i^g \mathcal{B}) \alpha_i \left(\frac{w_t l_t}{\Lambda_L^c} + \frac{(\Lambda_L^c - \Lambda_L^g)}{\Lambda_L^c} G_t \right).$$

Substituting this into the AD curve gives us the desired Keynesian cross.

F Appendix for Section 7

Proof of Proposition 10. We then define the Domar weights

$$\lambda_{\mathcal{I}} \equiv \frac{\sum_{i \in \mathcal{I}} p_i y_i}{Y} = \sum_{i \in \mathcal{I}} \lambda_i,$$

and

$$\lambda_{\mathcal{I}\mathcal{I}} \equiv \frac{p_{\mathcal{I}} x_{\mathcal{I}\mathcal{I}}}{Y} = \frac{\sum_{i \in \mathcal{I}} p_{\mathcal{I}} x_{i\mathcal{I}}}{Y} = \sum_{i \in \mathcal{I}} \lambda_{i\mathcal{I}}.$$

$$d \log p_{\mathcal{I}} \equiv \sum_{i \in \mathcal{I}} \lambda_i^{\mathcal{I},c} d \log p_i,$$

so that

$$d \log p_{\mathcal{I}} + d \log c_{\mathcal{I}} = \sum_{i \in \mathcal{I}} \lambda_i^{\mathcal{I},c} d \log p_i c_i,$$

where

$$\lambda_i^{\mathcal{I},c} = \frac{p_i c_i^{\mathcal{I}}}{\sum_{j \in \mathcal{I}} p_j c_j^{\mathcal{I}}}.$$

This implies

$$d \log p^{\mathcal{I}} = \left(I - \tilde{\Omega}^{\mathcal{I}\mathcal{I}} \right)^{-1} \left(-d \log A + d \log \mu + \tilde{\Omega}^{\mathcal{I}N\mathcal{I}} d \log p^{N\mathcal{I}} \right),$$

where $p^{\mathcal{I}}$, $d \log A$, $d \log \mu$, and $p^{N\mathcal{I}}$ are price vectors. Re-injecting, we get

$$d \log p_{\mathcal{I}} = \left(\lambda^{\mathcal{I},c} \right)' \left(I - \tilde{\Omega}^{\mathcal{I}\mathcal{I}} \right)^{-1} \left(-d \log A + d \log \mu + \tilde{\Omega}^{\mathcal{I}N\mathcal{I}} d \log p^{N\mathcal{I}} \right).$$

Combining, we get an expression for industry-level net productivity:

$$d \log c_{\mathcal{I}} - \sum_{j \notin \mathcal{I}} \tilde{\Lambda}_j^{\mathcal{I}} d \log x_{\mathcal{I}j} = \sum_{i \in \mathcal{I}} \tilde{\lambda}_i^{\mathcal{I}} d \log A_i - \sum_{i \in \mathcal{I}} \tilde{\lambda}_i^{\mathcal{I}} d \log \mu_i - \sum_{j \notin \mathcal{I}} \tilde{\Lambda}_j^{\mathcal{I}} d \log \Lambda_j^{\mathcal{I}}.$$

This equation has a similar interpretation as the aggregate TFP decomposition in Baqaee and Farhi (2017a). Specifically, suppose that we counterfactually keep the physical share of all the

inputs constant as the shocks hit the economy. Call this the passive allocation. We get that for the passive allocation

$$d \log c_{\mathcal{I}} - \sum_j \tilde{\Lambda}_j^{\mathcal{I}} d \log x_{\mathcal{I}j} = \sum_{i \in \mathcal{I}} \tilde{\lambda}_i^{\mathcal{I}} d \log A_i.$$

This gives an interpretation of the change in allocative efficiency as the gap with the passive allocation.

Appendix for Aggregating Industry-Level Aggregates

We have shown how to define and characterize industry aggregates for a given partition of producers into industries. A different question is whether these industry aggregates are sufficient statistics for the model to the first order, a property which we call first-order aggregation. We discuss two different notions by contrasting economic and accounting aggregation. Accounting aggregation generally holds while economic aggregation only holds for certain variables or in very special cases where the economy has an industry structure.

