B Proofs

Proof of Theorem 1. By Shephard’s lemma, we know that, to a first order, the productivity shock $A_k$ will change the prices of any industry who purchases inputs, either directly or indirectly, from industry $k$

$$\frac{\mathrm{d} \log p_i}{\mathrm{d} \log A_k} = -1(i = k) + \sum_f \Omega_{if} \frac{\mathrm{d} \log w_f}{\mathrm{d} \log A_k} + \sum_j \Omega_{ij} \frac{\mathrm{d} \log p_j}{\mathrm{d} \log A_k}. \quad (19)$$

Denote the $N \times F$ matrix corresponding to \(\tilde{\Omega}_{if}\) by $\tilde{\alpha}_{if}$. Then rewrite the equation above in matrix form to get

$$\frac{\mathrm{d} \log p}{\mathrm{d} \log A_k} = (I - \tilde{\Omega})^{-1}(\tilde{\alpha} \frac{\mathrm{d} \log w}{\mathrm{d} \log A_k} - e_k) = \tilde{\Psi}(\tilde{\alpha} \frac{\mathrm{d} \log w}{\mathrm{d} \log A_k} - e_k), \quad (20)$$

where $e_k$ is the $k$th standard basis vector. Let the household’s aggregate consumption good to be the numeraire, so that the household’s ideal price index $P_c$ is always equal to one. Then we know that

$$\frac{\mathrm{d} \log P_c}{\mathrm{d} \log A_k} = 0. \quad (21)$$

Combine this with the previous expression to get

$$-b'\tilde{\Psi} e_k + b'\tilde{\Psi} \tilde{\alpha} \frac{\mathrm{d} \log w}{\mathrm{d} \log A_k} = 0. \quad (22)$$

Note that, $b'\tilde{\Psi} = \tilde{\lambda}$ and $b'\tilde{\Psi} \tilde{\alpha} = \tilde{\Lambda}$. Hence,

$$-\tilde{\lambda}_k + \tilde{\Lambda} \frac{\mathrm{d} \log w}{\mathrm{d} \log A_k} = 0. \quad (23)$$

Now, note that

$$\Lambda_f = \frac{w_f L_f}{P_c C}. \quad (24)$$

From this, we know that

$$\frac{\mathrm{d} \log \Lambda_f}{\mathrm{d} \log A_k} = \frac{\mathrm{d} \log w_f}{\mathrm{d} \log A_k} + \frac{\mathrm{d} \log L_f}{\mathrm{d} \log A_k} - \frac{\mathrm{d} \log Y}{\mathrm{d} \log A_k}. \quad (25)$$

Substitute this into the previous expression to get

$$-\tilde{\lambda}_k + \tilde{\Lambda} \frac{\mathrm{d} \log \Lambda}{\mathrm{d} \log A_k} - \tilde{\Lambda} \frac{\mathrm{d} \log L_f}{\mathrm{d} \log A_k} + \frac{\mathrm{d} \log Y}{\mathrm{d} \log A_k} = 0, \quad (26)$$
where we use the fact that $\sum f \tilde{\Lambda}_f = 1$. Rearrange this to get the desired result. To get an explicit characterization for $d \log \Lambda_k / d \log A_k$ in terms of structural parameters of the model, without loss of generality, assume that each good is produced from a distinct primary factor (this can be achieved by relabelling the input-output matrix). Now note that

$$\lambda_i = b_i + \sum_j \mu_j^{-1} \omega_{ji} \lambda_j, \quad (27)$$

$$w_i = \frac{\alpha_i p_i y_i}{\mu_i}, \quad (28)$$

$$\lambda_i = \frac{p_i y_i}{P_c}, \quad (29)$$

$$p_i = \frac{1}{A_i} C_i (p_1, \ldots, p_N, w_i) \mu_i, \quad (30)$$

$$P_c = \frac{\sum_i p_i C_i}{C} = 1. \quad (31)$$

We can differentiate these to get to our answer. Denote the Moroshima elasticity of substitution between $k$ and $j$ for the total cost function of industry $i$ by $\rho_{ikj}^i$. Then we can write

