Monetary Policy, Bounded Rationality, and Incomplete Markets*

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January 2019

This paper extends the benchmark New-Keynesian model by introducing two frictions: (1) agent heterogeneity with incomplete markets, uninsurable idiosyncratic risk, and occasionally-binding borrowing constraints; and (2) bounded rationality in the form of level-k thinking. Compared to the benchmark model, we show that the *interaction* of these two frictions leads to a powerful mitigation of the effects of monetary policy, which is more pronounced at long horizons, and offers a potential rationalization of the "forward guidance puzzle". Each of these frictions, in isolation, would lead to no or much smaller departures from the benchmark model.

1 Introduction

The baseline New Keynesian setup is a workhorse model for monetary policy analysis. However, in its basic form, it also has implications that are controversial or unrealistic. For example, despite various concrete results that limit the number of equilibria, indeterminacy concerns remain. In addition, although the model provides a rationale for effective monetary policy, some view the power of monetary policy as too effective, and changes in future interest rates may be especially powerful—the so-called "forward guidance puzzle".[1] Finally, while the model can

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[1] A similar issue arises in the context of the fiscal policy: as shown by Christiano, Eichenbaum and Rebelo (2011) at the zero lower bound spending is very stimulative. As shown by Farhi and Werning (2016a), future spending is more powerful—a "fiscal forward guidance puzzle". In ongoing work we apply the framework we introduce here to fiscal policy and show that level-k thinking mitigates the inflation-output feedback loop which is responsible for these effects, and improves the realism of the model.

explain recessive effects arising at the zero lower bound or following from other contractive monetary policies, these effects seem excessive.[2]

These shortcomings follow from the extreme forward-looking nature of the model which is, in turn, due to the assumptions of complete markets and rational expectations. This paper studies the effects of monetary policy exploring two realistic departures from the benchmark model. Our first departure is to allow for heterogeneous agents and incomplete markets. Our second departure is to adopt a particular form of bounded rationality. As we will show, these two frictions interact and help make the model more realistic.

In standard New-Keynesian models changes in future real interest rates have the same effect on current output as changes in current real interest rates, a property that some have labeled the "forward guidance puzzle", as introduced by Del Negro et al. (2015).[3] In isolation, the departures that we consider potentially alter this property, but only moderately so. . However, the combination of both departures, significantly reduces the sensitivity of current output to future interest rate changes—we call this the mitigation effect—the more so, the further in the future they take place—we call this the horizon effect. In other words, incomplete markets and level-k bounded rationality are *complements*.

Our first deviation replaces the representative-agent, complete-market assumption with heterogenous agents making consumption decisions subject to idiosyncratic shocks to income. These shocks cannot be insured and borrowing is limited. These realisistic frictions hinder the capacity of households to smooth their consumption, potentially affecting the potency of forward guidance. Intuitively, if agents expect to be borrowing constrained in the near future, then changes in future interest rates should not greatly influence their current consumption decisions. This line of reasoning was put forth by McKay et al. (2016). However, as shown by Werning (2015), while incomplete markets always have an effect on the *level* of aggregate consumption, the way it affects its *sensitivity* to current and future interest rates is less clear. Indeed, this sensitivity is completely unchanged in some benchmark cases and may be enhanced in others. This implies that the power of forward guidance is not necessarily diminished by incomplete markets, at least not without adopting other auxiliary assumptions.[4] Here we adopt

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[2] For example, in deterministic models, recessionary forces become arbitrarily large as the duration of the liquidity trap lengthens, and in some stochastic models, when the probability of remaining in the liquidity trap is large enough, the equilibrium simply ceases to exist. These effects are exact manifestations of the forward guidance puzzle in reverse—applied to a situation where monetary policy is too contractionary over the horizon of the liquidity trap. Although we do not develop these applications explicitly in the paper, our resolution of the forward guidance puzzle also leads to a resolution of these liquidity trap paradoxes.

[3] The term "puzzle" is somewhat subjective. This standard disclaimer is perhaps especially relevant in this case given that standard empirical identification challenges are heightened when focusing on forward guidance shocks relative to standard monetary shocks.

[4] As highlighted in Werning (2015), two features that push to mitigate the impact of future interest rates relative to current interest rates on current aggregate consumption are: (i) procyclicality of income risk, making precautionary savings motives low during a recession; and (ii) countercyclicality of liquidity relative to income, making asset prices or lending fluctuate less than output. If one adopts the reverse assumptions, as a large literature does—so that recessions heighten risk, precautionary savings and are accompanied by large drops in asset prices

the benchmark cases that imply the neutral conclusion that incomplete markets have no effect on the sensitivity of aggregate consumption to interest rates.

Our second deviation drops the rational-expectations assumption in favor of a form of bounded rational expectations that we refer to as "level-k thinking", which describes how expectations react to a change in policy. Starting from a status quo rational-expectations equilibrium, agents form expectations about changes in future macroeconomic variables based on a finite deductive procedure about others' behavior, involving k iterations. This form of bounded rationality has received attention and support in game theory settings, both in theory and in lab experiments.[5] Closely related concepts have been employed in macroeconomic settings, such as the "calculation equilibrium" of Evans and Ramey (1992; 1995; 1998) and the "reflective equilibrium" of Garcia-Schmidt and Woodford (2019).

Our choice of level-k thinking amongst the "wilderness" of bounded rationality deserves further motivation. We believe it to be well suited for the economic scenarios that involve salient policy announcements. Especially during times of crises or at the zero lower bound, policy changes involve heightened attention by markets and announcements are widely communicated, scrutinized, and discussed. However, the effects that such policy shifts have on the economy are far from obvious. This is especially true if the policies are relatively rare and unusual, or if past data is noisy. In the context of monetary policy we consider the credible announcement of a new interest rate policy path that is swiftly and fully incorporated into the yield curve (Del Negro et al., 2015). However, unlike interest rates, output and inflation are not directly under the control of central banks, so agents must form expectations about them indirectly. Backward-looking learning approaches to the formation of such expectations seem inadequate, since agents realize that past experience offers little guidance. Similarly, approaches based on inattention or private information miss the salience of the policy announcement and would imply a dampened reaction of the yield curve. Moreover, both approaches fail to draw a distinction between exogenous policy variables (e.g. interest rates) and endogenous macroeconomic variables (e.g. output and inflation).

In contrast, our notion of level-k thinking is forward-looking and does not assume imperfect observation of policy announcements, but instead focuses on the limited capacity to foresee the implications of such announcements. Agents are aware of the new path of interest rates, which is thus fully reflected in the yield curve. Agents make an effort to deduce the implications for output and inflation, but stop short of achieving perfect foresight. Our main results show that

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or lending relative to GDP—then aggregate consumption becomes even more sensitive to future interest rates, relative to current interest rates. We adopt a neutral benchmark where risk and liquidity are acyclical.

[5] For evidence supporting level-k thinking in laboratory experiments in games of full information, see Stahl and Wilson (1994), Stahl and Wilson (1995), Nagel (1995), Costa-Gomes et al. (2001), Camerer et al. (2004), Costa-Gomes and Crawford (2006), Arad and Rubinstein (2012), Crawford et al. (2013), Kneeland (2015), and Mauersberger and Nagel (2018). Almost all the estimates point to low levels of reasoning (between 0 and 3) when the subjects are confronted with a new game but have had the rules explained to them.

	Complete Markets	Incomplete Markets
Rational Expectations	Benchmark	Zero or Modest Improvement
Bounded Rationality	Modest Improvement	Sizable Improvement

Table 1: Schematic summary of results illustrating the complementarity of bounded rationality and incomplete markets in mitigating the extreme effects of expected future interest rates (i.e. forward guidance puzzle) present in the benchmark New Keynesian model.

each departure from the standard model in isolation has moderate or zero effects, but that the combination of both incomplete markets and level-*k* bounded rationality has the potential to significantly dampen the reaction of current output to future interest rates. Given the empirical relevance of both departures from the representative agent rational expectations model, we believe that this provides a realistic resolution of the "forward guidance puzzle". Table 1 provides a schematic summary.

Our basic mechanism can be best appreciated under the simplifying assumption of prices and wages that are fully rigid, which we adopt for most of the paper (we relax this in Section 5). In our model, households care to forecast the path for aggregate income because of its effect on future household income. With full price rigidity, given the new interest rate path, this turns out to be the only endogenous macroeconomic variable that households need to forecast. They form these expectations according to the following iterative level-*k* iterative. Level-1 thinking assumes that agents expect the path for future output to remain as in the original rational-expectations equilibrium before the announced change in the path of interest rates. Given current assets and income, individuals choose consumption and savings, reacting to the new interest rate path, using the status quo expectations for future aggregate income. In equilibrium, aggregate output equals aggregate consumption in each period, and the economy is in (general) equilibrium. In the *k*-th deductive round, households take the path of future output to be the equilibrium path of output that obtains in the previous round, etc. This process converges to the rational-expectations equilibrium when the number of rounds *k* goes to ∞.[6]

We start in Section 2 by formally introducing our equilibrium concepts (temporary equilibrium, rational-expectations equilibrium, level-*k* equilibrium) with a general reduced-form aggregate consumption function. All the explicit models derived later in the paper can be seen as special cases yielding specific micro-foundations for the reduced-form aggregate consump-

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[6] An interesting advantage of working with level-*k* is that it sidesteps issues of indeterminacy, as argued forcefully by Garcia-Schmidt and Woodford (2019). Indeed, for any shift in the path of interest rates, the equilibrium outcome for any level *k* is unique. Indeed, one can see level-*k* thinking as a selection device which isolates a particular rational expectations equilibrium in the limit when *k* goes to ∞, without having to resort to policy rules or the Taylor principle. When prices are rigid, level-*k* converges to rational expectations when *k* goes to ∞. When prices have some degree of flexibility, each level-*k* equilibrium remains uniquely determined, but the convergence depends on the monetary policy rule. We obtain convergence with a Taylor rule.

tion function by aggregating individual consumption functions.

We offer a decomposition of the response of output to interest rate changes under rational expectations into a partial equilibrium effect and a general equilibrium effect. The partial equilibrium effect computes the change in aggregate consumption resulting from the change in the path of interest rates, holding the expected path for aggregate income unchanged. The general equilibrium effect then considers the change in aggregate consumption resulting from adjusting the expectations of future aggregate income. Under some conditions, the level-1 outcome coincides with the partial equilibrium response, since it keeps expectations about future aggregate income unchanged. As one increases the number of rounds of thinking, the level-k outcome incorporates to a greater extent the general equilibrium effects from increased future expected aggregate income, converging to the rational-expectations outcome as k goes to ∞.

Overall, our results in Sections 3-5 indicate that the interaction of bounded rationality and incomplete markets has the potential for significant mitigation and horizon effects in monetary policy, even when each element has only modest effects in isolation. There are two critical general mechanisms at play: level-k thinking mitigates general equilibrium effects, making the equilibrium response closer to the partial equilibrium response; and incomplete markets tends to mitigate the partial equilibrium response. Uncovering these mechanisms suggests that our conclusions are robust to the details of the market incompleteness.

Some intuition can be grasped by contrasting two tractable cases which are amenable to closed-form solutions: the complete-markets or representative-agent model, covered in Section 3; and a perpetual youth model with annuities where different lifespans can be re-interpreted as intervals between occasionally-binding borrowing constraints, covered in Section 4.

Consider first the complete-markets or representative-agent model. Bounded rationality affects the response of consumption to the path of interest rates. First, the effect of the current interest rate is equal to the one under rational expectation, but the effects of any future interest rate change on output are lower, implying that there is a mitigation effect. Second, this mitigation is stronger for interest rate changes occurring further out in the future, implying that there is a horizon effect.

Qualitative conclusions aside, our calculations show that the mitigation and horizon effects obtained with level-k bounded rationality and a representative agent are relatively modest. In particular, for level-1 we show that the response of current output to an interest rate change decreases exponentially with the horizon with an exponent equal to the interest rate. That is, the response is proportional to $e^{-r\tau}$ where τ is the horizon and r is the interest rate. For example, with an interest rate of 2%, the effect on output of an interest rate change in 4 years is a fraction 0.92 of the effect of a contemporaneous interest rate change, arguably a small amount of mitigation and horizon.

Consider now the perpetual youth model of occasionally-binding borrowing constraints. We specify the model with logarithmic utility to ensure that there are neither mitigation nor horizon effects under rational expectations. Intuitively, although binding borrowing constraints mitigate the substitution effect from changes in interest rates, it enhances the reaction of consumption to changes in income, i.e. increases marginal propensities to consume (MPCs). This larger income effect exactly offsets the smaller substitution effect for our baseline specification. The general point is that under rational expectations incomplete markets does not necessarily deliver a departure in the aggregate response of consumption to the path of interest rates, even though the underlying mechanism and intuition may be quite different.

Turning to level-k bounded rationality we obtain mitigation and horizon effects, but these effects are now amplified relative to those in the representative agent case. In particular, for level-1 the response of current output to an interest rate change decreases exponentially with the horizon at a rate equal to the interest rate, r, plus the rate of death, λ, which should be interpreted as the frequency of binding borrowing constraints. That is, the response is proportional to $e^{-(r+\lambda)\tau}$. For example, a frequency of binding borrowing constraints of 15% implies a response of current output to an interest rate change 4 years into the future of almost half of the effect of a contemporaneous interest rate change.

Section 4 also shows that this intuition extends to the standard Bewley-Aiyagari-Huggett model. This model combines occasionally-binding borrowing constraints (like the perpetual youth model) and precautionary savings due to uninsurable idiosyncratic uncertainty (unlike the perpetual youth model). As is well known, Bewley-Aiygari-Huggett models are not analytically tractable, so we must turn to numerical simulations. Our explorations show that this model delivers significant mitigation and horizon effects. Consistent with our earlier results, we find that these effects are especially strong when the model is parameterized to feature significant risk and binding borrowing constraints. Quantitatively, in our baseline calibration, we find that the effect on output of an interest rate change in 4 years is about 50% of the effect of a contemporaneous interest rate change; this number is similar to our perpetual youth example.

Finally, in Section 5, we study the role of inflation by departing from the assumption of fully rigid prices. Household must now also form expectations regarding future inflation. We modify the Bewley-Aiygari-Huggett model of Section 4 to incorporate monopolistic competition and staggered time-dependent pricing a la Calvo, as well as explicit labor supply and labor demand decisions. We find that our results survive and are even strengthened: The "forward guidance puzzle" is even worse than with rigid prices because an inflation-output feedback loop which is more powerful at longer horizons; and while the separate introduction of bounded rationality or of incomplete markets does not provide a quantitatively realistic solution of the "puzzle", the joint introduction of these frictions does.[7]

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[7] When prices are sticky but not entirely rigid, the introduction of bounded rationality significantly improves the "forward guidance puzzle" because it mitigates the inflation output feedback loop which makes the "puzzle" worse in the first place. However, this mitigation is not enough to provide a realistic resolution of the puzzle: even at level 1, it only produces the same limited mitigation and horizon effects as the model with rigid prices. Only the joint introduction of bounded rationality and incomplete markets provides a realistic solution of the "puzzle".

Related literature. The intellectual genealogy of the concept of level-k equilibrium is well described in García-Schmidt and Woodford (2015). More generally, and following a categorization proposed by Guesnerie (1992) and adopted by Woodford (2013), our approach belongs to the eductive class of deviations from rational expectations, where one assumes that agents correctly understand the model and form inferences about future outcomes through a process of reflection, independent of experience, and not necessarily occurring in real time.[8] This class of deviations from rational expectations is distinct from inductive approaches, which assume that the probabilities that people assign to possible future outcomes should not be too different from the probabilities with which different outcomes actually occur, given that experience should allow some familiarity with these probabilities, regardless of whether agents understand the way in which these outcomes are generated (models of incomplete information with econometrics learning and models of partially or approximately correct beliefs).

The concept of level-k equilibrium is related to the iterative algorithm proposed by Fair and Taylor (1983) to compute numerically rational-expectations equilibria of a dynamic economic models, with the successive iterations resembling the ones described in the construction of level-k equilibria. The difference that the concept of level-k equilibrium sees the different iterations not simply as steps towards the computation of rational-expectations equilibria, but as interesting equilibrium concepts per se that can be compared to the data. The concept of level-k equilibrium is closely related to the concept of calculation equilibrium of Evans and Ramey (1992; 1995; 1998), who advocate stopping after a few iterations owing to calculations costs. It is slightly different from the concept of "reflective equilibrium" in Garcia-Schmidt and Woodford (2019) who consider a continuous process whereby expectation are governed by a differential equation in the level of thought rather than by a discrete recursion as we do, but this difference is largely inconsequential.