Economic vs. Accounting Aggregation

We say that first-order economic aggregation holds for a variable of interest if the following holds: given the *initial* industry aggregates (input-output information at the industry level, industry markup/wedges), *changes* in industry productivities, and *changes* in industry markups/wedges are enough to compute *changes* in higher level aggregates of the variable of interest associated with coarser partitions of producers into industries, given full knowledge of the underlying disaggregated structure of the model but without knowledge of the changes in producer productivity and markups/wedge shocks that are giving rise to the changes in industry productivities and markups/wedges, and without knowing the initial allocation at a lower level of aggregation.

This notion of *economic* aggregation is different from the more common *accounting* notion of aggregation, which says the *initial values and changes* in industry aggregates are enough to compute *changes* in higher level aggregates of the variable of interest associated with coarser partitions of producers into industries.

Accounting aggregation holds in all the models that we have considered. By contrast, economic aggregation fails in general. First-order economic aggregation holds for productivity defined as changes in Solow residuals. It typically fails for other variables. For example first-order economic aggregation fails for markups/wedges, the levels of which can be computed from industry aggregates but not their changes. First-order economic aggregation only holds for aggregate output if Hulten's theorem holds, such as for example when the economy is efficient and factor supplies are inelastic, and even in that case, it does not hold for output at a strictly intermediate level of aggregation.

Economic Aggregation with an Industry Structure

We now discuss an interesting special case where first-order economic, and in fact, full nonlinear economic aggregation holds. We assume that the economy has an *industry structure*, by which we mean we can relabel the economy so that there exists a partition of producers into industries such that the following two properties hold. First all producers i in industry \mathcal{I} have the same production function up to productivity shocks so that they charge prices $p_i = (\mu_i/A_i)C_{\mathcal{I}}$ where $C_{\mathcal{I}}$ is a common marginal cost function. Second, all producers i in industry \mathcal{I} enter into the production functions of other producers and in final demand through a single CES industry aggregator

$$\frac{y_{\mathcal{I}}}{\bar{y}_{\mathcal{I}}} = \left(\sum_{i \in \mathcal{I}} \omega_{\mathcal{I}i} \left(\frac{y_i}{\bar{y}_i} \right)^{\frac{\theta_{\mathcal{I}}-1}{\theta_{\mathcal{I}}}} \right)^{\frac{\theta_{\mathcal{I}}}{\theta_{\mathcal{I}}-1}}.$$

This producer is a pure aggregator and is otherwise passive: it does not experience productivity shocks and charges no markup/wedge. We can compute the associated price index

$$p_{\mathcal{I}} = \left(\sum_{i \in \mathcal{I}} \omega_{\mathcal{I}i}^{\theta_{\mathcal{I}}} \left(\frac{\bar{y}_{\mathcal{I}}}{\bar{y}_i} \right)^{\theta_{\mathcal{I}}-1} p_i^{1-\theta_{\mathcal{I}}} \right)^{\frac{1}{1-\theta_{\mathcal{I}}}}.$$

These have the properties that in equilibrium, $p_{\mathcal{I}}y_{\mathcal{I}} = \sum_{i \in \mathcal{I}} p_i y_i$, $p_{\mathcal{I}}y_{\mathcal{I}}/Y = \lambda_{\mathcal{I}}$, and the log derivative of this index then coincides exactly with the definition of $d \log y_{\mathcal{I}}$ given above. The structure of the model also ensures that all the producers in the industry have the same production function, up to productivity shocks.