$$\frac{d \lambda_i}{d \log A_k} = \frac{d b_i}{d \log A_k} + \frac{d \tilde{\Omega}_{ij}}{d \log A_k} \mu_j^{-1} \lambda_j \sum_j \frac{d \mu_j}{d \log A_k}, \quad (32)$$

$$\frac{d b_i}{d \log A_k} = b_i \sum_{j \neq i} b_j \left( \frac{d \tilde{\Omega}_{ji}}{d \log A_k} \mu_j^{-1} \lambda_j \frac{d \mu_j}{d \log A_k} - \frac{d \mu_j}{d \log A_k} \right), \quad (33)$$

$$\frac{d \tilde{\Omega}_{ij}}{d \log A_k} = \tilde{\Omega}_{ij} \sum_{k \neq j} \tilde{\Omega}_{ik} \left( 1 - \frac{1}{\rho_{jk}^i} \right) \left( \frac{d p_j}{d \log A_k} - \frac{d p_j}{d \log A_k} \right) + \tilde{\Omega}_{ij} \tilde{\alpha}_i \left( 1 - \frac{1}{\rho_{ij}^k} \right) \left( \frac{d p_j}{d \log A_k} - \frac{d w_i}{d \log A_k} \right), \quad (34)$$

$$\frac{d \log w_i}{d \log A_k} = \mu_i^{-1} \tilde{\alpha}_i \sum_j \tilde{\Omega}_{ij} \left( 1 - \frac{1}{\rho_{ij}^k} \right) \left( \frac{d \tilde{\Omega}_{jk}}{d \log A_k} - \frac{d p_j}{d \log A_k} \right) + \frac{d p_i}{d \log A_k} + \frac{d y_i}{d \log A_k} - \frac{d \mu_i}{d \log A_k}, \quad (35)$$

$$\frac{d \log y_i}{d \log A_k} = \frac{1}{\lambda_i} \frac{d \lambda_i}{d \log A_k} - \frac{d \log p_i}{d \log A_k} + \frac{d \log P_c C}{d \log A_k}, \quad (36)$$

$$\frac{d \log p_i}{d \log A_k} = -1 (i = k) + \sum_j \tilde{\Omega}_{ij} \frac{d \log p_j}{d \log A_k} + (1 - \sum_j \tilde{\Omega}_{ij}) \frac{d \log w_i}{d \log A_k}. \quad (37)$$
\[ \frac{1}{\lambda_i} \frac{d \lambda_i}{d \log A_k} = \frac{d \log p_i}{d \log A_k} + \frac{d \log P_C}{d \log A_k}, \quad (38) \]

The proof for the case with markups is very similar.

Proof of Proposition 2. This is a special case of Proposition 4.

Proof of Proposition 3. This is a special case of Proposition 5.

Proof of Proposition 4. Denote the \( N \times F \) matrix corresponding to \( \tilde{\Omega}_{if} \) by \( \tilde{\alpha}_{if} \). By Shephard’s lemma,

\[ \frac{d \log p_i}{d \log A_k} = -\frac{1}{i = k} + \sum_j \tilde{\Omega}_{ij} \frac{d \log p_j}{d \log A_k} + \sum_f \tilde{\alpha}_{if} \frac{d \log w_f}{d \log A_k}. \quad (39) \]

Invert this system to get

\[ \frac{d \log p_i}{d \log A_k} = -\Psi_{ek} + \Psi_{if} \frac{d \log w}{d \log A_k}, \quad (40) \]

where \( \Psi_f = (I - \tilde{\Omega})^{-1} \tilde{\alpha} \) is \( \tilde{\Lambda} \).