The concept of level-k thinking has also been proposed to explain behavior in laboratory experiments with games of full information. Starting with Stahl and Wilson (1994), Stahl and Wilson (1995), and Nagel (1995), laboratory experiments have been carried out to test level-k thinking and estimate the level of k: see for example Costa-Gomes et al. (2001), Camerer et al. (2004), Costa-Gomes and Crawford (2006), Arad and Rubinstein (2012), Crawford et al. (2013), Kneeland (2015), and Mauersberger and Nagel (2018). Almost all the estimates point to low levels of reasoning (between 0 and 3) when the subjects are confronted with a new game but have had the rules explained to them. There are also estimates from field experiments (Östling et al., 2011; Batzilis et al., 2017) and estimates using biological methods (Wang et al., 2010) such as eye-tracking and pupil dilation to supplement choice data, all yielding values of k of no more than 2. In their recent survey Mauersberger and Nagel (2018) write: "The most important contribution of the level-k model comes from the large number of empirical

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[8] This terminology originates in the work of Guesnerie (1992) on "eductive stability".

observations that most subjects engage in no more than 3 levels."[9] These games are arguably considerably simpler than our model economies, suggesting lower levels of reasoning in our context.[10]

Our paper also belongs to the growing literature studying incomplete-markets models with nominal rigidities.[11] This literature has not deviated from rational expectations. Within this literature, our paper is closely related to Del Negro et al. (2015) and McKay et al. (2016), who study forward guidance in New Keynesian models with an overlapping generations structure and a Bewley-Aiyagari-Hugget structure, respectively.[12] Both papers argue that binding borrowing constraints and precautionary motives shorten the horizon of agents consumption behavior and reduce the effect that future interest rates have on current output. However, Werning (2015) shows that whether or not incomplete markets dampens or amplifies the power of forward guidance depends the cyclicality of risk and liquidity, which is in turn determined by delicate assumption on distributional impact on wages, employment and profits of expansions, as well as the available assets and the form of borrowing constraints. Indeed, for a neutral benchmark case, where risk and liquidity are acyclical relative to output, it is shown that an incomplete market economy behaves identically to a representative agent one with respect to monetary policy. If, instead, risk is countercyclical and liquidity is procyclical, as much of the finance literature assumes, then the power of forward guidance is amplified. Indeed, the dampening found in Del Negro et al. (2015) and McKay et al. (2016) simulations is driven by assumptions that lead risk to be procyclical and liquidity to be countercyclical.[13]

In this paper, we focus on the neutral benchmark neutral case mentioned above. This deliberate choice ensures that our results are not driven by incomplete markets, so that our analysis isolates the interactions between incomplete markets and bounded rationality. There are two critical general mechanisms at play: level-k thinking mitigates general equilibrium effects,

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[9] Moreover, some papers show that low levels of k are not only due to agents' bounded rationality, but also to their beliefs regarding their opponents' bounded rationality (Alaoui and Penta (2016, 2017), Fehr and Huck (2016), Friedenberg et al. (2017)), thereby further validating the level-k thinking model.

[10] Some recent papers explore the determination and stability of k (Georganas et al. 2015, Alaoui and Penta 2016, 2017, Fehr and Huck 2016). They find that k may be endogenous in the sense that raising game stakes or increasing beliefs on opponents' cognitive ability leads to higher levels of subjects' reasoning. However, models that seek to endogenize k suffer a form of "infinite regress". In the paper, we therefore take k to be exogenous in our model. We leave the development of a model with an endogenous determination of k to future research.

[11] Recent papers include Auclert (2017), Caballero and Farhi (2017), Eggertsson and Krugman (2012), Farhi and Werning (2016a,b, 2017), Gali et al. (2007), Guerrieri and Lorenzoni (2017), Kaplan and Violante (2014), Kaplan et al. (2016), Kekre (2016), Oh and Reis (2012), Ravn and Sterk (2016), and Sterk and Tenreyro (2013).

[12] Caballero and Farhi (2017) offer a rationalization of the forward guidance puzzle in a model with heterogenous risk aversion where risk-tolerant agents issue safe assets to risk-averse agents through a process of securitization of real risky assets hampered by a securitization constraint. When the securitization constraint is binding, the effectiveness of forward guidance is reduced because the constraint prevents it from increasing the supply of safe assets and hence reduces its ability to stimulate the economy.

[13] Specifically, in both Del Negro et al. (2015) and McKay et al. (2016) profits relative to output are countercyclical, due to the model's procyclicality of real wages and the absence of any fixed costs or labor hoarding. McKay et al. (2016) also assumes that the only asset available is a risk free bond which is kept in constant net supply by the government, implying that asset values relative to output drop in an expansion.

making the equilibrium response closer to the partial equilibrium response; and incomplete markets tends to mitigate the partial equilibrium response. Uncovering these mechanisms suggests that our conclusions are robust to the details of the market incompleteness. It also indicates that the strength of our conclusions depend on auxiliary assumptions, often overlooked, regarding the cyclicality of risk and liquidity. Indeed, the partial equilibrium response under incomplete markets is independent of these features, and is lower than under complete markets. On the other hand, the general equilibrium and total responses with incomplete markets depend on the cyclicality of risk and liquidity (Werning, 2015). Throughout the paper, we focus on a neutral case where risk and liquidity are acyclical where an *incomplete-markets irrelevance* result obtains under rational expectations. It provides a tractable benchmark where under rational expectations, monetary policy has the same effects with complete and incomplete markets. Differences in these effects between complete and incomplete markets arise solely from the interaction between incomplete markets and bounded rationality. If instead one assumed that risk were countercyclical (precautionary concerns rise in recessions), or that liquidity were procyclical (asset prices and credit rise relative to output in booms), then the total equilibrium response with incomplete markets would actually be higher than with complete markets under rational expectations. Given incomplete markets, introducing level-k thinking would make more of a difference; and introducing incomplete markets and level-k thinking together would make more of a difference. Conversely, if one assumed that risk were procylical and that liquidity were countercyclical, then the total equilibrium response with incomplete markets would be lower than with complete markets under rational expectations. Given incomplete markets, introducing level-k thinking would make less of a difference; and introducing incomplete markets and level-k thinking together would make less of a difference, but the overall response under level-k thinking with incomplete markets could still be weaker than with complete markets.

Most closely related to our paper are Garcia-Schmidt and Woodford (2019), Wiederholt (2016), Gabaix (2017), and Angeletos and Lian (2018), who study the effects of monetary policy, and in particular the limits of forward guidance, in standard New Keynesian models with either bounded rationality or full rationality and informational frictions the last two of these papers. An important difference between these papers and ours is that they maintain the assumption of complete markets while we study incomplete markets. Another important difference between our paper and Gabaix (2017), Angeletos and Lian (2018), and Wiederholt (2016) is that they rely on an inductive approach with informational frictions and full rationality, instead of an eductive approach with bounded rationality. In Gabaix (2017), agents are assumed to be inattentive to the interest rate. In Angeletos and Lian (2018), there is imperfect common knowledge because agents receive private signals about interest rate changes and must forecast the forecasts of others. Wiederholt (2016) also assumes informational frictions but of a different form, by positing that agents have sticky expectations a la Mankiw and Reis (2002) and receive information about interest rate changes after the realization of an idiosyncratic Poisson shock. In contrast to these models with full rationality and informational frictions, ours is one of full information with bounded rationality where agents know the path of interest rates but face difficulties in calculating the macroeconomic equilibrium consequences of changes in interest rates. We think that our approach is better suited to capture the limits of forward guidance in contexts where considerable efforts are made by central banks to communicate their policies and where indeed experience shows that the yield curve is very reactive to these announcements. Moreover, absent an extra assumption that consumers are less attentive or informed about interest rate changes when they occur in the more distant future, the response of output to changes in interest rates is always greater than the partial equilibrium response. Given that these models assume a representative agent, the partial equilibrium response is relatively large at standard horizons and features only weak horizon effects, and by implication, so is the response of output. Basically, the main channel through which these models change the impact of monetary policy is through the mitigation of the powerful feedback loop between output and inflation, and removing the effects of this feedback loop still leaves monetary policy relatively potent, even at relatively long horizons.

2 Level-k in a General Reduced-Form Model

We begin by introducing the basic concepts of level-k equilibrium within a general model building on a reduced-form aggregate consumption function. Various explicit disaggregated models can be explicitly reduced to this formulation. For example, representative-agent models, overlapping generations models, models with a fraction of permanent-income consumers and a fraction of hand-to-mouth consumers, and Bewley-Aiyagari-Huggett models of heterogenous agents with income fluctuation and incomplete markets, all give rise to an aggregate consumption function of the form considered below. We will make this mapping explicit for several of these models in future sections.

2.1 Baseline General Reduced-Form Model

We consider a simple model with one consumption good in every period and no investment. Time is discrete and the horizon is infinite with periods $t = 0, 1, \ldots$ We denote current and future real nominal interest rates by $\{R_{t+s}\}$, and current and future aggregate income by $\{Y_{t+s}\}$, where s runs from 0 to ∞. We focus for simplicity on the extreme case with perfectly rigid prices, where real interest rates equal nominal interest rates. We maintain this assumption in Sections 2-4.2. We take as given the path of nominal interest rates $\{R_{t+s}\}$ coincides with the path of real interest rates. Our goal is to solve for the equilibrium path of aggregate income $\{Y_{t+s}\}$. An alternative interpretation is that we are characterizing the response of the economy to different path of real interest rates, which are under the control of the monetary authority because of nominal rigidities. In any case, we relax the assumption of perfectly rigid prices in Section 5 where we consider sticky prices.

Aggregate consumption function. We postulate an aggregate consumption function

$$C_t = C^*(\{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}), \tag{1}$$

where $\{Y^e_{t+1+s}\}$ denotes future anticipated aggregate income.

The fact that the aggregate consumption function depends only on current and future interest rates, current income and future anticipated income is useful and merits brief discussion. With a representative agent such a formulation is straightforward, and we discuss this example below. Otherwise, the consumption function should be interpreted as performing an aggregation and consolidating any distributional effects, including solving out for wages and profits as a function of current Y_t. Implicitly we are also assuming there is no heterogeneity in beliefs about future income, $\{Y^e_{t+1+s}\}$, although one may extend the analysis to capture heterogeneity in beliefs.

In this formulation the consumption function is purely forward looking—it does not depend on the past or on any state variable that is affected by the past. This can accommodate various interesting and simple models, such as the representative agent, the perpetual youth overlapping generations model, and certain simple models with heterogeneity such as models fraction of hand-to-mouth agents. It does not fit all situations, however. In the next subsection we provide an extension with an aggregate state variable which allows us to capture standard Bewley-Aiyagari-Huggett models.

Temporary equilibria. We are interested in allowing for more general beliefs than rational expectations. We start by defining the notion of temporary equilibrium in the spirit of Hicks (1939) and Lindahl (1939), and further developed by Grandmont (1977; 1978). A temporary equilibrium takes as given a sequence of beliefs $\{Y^e_t\}$ and simply imposes that the goods market clear

$$Y_t = C_t. \tag{2}$$

Definition (Temporary equilibrium)**.** *Given a sequence of beliefs* $\{Y^e_t\}$*, a temporary equilibrium is a sequence* $\{R_t, Y_t\}$ *satisfying* (1) *and* (2) *for all* $t \geq 0$*.*

It is important to note that since there is no uncertainty and hence no revelation of information over time, there is only one sequence of beliefs, which is not updated over time: in other words, beliefs at date u about output at date t are given by Y^e_t for all dates $u < t$.

Start at some baseline temporary equilibrium $\{R_t, Y_t, Y^e_t\}$ and consider the one-time unexpected announcement at $t = 0$ of a new interest rate path $\{\hat{R}_t\}$. The equilibrium response depends on the adjustment of beliefs. We now describe two possible adjustments of beliefs: rational expectations and level-k thinking.

Rational-expectations equilibria. A rational-expectations equilibrium is a particular case of temporary equilibrium with the extra requirement of perfect foresight, i.e. that beliefs about future income coincide with actual future income

$$\{Y_t^e\} = \{Y_t\}. \tag{3}$$

Definition (Rational expectation equilibrium). *A rational-expectations equilibrium (REE) is a sequence* $\{R_t, Y_t, Y_t^e\}$ *such that* $\{R_t, Y_t\}$ *is a temporary equilibrium given beliefs* $\{Y_t^e\}$ *and which satisfies perfect foresight* (3) *for all* $t \geq 0$.

For notational convenience, we often denote a given REE by $\{R_t, Y_t\}$ instead of using the more cumbersome notation $\{R_t, Y_t, Y_t\}$.

Start at some baseline REE $\{R_t, Y_t\}$ and consider as above a one-time unexpected announcement at $t = 0$ of a new interest rate path $\{\hat{R}_t\}$ leading to a new REE $\{\hat{R}_t, \hat{Y}_t\}$. Under rational expectations, there is an issue about *selection* since there are typically several REEs for a given interest rate path $\{\hat{R}_t\}$. In our detailed applications, and for the considered interest rate paths, we will always be able to select a unique REE by imposing that the baseline and new REEs coincide in the long run:

$$\lim_{t\to\infty} \hat{Y}_t = \lim_{t\to\infty} Y_t.$$

From now on, we always use this selection.

Level-k equilibria. We now deviate from rational expectations and describe an alternative adjustment of expectations encapsulated in the notion of level-k thinking. We then introduce the notion of level-k equilibrium $\{R_t, \hat{Y}_t^k\}$ which is a temporary equilibrium with a sequence of beliefs $\{\hat{Y}_t^{e,k}\}$ indexed by k. As already explained above in the definition of temporary equilibria, given a level k, there is only one sequence of beliefs which is not updated over time since there is no uncertainty and no revelation of information over time. As above, we start at some baseline REE $\{R_t, Y_t\}$, and consider a one-time unexpected shock change in the path for the interest rate $\{\hat{R}_t\}$ at $t = 0$.

The level-1 equilibrium $\{\hat{R}_t, \hat{Y}_t^1\}$ is a temporary equilibrium given beliefs $\{\hat{Y}_t^{e,1}\} = \{Y_t\}$ corresponding to the aggregate income path of the original REE. In other words, expectations for future aggregate income are unchanged after the announced change in interest rates and equal to the original REE path. For each $t = 0, 1, \dots$, $\hat{Y}_t^1$ can be computed as the following fixed point equation

$$\hat{Y}_t^1 = C^*(\{\hat{R}_{t+s}\}, \hat{Y}_t^1, \{Y_{t+1+s}\}).$$

The level-1 equilibrium captures a situation where agents take into account the new announced path for interest rates and observe present income, but do not adjust their expectations about future income. However, actual realized income is affected.

The level-2 equilibrium $\{\hat{R}_t, \hat{Y}_t^2\}$ is a temporary equilibrium given beliefs $\{\hat{Y}_t^{e,2}\} = \{\hat{Y}_t^1\}$ corresponding to the aggregate income path from level-1. For every $t \geq 0$, $\hat{Y}_t^2$ can be computed as the following fixed point equation

$$\hat{Y}_t^2 = C^*(\{\hat{R}_{t+s}\}, \hat{Y}_t^2, \{\hat{Y}_{t+1+s}^1\}).$$

Here agents update their beliefs to take into account that the change in aggregate spending (by all other agents) associated with level-1 thinking has an effect on aggregate income (and hence on their own income). In other words, level-2 thinking incorporates the general equilibrium effects of future income from level 1.

Continuing, the level-k equilibrium $\{\hat{R}_t, \hat{Y}_t^k\}$ is defined as a temporary equilibrium given beliefs $\{\hat{Y}_t^{e,k}\} = \{\hat{Y}_t^{k-1}\}$ corresponding to the aggregate income path of the level-$k-1$ equilibrium in a similar manner. Thus, $\hat{Y}_t^k$ solves the fixed point equation

$$\hat{Y}_t^k = C^*(\{\hat{R}_{t+s}\}, \hat{Y}_t^k, \{\hat{Y}_{t+1+s}^{k-1}\}).$$

We emphasize that this process of expectation formation regarding the future path of output takes place once and for all when the new interest rate path is announced.

Definition (Level-k equilibrium)**.** *Given an initial REE* $\{R_t, Y_t\}$ *and a new interest rate path* $\{\hat{R}_t\}$*, the level-k equilibrium* $\{R_t, \hat{Y}_t^k\}$ *is defined by a recursion indexed by* $k \geq 0$ *with initial condition* $\{\hat{Y}_t^0\} = \{Y_t\}$*, and such that* $\{\hat{R}_t, \hat{Y}_t^k\}$ *is a temporary equilibrium given beliefs* $\{\hat{Y}_t^{e,k}\} = \{\hat{Y}_t^{k-1}\}$.

In the definitions of temporary and level-k equilibria, we include the actual present aggregate income, instead of some expectation over current aggregate income. This implies that markets clear in the present period and that basic macroeconomic identities hold. This impact of current aggregate income, however, will vanish in some cases in continuous time.

Note that in contrast to rational-expectations equilibria, there is no issue of equilibrium selection in level-k equilibria. The initial REE equilibrium $\{R_t, Y_t\}$ acts as an anchor which ensures that the construction of the level-k equilibrium associated with a new interest rate path $\{\hat{R}_t\}$ is *determinate*. Of course, this result is conditional on a particular choice for the initial condition for beliefs $\{\hat{Y}_t^0\}$, which we have taken to be the status quo $\{Y_t\}$. In principle we could have specified a different initial condition for beliefs, and this would have led to different level-k equilibria. In this sense, once we open up the possibility of choosing different initial conditions for beliefs, there is also indeterminacy under level-k. In our view, taking the status quo as the initial condition for beliefs is psychologically natural and is an integral part of the level-k formulation, while no such natural principle underpins the requirement that output be the same as under the status quo. It is this perspective that leads us to state that level-*k* equilibria are determinate while REE are not. However, we recognize that one could also argue that we have only resolved the indeterminacy of level-*k* equilibria by taking the status quo to be the initial condition for beliefs, in a similar way that we have resolved the indeterminacy under REE by requiring that the long-run level of output be the same as under the status quo.

Decomposing equilibrium changes: PE and GE. Start at some baseline REE $\{R_t, Y_t\}$ and consider as above an one-time unexpected announcement at $t = 0$ of a new interest rate path $\{\hat{R}_t\}$.