We can then define an index of industry productivity

$$A_{\mathcal{I}} \equiv \mu_{\mathcal{I}} \left(\sum_{i \in \mathcal{I}} \omega_{\mathcal{I}i}^{\theta_{\mathcal{I}}} \left(\frac{\bar{y}_{\mathcal{I}}}{\bar{y}_i} \right)^{\theta_{\mathcal{I}}-1} \left(\frac{A_i}{\mu_i} \right)^{\theta_{\mathcal{I}}-1} \right)^{\frac{1}{\theta_{\mathcal{I}}-1}} = \bar{A}_{\mathcal{I}} \frac{\mu_{\mathcal{I}}}{\bar{\mu}_{\mathcal{I}}} \left(\sum_{i \in \mathcal{I}} \frac{\bar{\lambda}_i}{\bar{\lambda}_{\mathcal{I}}} \left(\frac{A_i \bar{\mu}_i}{\bar{A}_i \mu_i} \right)^{\theta_{\mathcal{I}}-1} \right)^{\frac{1}{\theta_{\mathcal{I}}-1}},$$

where $\mu_{\mathcal{I}}$ is the industry markup/wedges defined above

$$\mu_{\mathcal{I}} = \left(\sum_{i \in \mathcal{I}} \frac{\lambda_i}{\lambda_{\mathcal{I}}} \mu_i^{-1} \right)^{-1}.$$

These definitions are chosen to match the profit share $1 - \mu_{\mathcal{I}}^{-1} = \sum_{i \in \mathcal{I}} (\lambda_i/\lambda_{\mathcal{I}})(1 - \mu_i^{-1})$ of the industry as above and in addition the industry price cost margin relationship $p_{\mathcal{I}} = (\mu_{\mathcal{I}}/A_{\mathcal{I}})C_{\mathcal{I}}$. It is then easy to see that

$$d \log \hat{S}R_{\mathcal{I}}^{\text{gross}} = d \log A_{\mathcal{I}},$$

so that the corrected gross Solow residual recovers the change in productivity index.

Note that just like the industry markup, industry productivity depends on the endogenous propagation mechanisms of the model because $\mu_{\mathcal{I}}$, λ_i , and $\lambda_{\mathcal{I}}$ do. Indeed, we have

$$\begin{aligned} d \log \mu_{\mathcal{I}} &= \sum_{i \in \mathcal{I}} \frac{\lambda_i \mu_{\mathcal{I}}}{\lambda_{\mathcal{I}} \mu_i} d \log \mu_i - \sum_{i \in \mathcal{I}} \frac{\lambda_i \mu_{\mathcal{I}}}{\lambda_{\mathcal{I}} \mu_i} d \log \left(\frac{\lambda_i}{\lambda_{\mathcal{I}}} \right), \\ d \log A_{\mathcal{I}} &= d \log \mu_{\mathcal{I}} + \sum_{i \in \mathcal{I}} \frac{\lambda_i}{\lambda_{\mathcal{I}}} (d \log A_i - d \log \mu_i), \end{aligned}$$

which combined with the shares propagation equations, gives a full structural characterization of the elasticities of industry markups and productivity to the different productivities and markups/wedges:

$$d \log \mu_{\mathcal{I}} = \sum_{i \in \mathcal{I}} (\theta_{\mathcal{I}} - 1) \frac{\lambda_i}{\lambda_{\mathcal{I}}} \left(1 - \frac{\mu_{\mathcal{I}}}{\mu_i} \right) d \log A_i + \sum_{i \in \mathcal{I}} \left[\frac{\lambda_i}{\lambda_{\mathcal{I}}} + \theta_{\mathcal{I}} \frac{\lambda_i}{\lambda_{\mathcal{I}}} \left(\frac{\mu_{\mathcal{I}}}{\mu_i} - 1 \right) \right] d \log \mu_i, \quad (42)$$

$$d \log A_{\mathcal{I}} = \sum_{i \in \mathcal{I}} \left[\frac{\lambda_i}{\lambda_{\mathcal{I}}} + (\theta_{\mathcal{I}} - 1) \frac{\lambda_i}{\lambda_{\mathcal{I}}} \left(1 - \frac{\mu_{\mathcal{I}}}{\mu_i} \right) \right] d \log A_i + \sum_{i \in \mathcal{I}} \theta_{\mathcal{I}} \frac{\lambda_i}{\lambda_{\mathcal{I}}} \left(\frac{\mu_{\mathcal{I}}}{\mu_i} - 1 \right) d \log \mu_i. \quad (43)$$

The model can then be economically aggregated. At higher levels of aggregation corresponding to coarser partitions, we can represent the economy as being composed of fictitious industry producers with markups and productivities given by $A_{\mathcal{I}}$ and $\mu_{\mathcal{I}}$. As a consequence, first-order economic aggregation obtains: knowing the initial industry aggregates and the changes in industry productivities and markups $d \log A_{\mathcal{I}}$ and $d \log \mu_{\mathcal{I}}$ is enough to compute the changes in the equilibrium allocation at higher levels of aggregation, without knowing what the changes in productivities and markups are at a lower level of aggregation, and without knowing what the initial allocation is at lower levels of aggregation.