Now consider a factor \( L \), we have

\[
\frac{d \Lambda_L}{d \log A_k} = \sum_i b_i (1 - \theta_0) [-\tilde{\Psi}_{ik} + \sum_f \Psi_{if} \frac{d \log w_f}{d \log A_k}] \Psi_{il}, \\
+ \sum_j (1 - \theta_j) \tilde{\Lambda}_{ij} \sum_i \tilde{\Omega}_{ij} [-\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if} \frac{d \log w_f}{d \log A_k}] + \sum_f \tilde{\Psi}_{if} \frac{d \log w_f}{d \log A_k} \Psi_{il}, \\
+ (\theta_k - 1) \tilde{\Lambda}_{ij} \sum_i \tilde{\Omega}_{ki} \Psi_{il}.
\]

Simplify this to

\[
d \Lambda_L = (\theta_0 - 1) \left( \sum_i b_i \tilde{\Psi}_{ik} \Psi_{il} - \sum_i b_i \Psi_{il} \sum_f \tilde{\Psi}_{if} \frac{d \log w_f}{d \log A_k} \right), \\
+ \sum_j (\theta_j - 1) \tilde{\Lambda}_{ij} \sum_i \tilde{\Omega}_{ij} \tilde{\Psi}_{ik} \Psi_{il} - \sum_i \tilde{\Omega}_{ij} \Psi_{ik} \Psi_{il}, \\
+ \sum_j (1 - \theta_j) \tilde{\Lambda}_{ij} \sum_i \tilde{\Omega}_{ij} \sum_f (\Psi_{if} - \tilde{\Psi}_{if}) \frac{d \log w_f}{d \log A_k} \Psi_{il}, \\
+ (\theta_k - 1) \tilde{\Lambda}_{ij} \sum_i \tilde{\Omega}_{ki} \Psi_{il},
\]

\[ = (\theta_0 - 1) \left( \sum_i b_i \tilde{\Psi}_{ik} \Psi_{il} - \sum_i b_i \Psi_{il} \sum_f \tilde{\Psi}_{if} \frac{d \log w_f}{d \log A_k} \right), \]

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\[ \sum_j (\theta - 1) \lambda_j \mu_j \left[ \sum_i \tilde{\Omega}_{ji} \psi_{ik} \psi_{il} - \left( \sum_i \tilde{\Omega}_{ji} \psi_{il} \right) \left( \sum_i \tilde{\Omega}_{ji} \psi_{ik} \right) \right] \]
\[ + \sum_j (1 - \theta_j) \lambda_j \mu_j \sum_i \sum_f \left( \psi_{ij} - \psi_{jf} \right) \frac{d \log w_f}{d \log A_k} \psi_{il} \]
\[ = (\theta - 1) \left( \sum_i b_i \psi_{ik} \psi_{il} - \sum_i b_i \psi_{il} \sum_f \psi_{if} \frac{d \log w_f}{d \log A_k} \right) \]
\[ + \sum_j (1 - \theta_j) \lambda_j \mu_j \sum_i \sum_f \left[ \sum_i \tilde{\Omega}_{ji} \psi_{ij} \psi_{il} - \left( \sum_i \tilde{\Omega}_{ji} \psi_{il} \right) \left( \sum_i \tilde{\Omega}_{ji} \psi_{ij} \right) \right] \frac{d \log w_f}{d \log A_k} \]
\[ = (\theta - 1) \left( \sum_i b_i \psi_{ik} \psi_{il} - \sum_i b_i \psi_{il} \sum_f \psi_{if} \frac{d \log w_f}{d \log A_k} \right) \]
\[ + \sum_j (1 - \theta_j) \lambda_j \mu_j \sum_i \sum_f \text{Cov}_{\tilde{\Omega}_{ij}}(\tilde{\psi}_{(k)}, \psi_{(l)}) \frac{d \log w_f}{d \log A_k} \]
\[ = (\theta - 1) \left( \sum_i \sum_f \psi_{if} \frac{d \log w_f}{d \log A_k} \right) \]
\[ + \sum_j (1 - \theta_j) \lambda_j \mu_j \sum_i \sum_f \left( \tilde{\psi}_{(k)} - \sum_f \psi_{(f)} \frac{d \log w_f}{d \log A_k} \right) \psi_{(l)} \]
\[ = (\theta - 1) \text{Cov}_{\psi}(\tilde{\psi}_{(k)} - \sum_f \psi_{(f)} \frac{d \log w_f}{d \log A_k}, \psi_{(l)}) + (\theta - 1) \left( \bar{\lambda}_k - \sum_f \bar{\lambda}_f \frac{d \log w_f}{d \log A_k} \right) \lambda_L, \]
\[ + \sum_j (1 - \theta_j) \lambda_j \mu_j \sum_i \sum_f \left( \tilde{\psi}_{(k)} - \sum_f \psi_{(f)} \frac{d \log w_f}{d \log A_k} \right) \psi_{(l)}. \]