Under rational expectations, the new equilibrium $\{\hat{R}_t, \hat{Y}_t\}$ is an REE. We can decompose the change in aggregate income

$$\Delta Y_t = \hat{Y}_t - Y_t$$

as

$$\Delta Y_t = \Delta Y_t^{PE} + \Delta Y_t^{GE},$$

where

$$\begin{aligned}
\Delta Y_t^{PE} &= C^*(\{\hat{R}_{t+s}\}, Y_t, \{Y_{t+1+s}\}) - C^*(\{R_{t+s}\}, Y_t, \{Y_{t+1+s}\}), \\
\Delta Y_t^{GE} &= C^*(\{\hat{R}_{t+s}\}, \hat{Y}_t, \{\hat{Y}_{t+1+s}\}) - C^*(\{\hat{R}_{t+s}\}, Y_t, \{Y_{t+1+s}\}).
\end{aligned}$$

The term ΔY_t^{PE} can be interpreted as a partial equilibrium effect considering only the change in interest rates, holding constant current and future income. The term ΔY_t^{GE} captures the general equilibrium effects from changing current and future expected income, holding interest rates fixed at their new level.

Under level-*k* thinking, we denote the change in aggregate income by

$$\Delta Y_t^k = \hat{Y}_t^k - Y_t.$$

We can again use a decomposition

$$\Delta Y_t^k = \Delta Y_t^{PE} + \Delta Y_t^{k,GE},$$

with

$$\Delta Y_t^{k,GE} = C^*(\{\hat{R}_{t+s}\}, \hat{Y}_t^k, \{\hat{Y}_{t+1+s}^{k-1}\}) - C^*(\{\hat{R}_{t+s}\}, Y_t, \{Y_{t+1+s}\}).$$

In particular, since $\{\hat{Y}_t^0\} = \{Y_t\}$, the only reason why $\Delta Y_t^{1,GE} = C^*(\{\hat{R}_{t+s}\}, \hat{Y}_t^1, \{Y_{t+1+s}\}) - C^*(\{\hat{R}_{t+s}\}, Y_t, \{Y_{t+1+s}\})$ is not zero is due to the effect of the adjustment of current income $\hat{Y}_t^1$. As we shall see, this difference vanishes in some cases in continuous time. In these cases, level-1 thinking coincides exactly with the partial equilibrium effect.

Effects of monetary policy at different horizons. To summarize the effects of monetary policy at different horizons, we study the elasticity of output at date 0 of an interest rate change at horizon τ. We therefore use level-k equilibria only to inform the way expectations are formed, not to describe the way the economy actually responds over time to the monetary policy announcement.

We consider an initial REE $\{R_t, Y_t\}$ which for simplicity we assume is a steady state with $R_t = R$ and $Y_t = Y$ for all $t \geq 0$. We consider a change $\{\hat{R}_t\}$ in the path for the interest rate ΔR_τ at date τ so that $\hat{R}_\tau = R + \Delta R_\tau$ and $\hat{R}_t = R_t$ for $t \neq \tau$. The rational-expectations elasticity is defined as

$$\epsilon_\tau = \lim_{\Delta R_\tau \to 0} -\frac{R_\tau}{Y}\frac{\Delta Y_0}{\Delta R_\tau},$$

and can be decomposed as

$$\epsilon_\tau = \epsilon_\tau^{PE} + \epsilon_\tau^{GE},$$

where

$$\epsilon_\tau^{PE} = \lim_{\Delta R_\tau \to 0} -\frac{R_\tau}{Y}\frac{\Delta Y_0^{PE}}{\Delta R_\tau},$$

$$\epsilon_\tau^{GE} = \lim_{\Delta R_\tau \to 0} -\frac{R_\tau}{Y}\frac{\Delta Y_0^{GE}}{\Delta R_\tau}.$$

Similarly, the level-k elasticity is defined as

$$\epsilon_\tau^{k} = \lim_{\Delta R_\tau \to 0} -\frac{R_\tau}{Y}\frac{\Delta Y_0^{k}}{\Delta R_\tau}.$$

2.2 Extended Model with an Aggregate State Variable

The previous analysis is sufficient for the simplest cases, such as the representative agent and the perpetual youth overlapping generations models. Aggregate consumption is purely forward looking in these cases. However, in an incomplete-markets Bewley-Aiyagari-Huggett economy, the distribution of wealth induces a backward looking component. To incorporate these effects we now extend the analysis to include an aggregate state variable.

Suppose that aggregate consumption is given by

$$C_t = C^*(\{R_{t+s}\}, Y_t, \{Y_{t+1+s}^e\}, \Psi_t), \tag{4}$$

where the state variable Ψ_t is potentially of a large dimension and evolves according to some equilibrium law of motion

$$\Psi_{t+1} = M(\{R_{t+s}\}, Y_t, \{Y_{t+1+s}^e\}, \Psi_t). \tag{5}$$

The initial state Ψ_0 is taken as given. In incomplete-markets economies, Ψ_t may capture the distribution of wealth and M the evolution of the wealth distribution. The important point is that the aggregate consumption function is no longer purely forward looking.

We can easily extend all our definitions. A temporary equilibrium given beliefs $\{Y_t^e\}$ is a set of sequences $\{R_t, Y_t, \Psi_t\}$ satisfying (2), (4), and (5) for all $t \geq 0$. An REE is a set of sequences $\{R_t, Y_t, Y_t^e, \Psi_t\}$ such that $\{R_t, Y_t, \Psi_t\}$ is a temporary equilibrium given beliefs $\{Y_t^e\}$ and which satisfies perfect foresight (3) for all $t = 0, 1, \ldots$ Given a baseline REE and a one-time unexpected announced at $t = 0$ of a new interest rate path $\{\hat{R}_t\}$, level-k equilibria $\{\hat{R}_t, \hat{Y}_t^k, \hat{\Psi}_t^k\}$ are defined by a recursion indexed by $k \geq 0$ with initial condition $\{\hat{Y}_t^0\} = \{Y_t\}$, and such that $\{\hat{R}_t, \hat{Y}_t^k, \hat{\Psi}_t^k\}$ is a temporary equilibrium given beliefs $\{\hat{Y}_t^{e,k}\} = \{\hat{Y}_t^{k-1}\}$. Armed with these definitions, it is straightforward to extend the definitions of the elasticities ϵ_τ, ϵ_τ^{PE}, ϵ_τ^{GE}, and ϵ_τ^k.

3 Representative Agent

In this section, we consider the particular case of a representative-agent model. The model is a particular case of the general reduced-form model of Section 2, with an explicit microfoundation for the aggregate consumption function.

We consider a Lucas tree economy with a unit supply of Lucas trees with time-t value V_t capitalizing a stream δY_t of dividends and with non-financial (labor) income given by $(1 - \delta) Y_t$. The representative agent can invest in Lucas trees and also borrow and lend in short-term risk-free bonds with the sequence of interest rates $\{R_t\}$. At every point in time t, the agent has beliefs $\{Y_{t+1+s}^e, V_{t+1+s}^e\}$ about future aggregate income and values of Lucas trees. It is important to note that $\{V_t^e\}$, just like $\{Y_t^e\}$, represents on sequence which is not updated over time. The need to distinguish between V_t and V_t^e is because the value of Lucas trees is defined inclusive of dividends, and that current dividends are observed whereas future dividends are only expected.

In Section 3.1, we show how to derive the reduced-form aggregate consumption function from the consumption policy function of an individual problem using the asset market clearing condition for a general utility function. We then leverage all the definitions of Section 2.1: temporary equilibria, rational-expectations equilibria, level-k equilibria, and the corresponding interest rate elasticities. In Section 3.2, we specialize the model to the case of an isoelastic utility function and derive analytical results.

3.1 The General Representative-Agent Model

In this section, we consider a general model with a per-period utility function U.

Individual problem. Consider sequences $\{R_t, Y_t, Y_t^e, V_t, V_t^e\}$. An individual takes these sequences as given. Agent consumption c_t is determined as a function of past bond and Lucas tree holdings b_{t-1} and x_{t-1} via the individual consumption function

$$c_t = c^*(b_{t-1}, x_{t-1}; \{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}, V_t, \{V^e_{t+1+s}\}).$$

This individual consumption functions at time t are derived from the following individual problem at time t, given b_{t-1} and x_{t-1}:

$$\max_{\{\tilde{c}_{t+s}, \tilde{b}_{t+s}, \tilde{x}_{t+s}\}} \sum_{s=0}^{\infty} \beta^s U(\tilde{c}_{t+s})$$

subject to the current actual budget constraint

$$\tilde{c}_t = (1-\delta)Y_t + x_{t-1}V_t + b_{t-1}R_{t-1} - \tilde{x}_t(V_t - \delta Y_t) - \tilde{b}_t,$$

and future expected budget constraints

$$\tilde{c}_{t+1+s} = (1-\delta)Y^e_{t+1+s} + \tilde{x}_{t+s}V^e_{t+1+s} + \tilde{b}_{t+s}R_{t+s} - \tilde{x}_{t+1+s}(V^e_{t+1+s} - \delta Y^e_{t+1+s}) - \tilde{b}_{t+1+s} \quad \forall s \geq 0.$$

Here we use variables with a tilde to capture the arguments in the optimization of the individual problem at date t, given b_{t-1} and x_{t-1}. They should in principle be indexed by t, b_{t-1}, and x_{t-1}, but we suppress this dependence to streamline the notation. The individual consumption function $c^*(b_{t-1}, x_{t-1}; \{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}, V_t, \{V^e_{t+1+s}\})$ is the policy function for $\tilde{c}_t$.

We now simplify these steps by imposing the no-arbitrage condition that the expected return on the Lucas tree between t and $t+1$ is equal to that of the risk-free bond:

$$\frac{V^e_{t+1}}{V_t - \delta Y_t} = R_t \quad \forall t \geq 0.$$

These no-arbitrage conditions are necessary for the individual problems to have a solution. They imply that actual and expected asset values are given by:[14]

$$V_t = \delta Y_t + \frac{V^e_{t+1}}{R_t} = \delta Y_t + \sum_{s=0}^{\infty} \frac{\delta Y^e_{t+1+s}}{\Pi^s_{u=0} R_{t+u}} \quad \forall t \geq 0,$$
$$V^e_t = \delta Y^e_t + \frac{V^e_{t+1}}{R_t} = \delta Y^e_t + \sum_{s=0}^{\infty} \frac{\delta Y^e_{t+1+s}}{\Pi^s_{u=0} R_{t+u}} \quad \forall t \geq 0. \tag{6}$$

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[14] The need to distinguish between V_t and V_t^e is because the value of Lucas trees is defined inclusive of dividends, and that current dividends are observed whereas future dividends are only expected. In continuous time, the value of current dividend accounts for a negligible fraction of the price of the Lucas tree and this distinction becomes irrelevant.

The first equality in the first equation states that the value of Lucas tree at t is the sum of the dividend δY_t and of the discounted expected value of the Lucas tree at date $t+1$. This guarantees that there is no arbitrage between Lucas trees and risk-free bonds between t and $t+1$ since the expected returns on holding a Lucas tree and a risk-free bond are equalized at R_t. The second equality in this first equation is just using the fact that there must be no arbitrage in the economy that the agent expects to occur from the next period onwards in order to pin down the expected value of Lucas trees. Of course, there can be a gap between the value of the Lucas tree that the agent expects at $t+1$ and its true value. This implies that the actual return on a Lucas tree $V_{t+1}/(V_t - \delta Y_t) = R_t + \delta(Y_{t+1} - Y^e_{t+1})/(V_t - \delta Y_t)$ might be different from its expected return $V^e_{t+1}/(V_t - \delta Y_t) = R_t$.

Given no arbitrage, an individual agents is indifferent between bonds and Lucas trees, and the composition of his portfolio is indeterminate. Accordingly, we define a new variable $a_t = b_{t-1}R_{t-1} + x_{t-1}(\delta Y_t + V_t)$ denoting financial wealth at time t. We can then simplify the individual problem at time t:

$$\max_{\{\tilde{c}_t, \tilde{a}_{t+1+s}\}} \sum_{s=0}^{\infty} \beta^s U(\tilde{c}_{t+s})$$

subject to the current actual budget constraint

$$\tilde{c}_t = (1-\delta)Y_t + a_t - \frac{\tilde{a}_{t+1}}{R_t},$$

and future expected budget constraint

$$\tilde{c}_{t+1+s} = (1-\delta)Y^e_{t+1+s} + \tilde{a}_{t+1+s} - \frac{\tilde{a}_{t+2+s}}{R_{t+1+s}} \quad \forall s \geq 0.$$

The individual consumption function $c^*(a_t; \{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\})$ is the policy functions for $\tilde{c}_t$. Note that V_t and $\{V^e_{t+1+s}\}$ are no longer arguments of this policy function, a very convenient simplification.

Reduced-form aggregate consumption function. The reduced-form aggregate consumption is obtained from the individual consumption function

$$C(\{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}) = c(a_t; \{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\})$$

by imposing the asset market clearing condition $a_t = V_t$, where V_t is given by the no-arbitrage condition (6). This yields

$$C(\{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}) = c^*(\delta Y_t + \sum_{s=1}^{\infty} \frac{\delta Y^e_{t+1}}{\Pi^s_{u=0} R_{t+u}}; \{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}).$$

We can then use this reduced-form aggregate consumption function to go through all the definitions given in Section 2: temporary equilibria, rational-expectations equilibria, level-k equilibria, and the corresponding interest rate elasticities.

3.2 Isoelastic Utility Function

In this section, we specialize the model to the case of an isoelastic utility function with intertemporal elasticity of substitution σ:

$$U(c) = \begin{cases} \frac{c^{1-\frac{1}{\sigma}}-1}{1-\frac{1}{\sigma}} & \text{if} \quad \sigma \neq 1, \\ \log(c) & \text{if} \quad \sigma = 1. \end{cases}$$

It is then easy to see that the individual consumption function is

$$c^*(a_t; \{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}) = \frac{a_t + (1-\delta)Y_t + \sum_{s=0}^{\infty} \frac{(1-\delta)Y^e_{t+1+s}}{\Pi_{u=0}^{s} R_{t+u}}}{1 + \sum_{s=0}^{\infty} \frac{\beta^{\sigma(1+s)}}{\Pi_{u=0}^{s} {R_{t+u}}^{1-\sigma}}},$$

so that the aggregate reduced-form consumption function is

$$C(\{R_{t+s}\}, Y_t, \{Y^e_{t+1+s}\}) = \frac{Y_t + \sum_{s=0}^{\infty} \frac{Y^e_{t+1+s}}{\Pi_{u=0}^{s} R_{t+u}}}{1 + \sum_{s=0}^{\infty} \frac{\beta^{\sigma(1+s)}}{\Pi_{u=0}^{s} {R_{t+u}}^{1-\sigma}}}.$$

Equilibrium characterization. For concreteness, we briefly characterize the various equilibria in the context of this particular model. Given beliefs $\{Y^e_t\}$, and given the path for interest rates $\{R_t\}$, $\{R_t, Y_t\}$ is a temporary equilibrium if and only if the path for aggregate income $\{Y_t\}$ is given by

$$Y_t = \frac{\sum_{s=0}^{\infty} \frac{Y^e_{t+1+s}}{\Pi_{u=0}^{s} R_{t+u}}}{\sum_{s=0}^{\infty} \frac{\beta^{\sigma(1+s)}}{\Pi_{u=0}^{s} {R_{t+u}}^{1-\sigma}}} \quad \forall t \geq 0.$$

Similarly, given the path for interest rates $\{R_t\}$, $\{R_t, Y_t\}$ is an REE if and only if the path for aggregate income $\{Y_t\}$ satisfies the fixed point

$$Y_t = \frac{\sum_{s=0}^{\infty} \frac{Y_{t+1+s}}{\Pi_{u=0}^{s} R_{t+u}}}{\sum_{s=0}^{\infty} \frac{\beta^{\sigma(1+s)}}{\Pi_{u=0}^{s} {R_{t+u}}^{1-\sigma}}} \quad \forall t \geq 0.$$

Finally given an initial REE $\{R_t, Y_t\}$ and a new interest rate path $\{\hat{R}_t\}$, the level-k equilibria $\{\hat{R}_t, \hat{Y}_t^k\}$ satisfy the following recursion over $k \geq 0$:

$$\hat{Y}_t^k = \frac{\sum_{s=0}^{\infty} \frac{\hat{Y}_{t+1+s}^{k-1}}{\Pi_{u=0}^{s} \hat{R}_{t+u}}}{\sum_{s=0}^{\infty} \frac{\beta^{\sigma(1+s)}}{\Pi_{u=0}^{s} \hat{R}_{t+u}{}^{1-\sigma}}} \quad \forall t \geq 0,$$

with the initialization that $\hat{Y}_t^0 = Y_t$ for all $t \geq 0$.

We now turn to the computation of the different interest rate elasticities of output around a steady state REE $\{R_t, Y_t\}$ with $R_t = R = \beta^{-1} > 1$ and $Y_t = Y > 0$ for all $t \geq 0$.

Monetary policy at different horizons under RE. We start with the RE case, where as discussed above, we use the selection that $\lim_{t \to \infty} Y_t = Y$ as we perform the comparative statics underlying the computation of the interest rate elasticities of output at impact.