Consider the following simple example of an economy with an industry structure. We assume that there are different industry indexed by \mathcal{I} . Final demand is represented by a CES aggregator of the different industry composite goods with an elasticity of substitution θ . Industry composite goods are CES aggregates elasticity of substitution $\theta_{\mathcal{I}}$ of the goods produced by the different producers in the industry. Producers produce linearly from labor, have different productivities, and charge different markups. Basically, each industry is a version of the horizontal economy studied in Section E.3.1. The important difference is that labor can be reallocated across industries. Economic aggregation obtains. For example, we get

$$d \log \mu = \sum_{\mathcal{I}} (\theta - 1) \lambda_{\mathcal{I}} \left(1 - \frac{\mu}{\mu_{\mathcal{I}}} \right) d \log A_{\mathcal{I}} + \sum_{\mathcal{I}} \left[\lambda_{\mathcal{I}} + \theta \lambda_{\mathcal{I}} \left(\frac{\mu}{\mu_{\mathcal{I}}} - 1 \right) \right] d \log \mu_{\mathcal{I}}, \quad (44)$$

$$d \log A = \sum_{\mathcal{I}} \left[\lambda_{\mathcal{I}} + (\theta - 1) \lambda_{\mathcal{I}} \left(1 - \frac{\mu}{\mu_{\mathcal{I}}} \right) \right] d \log A_i + \sum_{i \in \mathcal{I}} \theta \lambda_{\mathcal{I}} \left(\frac{\mu}{\mu_{\mathcal{I}}} - 1 \right) d \log \mu_{\mathcal{I}}, \quad (45)$$

where $d \log A_{\mathcal{I}}$ and $d \log \mu_{\mathcal{I}}$ are given by equations (42) and (43).

G Appendix: Non-homothetic Final Demand

In this section, we show how non-homothetic final demand, even with a representative consumer, affects our results and breaks symmetric propagation.

To that end, consider household utility function used by Comin, Lashkari, and Mestieri (2015). Then final-demand expenditure for good i is given by

$$b_i = \bar{b}_i \left(\frac{p_i}{E} \right)^{1-\theta_0} C^{\varepsilon_i},$$

where C is utility and E is the expenditure function of the household. Note that the fact that $\varepsilon_i \neq \varepsilon_j$ means that Engel curves can have different slopes. The expenditure function is given by

$$E(p, C) = \left(\sum_i \bar{b}_i C^{\varepsilon_i} p_i^{1-\theta_0} \right)^{\frac{1}{1-\theta_0}} = \sum_i p_i c_i,$$

which implicitly defines the consumption aggregator, or utility, C . Note that due to non-homotheticity, changes in utility and changes in real GDP may not be the same.

Proposition 16. *Consider a one-factor model with a representative consumer whose preferences are of the Comin, Lashkari, and Mestieri (2015) type. Suppose that the production functions of the goods producers in the economy belong to the nested-CES class, written in standard form. Finally, assume that the primary factor is inelastically supplied and that the model is efficient. Then*

$$\frac{d \log Y}{d \log A_k} = \lambda_k, \quad \frac{d \log C}{d \log A_k} = \frac{(1 - \theta_0)}{\bar{\varepsilon}} \lambda_k,$$

and

$$\frac{d \lambda_m}{d \log A_k} = \sum_j (\theta_j - 1) \lambda_j \text{Cov}_{\Omega(j)}(\Psi_{(m)}, \Psi_{(k)}) - \frac{(\theta_0 - 1)}{\bar{\varepsilon}} \lambda_k \text{Cov}_b(\Psi_{(m)}, \varepsilon),$$

where $\bar{\varepsilon} = \sum_i b_i \varepsilon_i$.