Hence, for a productivity shock \( d \log A_k \), letting \( \Lambda_L \) be demand for factor \( L \), and indexing
all factors by \( f \), we have
\[
\frac{d \Lambda_L}{d \log A_k} = (\theta_0 - 1) \text{Cov}_b \left( \Psi_{(k)} - \sum_f \Psi_{(f)} \frac{d \log w_f}{d \log A_k}, \Psi_{(L)} \right)
\]
\[
+ \sum_j (\theta_j - 1) \mu_j^{-1} \lambda_j \text{Cov}_{\tilde{\Omega}} \left( \Psi_{(k)} - \sum_f \Psi_{(f)} \frac{d \log w_f}{d \log A_k}, \Psi_{(L)} \right)
\]
\[
+ (\theta_0 - 1) \left( \tilde{\lambda}_k - \sum_f \tilde{\lambda}_f \frac{d \log w_f}{d \log A_k} \right) \lambda_L.
\] (41)

Combine this with the observation that
\[
\Lambda_L (d \log w_L + d \log L_L - d \log Y) = d \Lambda_L.
\] (42)
and the fact that, from Theorem 1,
\[
d \log Y = \tilde{\lambda}_k + \sum_f \tilde{\lambda}_f d \log \lambda_f.
\] (43)
Set \( d L_L = 0 \), since factors are inelastically supplied, and we have a linear system with \( F + 1 \) equations and \( F + 1 \) unknowns where \( F \) is the total number of factors. Substitute
\[
\frac{1}{\Lambda_f} \frac{d \Lambda_f}{d \log A_k} + \frac{d \log Y}{d \log A_k} = \frac{d \log w_f}{d \log A_k}
\] (44)
back into network formula to get
\[
\Lambda_L d \log \Lambda_L = (\theta_0 - 1) \text{Cov}_b \left( \Psi_{(k)} - \sum_f \Psi_{(f)} \frac{d \log \Lambda_f}{d \log A_k}, \Psi_{(L)} \right)
\]
\[
+ \sum_j (\theta_j - 1) \mu_j^{-1} \lambda_j \text{Cov}_{\tilde{\Omega}} \left( \Psi_{(k)} - \sum_f \Psi_{(f)} \frac{d \log \Lambda_f}{d \log A_k}, \Psi_{(L)} \right)
\]
\[
+ (\theta_0 - 1) \left( \tilde{\lambda}_k - \sum_f \tilde{\lambda}_f \frac{d \log \Lambda_f}{d \log A_k} - d \log Y \right) \lambda_L.
\] (45)

Use Theorem 1 to further simplify this
\[
\Lambda_L d \log \lambda_L = (\theta_0 - 1) \text{Cov}_b \left( \Psi_{(k)} - \sum_f \Psi_{(f)} d \log \lambda_f, \Psi_{(L)} \right)
\]
\[ + \sum_{j} (\theta_j - 1)\mu_j^{-1}\lambda_j \text{Cov}_{\Omega_0} \left( \Psi_{(k)} - \sum_{j} \tilde{\Psi}_{(j)} \frac{d \log \Lambda_{(j)}}{d \log \Lambda_{(k)}} \Psi_{(k)} \right). \] (46)