Proposition 1 (Representative agent, isoelastic utility, RE). *Consider the representative-agent model with isoelastic utility and rational expectations. The interest rate elasticities of output at impact do not depend on the horizon τ and are given by*

$$\epsilon_\tau = \sigma.$$

They can be decomposed as $\epsilon_\tau = \epsilon_\tau^{PE} + \epsilon_\tau^{GE}$ into PE and GE elasticities which are given by

$$\epsilon_\tau^{PE} = \sigma \frac{1}{R^{\tau+1}} \quad \textit{and} \quad \epsilon_\tau^{GE} = \sigma(1 - \frac{1}{R^{\tau+1}}).$$

The total interest rate elasticity of output is equal to the intertemporal elasticity of substitution $\epsilon_\tau = \sigma$, independently of the horizon τ. This lack of horizon effect is a version of the "forward guidance puzzle", which refers to the extreme effectiveness of forward guidance (interest rate changes in the future) in standard New-Keynesian models compared to its apparently more limited effectiveness in the data.

To understand this result, it it useful to go back to the decomposition into PE and GE effects. The lack of *horizon* effect

$$\frac{\partial \epsilon_\tau}{\partial \tau} = 0$$

can be understood as follows, where, slightly abusing notation, we write $\frac{\partial \epsilon_\tau}{\partial \tau}$ for $\epsilon_{\tau+1} - \epsilon_\tau$. The PE effect does feature a horizon effect so that ϵ_τ^{PE} is decreasing with the horizon τ with

$$\frac{\partial \epsilon_\tau^{PE}}{\partial \tau} = -\log(R)\epsilon_\tau^{PE} < 0.$$

This is because for a given path of output, a cut in interest rates is more discounted, and hence leads to a smaller partial equilibrium consumption increase, the further into the future the interest rate cut takes place. But the GE effect features an exactly offsetting anti-horizon effect so that ϵ_τ^{GE} increases with the horizon τ with

$$\frac{\partial \epsilon_\tau^{GE}}{\partial \tau} = -\frac{\partial \epsilon_\tau^{PE}}{\partial \tau} > 0.$$

This is because in general equilibrium, output increases for a longer time, up until the horizon of the interest rate cut, leading to a higher increase in human and financial wealth, the further into the future the interest rate cut takes place, and hence leads to a larger consumption increase. As a result, the relative importance of the GE effect increases with the horizon, and that of the PE effect correspondingly decreases with the horizon, but the two effects always sum up to a constant total effect.

Monetary policy at different horizons under level-*k*. We now turn to the level-*k* case. We start by defining the function

$$\mathcal{E}^k(R-1,\tau) = \sum_{m=0}^{k-1} (R-1)^m \sum_{s_0=0}^{\tau-1} \sum_{s_1=0}^{\tau-1-s_0} \cdots \sum_{s_{m-1}=0}^{\tau-1-s_{m-2}} 1.$$

The function $\mathcal{E}^k$ is increasing in k with $\mathcal{E}^1(R-1,\tau) = 1$ and $\lim_{k\to\infty} \mathcal{E}^k(R-1,\tau) = R^\tau$.[15]

Proposition 2 (Representative agent, level-*k*)**.** *Consider the representative-agent model with isoelastic utility and level-k thinking. The interest rate elasticities of output at impact depend on the horizon τ and are given by*

$$\epsilon_\tau^k = \sigma \frac{\mathcal{E}^k(R-1,\tau)}{R^\tau}.$$

To begin with, note that the interest rate elasticity of output with level-*k* thinking converges to its rational-expectations counterpart in the limit $k \to \infty$:

$$\lim_{k\to\infty} \epsilon_\tau^k = \epsilon_\tau.$$

The rational-expectations case can therefore be seen as a limit case of level-*k* thinking as the number of rounds k goes to ∞. Recall that the treatment of rational expectations required an equilibrium selection, whereas that of the level-*k* case did not. Hence one can also see the con-

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[15] It is also useful to compute a few other examples explicitly. We have

$$\begin{aligned}
&\mathcal{E}^1(R-1,\tau) = 1,\\
&\mathcal{E}^2(R-1,\tau) = 1 + (R-1)\tau,\\
&\mathcal{E}^3(R-1,\tau) = 1 + (R-1)\tau + \frac{(R-1)^2\tau(\tau-1)}{2}.
\end{aligned}$$

vergence of the level-k equilibrium to the particular rational-expectations limit as a validation of the equilibrium selection that underpinned its construction.

Next recall that the PE effect is always the same under rational expectations and under level-k thinking at ϵ_τ^{PE}. The level-1 elasticity is always higher than the PE effect by a factor of R since

$$\epsilon_\tau^1 = \sigma \frac{1}{R^\tau} = R\epsilon_\tau^{PE} > \epsilon_\tau^{PE},$$

but as we shall see below, the difference $\epsilon_\tau^{1,GE} = \epsilon_\tau^1 - \epsilon_\tau^{PE}$ vanishes in the continuous time limit where time periods become infinitesimal so that the per-period interest rate R shrinks to 1. The interest rate elasticity of output with level-k thinking is lower than under rational expectations

$$\epsilon_\tau^k < \epsilon_\tau,$$

but increases with the level k of thought

$$\frac{\partial \epsilon_\tau^k}{\partial k} > 0,$$

and as noted above, converges monotonically to its rational-expectations counterpart in the limit when k goes to ∞, where, slightly abusing notation, we write $\frac{\partial \epsilon_\tau^k}{\partial k}$ for $\epsilon_\tau^{k+1} - \epsilon_\tau^{k+1}$. The *mitigation* effect $\epsilon_\tau^k < \epsilon_\tau$ is entirely due to a mitigation of the GE effect $\epsilon_\tau^{k,GE} < \epsilon_\tau^{GE}$. Similarly, the monotonically increasing *convergence* $\lim_{k\to\infty} \epsilon_\tau^k = \epsilon_\tau$ is entirely due to the monotonically increasing convergence of the GE effect $\lim_{k\to\infty} \epsilon_\tau^{k,GE} = \epsilon_\tau^{GE}$.

In addition, for any $k > 0$, in contrast to the rational-expectations case, there is now a *horizon* effect of monetary policy

$$\frac{\partial \epsilon_\tau^k}{\partial \tau} < 0,$$

so that the effects of monetary policy decrease with its horizon. This horizon effect disappears in the rational-expectations limit when k goes to ∞.

However the mitigation and horizon effects are rather *weak*. To see this focus on the case $k = 1$. Then $\epsilon_\tau^1 = \sigma \frac{1}{R^\tau}$ and so

$$\frac{\partial \epsilon_\tau^1}{\partial \tau} = -\log(R)\epsilon_\tau^1.$$

Hence $\epsilon_\tau^1 = \epsilon_\tau$ when the interest rate change is contemporaneous $\tau = 0$, and then ϵ_τ^1 decreases with the horizon τ at the exponential rate $\log(R)$ while $\epsilon_\tau = \sigma$ stays constant. We call $\log(R)$ the strength of the horizon effect. If the annual interest rate is 5%, the effects of monetary policy decrease at rate 5% per year with a half life of 14 years; if the annual interest rate is 1%, the effects of monetary policy decrease at rate 1% per year with a half life of 69 years.

There is a simple intuition for all these results in terms of the decomposition of the effects of monetary policy into PE and GE effects. The PE effect features mitigation—the effect of interest rate changes is lower than the full effect under rational expectations because the latter is the sum of the GE and the PE effect. It also features horizon—for a fixed path of output, interest rate changes affect partial equilibrium consumption less, the further in the future they are. These effects are weak for reasonable values of R. As we shall see below, this last conclusion can be overturned in models with heterogenous agents and incomplete markets.

Under rational expectations, the GE effect eliminates the mitigation effect by adding to the PE effect, and eliminates the horizon effect because the GE effect features an anti-horizon effect. At round $k = 1$, monetary policy almost (exactly in the continuous time limit) coincides with the PE effect and features weak mitigation and weak horizon. In the rational-expectations limit when k goes to ∞, the mitigation and horizon effects disappear. Intermediate values of k interpolate smoothly and monotonically between these two extremes.

It is also interesting to note that the various interest rate elasticities of output are all independent of the amount of outside liquidity δ. This is because human and financial wealth play very similar roles in this representative-agent model. As we shall see shortly, this equivalence breaks down in heterogenous agents models with incomplete markets.

3.3 Continuous-Time Limit

We now explain how the results can be adapted in continuous time. This can be done either directly by setting up the model in continuous time, or by taking the continuous-time limit of the discrete time model. In Section 4.1, we follow the former approach. In this section instead, we follow the latter.

The continuous-time limit involves considering a sequence of economies indexed by $n \geq 0$, where the calendar length λ_n of a period decreases with n. For example, we can take $\lambda_n = \frac{1}{n}$. We keep the discount factor constant per unit of calendar time as we increase n requires by imposing that the discount factor per period equal $\beta_n = e^{\rho \lambda_n}$ for some instantaneous discount rate ρ. The steady-state interest rate is then constant per unit of calendar time as we increase n, but the interest rate per period is $R_n = e^{r\lambda_n}$ for the instantaneous interest rate $r = \rho$. This naturally implies that $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} R_n = 1$. Note that a given calendar date t corresponds to a different period number $t_n(t) = \frac{t}{\lambda_n}$ for different values of n.

We can then apply our definitions from the previous sections for every value of n and take the limit as n goes to ∞. For a fixed calendar date τ, we can compute the limits of $\epsilon_{t_n(\tau)}$, $\epsilon^{PE}_{t_n(\tau)}$, $\epsilon^{GE}_{t_n(\tau)}$, $\epsilon^{k}_{t_n(\tau)}$, and $\epsilon^{k,GE}_{t_n(\tau)}$ when n goes to ∞. We denote these limits by ϵ_τ, ϵ^{PE}_τ, ϵ^{GE}_τ, ϵ^{k}_τ, and $\epsilon^{k,GE}_\tau$. They represent the elasticities of output at date 0 to a localized cumulated interest rate change Δr_τ at date τ, by which we mean a change in the interest rate path $\{\hat{r}_t\}$ given by $\hat{r}_t = r + \Delta r_\tau \delta_\tau(t)$ where δ_τ is the Dirac function so that $\int_0^t (\hat{r}_u - r)du = 0$ for $t < \tau$ and $\int_0^t (\hat{r}_u - r)du = \Delta r_\tau$ for $t > \tau$.

We also define the continuous-time analogue $\mathcal{E}^k_{ct}(r\tau)$ of $\mathcal{E}^k(R-1,\tau)$:

$$\mathcal{E}^k_{ct}(r\tau) = \sum_{m=0}^{k-1} \frac{(r\tau)^m}{m!},$$

where $\mathcal{E}^k_{ct}(r\tau)$ is increasing in k with $\mathcal{E}^1_{ct}(r\tau) = 1$ and $\lim_{k\to\infty} \mathcal{E}^k_{ct}(r\tau) = e^{r\tau}$.

Proposition 3 (Representative agent, continuous time)**.** *Consider the representative-agent model with isoelastic utility and either rational expectations or level-k thinking. The interest rate elasticities of output at impact are given by*

$$\epsilon_\tau = \sigma, \quad \epsilon^{PE}_\tau = \sigma e^{-r\tau}, \quad \epsilon^{GE}_\tau = \sigma[1-e^{-r\tau}],$$

$$\epsilon^k_{t,\tau} = \sigma e^{-r\tau}\mathcal{E}^k_{ct}(r\tau).$$

All of our other results go through and the intuitions are identical. In particular, level-k thinking features (weak) mitigation $\epsilon^k_\tau < \epsilon_\tau$, and monotonic convergence with $\frac{\partial \epsilon^k_\tau}{\partial k} > 0$ and $\lim_{k\to\infty} \epsilon^k_\tau = 1$. Compared to the discrete-time case, a useful simplification occurs for $k = 1$ since now have

$$\epsilon^1_\tau = \epsilon^{PE}_\tau = \sigma e^{-r\tau},$$

so that level-1 now coincides exactly (and not just approximately) with the PE effect. This is because in continuous time, the impact of current income on current consumption vanishes, since it becomes a vanishing fraction of permanent income. As a result, the (weak) horizon effect is now given by

$$\frac{\partial \epsilon^1_\tau}{\partial \tau} = -r\epsilon^1_\tau,$$

so that its strength is simply r.

4 Heterogeneous Agents and Incomplete Markets

In this section, we introduce heterogeneous agents, borrowing constraints, and incomplete markets. We study both rational expectations and level-k thinking. We proceed in two stages. First, in Section 4.1, we study a simple “perpetual youth” model featuring occasionally-binding borrowing constraints but no precautionary savings which can be solved in closed form. Second, in Section 4.2, we consider a standard Bewley-Aiyagari-Huggett model of incomplete markets, which features not only occasionally binding borrowing constraints, but also precautionary savings, but which can only be solved numerically. Both models are particular cases of the general reduced-form model of Section 2, with different explicit microfoundations for the aggregate consumption function.

The "perpetual youth" model can be solved in closed form precisely because it features no precautionary savings, and this is turn is due to the fact that there are annuities. As a result, individual consumption functions are linear, and the wealth distribution does not enter as a relevant aggregate state variable in the aggregate consumption function. By contrast, the Bewley-Aiyagari-Huggett model does feature precautionary savings. As a result, individual consumption functions are nonlinear (concave), and the wealth distribution does enter as a relevant aggregate state variable in the aggregate consumption function. The wealth distribution must be tracked and solved for along the lines of the extension described in Section 2.2.

4.1 The Perpetual-Youth Model of Borrowing Constraints

In this section we introduce a standard overlapping generations model of the "perpetual youth" variety a la Yaari (1965) and Blanchard (1985). As is well known, overlapping generations models can be reinterpreted as models with heterogenous agents subject to borrowing constraints (see e.g. Woodford, 1990, Kocherlakota 1992). The death event under the finite lifetime interpretation represents a binding borrowing constraint in the other interpretation. The important common property is that horizons are shortened in that consumption is only smoothed over a limited intervals of time.

We offer an explicit interpretation along these lines. The perpetual youth setup with homothetic preferences and annuities allows us to neatly isolate the impact of occasionally binding borrowing constraints while getting rid of precautionary savings. It also implies that the model aggregates linearly, and therefore, that no extra aggregate state variable capturing the wealth distribution is required to characterize the aggregate equilibrium.

We set up the model directly in continuous time for tractability. The economy is populated by infinitely-lived agents randomly hit by idiosyncratic discount factor shocks that make borrowing constraints bind according to a Poisson process. There is unit mass of ex-ante identical atomistic agents indexed by i which is uniformly distributed over $[0,1]$.

We assume that per-period utility U is isoelastic with a unitary intertemporal elasticity of substitution $\sigma = 1$ which simplifies the analysis. We refer the reader to the appendix for the case $\sigma \neq 1$.

We allow for positive outside liquidity in the form of Lucas trees in unit-supply with time-t value V_t capitalizing a stream δY_t of dividends, the ownership of which at date 0 is uniformly distributed across agents. Non-financial (labor) income given by $(1-\delta)Y_t$. At every date, non-financial income is distributed uniformly across the population.

Agents can borrow and lend subject to borrowing constraints. We assume that the borrowing contracts have the same form as the Lucas trees. This assumption obviates the need to resolve indeterminacy in agent's portfolios to capture the evolution of the wealth distribution when agents are indifferent between different assets for which they expect identical returns (such as Lucas trees and short-term risk-free bonds) but which do not actually have identical returns. We only introduce risk-free bonds with a sequence of instantaneous interest rates $\{r_t\}$ at the margin and make sure that agents wouldn't be better off by including risk-free bonds in their portfolios.[16] Agents can also purchase actuarially fair annuities.

Individual problem. We first describe the individual problem. We proceed as in Section 3.1 to formulate the individual problem given the aggregate paths $\{Y_t, Y_t^e, r_t\}$ directly in terms of total financial wealth a_t^i as long as Lucas trees satisfy the no-arbitrage conditions

$$V_t = \int_0^{\infty} \delta Y_{t+s}^e e^{-\int_0^s r_{t+u} du} ds. \tag{7}$$

Agents are hit by idiosyncratic Poisson shocks with intensity λ. The life of an agent i is divided into "periods" by the successive realizations n of his idiosyncratic Poisson process occurring at the stopping times τ_n^i, with the convention $\tau_0^i = 0$. The agent has a low discount factor $\beta < 1$ between the different "periods" and an instantaneous discount rate ρ within each "period". Importantly, the agent cannot borrow against his future non-financial or human wealth accruing in any future "period". In other words, for $\tau_n^i \leq t < \tau_{n+1}^i$, agent i cannot borrow against any future non-financial income or human wealth accruing after τ_{n+1}^i. We assume that the discount factor $\beta < 1$ is sufficiently low that agents are up against their borrowing constraints between two "periods", so that in equilibrium, agents always choose not to bring in any financial wealth from one "period" to the next and hence that $a_{\tau_{n+1}^i}^i = 0$ for all $n \geq 0$ and $i \in [0,1]$, where a_t^i denotes the financial wealth of agent i at time t. The parameter λ can then be thought of as indexing the frequency of binding borrowing constraints.