Clearly, the final term is non-symmetric in k and m . A shock to k moves aggregate income by λ_k whereas a shock to m moves aggregate income by λ_m (following Hulten's theorem). Given a change in aggregate income, final demand shifts across different inputs. Changes in final demand, therefore, change the Domar weight of different goods. The way changes in final demand affect demand λ_m for m depends on the covariance between the income elasticity of the Engel curves ε and the exposure $\Psi_{(m)}$ of final goods to m . As the household becomes richer, it redirects expenditures in favor of those goods with relatively higher income elasticities ε_i . As

it does this substitution, this redirects demand away from or towards $\Psi_{(m)}$, depending on the covariance between these $\Psi_{(m)}$ and ε . In the extreme case when all income elasticities are the same, then we recover symmetric propagation, since in that case final demand is homothetic.

Example: Hump-shaped Structural Change

A common objective of the literature on structural transformation is to generate hump-shaped pattern in manufacturing as a function of productivity, so that initially, as manufacturing becomes more productive, its share in the economy expands, but eventually this pattern reverses. Comin, Lashkari, and Mestieri (2015) show that their specification is capable of generating such patterns. We can recover their insight via Proposition 16. Consider an economy with one factor. Suppose that household consumption has the Comin, Lashkari, and Mestieri (2015) form, and further, suppose that goods are substitutes for the household so that $\theta_0 < 1$. For simplicity, assume that $\theta_j = 1$ for all $j \neq 1$, so that all other production functions are Cobb-Douglas.

It's easy to check that for any good i ,

$$\frac{d \log \lambda_i}{d \log C} = (\theta_0 - 1) \frac{d \log E}{d \log C} + \varepsilon_i = \varepsilon_i - \bar{\varepsilon},$$

so that λ_i is declining in utility if $\varepsilon_i < \bar{\varepsilon}$.

Let one of the goods be manufacturing, indexed by m . Then, consider an aggregate labor-augmenting productivity shock $d \log A_L$. An almost immediate application of our formula gives

$$\frac{d \log \lambda_m}{d \log A_L} = (\theta_0 - 1) \text{Cov}_b(\Psi_{(m)}, \mathbf{1}) - \frac{(\theta_0 - 1)}{\bar{\varepsilon}} \text{Cov}_b(\Psi_{(m)}, \varepsilon) = -\frac{(\theta_0 - 1)}{\bar{\varepsilon}} \text{Cov}_b(\Psi_{(m)}, \varepsilon).$$

In the case where demand is homothetic, this gives zero. When demand is non-homothetic, the effect depends on the covariance between exposure $\Psi_{(m)}$ of goods to m and the income-effects ε_i . An especially simple case is the horizontal economy in Figure 1b, where there are no intermediate inputs and all goods are directly produced from labor, where $\Psi_{(m)}$ is just the m th standard basis vector. Then

$$\frac{d \log \lambda_m}{d \log A_L} = \frac{(1 - \theta_0)}{\bar{\varepsilon}} \lambda_m (\varepsilon_m - \bar{\varepsilon}).$$

Comin, Lashkari, and Mestieri (2015) argue that for broad categories, like agriculture, manufacturing, and services, $\theta_0 < 1$ so that goods are gross complements. They also find that $\varepsilon_a < \varepsilon_m < \varepsilon_s$, where a, m, s correspond to agriculture, manufacturing, and services. Then, it's clear that when the share of services and manufacturing is low, $\varepsilon_m > \bar{\varepsilon}$, and hence, $d \lambda_m$ is increasing in aggregate labor productivity. However, eventually, as λ_m and λ_s get large enough,

$\varepsilon_m < \bar{\varepsilon}$ and this effect reverses. This gives a hump-shaped path of λ_m as a function of aggregate productivity growth A_L .