Note that the final demand consumer needs to be labelled as producer 0 to complete the proof.

\[ \square \]

**Proof of Example 3.1.** We start with the matrices listed in (15) and (16), so that

\[ \Gamma = \begin{pmatrix} (1 - \theta_0)\text{Cov}_b(\tilde{\Psi}_{(i)}, \Psi_{(i)}) & (1 - \theta_0)\text{Cov}_b(\tilde{\Psi}_{(k)}, \Psi_{(k)}) \\ (1 - \theta_0)\text{Cov}_b(\Psi_{(k)}, \Psi_{(l)}) & (1 - \theta_0)\text{Cov}_b(\Psi_{(k)}, \Psi_{(k)}) \end{pmatrix}. \] (47)

\[ \delta_{(i)} = \begin{pmatrix} (1 - \theta_0)\text{Cov}_b(\tilde{\Psi}_{(i)}, \Psi_{(i)}) \\ (1 - \theta_0)\text{Cov}_b(\tilde{\Psi}_{(i)}, \Psi_{(i)}) \end{pmatrix}. \] (48)

From the structure of the problem, we can explicitly write the value of \( \Gamma \) as follows:

\[ \Gamma = (1 - \theta_0) \begin{pmatrix} b_3(b_1\mu_1^{-1} + b_2\mu_2^{-1}) & -b_3(b_1\mu_1^{-1} + b_2\mu_2^{-1}) \\ -b_3(1 - b_3)\mu_3^{-1} & b_3(1 - b_3)\mu_3^{-1} \end{pmatrix}. \] (49)

First we look at the cases for \( i = 1, 2 \) (the case where a factor of production is shared). We note that the case is symmetric for \( i = 1, 2 \) by the structure of the network and the problem. For these values, we have that

\[ \delta_{(1)} = (1 - \theta_0) \begin{pmatrix} b_1(\mu_1^{-1} - (b_1\mu_1^{-1} + b_2\mu_2^{-1})) \\ -b_1b_3\mu_3^{-1} \end{pmatrix}. \] (50)

Generally (for all cases), in order to solve the system in equation (14), write

\[ (\Lambda - \Gamma) = \begin{pmatrix} (b_1\mu_1^{-1} + b_2\mu_2^{-1})(1 - (1 - \theta_0)b_3) & (1 - \theta_0)b_3(\mu_1^{-1}b_1 + b_2\mu_2^{-1}) \\ (1 - \theta_0)b_3(1 - b_3)\mu_3^{-1} & b_3\mu_3^{-1}(1 - (1 - \theta_0)(1 - b_3)) \end{pmatrix}. \] (51)

Invert this to get

\[ (\Lambda - \Gamma)^{-1} = \frac{1}{\det \Gamma} \begin{pmatrix} b_3\mu_3^{-1}(1 - (1 - \theta_0)(1 - b_3)) & -(1 - \theta_0)b_3(\mu_1^{-1}b_1 + b_2\mu_2^{-1}) \\ -(1 - \theta_0)b_3(1 - b_3)\mu_3^{-1} & (b_1\mu_1^{-1} + b_2\mu_2^{-1})(1 - (1 - \theta_0)b_3) \end{pmatrix}. \] (52)

The determinant is

\[ \det \Gamma = (b_1\mu_1^{-1} + b_2\mu_2^{-1})(1 - (1 - \theta_0)b_3)b_3\mu_3^{-1}(1 - (1 - \theta_0)(1 - b_3)) - (1 - \theta_0)^2b_3^2(1 - b_3)\mu_3^{-1}(\mu_1^{-1}b_1 + b_2\mu_2^{-1}), \]