The problem of an individual agent at date t with financial wealth a_t^i and who is in "period" n_t is therefore given by

$$\max_{\{\tilde{c}_{t+s}^i, \tilde{a}_{t+s}^i\}} \mathbb{E}_t \sum_{n=0}^{\infty} \beta^n \int_{\tau_{n_t+n}^i}^{\tau_{n_t+n+1}^i} \log(\tilde{c}_{t+s}^i) e^{-\rho s} ds,$$

subject to the future expected budget constraints

$$\frac{d\tilde{a}_{t+s}^i}{ds} = (r_{t+s} + \lambda)\tilde{a}_{t+s}^i + (1-\delta)Y_{t+s}^e - \tilde{c}_{t+s}^i \quad \text{for} \quad \tau_{n_t+n}^i \leq t+s < \tau_{n_t+n+1}^i,$$

the initial condition

$$\tilde{a}_t^i = a_t^i,$$

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[16] These issues were moot in the representative-agent model of Section 3 since the representative agent only holds Lucas trees in equilibrium.

and the borrowing constraints

$$\tilde{a}^i_{\tau_{n_t+n+1}} = 0 \quad \forall n \geq 0.$$

The individual consumption function is the policy function for consumption at date t and is given by

$$c^*(a^i_t; \{r_{t+s}\}, \{Y^e_{t+s}\}) = (\rho + \lambda)[a^i_t + \int_0^\infty (1-\delta) Y^e_{t+s} e^{-\int_0^s (r_{t+u}+\lambda) du} ds].$$

Note that this policy function is independent of the "period" n because the idiosyncratic Poisson process is memoryless. It depends only on expected future income $\{Y^e_{t+s}\}$ but not on current income Y_t because of the continuous time assumption.

The law of motion for a^i_t is given by the actual (as opposed to expected) budget constraints

$$\frac{da^i_t}{dt} = (r_t + \frac{\delta(Y_t - Y^e_t)}{V_t} + \lambda) a^i_t + (1-\delta) Y_t - c^*(a^i_t; \{r_{t+s}\}, \{Y^e_{t+s}\}) \quad \text{for} \quad \tau^i_{n_t+n} \leq t+s < \tau^i_{n_t+n+1},$$

the initial condition

$$a^i_0 = V_t,$$

and the borrowing constraints

$$a^i_{\tau_n} = 0 \quad \forall n \geq 1.$$

Here $r_t + \delta(Y_t - Y^e_t)/V_t = (\delta Y_t + dV_t/dt)/V_t$ is the actual return on a Lucas tree, which may differ from the return $r_t = (\delta Y^e_t + dV_t/dt)/V_t$ that the agent expects to receive since the agent's beliefs about future dividends may be incorrect.

Aggregate state variable. The model also features an aggregate state variable as in Section 2.2: the wealth distribution $\Psi_t = \{a^i_t\}$. The law of motion for Ψ_t is entirely determined by the laws of motion for individual financial wealth a^i_t. However as we shall see below, this aggregate state variable is not required to characterize the aggregate equilibrium.

Reduced-form aggregate consumption function. The reduced-form aggregate consumption function is obtained by aggregating over i the individual consumption function $C(\{r_{t+s}\}, \{Y^e_{t+s}\}) = \int_0^1 c^*(a^i_t; \{r_{t+s}\}, \{Y^e_{t+s}\}) di$ and imposing the asset market clearing condition $\int a^i_t di = V_t$, where V_t is given by the no-arbitrage condition (7). This yields

$$C(\{r_{t+s}\}, \{Y^e_{t+s}\}) = (\rho + \lambda)[\int_0^\infty \delta Y^e_{t+s} e^{-\int_0^s r_{t+u} du} ds + \int_0^\infty (1-\delta) Y^e_{t+s} e^{-\int_0^s (r_{t+u}+\lambda) du} ds].$$

Just like the individual consumption function, and for the same reason, the reduced-form aggregate consumption function depends only on expected future income $\{Y^e_{t+s}\}$ but not on current income Y_t. More importantly, the aggregate consumption function is independent of the aggregate state variable $\Psi_t = \{a_t^i\}$.

Remarkably, the only difference in the reduced form aggregate consumption function compared to the representative-agent model analyzed in Sections 3.2-3.3 is that future expected aggregate non-financial income $(1-\delta)Y_{t+s}^e$ is discounted at rate $e^{-\int_0^s (r_{t+u}+\lambda)du}$ instead of $e^{-\int_0^s r_{t+u}du}$. Future expected aggregate financial income δY_{t+s}^e, incorporated in the value of Lucas trees V_t, is still discounter at rate $e^{-\int_0^s r_{t+u}du}$. This is intuitive since borrowing constraints limit the ability of agents to borrow against future non-financial income but does not prevent them from selling their assets when they are borrowing constrained.[17]The representative-agent model can be obtained as the limit of this model when the frequency λ of binding borrowing constraints goes to zero.

Equilibrium characterization. For concreteness, we briefly characterize the various equilibria in the context of this particular model. Given beliefs $\{Y_t^e\}$, and given the path for interest rates $\{r_t\}$, $\{r_t, Y_t\}$ is a temporary equilibrium if and only if the path for aggregate income $\{Y_t\}$ is given by

$$Y_t = (\rho+\lambda)[\int_0^\infty \delta Y_{t+s}^e e^{-\int_0^s r_{t+u}du} ds + \int_0^\infty (1-\delta) Y_{t+s}^e e^{-\int_0^s (r_{t+u}+\lambda)du} ds] \quad \forall t \geq 0.$$

Similarly, given the path for interest rates $\{r_t\}$, $\{r_t, Y_t\}$ is an REE if and only if the path for aggregate income $\{Y_t\}$ satisfies the fixed point

$$Y_t = (\rho+\lambda)[\int_0^\infty \delta Y_{t+s} e^{-\int_0^s r_{t+u}du} ds + \int_0^\infty (1-\delta) Y_{t+s} e^{-\int_0^s (r_{t+u}+\lambda)du} ds] \quad \forall t \geq 0.$$

Finally given an initial REE $\{r_t, Y_t\}$ and a new interest rate path $\{\hat{r}_t\}$, the level-k equilibria $\{\hat{r}_t, \hat{Y}_t^k\}$ satisfy the following recursion over $k \geq 0$:

$$\hat{Y}_t^k = (\rho+\lambda)[\int_0^\infty \delta \hat{Y}_{t+s}^{k-1} e^{-\int_0^s r_{t+u}du} ds + \int_0^\infty (1-\delta) \hat{Y}_{t+s}^{k-1} e^{-\int_0^s (r_{t+u}+\lambda)du} ds] \quad \forall t \geq 0.$$

with the initialization that $\hat{Y}_t^0 = Y_t$ for all $t \geq 0$.

We now turn to the computation of the different interest rate elasticities of output around a steady state REE $\{R_t, Y_t\}$ $Y_t = Y > 0$ and $r_t = r$ for all $t \geq 0$ with

$$1 = (1-\delta)\frac{\rho+\lambda}{r+\lambda} + \delta\frac{\rho+\lambda}{r}.$$

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[17] Note that this requires financial assets to be liquid. Financial income (dividends) from partly illiquid assets should be discounted at a higher rate. For example, suppose that a fraction of trees can be sold while others cannot (or at a very large cost). Illiquid trees should then be treated like non-financial income. The financial income of illiquid trees should be discounted at rate $e^{-\int_0^s (r_{t+u}+\lambda)du}$ while that of liquid trees should be discounted at rate $e^{-\int_0^s r_{t+u}du}$. In essence, introducing illiquid trees is isomorphic to a reduction in δ.

Later when we derive comparative statics with respects to variations in λ, we vary ρ at the same time to keep the interest rate constant at r.

Monetary policy at different horizons under RE. We start with the RE case, where we use the selection $\lim_{t\to\infty} Y_t = Y$ as we perform the comparative statics underlying the computation of the interest rate elasticities of output.

Proposition 4 (Perpetual youth model of borrowing constraints, RE)**.** *Consider the perpetual youth model of borrowing constraints with logarithmic utility $\sigma = 1$ and rational expectations. The interest rate elasticities of output at impact do not depend on the horizon τ and are given*

$$\epsilon_\tau = 1.$$

They can be decomposed as $\epsilon_\tau = \epsilon_\tau^{PE} + \epsilon_\tau^{GE}$ into PE and GE elasticities which are given by

$$\epsilon_\tau^{PE} = (1-\delta)\frac{\rho+\lambda}{r+\lambda}e^{-(r+\lambda)\tau} + \delta\frac{\rho+\lambda}{r}e^{-r\tau}$$
$$\epsilon_\tau^{GE} = (1-\delta)\frac{\rho+\lambda}{r+\lambda}[1-e^{-(r+\lambda)\tau}] + \delta\frac{\rho+\lambda}{r}[1-e^{-r\tau}].$$

A remarkable result in this proposition is that the interest rate elasticity of output ϵ_τ is completely independent of the frequency λ of binding borrowing constraints

$$\frac{\partial \epsilon_\tau}{\partial \lambda} = 0,$$

and is therefore exactly identical to its counterpart in the representative-agent model as described in Proposition 1 adapted to continuous time in Proposition 3. In other words, the incompleteness of markets introduced in the perpetual youth model of borrowing constraints is irrelevant for the aggregate effects of monetary policy. This is a version of the incomplete-markets irrelevance result under rational expectations. Although the result also holds for any $\delta > 0$, the intuition is conveyed most transparently in the case of no outside liquidity $\delta = 0$ because in this case $\rho = r$ is independent of λ (otherwise we have to vary ρ so as to keep r constant when we vary λ). The PE effect is weaker, the higher is λ, so that

$$\frac{\partial \epsilon_\tau^{PE}}{\partial \lambda} = -\tau e^{-(r+\lambda)\tau} < 0.$$

This is because for a given path of output, a higher frequency λ of borrowing constraints leads to more discounting of future interest rate cuts, and hence to a response of consumption to a future interest rate cut in partial equilibrium. But the GE effect is stronger, the higher is λ, leading to a complete offset

$$\frac{\partial \epsilon_\tau^{GE}}{\partial \lambda} = -\frac{\partial \epsilon_\tau^{PE}}{\partial \lambda} > 0.$$

This is because the aggregate marginal propensity to consume $\rho + \lambda = r + \lambda$ increases with the frequency λ of borrowing constraints, and hence so does the general equilibrium Keynesian multiplier.[18]

Monetary policy at different horizons under level-*k*. We now turn to the level-k case.

Proposition 5 (Perpertual youth model of borrowing constraints, level-k)**.** *Consider the perpetual youth model of borrowing constraints with logarithmic utility $\sigma = 1$ and level-k thinking. The interest rate elasticities of output at impact depend on the horizon τ. In the extreme cases of no outside liquidity $\delta = 0$ and very abundant outside liquidity when δ goes to 1, they are given by:*

$$\begin{aligned} \epsilon_\tau^k &= e^{-(r+\lambda)\tau} \mathcal{E}_{ct}^k((r+\lambda)\tau) \quad \textit{when} \quad \delta = 0, \\ \epsilon_\tau^k &= e^{-r\tau} \mathcal{E}_{ct}^k(r\tau) \quad \textit{when} \quad \delta \to 1. \end{aligned}$$

Unlike in the rational-expectations case, under level-k, the interest rate elasticity of output ϵ_τ^k depends of the frequency λ of binding borrowing constraints, breaking incomplete-markets irrelevance. Indeed, there are now similarities but also important differences between Proposition 5 and its counterpart in the representative-agent model as described in Proposition 2 adapted to continuous time in Proposition 3.

With incomplete markets like in the representative-agent case, level-k thinking features mitigation $\epsilon_\tau^k < \epsilon_\tau$, and monotonic convergence with $\frac{\partial \epsilon_\tau^k}{\partial k} > 0$ and $\lim_{k\to\infty} \epsilon_\tau^k = 1$. In addition, level-1 coincides exactly with the PE effect $\epsilon_\tau^1 = \epsilon_\tau^{PE}$.

But ϵ_τ^k now depends on the frequency λ of binding borrowing constraints as long as $\delta < 1$, and as a result differs from its value in the rational-expectations case, where we vary ρ to keep the interest rate r constant as we vary λ. For simplicity, we focus on the case with no outside liquidity $\delta = 0$ where $r = \rho$, which leads to very transparent formulas. For any k, ϵ_τ^k decreases with λ so that more frequent borrowing constraints lead to stronger *mitigation* of the effects of monetary policy

$$\frac{\partial \epsilon_\tau^k}{\partial \lambda} = -e^{-(r+\lambda)\tau} \frac{(r+\lambda)^{k-1}\tau^k}{(k-1)!} < 0.$$

Moreover, for any k, $\frac{\partial \epsilon_\tau^k}{\partial \tau}$ decreases with λ so that more frequent borrowing constraints lead to

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[18] Note that this property holds despite the existence of a countervailing effect that arises because the increase in human wealth associated with the general equilibrium increase in output is lower when λ is higher because human wealth is more discounted.

stronger *horizon* effects of monetary policy for small enough horizons

$$\frac{\partial^2 \epsilon_\tau^k}{\partial \lambda \partial \tau} = \epsilon_\tau^k \frac{(r+\lambda)\tau - k}{\tau} < 0 \quad \text{for} \quad \tau < \frac{k}{r+\lambda}.$$

These effects disappear in the rational-expectations case which obtains in the limit where k goes to ∞. These effects also disappear when outside liquidity is very abundant in the limit where δ goes to 1 since then $\epsilon_\tau^k = e^{-r\tau}\mathcal{E}_{ct}^k(r\tau)$ is independent of λ.

This can be seen most clearly in the case $k = 1$ where when $\delta = 0$, we get

$$\epsilon_\tau^1 = e^{-(r+\lambda)\tau}, \quad \text{and} \quad \frac{\partial \epsilon_\tau^1}{\partial \tau} = -(r+\lambda)\epsilon_\tau^1,$$

so that the strength of the mitigation and horizon effects is $r + \lambda$ instead of r in the representative-agent case. As a result, the mitigation and horizon effects are plausibly much stronger than in the representative-agent case, even if the interest rate is very low. If the annual interest rate is $r = 5\%$, then the effects of monetary policy decrease at rate 5% per year with a half life of 14 years if $\lambda = 0$ as in the representative-agent case, but decrease at rate 15% per year with a half life of 5 years if $\lambda = 10\%$; if the annual interest rate is 1% the effects of monetary policy decrease at rate 11% per year with a half life of 69 years if $\lambda = 0$ as in the representative-agent case, but decrease at rate 11% per year with a half life of 6 years if $\lambda = 10\%$. In the limit of very abundant outside liquidity when δ goes to 1 instead, we have $\epsilon_\tau^1 = e^{-r\tau}$ and $\frac{\partial \epsilon_\tau^1}{\partial \tau} = -r\epsilon_\tau^1$ as in the representative-agent case and independently of λ.

The results for a finite k are in striking contrast to the rational-expectations benchmark, which obtains in the limit where k goes to ∞. Level-k thinking leads to a *mitigation* of the effects of monetary policy so that interest rate changes have less of an effect on output. Level-k thinking also leads to a *horizon* effect of monetary policy so that interest rate changes have less of an effect on output, the further in the future they take place. The mitigation and horizon effects that arise with level-k thinking are stronger, the more frequent are borrowing constraints, i.e. the higher is λ. This illustrates a profound *interaction* between level-k thinking and incomplete markets. This interaction disappears in the limit where outside liquidity is very abundant when δ goes to 1.

There is a simple intuition for all these results in terms of the decomposition of the effects of monetary policy into PE and GE effects. As already explained in Section 3, the PE effect features mitigation—the effect of interest rate changes is lower than the full effect under rational expectations because the latter is the sum of the GE and the PE effect. It also features horizon—for a fixed path of output, interest rate changes affect partial equilibrium consumption less, the further in the future they are. Under rational expectations, the GE effect eliminates the mitigation effect by adding to the PE effect, and also eliminates the horizon effect because the GE effect features an anti-horizon effect. With level-1 thinking, monetary policy coincides with the PE effect and features mitigation and horizon. In the rational-expectations limit when k goes to ∞, the mitigation and horizon effects disappear. Intermediate values of k interpolate smoothly and monotonically between these two extremes.

The effects of the frequency λ of binding borrowing constraints can be understood as follows. The horizon and mitigation effects of the PE effect are stronger, the higher is λ because of higher discounting of non-financial (human) wealth. Under rational expectations, the GE effect offsets this dependence on λ because the aggregate marginal propensity to consume $\rho + \lambda$ and hence the Keynesian multiplier increase with λ. At level-1, monetary policy coincides with the PE effect and the horizon and mitigation features are stronger, the higher is λ. In the rational-expectations limit where k goes to ∞, the dependence of the mitigation and horizon effects on λ disappears. Intermediate values of k interpolate smoothly and monotonically between these two extremes. This also explains why the interaction between bounded rationality and incomplete markets disappears in the limit where outside liquidity is very abundant when δ goes to 1, since it is only non-financial (human) wealth which is more discounted when borrowing constraints bind more often, but not the dividends promised by the Lucas trees.

4.2 The Bewley-Aiyagari-Huggett Model of Borrowing Constraints and Precautionary Savings

In this section, we consider a standard Bewley-Aiyagari-Huggett model of incomplete markets. This model features not only occasionally binding borrowing constraints like the perpetual youth model of borrowing constraints developed in Section 4.1 but also precautionary savings. As a result, individual consumption functions are no longer linear but are instead concave, linear aggregation does not obtain, and the wealth distribution becomes a relevant aggregate state variable.

There is a unit mass of infinitely-lived agents indexed by i distributed uniformly over $[0, 1]$. Time is discrete with a period taken to be a quarter. Agents have logarithmic utility $\sigma = 1$ and discount factor β.