Comin, Lashkari, and Mestieri (2015) also emphasize that manufacturing productivity has increased more quickly than services productivity, but the quantity of services has increased over time relative to manufacturing. A homothetic function is not able to generate such a pattern. To see this, consider improvements in manufacturing productivity $d \log A_m$. Then, an application of Proposition 16 gives

$$\frac{d \log \lambda_m}{d \log A_m} = (\theta_0 - 1) \text{Var}_b(\Psi_{(m)}) - \frac{(\theta_0 - 1)}{\bar{\varepsilon}} \lambda_m \text{Cov}_b(\Psi_{(m)}, \varepsilon).$$

An especially simple case is the horizontal economy in Figure 1b, where there are no intermediate inputs and all goods are directly produced from labor. In that case, our expression simplifies to

$$\frac{d \log \lambda_m}{d \log A_m} = (\theta_0 - 1) \lambda_m (1 - \lambda_m) + \frac{(\theta_0 - 1)}{\bar{\varepsilon}} \lambda_m^2 (\bar{\varepsilon} - \varepsilon_m).$$

The quantity of manufacturing is given by

$$\frac{d \log y_m}{d \log A_m} = (\theta_0 - 1) \lambda_m (1 - \lambda_m) + 1 + \frac{(\theta_0 - 1)}{\bar{\varepsilon}} \lambda_m^2 (\bar{\varepsilon} - \varepsilon_m).$$

This follows from $d \log y_m = d \log \lambda_m - d \log p_m + d \log(P_Y Y) = d \log \lambda_m - d \log p_m$.

Clearly, when demand is homothetic, $d \log y_m / d \log A_m > 0$, since $\lambda_m(1 - \lambda_m) < 1$. However, with non-homotheticities, it is possible to have $d \log y_m / d \log A_m$ be negative as long as $\bar{\varepsilon} \gg \varepsilon_m$.

Proof of Proposition 16. Note that

$$\frac{\partial \log E}{\partial \log C} = \frac{1}{1 - \theta_0} \sum_i b_i \varepsilon_i.$$

Then, setting $E = 1$ to be the numeraire, we have

$$\frac{d \log E}{d \log p_i} = \frac{\partial \log E}{\partial \log p_i} + \frac{\partial \log E}{\partial \log C} \frac{d \log C}{d \log p_i} = 0.$$

This implies that

$$\frac{\partial \log C}{\partial \log p_i} = - \frac{\frac{\partial E}{\partial \log p_i}}{\frac{\partial \log E}{\partial \log C}}.$$

Shephard's Lemma then implies that

$$\frac{\partial \log C}{\partial \log p_i} = -\frac{b_i}{\frac{\partial \log E}{\partial \log C}} = (\theta_0 - 1) \frac{b_i}{\sum_j b_j \varepsilon_j}.$$

On the other hand, we have

$$\begin{aligned} \frac{d b_i}{d \log p_i} &= (1 - \theta_0) b_i + \varepsilon_i b_i \frac{d \log C}{d \log p_i} - (1 - \theta_0) b_i \frac{d \log E}{d \log p_i}, \\ &= (1 - \theta_0) b_i - (1 - \theta_0) \frac{\varepsilon_i}{\sum_j b_j \varepsilon_j} b_i - 0, \\ &= (1 - \theta_0) b_i \left(1 - \frac{\varepsilon_i}{\sum_j b_j \varepsilon_j} b_i \right). \end{aligned}$$

Where restore homotheticity when $\varepsilon_i = \varepsilon$ for all i . To see how symmetric propagation fails to hold, consider a model with a single factor. Using this demand system, we get

$$\frac{d \lambda_m}{d \log A_k} = (\theta_0 - 1) \sum_i b_i \psi_{ik} \psi_{im} + \sum_i \varepsilon_i b_i \psi_{im} \frac{d \log C}{d \log A_k} + \sum_j (\theta_j - 1) \lambda_j \text{Cov}_{\Omega^{(j)}}(\Psi_{(m)}, \Psi_{(k)}).$$