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Plug this back into (52) and simplify,

\[
(A - \Gamma)^{-1} = \frac{1}{\Lambda_L \Lambda_k \theta_0} \left( \begin{array}{ccc}
\Lambda_k (1 - (1 - \theta_0)(1 - b_3)) & -(1 - \theta_0) b_3 \Lambda_L \\
-(1 - \theta_0)(1 - b_3) & \Lambda_L (1 - (1 - \theta_0)b_3)
\end{array} \right) .
\] (53)

Returning to the specific case where \( i = 1 \),

\[
(A - \Gamma)^{-1} \delta_{(1)} = \frac{1}{\theta_0} \left( \begin{array}{c} b_1 (\theta_0 - 1) \left( \frac{\mu_1^{-1}}{\Lambda_L} - 1 \right) \theta_0 + b_3 (1 - \theta_0) \frac{\mu_1^{-1}}{\Lambda_L} + b_3 (\theta_0 - 1) - b_3 (\theta_0 - 1) \right) \\
(\theta_0 - 1)(1 - b_3) \left( \frac{\mu_1^{-1}}{\Lambda_L} - 1 \right) - (1 - (1 - \theta_0)b_3)
\end{array} \right) .
\] (54)

Multiplying the values in (54), and using the identity in (14),

\[
d \log \Lambda = (A - \Gamma)^{-1} \delta_{(1)},
\]

\[
= \frac{b_1 (\theta_0 - 1)}{\theta_0} \left( \left( \frac{\mu_1^{-1}}{\Lambda_L} - 1 \right) \theta_0 + b_3 (1 - \theta_0) \frac{\mu_1^{-1}}{\Lambda_L} + b_3 (\theta_0 - 1) - b_3 (\theta_0 - 1) \right) \\
(\theta_0 - 1)(1 - b_3) \left( \frac{\mu_1^{-1}}{\Lambda_L} - 1 \right) - (1 - (1 - \theta_0)b_3)
\]

Combine this with (2) gives

\[
d \log \gamma = b_1 - (1 - b_3) b_1 \frac{\theta_0 - 1}{\theta_0} \left( \left( \frac{\mu_1^{-1}}{\Lambda_L} - 1 \right) \theta_0 + b_3 (1 - \theta_0) \frac{\mu_1^{-1}}{\Lambda_L} \right) \\
- b_3 b_1 \frac{\theta_0 - 1}{\theta_0} \left[ (\theta_0 - 1)(1 - b_3) \left( \frac{\mu_1^{-1}}{\Lambda_L} - 1 \right) - (1 - (1 - \theta_0)b_3), \right].
\]

which further simplifies to

\[
\frac{d \log \gamma}{d \log \lambda_1} = b_1 + b_1 (\theta_0 - 1) \left[ 1 - (1 - b_3) \frac{\mu_1^{-1}}{\Lambda_L} \right].
\] (55)

This gives the desired result for firm 1. Note, as mentioned before, a symmetric result holds for firm 2. In the case of firm 3, we have that

\[
\delta_{(3)} = (\theta_0 - 1) \left( -b_3 (b_1 \mu_1^{-1} + b_2 \mu_2^{-1}) \right) .
\] (56)
The results from (53) give the blueprint for solving for the value of \( \frac{d \log Y}{d \log A_3} \) as well. From this, we can conclude that

\[
\frac{d \log Y}{d \log A_3} = b_3 - \left( b_1 + b_2 \right) \left( \frac{1 - (1 - \theta_0)(1 - b_3)}{\theta_0 \Lambda_i} \right) \left( \frac{(1 - \theta_0) \Lambda_k}{1 - (1 - \theta_0) b_3} \right) \left( -b_3 \frac{1}{b_3} \right)
\]

which further simplifies to

\[
= b_3 - \left( 1 - b_3 \right) \left( \frac{b_3 (1 - \theta_0) [1 - (1 - \theta_0)(1 - b_3)] + (\theta_0 - 1)^2 b_3 (1 - b_3)}{(1 - \theta_0)^2 b_3 (1 - b_3) + [1 - (1 - \theta_0)(1 - b_3)] \theta_0 \Lambda_i} \right).
\]