Agents face idiosyncratic non-financial income risk $y_t^i(1-\delta)Y_t$. There is a unit supply of Lucas trees capitalizing the flow of dividends δY_t. The idiosyncratic income process is $\log(y_t^i) = \rho_\epsilon \log(y_{t-1}^i) + \epsilon_t^i$, where ϵ_t^i is i.i.d. over time, independent across agents and follows a normal distribution with variance σ_ϵ^2 and mean $\mathbb{E}[\epsilon_t^i] = -\sigma_\epsilon^2(1-\rho_\epsilon^2)^{-1}/2$ so that $\int y_t^i di = 1$.

Agents can borrow and lend subject to borrowing constraints. Like in Section 4.1 and partly for the same reason, we assume that the borrowing contracts have the same form as the Lucas trees. We also assume that the borrowing constraints take a simple form, namely that agents cannot have a negative asset position. These choices ensure that under rational expectations, the incomplete-markets irrelevance result holds, and the interest rate elasticity of output coincides with that of a complete-markets or representative agent model $\epsilon_{t,\tau} = 1$. We only introduce risk-free bonds with a sequence of interest rates $\{R_t\}$ at the margin and make sure that agents wouldn't be better off by including risk-free bonds in their portfolios.

Individual problem. We first describe the individual problem. We proceed as in Section 3.1 to formulate the individual problem given the aggregate paths $\{Y_t, Y_t^e, R_t\}$ directly in terms of total financial wealth a_t^i as long as Lucas trees satisfy the no-arbitrage conditions (6).

The problem at date t with financial wealth a_t^i is

$$\max_{\{\tilde{c}_{t+s}^i, \tilde{a}_{t+1+s}^i\}} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \log(\tilde{c}_{t+s}^i) ds,$$

subject to the current actual budget constraint

$$\tilde{c}_t^i = (1-\delta) y_t^i Y_t + a_t^i - \frac{\tilde{a}_{t+1}^i}{R_t},$$

the future expected budget constraints

$$\tilde{c}_{t+1+s}^i = (1-\delta) y_{t+1+s}^i Y_{t+1+s}^e + \tilde{a}_{t+1+s}^i - \frac{\tilde{a}_{t+2+s}^i}{R_{t+1+s}} \quad \forall s \geq 0,$$

and the borrowing constraints

$$\tilde{a}_{t+1+s}^i \geq 0 \quad \forall s \geq 0.$$

The the individual consumption function $c^*(a_t^i, y_t^i; \{R_{t+s}\}, Y_t, \{Y_{t+s}^e\})$ is the policy function for $\tilde{c}_t^i$. The law of motion for a_t^i is given by the actual (as opposed to expected) budget constraint

$$a_{t+1}^i = [R_t + \frac{\delta(Y_t - Y_t^e)}{V_t - \delta Y_t}][(1-\delta) y_t^i Y_t + a_t^i - c^*(a_t^i, y_t^i; \{R_{t+s}\}, Y_t, \{Y_{t+s}^e\})],$$

where $R_t + \delta(Y_t - Y_t^e)/(V_t - \delta Y_t) = V_{t+1}/(V_t - \delta Y_t)$ is the actual return on Lucas tree, which may differ from the return $R_t = V_{t+1}^e/(V_t - \delta Y_t)$ that the agent expects since the agent's beliefs about future dividends may be incorrect.

Aggregate state variable. The model also features an aggregate state variable as in Section 2.2: the joint distribution of wealth and income shocks $\Psi_t = \{a_t^i, y_t^i\}$. The law of motion for Ψ_t is entirely determined by the laws of motion for individual financial wealth and income shocks given an initial condition Ψ_0 with $\int a_0 d\Psi(a_0, y_0) = V_0$, where V_0 is given by the no-arbitrage condition (6) and $\int y_0 d\Psi(a_0, y_0) = 1$. In contrast to the perpetual youth model of borrowing constraints developed in Section 4.1, this aggregate state variable is required to characterize the aggregate equilibrium.

Reduced-form aggregate consumption function. The reduced-form aggregate consumption function is obtained by aggregating over i the individual consumption function

$$C(\{R_{t+s}\}, Y_t, \{Y^e_{t+s}\}, \Psi_t) = \int_0^1 c^*(a_t, y_t; \{R_{t+s}\}, Y_t, \{Y^e_{t+s}\}) d\Psi_t(a_t, y_t).$$

Temporary equilibria, RE equilibria, and level-k equilibria are then defined exactly as in the general reduced form model described in Section 2.

Monetary policy at different horizons. This model cannot be solved analytically, and so we rely on simulations instead. We consider a steady state $\{Y, R, \Psi\}$ of the model with a 2% annual interest rate and a corresponding quarterly interest rate of $R = 1.005$. We take $\rho_\epsilon = 0.966$, and $\sigma^2_\epsilon = 0.017$ for the idiosyncratic income process as in McKay et al. (2016) and Guerrieri and Lorenzoni (2015). For our baseline economy, we take $\frac{V}{Y} = 1.44$ for the fraction of outside liquidity to output, exactly as in McKay et al. (2016).[19] The values of $\beta = 0.988$ and $\delta = 0.035$ are calibrated to deliver these values of R and $\frac{V}{Y}$. The fraction of borrowing-constrained agents in the steady state is then 14.7%.

Figure 1 depicts the proportional output response of the economy to a 1% interest rate cut at different horizons, or in other words, the interest rate elasticity of output ϵ^k_τ at different horizons τ, for different values of k, comparing the incomplete-markets baseline economy with the complete-markets or representative-agent version of the same economy. The figure illustrates the strong mitigation and horizon effects brought about by the interaction of incomplete markets and bounded rationality, by comparing the economy with $k = 1$ and incomplete markets, the economy with $k = 1$ and complete markets, and the economy with rational expectations which obtains in the limit when k goes to ∞ where the degree of market incompleteness becomes irrelevant by construction. It also shows how these mitigation and horizon effects dissipate as we increase the level of reasoning k, moving towards rational expectations. In the present simulation, the convergence to rational expectations is quite fast, resulting in outcomes that are close to rational expectations for values $k \geq 2$. However, the simulation is only illustrative and fast convergence is not a general property. In addition, very low levels of k, including $k = 1$, are perhaps realistic as descriptions of household behavior when confronted with unusual monetary policy announcements.[20]

Figure 2 illustrates how these effects change as we move away from the baseline economy by varying the discount factor β and the amount of liquidity δ while keeping the steady-state annual interest rate constant at 2%. These different calibrations can be understood as repre-

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[19] This value for the fraction of outside liquidity to output $\frac{V}{Y} = 1.44$ is meant to capture the value of liquid (as opposed to illiquid) wealth in the data.

[20] We are suggesting that level-k thinking may be higher in other contexts. For example, in financial markets, deeply invested traders may undertake much higher rounds of thinking. In the present model it is consumption decisions by households that matter, making low levels of k more relevant.

senting different degrees of market incompleteness since they lead to different values for the fraction of borrowing-constrained agents in the steady state and for the aggregate marginal propensity to consume. The model approximates the complete-markets model in the limit where this fraction goes to zero.

Once again the figure powerfully illustrates the strong interaction of incomplete markets and bounded rationality: For a given finite value of k, the mitigation and horizon effects are much stronger when markets are more incomplete in the sense that the steady-state fraction of borrowing-constrained agents is higher; furthermore, the convergence to rational expectations is slower when markets are more incomplete.[21]

Overall, in this calibrated Bewley-Aiyagari-Huggett economy with occasionally borrowing constraints and precautionary savings, there are powerful interactions between bounded rationality and incomplete markets. This reinforces the analytical results that we obtained in the perpetual youth model of borrowing constraints developed in Section 4.1 which features borrowing constraints but no precautionary savings.

5 Sticky Prices and Inflation

So far, we have abstracted from inflation by assuming that prices are fully rigid, or equivalently by focusing on the response of the economy to changes in the path of real interest rates. In this section, we depart from the assumption of fully rigid prices and study the role of inflation.

We modify the model of Section 4.2 to incorporate monopolistic competition and sticky prices a la Calvo, as well as explicit labor supply and labor demand decisions in the presence of idiosyncratic productivity shocks. To highlight the interaction between incomplete markets and level-k expectations, we maintain a specification which guarantees that under rational expectations, the responses of output and inflation to a monetary policy shock are equivalent to that of a representative agent model.

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[21] Finally, in Figure 5, we illustrate the consequences of deviating from log utility by reporting ϵ_{16}^{k} for $k = 1$ and $k = \infty$ for different values of the intertemporal elasticity of substitution σ (which is also the inverse of the relative risk aversion coefficient γ) for both the incomplete-markets economy with heterogeneous agents and the complete-markets economy with a representative agent. We vary the discount factor β across the two economies to keep the interest rate constant at 2%. With rational expectations ($k = \infty$), the output effects of monetary policy are larger with incomplete markets than with complete markets when $\sigma < 1$, smaller when $\sigma > 1$, and identical when $\sigma = 1$. These results are consistent with those in Werning (2015) who shows that incomplete markets amplify the effects of monetary policy when liquidity is procyclical and mitigates them when it is countercylical, since liquidity (the ratio of the value of assets to aggregate output) turns from procyclical to countercyclical (in response to monetary policy shocks) when σ crosses one from below. However, for realistic values, the difference is small. Comparing $k = \infty$ with $k = 1$ and complete markets with incomplete markets shows the robustness of our finding that bounded rationality and incomplete markets interact to powerfully mitigate the effects of monetary policy.

Model summary. For brevity, we only offer a short summary of the main differences that arise in this setup. The full description of the model and of its solution can be found in the appendix.

First, individual agents now make labor supply decisions, in addition to consumption and savings decisions. Their income is the product of their work effort and of their wage. The latter is proportional to their idiosyncratic productivity which follows a geometric Gaussian AR(1) process (with parameters ρ_ϵ and σ_ϵ). Utility is separable between consumption and leisure, logarithmic in consumption and isoelastic in labor (with Frisch elasticity of labor supply $1/\gamma$).

Second, final goods are produced by a competitive sector using a CES aggregate of intermediate goods (with elasticity θ). Intermediate goods are produced linearly from upstream goods by monopolistically competitive firms who get to change their price with some probability (with per-period probability of a price change λ). This leads to forward-looking pricing decisions that require expectations of future aggregate variables. Upstream goods are produced competitively with a Cobb-Douglas technology (with shares δ and $1-\delta$) from capital and labor. Capital is assumed to be in fixed supply and its profits are capitalized in a Lucas tree which is traded across agents and provides liquidity.

Third, for reasons that we discuss below, we describe monetary policy as a path of interest rate rules specifying nominal interest rates in any given period as a function of the inflation rate in that period, and changes in monetary policy as changes in this path.[22]

Relative to our model with rigid prices, additional aggregate state variables now affect households and monopolistic firms. In particular, we now need to track not only the paths of nominal interest rates, output, and beliefs about output, but also profits, wages, and prices as well as beliefs about these variables. Moreover, aggregation requires keeping track not just of the wealth distribution but also of the distribution of prices.

Effects of monetary policy at different horizons. We compute the effects of monetary policy at different horizons around a steady state with no inflation $\Pi = 1$ and with constant nominal interest rate R. We consider perturbations of monetary policy. We hold interest rates fixed at R before τ, change the interest rate at τ by ΔR_τ, and allow the interest to vary according to

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[22] With sticky but imperfectly rigid prices, specifying a path for nominal interest rates still leads to a unique level-k equilibrium for any value of k, but is no longer sufficient to ensure the convergence of the sequence of level-k equilibria to a rational-expectations equilibrium when k goes to ∞. Specifying a path of reactive enough rules for the nominal interest rate fixes that problem. Interest rate rules play a different role in level-k equilibria compared to their traditional role with rational expectations. In rational expectations equilibria, sufficiently reactive interest rate rules ensure local determinacy of the equilibrium by ensuring that alternative candidate equilibria feature explosive dynamics and hence do not remain in the vicinity of the equilibrium. Essentially, what matters for local determinacy is the off-equilibrium commitment of the central bank. In contrast, in level-k equilibria, there is a unique global equilibrium for any k. The interest rate rule results in an endogenously different nominal interest rate path for different values of k. If the interest rate rule is responsive enough, then the equilibrium converges to a rational expectations equilibrium when k goes to infinity. What matters is the interest rate path in equilibrium for different values of k, instead of the off-equilibrium commitment.

the unchanged standard policy rule after τ (consistent with the steady state). That is, we set $R_t(\Pi_t) = R$ for $t < \tau$; $R_\tau(\Pi_t) = R + \Delta R_\tau$; and $R_t(\Pi_t) = R\Pi_t^\phi$ for $t > \tau$. We then compute $\epsilon_\tau = \lim_{\Delta R_\tau \to 0} -(R/Y)(\Delta Y_0/\Delta R_\tau)$ and $\epsilon_\tau^\Pi = \lim_{\Delta R_\tau \to 0} -(R/\Pi)(\Delta \Pi_1/\Delta R_\tau)$.

The model cannot be solved analytically, and so we rely on simulations. We consider the same parameter values for R, ρ_ϵ, σ_ϵ, δ, and β as in Section 4.2. We take $\gamma = 2$ to match a Frisch elasticity of labor supply of 0.5. We set $\theta = 6$ to generate a desired markup of 1.2 and $\lambda = 0.85$, which implies an average price duration of about 6 quarters as in Christiano et al. (2011). Finally we pick the coefficient in the interest rate rule to be $\phi = 1.5$.

Figure 3 depicts the proportional output and inflation responses to a 1% interest rate cut at different horizons, or in other words, the interest rate elasticities at impact of output ϵ_τ^k and inflation $\epsilon_\tau^{\Pi,k}$ at different horizons τ. The figure shows our incomplete-markets economy as well as the complete-markets (representative-agent) version of the same economy, under level-k bounded rationality for different values of k.

Comparing Figures 1 and 3, we verify numerically the analytical result that for $k = 1$, the response of output is identical when prices are rigid and when they are sticky, simply because at this level of reasoning, agents do not expect any inflation even if prices are sticky. The level-1 response of output features mitigation and horizon effects. With sticky prices, the response of inflation also features mitigation and horizon effects. These effects are stronger for the incomplete-markets economy than for the complete-markets economy.

As k increases, the responses of output and inflation converge monotonically to their rational-expectations counterparts. In the rational-expectations limit when k goes to ∞, the responses of output and inflation for the incomplete-markets and complete-markets economy coincide. Comparing low values of k and especially $k = 1$ with high values of k therefore demonstrates that the complementarity between incomplete markets and bounded rationality that we uncovered in the case with rigid prices is robust to the introduction of inflation.

For high enough values of k, these responses acquire anti-horizon effects in the sense that the response of current output and inflation increase with the horizon of monetary policy. This is unlike the case with rigid prices where the rational-expectations equilibrium features no horizon effect. These anti-horizon effects arise because of a feedback loop between output and inflation whereby higher output now and in the future up until the horizon of monetary policy generates higher inflation now and in the future, which reduces real interest rates, which further increases output now and in the future, etc. The higher the level of reasoning k, the more rounds in the feedback loop, and the stronger its effects. And for a given k, the longer the horizon of monetary policy, the longer the time horizon over which this feedback loop plays out, and the stronger its effects.

Figure 4 illustrates how these effects change as we move away from the baseline economy by varying the discount factor β and amount of liquidity δ while keeping the steady-state annual interest rate constant at 2%. These different calibrations lead to different values for the fraction of borrowing-constrained agents in the steady state, and thus affect the average marginal propensity to consume.[23]

Once again the figure powerfully illustrates the strong interaction of incomplete markets and bounded rationality: For a given finite value of k, the mitigation and horizon effects are much stronger when the steady-state fraction of borrowing-constrained agents is higher; furthermore, the convergence to rational expectations is also slower. In fact, comparing Figures 2 and 4 for intermediate values of k shows that this complementarity is amplified when moving from rigid to sticky prices.

Overall, incorporating sticky prices and inflation worsens the "forward guidance puzzle" under rational expectations: The rational-expectations equilibrium features anti-horizon effects with the effects of monetary policy on output and inflation strongly increasing with the horizon of monetary policy. Incomplete markets alone does not change these properties since the aggregate properties of our model are invariant to the degree of market incompleteness under rational expectations. Level-k bounded rationality alone mitigates and for low values of k reverses these effects. But even for $k = 1$, the horizon effects remain very weak, exactly as in the case of rigid prices considered in Section 4.2. Level-k bounded rationality and incomplete markets together generate powerful horizon effects, exactly as in the case of rigid prices considered in Section 4.2. The complementarity between incomplete markets and bounded rationality that we identified in the case of rigid prices remains and is even strengthened with sticky prices.

6 Conclusion

We have demonstrated a strong interaction between two forms of frictions, bounded rationality and incomplete markets. In economies with nominal rigidities, this interaction has important implications for the transmission of monetary policy, by mitigating its effects, the more so, the further in the future that monetary policy change takes place. This offers a possible rationalization of the so-called "forward guidance puzzle". We conjecture that these conclusions generalize to other shocks and policies. We pursue these directions in ongoing work.

An important direction for future research is to generalize level-k thinking to a dynamic stochastic environment. Indeed, we have proceeded by assuming that a one-time unanticipated shocks hits an economy in a deterministic equilibrium.[24] In this context, it is natural to make the initial equilibrium the status quo on which to anchor beliefs in the initialization of the level-k recursion. In principle, in a dynamic stochastic environment, one need to specify what contingencies agents foresee when they form their expectations, when agents undertake

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[23] We also vary the inverse Frisch elasticity γ so that $\frac{\gamma+\delta}{1-\delta}$ remains unchanged from its baseline level. This guarantees that the rational expectations elasticity ϵ_τ is the same across simulations.