Noting that

$$\begin{aligned} \frac{d \log E}{d \log A_k} &= \sum_i \frac{\partial \log E}{\partial \log p_i} \frac{d \log p_i}{d \log A_k} + \frac{\partial \log E}{\partial \log C} \frac{d \log C}{d \log A_k}, \\ &= -\sum_i b_i \psi_{ik} + \frac{1}{1 - \theta_0} \left(\sum_i b_i \varepsilon_i \right) \frac{d \log C}{d \log A_k} = 0. \end{aligned}$$

Rearrange this to get

$$\frac{d \log C}{d \log A_k} = \frac{(1 - \theta_0)}{\sum_i b_i \varepsilon_i} \lambda_k.$$

Substitute this back into the earlier expression to get

$$\frac{d \lambda_m}{d \log A_k} = (\theta_0 - 1) \sum_i b_i \psi_{ik} \psi_{im} - (\theta_0 - 1) \frac{(\sum_i \varepsilon_i b_i \psi_{im})}{\sum_i b_i \varepsilon_i} \lambda_k + \sum_j (\theta_j - 1) \lambda_j \text{Cov}_{\Omega^{(j)}}(\Psi_{(m)}, \Psi_{(k)}).$$

In other words,

$$\frac{d \lambda_m}{d \log A_k} = (\theta_0 - 1) \text{Cov}_b(\Psi_{(m)}, \Psi_{(k)}) + \sum_j (\theta_j - 1) \lambda_j \text{Cov}_{\Omega^{(j)}}(\Psi_{(m)}, \Psi_{(k)}) + (\theta_0 - 1) \lambda_k \left(\sum_i b_i \psi_{im} \left(1 - \frac{\varepsilon_i}{\varepsilon} \right) \right).$$

or

$$\frac{d \lambda_m}{d \log A_k} = (\theta_0 - 1) \text{Cov}_b(\Psi_{(m)}, \Psi_{(k)}) + \sum_j (\theta_j - 1) \lambda_j \text{Cov}_{\Omega^{(j)}}(\Psi_{(m)}, \Psi_{(k)}) - \frac{(\theta_0 - 1)}{\bar{\varepsilon}} \lambda_k \text{Cov}_b(\Psi_{(m)}, \varepsilon).$$

This can be written more compactly as

$$\frac{d \lambda_m}{d \log A_k} = (\theta_0 - 1) \text{Cov}_b(\Psi_{(m)}, \Psi_{(k)} - \lambda_k \varepsilon / \bar{\varepsilon}) + \sum_j (\theta_j - 1) \lambda_j \text{Cov}_{\Omega^{(j)}}(\Psi_{(m)}, \Psi_{(k)}).$$

$$\frac{d \lambda_m}{d \log A_k} = (\theta_0 - 1) \text{Cov}_b(\Psi_{(m)}, \Psi_{(k)} - \lambda_k \varepsilon / \bar{\varepsilon}) + \sum_j (\theta_j - 1) \lambda_j \text{Cov}_{\Omega^{(j)}}(\Psi_{(m)}, \Psi_{(k)}).$$

□

H Appendix: Heterogeneous Agents and Distortions

When there are heterogeneous consumers and there are wedges, we have to track how revenues earned by wedges are distributed across consumers. To do this, it is convenient to introduce “fictitious” factors corresponding to revenues earned by consumer c from taxes. These fictitious factors have a cost-based Domar weight of zero, but a positive revenue-based Domar weight. Define revenue-based and cost-based variables in the usual way. Let F be the set of real factors, and let F^* be the set of real and fictitious factors.

Proposition 17. *In response to some increase in productivity, we have*

$$\begin{aligned} \frac{d \log \lambda_i}{d \log A_k} &= \sum_j (1 - \theta_j) \frac{\lambda_j}{\lambda_i} \mu_j^{-1} \text{Cov}(\tilde{\Psi}_{(k)}, \Psi_{(i)}) + \sum_{g \in F} \sum_j (1 - \theta_j) \frac{\lambda_j}{\lambda_i} \mu_j^{-1} \text{Cov}(\tilde{\Psi}_g, \Psi_{(i)}) \frac{d \log \Lambda_g}{d \log A_k} \\ &+ \sum_c \chi_c \frac{\lambda_i^c}{\lambda_i} \frac{d \log \chi_c}{d \log A_k}. \end{aligned}$$