This simplifies to give the required result

\[
\frac{d \log Y}{d \log A_3} = b_3.
\]

Proof of Proposition 5. From Theorem 1, we need only to characterize \( \frac{d \log \Lambda}{d \log \mu_k} \). For a markup shock, we can differentiate demand for a quantity \( m \) to get

\[
\frac{d \lambda_m}{d \log \mu_k} = \sum_i b_i (1 - \theta_0)(d \log p_i) + \sum_j \tilde{\Omega}_{ij} \mu_j^{-1} (1 - \theta_j) \lambda_j [d \log p_i - d \log p_j] - \Theta_k \theta_k.
\]

By Shephard’s lemma,

\[
d \log p_i = 1(i = k) + \sum_j \tilde{\Omega}_{ij} d \log p_j + \sum_f \tilde{\alpha}_{if} \frac{d \log w_f}{d \log \mu_k}.
\]

Invert this system to get

\[
d \log p_i = \Psi_{ek} + \Psi_f d \log w,
\]

where \( \Psi_f = (I - \tilde{\Omega})^{-1} \tilde{\alpha} \) is a \( N \times K \) matrix of network-adjusted factor intensities by industry. Substituting this back into (60) and set \( m = L \) to get

\[
\frac{d \Lambda_L}{d \log \mu_k} = \sum_i b_i (1 - \theta_0) [\Psi_{ik} + \sum_f \Psi_{if} \frac{d \log w_f}{d \log \mu_k}] \Psi_{iL},
\]

\[
+ \sum_j (1 - \theta_j) \lambda_j \mu_j^{-1} \sum_i \tilde{\Omega}_{ij} [\Psi_{ik} + \sum_f \Psi_{if} \frac{d \log w_f}{d \log \mu_k}] - \Psi_{jk} - \sum_f \tilde{\Psi}_{jf} \frac{d \log w_f}{d \log \mu_k} \Psi_{iL}.
\]
\[-\theta_k \mu_k^{-1} \lambda_k \sum_i \Omega_{ki} \Psi_{il}.\]

Simplify this to

\[
d \Lambda_L = (1 - \theta_0) \left( \sum_i b_i \bar{\Psi}_{ik} \Psi_{il} \right) + (1 - \theta_0) \left( \sum_i b_i \Psi_{il} \sum_f \bar{\Psi}_{if} \frac{d \log w_f}{d \log \mu_k} \right),
\]

\[
+ \sum_j (1 - \theta_j) \lambda_j \mu_j^{-1} \left[ \sum_i \bar{\Omega}_{ji} \Psi_{il} \sum_i \bar{\Omega}_{ji} \Psi_{jk} \right]
\]

\[
+ \sum_j (1 - \theta_j) \lambda_j \mu_j^{-1} \left[ \sum_i \bar{\Omega}_{ji} \sum_f \Psi_{if} \frac{d \log w_f}{d \log \mu_k} \Psi_{il} - \sum_i \bar{\Omega}_{ji} \Psi_{il} \sum_f \Psi_{if} \frac{d \log w_f}{d \log \mu_k} \right]
\]

\[- \theta_k \lambda_k \mu_k^{-1} \Psi_{kl},
\]

\[
=(1 - \theta_0) \left( \text{Cov}_b(\Psi_{(k)}, \Psi_{(l)}) + \bar{\lambda}_k \Lambda_L \right)
\]

\[
+ \sum_j (1 - \theta_j) \lambda_j \mu_j^{-1} \left( \text{Cov}_{\bar{\Omega}_{(j)}}(\bar{\Psi}_{(k)}, \Psi_{(l)}) - 1(j = k) \Psi_{jl} \right)
\]

\[
+ (1 - \theta_0) \left( \text{Cov}_b \left( \sum_f \bar{\Psi}_{(f)} \frac{d \log w_f}{d \log \mu_k}, \Psi_{(l)} \right) + \sum_f \bar{\Lambda}_f \frac{d \log w_f}{d \log \mu_k} \Lambda_L \right)
\]