[24] This simplification is also present in other deviations approaches proposition to deviate from rational expectations such as Gabaix (2017) for example.

to form new expectations, and how to update the status quo on which they initialize their level-*k* recursion. Many choices are possible, and, in the absence of further empirical evidence, it is a priori not clear that one is more natural than the other. For example, one possibility is to consider a regime-switching process, where agents foresee the possibility of future regime switches, a regime switch triggers the formation of new expectations, and the new status quo is given by some combination of a new exogenous post-switch heuristics, pre-switch expectations, and past equilibrium realizations. Regime switches could be triggered by the realization of large shocks, or at regular possibly stochastic intervals. It is our opinion that exploring the implications of these different specifications and mustering empirical evidence to help sort through the different theoretical possibilities is a demanding but promising agenda.

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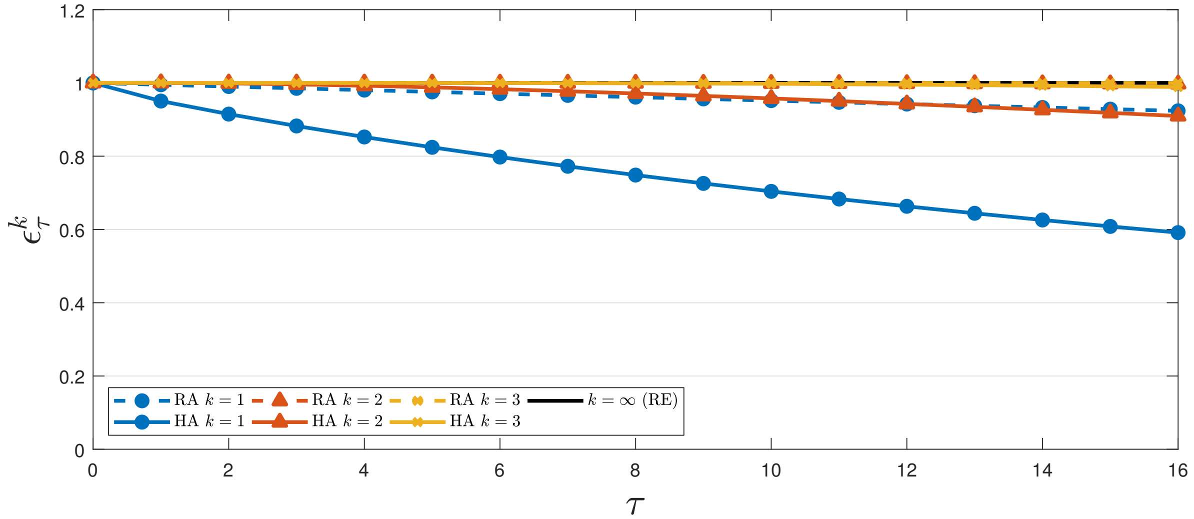

Figure 1: Proportional output response ϵ_τ^k at date 0 to a 1% interest rate cut at different horizons τ for the baseline incomplete-markets economy (solid lines) and the complete-markets or representative-agent economy (dashed lines), in the model with rigid prices of Section 4.2. Different colors represent equilibrium output under level-k thinking with different values of k.

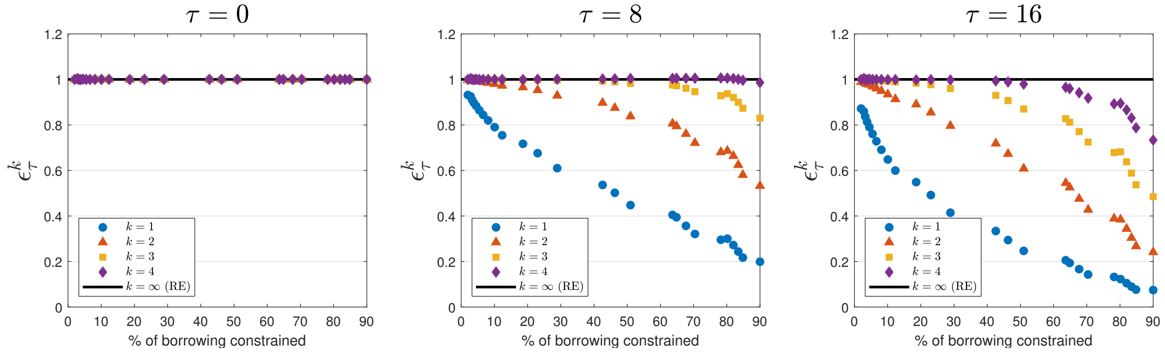

Figure 2: Proportional output response ϵ_τ^k at date 0 to a 1% interest rate cut at a horizon of $\tau = 0$, $\tau = 8$ quarters, and $\tau = 16$ quarters, in the model with rigid prices of Section 4.2. Different colors represent equilibrium output under level-k thinking with different values of k. Different dots of the same color correspond to economies with different fractions of borrowing-constrained agents in steady state. This variation is achieved by varying the discount factor β and amount of liquidity δ and keeping the steady-state annual interest rate constant at 2%.

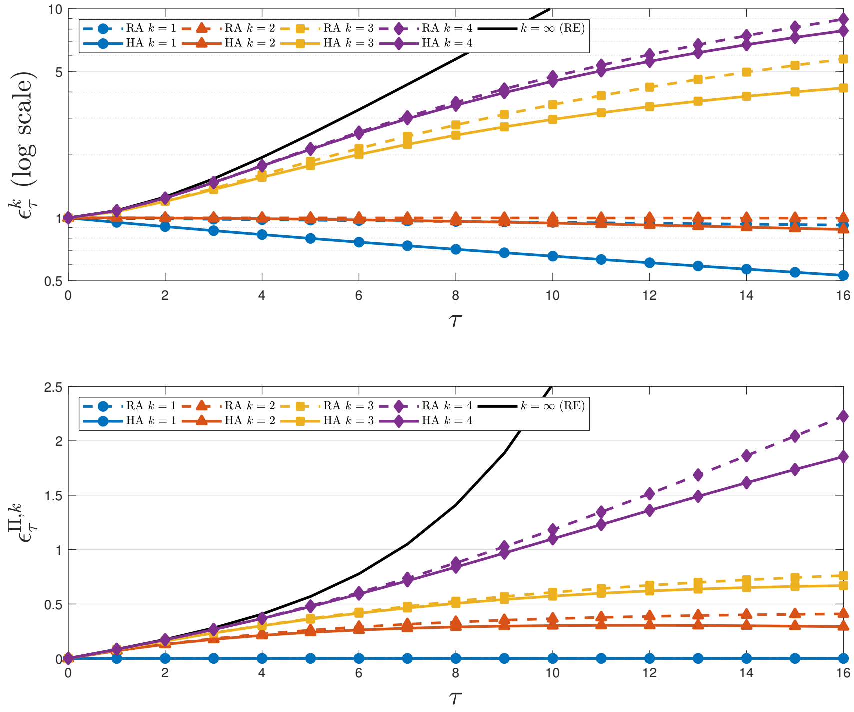

Figure 3: Proportional output response ϵ_τ^k and inflation response $\epsilon_\tau^{\Pi,k}$ to a 1% interest rate cut at different horizons τ for the baseline incomplete-markets economy (dashed lines) and the complete-markets or representative-agent economy (solid lines), in the model with sticky prices of Section 5. Different colors represent equilibrium output under level-k thinking with different values of k. The top panel uses a log scale, and the bottom panel a level scale.

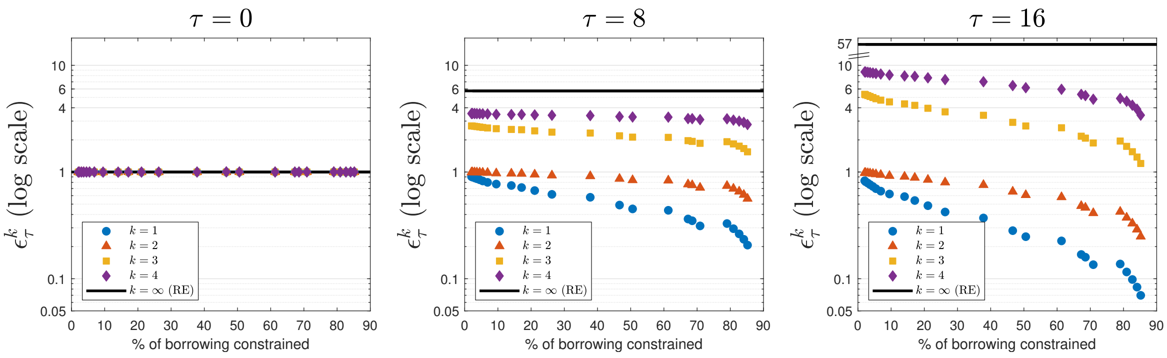

Figure 4: Proportional output response ϵ_τ^k at date 0 to a 1% interest rate cut at a horizon of $\tau = 0$, $\tau = 8$ quarters, and $\tau = 16$ quarters, in the model with sticky prices of Section 5. Different colors represent equilibrium output under level-k thinking with different values of k. Different dots of the same color correspond to economies with different fractions of borrowing-constrained agents in steady state. This variation is achieved by varying the discount factor β and amount of liquidity δ and keeping the steady-state annual interest rate constant at 2%. The panels use a log scale.

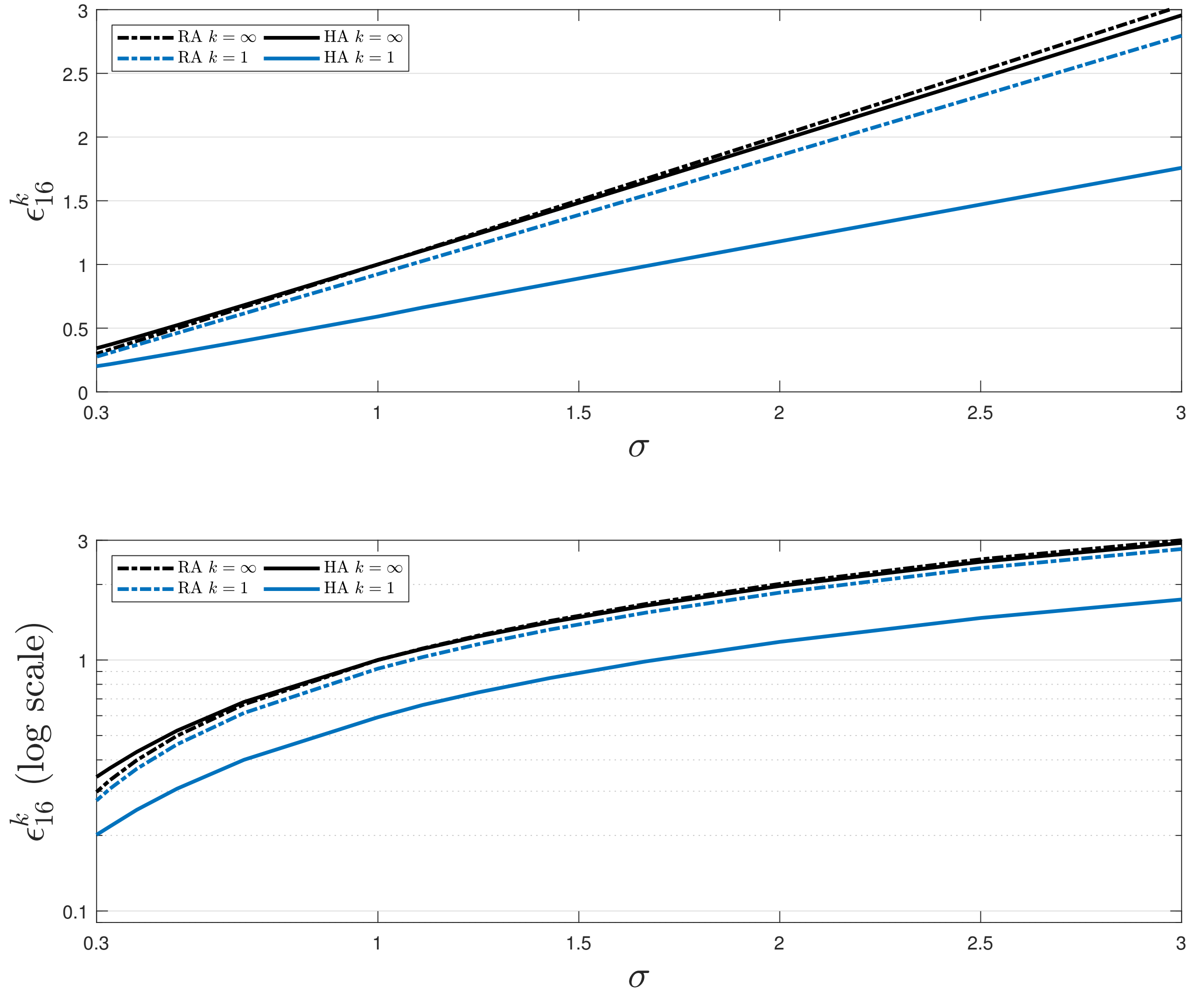

Figure 5: Proportional output response ϵ_{16}^k at date 0 to a 1% interest rate cut at a horizon of $\tau = 16$ quarters, in the model with rigid prices of Section 4.2 for different values of the intertemporal elasticity of substitution $\sigma \in [0.3, 3]$. Different colors represent equilibrium output under level-k thinking with different values of k. Solid lines correspond to the incomplete-markets economy with heterogeneous agents, and dashed-doted lines to the complete-markets economy with a representative agent. We vary the discount factor β in the two economies to keep the steady-state annual interest rate constant at 2%. The top panel uses a level scale and the bottom panel a log scale.

7 Appendix for Online Publication

7.1 Proofs of Propositions 1 and 2

We consider an initial REE $\{R_t, Y_t\}$ which is a steady state with $R_t = R$ and $Y_t = Y$ for all $t \geq 0$. This only requires that $\beta R = 1$. We consider a change $\{\hat{R}_t\}$ in the path for the interest rate ΔR_τ at date τ so that $\hat{R}_\tau = R + \Delta R_\tau$ and $\hat{R}_t = R_t$ for $t \neq \tau$.

We start by computing the new new REE $\{\hat{R}_t, \hat{Y}_t\}$. Because the aggregate model is purely forward looking, we can immediately conclude that for $t > \tau$, $\hat{Y}_t = Y$ and so $\Delta \hat{Y}_t = 0$. And we guess and verify that for $t \leq \tau$, $\hat{Y}_t = Y(1 + \frac{\Delta R}{R})^{-\sigma}$ and so

$$\Delta \hat{Y}_t = Y[(1 + \frac{\Delta R}{R})^{-\sigma} - 1].$$

This immediately implies that

$$\epsilon_\tau = \sigma.$$

We can perform the decomposition into a partial equilibrium effect and a general equilibrium effect. For $t > \tau$, we have $\Delta \hat{Y}_t^{PE} = \Delta \hat{Y}_t^{GE} = 0$, and for $t \leq \tau$, we have

$$\Delta \hat{Y}_t^{PE} = Y \frac{\frac{(1+\frac{\Delta R}{R})^{-1} - (1+\frac{\Delta R}{R})^{\sigma-1}}{R^{\tau-t+1}}}{1 + \frac{(1+\frac{\Delta R}{R})^{\sigma-1} - 1}{R^{\tau-t+1}}},$$

$$\Delta \hat{Y}_t^{GE} = Y[(1 + \frac{\Delta R}{R})^{-\sigma} - 1] - Y \frac{\frac{(1+\frac{\Delta R}{R})^{-1} - (1+\frac{\Delta R}{R})^{\sigma-1}}{R^{\tau-t+1}}}{1 + \frac{(1+\frac{\Delta R}{R})^{\sigma-1} - 1}{R^{\tau-t+1}}}.$$

This immediately implies that

$$\epsilon_{t,\tau}^{PE} = \sigma \frac{1}{R^{\tau-t+1}},$$

$$\epsilon_{t,\tau}^{GE} = \sigma(1 - \frac{1}{R^{\tau-t+1}}).$$

Next we compute the level-k equilibria $\{\hat{R}_t, \hat{Y}_t^k\}$. We have

$$\hat{Y}_t^k = \frac{\sum_{s=0}^{\tau-t-1} \frac{\hat{Y}_{t+1+s}^{k-1}}{R^{1+s}} + (1 + \frac{\Delta R}{R})^{-1} \sum_{s=\tau-t}^{\infty} \frac{\hat{Y}_{t+1+s}^{k-1}}{R^{1+s}}}{\frac{1}{R} \frac{1 - \frac{1}{R^{\tau-t}}}{1 - \frac{1}{R}} + (1 + \frac{\Delta R}{R})^{\sigma-1} \frac{\frac{1}{R^{\tau-t+1}}}{1 - \frac{1}{R}}}.$$

This implies that

$$\Delta\hat{Y}_t^k = \frac{\sum_{s=0}^{\tau-t-1} \frac{\Delta\hat{Y}_{t+1+s}^{k-1}}{R^{1+s}} + (1+\frac{\Delta R}{R})^{-1} \sum_{s=\tau-t}^{\infty} \frac{\Delta\hat{Y}_{t+1+s}^{k-1}}{R^{1+s}} + Y \frac{(1+\frac{\Delta R}{R})^{-1} - (1+\frac{\Delta R}{R})^{\sigma-1}}{1-\frac{1}{R}} \frac{1}{R^{\tau-t+1}}}{\frac{1}{R}\frac{1-\frac{1}{R^{\tau-t}}}{1-\frac{1}{R}} + \frac{(1+\frac{\Delta R}{R})^{\sigma-1}}{1-\frac{1}{R}} \frac{1}{R^{\tau-t+1}}}.$$

We get

$$\epsilon_\tau^1 = \sigma \frac{1}{R^\tau},$$

$$\epsilon_\tau^2 = \sigma \frac{1}{R^\tau} \left[1 + (R-1)\tau\right],$$

$$\epsilon_\tau^3 = \sigma \frac{1}{R^\tau} \left[1 + (R-1)\tau + (R-1)^2 \frac{\tau(\tau-1)}{2}\right],$$

and more generally

$$\epsilon_\tau^k = \sigma \frac{1}{R^\tau} \left[\sum_{n=0}^{k} (R-1)^n \sum_{s_0=0}^{\tau-1} \sum_{s_1=0}^{\tau-1-s_0} \cdots \sum_{s_{n-2}=0}^{\tau-1-s_{n-3}} 1\right].$$

7.2 The Perpetual Youth Model of Borrowing Constraints with $\sigma \neq 1$

Individual consumption function. When $\sigma \neq 1$, the individual consumption function is given by

$$c^*(a_t^i; \{r_{t+s}\}, \{Y_{t+s}^e\}) = \frac{a_t^i + \int_0^\infty (1-\delta) Y_{t+s}^e e^{-\int_0^s (r_{t+u}+\lambda) du} ds}{\int_0^\infty e^{-\int_0^s [(1-\sigma)(r_{t+u}+\lambda) + \sigma(\rho+\lambda)] du} ds}.$$

Aggregate state variable. Exactly as in the case $\sigma = 1$ treated in Section 4.1, the aggregate state variable Ψ_t (the wealth distribution) is not required to characterize the aggregate equilibrium since the reduced-form aggregate consumption function is independent of Ψ_t.