Income changes are given by

$$\frac{d \log \chi_c}{d \log A_k} = \left(\sum_{g \in F^*} \Phi_{cg} \Lambda_g \frac{d \log \Lambda_g}{d \log A_k} \right),$$

where g sums over real and fictitious factors. The changes in Domar weight of real factors is given by

$$\begin{aligned} \frac{d \log \Lambda_g}{d \log A_k} &= \sum_j (1 - \theta_j) \frac{\lambda_j}{\Lambda_g} \mu_j^{-1} \text{Cov}(\tilde{\Psi}_{(k)}, \Psi_{(g)}) + \sum_{g \in F} \sum_j (1 - \theta_j) \frac{\lambda_j}{\Lambda_g} \mu_j^{-1} \text{Cov}(\tilde{\Psi}_g, \Psi_{(g)}) \frac{d \log \Lambda_g}{d \log A_k} \\ &+ \frac{1}{\Lambda_g} \sum_c \chi_c \left(\Lambda_g^c - \Lambda_g \right) \frac{d \log \chi_c}{d \log A_k}. \end{aligned}$$

The changes in the Domar weight of fictitious factors k^* is given by

$$\frac{d \log \Lambda_i^*}{d \log \mu_k} = d \log \lambda_i,$$

where Λ_i^* is the income earned by the markup placed on the sales of the i th good.

Proposition 18. In response to some increase in a markup/wedge, we have

$$\begin{aligned} \frac{d \log \lambda_i}{d \log \mu_k} &= \sum_j (1 - \theta_j) \frac{\lambda_j}{\lambda_i} \mu_j^{-1} \text{Cov}(\tilde{\Psi}_{(k)}, \Psi_{(i)}) + \sum_{g \in F} \sum_j (1 - \theta_j) \frac{\lambda_j}{\lambda_i} \mu_j^{-1} \text{Cov}(\tilde{\Psi}_g, \Psi_{(i)}) \frac{d \log \Lambda_g}{d \log \mu_k} \\ &\quad - \frac{\lambda_k}{\lambda_i} (\Psi_{ki} - \mathbf{1}(k = i)) + \sum_c \chi_c \frac{\lambda_i^c}{\lambda_i} \frac{d \log \chi_c}{d \log \mu_k}. \end{aligned}$$

Income changes are given by

$$\frac{d \log \chi_c}{d \log \mu_k} = \left(\sum_{g \in F^*} \Phi_{cg} \Lambda_g \frac{d \log \Lambda_g}{d \log \mu_k} \right),$$

where g sums over real and fictitious factors. The changes in Domar weight of real factors is given by

$$\begin{aligned} \frac{d \log \Lambda_g}{d \log \mu_k} &= \sum_j (1 - \theta_j) \frac{\lambda_j}{\Lambda_g} \mu_j^{-1} \text{Cov}(\tilde{\Psi}_{(k)}, \Psi_{(g)}) + \sum_{g \in F} \sum_j (1 - \theta_j) \frac{\lambda_j}{\Lambda_g} \mu_j^{-1} \text{Cov}(\tilde{\Psi}_g, \Psi_{(g)}) \frac{d \log \Lambda_g}{d \log \mu_k} \\ &\quad - \frac{\lambda_k}{\Lambda_g} (\Psi_{ki} - \mathbf{1}(k = i)) + \frac{1}{\Lambda_g} \sum_c \chi_c (\Lambda_g^c - \Lambda_g) \frac{d \log \chi_c}{d \log \mu_k}. \end{aligned}$$

The changes in the Domar weight of fictitious factors k^* is given by

$$\frac{d \log \Lambda_i^*}{d \log \mu_k} = d \log \lambda_i + \frac{1}{\mu_i - 1} \mathbf{1}(i = k),$$

where Λ_i^* is the income earned by the markup placed on the sales of the i th good.

To recover changes in prices and quantities, we can apply the same results as in Section 6.