Reduced-form aggregate consumption function. The reduced-form aggregate consumption function is given by

$$C(\{r_{t+s}\}, \{Y_{t+s}^e\}) = \frac{\int_0^\infty \delta Y_{t+s}^e e^{-\int_0^s r_{t+u} du} ds + \int_0^\infty (1-\delta) Y_{t+s}^e e^{-\int_0^s (r_{t+u}+\lambda) du} ds}{\int_0^\infty e^{-\int_0^s [(1-\sigma)(r_{t+u}+\lambda) + \sigma(\rho+\lambda)] du} ds}.$$

Equilibrium characterization. For concreteness, we briefly characterize the various equilibria in the context of this particular model. Given beliefs $\{Y_t^e\}$, and given the path for interest rates $\{r_t\}$, $\{r_t, Y_t\}$ is a temporary equilibrium if and only if the path for aggregate income $\{Y_t\}$ is given by

$$Y_t = \frac{\int_0^\infty \delta Y^e_{t+s} e^{-\int_0^s r_{t+u} du} ds + \int_0^\infty (1-\delta) Y^e_{t+s} e^{-\int_0^s (r_{t+u}+\lambda) du} ds}{\int_0^\infty e^{-\int_0^s [(1-\sigma)(r_{t+u}+\lambda)+\sigma(\rho+\lambda)] du} ds} \quad \forall t \geq 0.$$

Similarly, given the path for interest rates $\{r_t\}$, $\{r_t, Y_t\}$ is an REE if and only if the path for aggregate income $\{Y_t\}$ satisfies the fixed point

$$Y_t = \frac{\int_0^\infty \delta Y_{t+s} e^{-\int_0^s r_{t+u} du} ds + \int_0^\infty (1-\delta) Y_{t+s} e^{-\int_0^s (r_{t+u}+\lambda) du} ds}{\int_0^\infty e^{-\int_0^s [(1-\sigma)(r_{t+u}+\lambda)+\sigma(\rho+\lambda)] du} ds} \quad \forall t \geq 0.$$

Finally given an initial REE $\{r_t, Y_t\}$ and a new interest rate path $\{\hat{r}_t\}$, the level-k equilibria $\{\hat{r}_t, \hat{Y}^k_t\}$ satisfy the following recursion over $k \geq 0$:

$$\hat{Y}^k_t = \frac{\int_0^\infty \delta \hat{Y}^{k-1}_{t+s} e^{-\int_0^s r_{t+u} du} ds + \int_0^\infty (1-\delta) \hat{Y}^{k-1}_{t+s} e^{-\int_0^s (r_{t+u}+\lambda) du} ds}{\int_0^\infty e^{-\int_0^s [(1-\sigma)(r_{t+u}+\lambda)+\sigma(\rho+\lambda)] du} ds} \quad \forall t \geq 0.$$

with the initialization that $\hat{Y}^0_t = Y_t$ for all $t \geq 0$.

We now turn to the computation of the different interest rate elasticities of output around a steady state REE $\{R_t, Y_t\}$ $Y_t = Y > 0$ and $r_t = r$ for all $t \geq 0$, where the steady-state interest rate r is given by

$$1 = [(1-\sigma)(r+\lambda) + \sigma(\rho+\lambda)][\frac{\delta}{r} + \frac{1-\delta}{r+\lambda}],$$

so that $r = \rho$ in the limit where the frequency of binding borrowing constraints λ goes to 0.

Monetary policy at different horizons under RE. The expressions for the interest rate elasticities of output ϵ_τ and their decompositions $\epsilon_{t,\tau} = \epsilon^{PE}_\tau + \epsilon^{GE}_\tau$ into PE and GE effects can be simplified in three special cases. The first case is when $\sigma = 1$ and is treated in the main body of the paper.

The second case is when the frequency of binding borrowing constraints λ goes to 0, where we get $r = \rho$ and

$$\epsilon_\tau = \sigma, \quad \epsilon^{PE}_\tau = \sigma e^{-r\tau}, \quad \epsilon^{GE}_\tau = \sigma[1 - e^{-r\tau}].$$

The third case is when there is no outside liquidity $\delta = 0$, where we get $r = \rho$ and

$$\epsilon_\tau = \sigma, \quad \epsilon^{PE}_\tau = \sigma e^{-(r+\lambda)\tau}, \quad \epsilon^{GE}_\tau = \sigma[1 - e^{-(r+\lambda)\tau}].$$

7.3 Details for the Model with Sticky Prices and Inflation in Section 5

This section fleshes out the details of the model with sticky prices and inflation in Section 5.

Monetary policy. Monetary policy is described by a path of interest rate rules $R_t(\Pi_t)$ specifying nominal interest rates as a function of the inflation rate $\Pi_t = P_t/P_{t-1}$. In what follows, we often simply write $\{R_t\}$ to denote the path of interest rate rules.

Aggregate variables and beliefs. These modifications change the relevant aggregate variables. In particular, we now need to track not only the path of nominal interest rates $\{R_t\}$ and the paths of output $\{Y_t\}$ and beliefs about output $\{Y_t^e\}$, but also the paths of aggregate real profits $\{X_t\}$ and beliefs about profits $\{X_t^e\}$, the paths of wages $\{W_t\}$ and beliefs about wages $\{W_t^e\}$, the paths of prices of final goods $\{P_t\}$ and beliefs about prices of final goods $\{P_t^e\}$, as well as the paths of prices of upstream goods $\{\hat{P}_t\}$ and beliefs about these prices $\{\hat{P}_t^e\}$. We define $\Omega_t = (Y_t, X_t, W_t, P_t, \hat{P}_t)$, and $\Omega_t^e = (Y_t^e, X_t^e, W_t^e, P_t^e, \hat{P}_t^e)$.

We assume that at every date t, beliefs about future wages, prices of final goods, and prices of upstream goods at date $t+s$ are scaled by P_t/P_t^e so that they are given by $W^e_{t+s}(P_t/P_t^e)$, $P^e_{t+s}(P_t/P_t^e)$, and $\hat{P}^e_{t+s}(P_t/P_t^e)$. This scaling allows the agents to incorporate the accumulated surprise inflation differential P_t/P_t^e that has already been realized but leaves unchanged beliefs about future relative prices $\hat{P}^e_{t+s}/P^e_{t+s}$ and wages W^e_{t+s}/P^e_{t+s} as well as beliefs about future inflation Π^e_{t+s}.

Technology. Final output is produced from intermediates by competitive firms indexed by $h \in [0,1]$ according to $y_t^h = \left(\int y_t^{hj\frac{\theta-1}{\theta}} dj\right)^{\frac{\theta}{\theta-1}}$. The different varieties of intermediates are produced using the upstream good by monopolisticcally competitive firms indexed by $j \in [0,1]$ according to $y_t^j = \hat{y}_t^j$. Finally, the upstream good is produced competitively from effective labor according to $\hat{Y}_t = N_t^{1-\delta}$, where $N_t = \int z_t^i n_t^i di$ is aggregate effective labor and δ is a measure of decreasing returns to scale. Decreasing returns to scale can be thought as arising from an underlying constant returns production function featuring capital and intermediate goods with strong frictions to the adjustment of capital, a standard assumption in the New Keynesian literature.

Individual firm price setting. The monopolistic firms producing the different varieties of intermediate goods are subject to a price setting friction à la Calvo. They only get a chance to change their price with probability $1-\lambda$ at every date, and these opportunities are independent across firms. A firm that gets a chance to change its price at date $t-1$ can change its price from date t onwards, and then chooses so set it to the following reset price

$$p_t^*(\{R_{t-1+s}\}, \Omega_{t-1}, \{\Omega^e_{t-1+s}\}) = \frac{\theta}{\theta-1} \frac{\sum_{s=0}^{\infty} \frac{\lambda^s}{\prod_{u=0}^{s-1}\left[R_{t+u}(\Pi^e_{t+u})\right]} Y^e_{t+s} (P^e_{t+s})^{\theta} \hat{P}^e_{t+s}}{\sum_{s=0}^{\infty} \frac{\lambda^s}{\prod_{u=0}^{s-1}\left[R_{t+u}(\Pi^e_{t+u})\right]} Y^e_{t+s} (P^e_{t+s})^{\theta}},$$

where $\theta/(\theta-1) > 1$ is the desired markup, $P_t = [\int (p_t^j)^{1-\theta} dj]^{1/(1-\theta)}$ is the aggregate price index and $\hat{P}_t$ is the price index for the upstream good.

Profits and Lucas trees. Real aggregate profits from the monopolistic intermediate good sector are given by $X_t = Y_t - \frac{\hat{P}_t}{P_t}\hat{Y}_t$. A fraction δX_t are capitalized by Lucas trees, and the remainder $(1-\delta)X_t$ is directly distributed to households in every period. The real aggregate profits $\delta\frac{\hat{P}_t}{P_t}\hat{Y}_t$ of the competitive upstream sector are also capitalized by Lucas trees and can be thought of as the rental income of capital. The trees real fruit for each period is thus given by $\delta X_t + \delta\frac{\hat{P}_t}{P_t}\hat{Y}_t = \delta Y_t$. The value of the trees can be calculated by no arbitrage

$$\begin{aligned} V_t &= \delta Y_t + \frac{\Pi^e_{t+1}}{R_t(\Pi_t)} V^e_{t+1} \quad \forall t \geq 0, \\ V^e_t &= \delta Y^e_t + \sum_{s=0}^{\infty} \prod_{u=0}^{s} \left[\frac{\Pi^e_{t+1+u}}{R_{t+u}(\Pi^e_{t+u})} \right] \delta Y^e_{t+1+s} \quad \forall t \geq 0. \end{aligned} \tag{8}$$

Individual agent problem. We first describe the individual problem. The problem at date t with real financial wealth a^i_t

$$\max_{\{\tilde{c}^i_{t+s}, \tilde{n}^i_{t+s}, \tilde{a}^i_{t+1+s}\}} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s [\log(\tilde{c}^i_{t+s}) - \frac{(\tilde{n}^i_t)^{1+\gamma}}{1+\gamma}],$$

subject to the current actual budget constraint

$$\tilde{c}^i_t = \frac{W_t}{P_t} z^i_t \tilde{n}^i_t + \alpha^i_t X_t + a^i_t - \frac{\Pi^e_{t+1}}{R_t(\Pi_t)} \tilde{a}^i_{t+1},$$

the future expected budget constraints

$$\tilde{c}^i_{t+1+s} = \frac{W^e_{t+1+s}}{P^e_{t+1+s}} z^i_{t+1+s} \tilde{n}^i_{t+1+s} + \alpha^i_{t+1+s} X^e_{t+1+s} + \tilde{a}^i_{t+1+s} - \frac{\Pi^e_{t+2+s}}{R_{t+1+s}(\Pi^e_{t+1+s})} \tilde{a}^i_{t+2+s} \quad \forall s \geq 0,$$

and the borrowing constraints

$$\tilde{a}_{t+1+s} \geq 0 \quad \forall s \geq 0,$$

where z^i_t is an idiosyncratic productivity shock and $z^i_t \tilde{n}^i_t$ is effective labor. We assume that this shock follows the process $\log(z^i_t) = \rho_\epsilon \log(z^i_{t-1}) + \epsilon^i_t$ where ϵ^i_t is i.i.d. over time, independent across agents, and follows a normal distribution with variance σ^2_ϵ and mean $\mathbb{E}[\epsilon^i_t] = -\sigma^2_\epsilon(1-\rho^2_\epsilon)^{-1}/2$.

We assume that the share α^i_t of aggregate real profits X_t from the monopolistic intermediate goods sector received by any given agent is proportional to its equilibrium labor income $z^i_t n^i_t$. This means that profits are rebated lump sum so that agents take the profits accruing to them as given when they make their labor supply decisions, since deviations from equilibrium leave α_t^i unchanged. As in Section 4.2, we assume that the borrowing contracts have the same form as the Lucas trees, and that agents cannot borrow. Taken together, these choices ensure that under rational expectations, the incomplete-markets irrelevance result holds, and the interest rate elasticity of output and inflation coincide with those of a complete-markets or representative-agent model.

We denote the policy function for consumption by $c^*(a_t^i, z_t^i; \{R_{t+s}\}, \Omega_t, \{\Omega_{t+s}^e\})$ and the policy function for labor by $n^*(a_t^i, z_t^i; \{R_{t+s}\}, \Omega_t, \{\Omega_{t+s}^e\})$. By analogy with Section 4.2, the law of motion for a_t^i is given

$$a_{t+1}^i = [\frac{R_t(\Pi_t)}{\Pi_{t+1}^e} + \frac{\delta(Y_t - Y_t^e)}{V_t - \delta Y_t}][\frac{W_t}{P_t} z_t^i n^*(a_t^i, z_t^i; \{R_{t+s}\}, \Omega_t, \{\Omega_{t+s}^e\}) + \alpha_t^i X_t + a_t^i - c^*(a_t^i, z_t^i; \{R_{t+s}\}, \Omega_t, \{\Omega_{t+s}^e\})$$

Temporary, RE, and level-*k* equilibria. We denote by $\Psi_t = \{a_t^i, z_t^i\}$ the joint distribution of wealth and productivity shocks. The law of motion for Ψ_t is entirely determined by the laws of motion for individual financial wealth and income shocks given an initial condition Ψ_0 with $\int z_0 d\Psi(a_0, z_0) = 1$ and $\int a_0 d\Psi(a_0, z_0) = V_0$, where V_0 is given by the no-arbitrage conditions (8).

Temporary equilibria, RE equilibria, and level-*k* equilibria are defined in a similar way as in the the general reduced form model described in Section 2. The main differences are that in each of these constructions, we must ensure not only that the goods market clears

$$Y_t = \int_0^1 c^*(a_t, z_t; \{R_{t+s}\}, \Omega_t, \{\Omega_{t+s}^e\}) d\Psi_t(a_t, z_t),$$

but also that the labor market clears

$$N_t = \int_0^1 z_t n^*(a_t, z_t; \{R_{t+s}\}, \Omega_t, \{\Omega_{t+s}^e\}) d\Psi_t(a_t, z_t).$$

We must solve not only for aggregate output $Y_t = \int y_t^h dh$ but also for aggregate effective labor $N_t = \int z_t^i n_t^i di$, the wage W_t, the price of final goods P_t, and the price of upstream goods $\hat{P}_t$. Because it aggregates the prices of intermediate goods producers, the aggreagte price index must follow the difference equation

$$P_t = [(1-\lambda)(p_t^*(\{R_{t-1+s}\}, \Omega_{t-1}, \{\Omega_{t-1+s}^e\}))^{1-\theta} + \lambda(P_{t-1})^{1-\theta}]^{\frac{1}{1-\theta}},$$

with initial condition $P_0 = P$. In addition, because of the optimality condition of the upstream goods producers, we must have

$$\hat{P}_t = W_t \frac{N_t^{\delta}}{1-\delta},$$

and

$$\Delta_t Y_t, = N_t^{1-\delta}$$

where Δ_t is an index of price dispersion which satisfies the difference equation

$$\Delta_t = \lambda \Pi_t^{\theta} \Delta_{t-1} + (1-\lambda) \left[\frac{1 - \lambda \Pi_t^{\theta-1}}{1-\lambda} \right]^{\frac{\theta}{\theta-1}},$$

with initial condition $\Delta_0 = 0$, which encapsulates the efficiency costs of misallocation arising from inflation.

The changes required to handle these differences involve the definition of a reduced-form aggregate consumption function and of a reduced-form aggregate effective labor supply function along the lines of the above equations. They also involve the definition of a reduced-form price of upstream goods function, of a reduced-form aggregate price of final goods function, and of a reduced-form aggregate wage function, along the lines of the above equations. The necessary steps are somewhat tedious but conceptually straightforward and so we omit them in the interest of space.