\[
+ \sum_j (1 - \theta_j) \lambda_j \mu_j^{-1} \left( \text{Cov}_{\bar{\Omega}_{(j)}} \left( \sum_f \bar{\Psi}_{(f)} \frac{d \log w_f}{d \log \mu_k}, \Psi_{(l)} \right) - 1(j = k) \Psi_{jl} \right)
\]

\[- \theta_k \lambda_k \mu_k^{-1} \Psi_{kl},
\]

\[
=(1 - \theta_0) \left( \text{Cov}_b(\Psi_{(k)} + \sum_f \bar{\Psi}_{(f)} \frac{d \log w_f}{d \log \mu_k}, \Psi_{(l)}) \right)
\]

\[
+ \sum_j (1 - \theta_j) \lambda_j \mu_j^{-1} \left( \text{Cov}_{\bar{\Omega}_{(j)}}(\Psi_{(k)} + \sum_f \bar{\Psi}_{(f)} \frac{d \log w_f}{d \log \mu_k}, \Psi_{(l)}) - 1(j = k) \Psi_{jl} \right)
\]

\[
+ (1 - \theta_0) \left( \bar{\lambda}_k + \sum_f \bar{\Lambda}_f \frac{d \log w_f}{d \log \mu_k} \right) \Lambda_L - \lambda_k \mu_k^{-1} \Psi_{kl},
\]

\[
=(1 - \theta_0) \left( \text{Cov}_b(\Psi_{(k)} + \sum_f \bar{\Psi}_{(f)} \frac{d \log w_f}{d \log \mu_k}, \Psi_{(l)}) \right)
\]

\[
+ \sum_j (1 - \theta_j) \lambda_j \mu_j^{-1} \left( \text{Cov}_{\bar{\Omega}_{(j)}}(\Psi_{(k)} + \sum_f \bar{\Psi}_{(f)} \frac{d \log w_f}{d \log \mu_k}, \Psi_{(l)}) \right)
\]

\[- \lambda_k \Psi_{kl}.
\]
The final line follows from the fact that \( \left( \tilde{\lambda}_k + \sum_f \tilde{\lambda}_f \frac{d \log w_f}{d \log \mu_k} \right) = b' d \log p = d \log P_e = 0 \). Finally, substitute \( d \log w_f = d \log \Lambda_f + d \log Y \) into the expression above to get

\[
\frac{d \Lambda_L}{d \log \mu_k} = (1 - \theta_0) \left( \text{Cov}_b(\tilde{\Psi}(k) + \sum_f \tilde{\Psi}(f) (d \log \Lambda_f + d \log Y), \Psi(L)) \right)
+ \sum_j (1 - \theta_j) \lambda_j \mu_j^{-1} \left( \text{Cov}_{\Omega(j)}(\tilde{\Psi}(k) + \sum_f \tilde{\Psi}(f) (d \log \Lambda_f + d \log Y), \Psi(L)) \right)
- \lambda_k \Psi_{kL}.
\]

To complete the proof, note that

\[
d \log Y \sum_f \tilde{\Psi}(f) = d \log Y 1. \tag{63}
\]

In other words, this is a vector of all ones multiplied by the scalar \( d \log Y \), and hence it drops out of the covariance operators, since the covariance of a vector of ones with any other vector under any probability distribution is always equal to zero. Hence,

\[
\frac{d \Lambda_L}{d \log \mu_k} = (1 - \theta_0) \left( \text{Cov}_b(\tilde{\Psi}(k) + \sum_f \tilde{\Psi}(f) d \log \Lambda_f, \Psi(L)) \right)
+ \sum_j (1 - \theta_j) \lambda_j \mu_j^{-1} \left( \text{Cov}_{\Omega(j)}(\tilde{\Psi}(k) + \sum_f \tilde{\Psi}(f) d \log \Lambda_f, \Psi(L)) \right)
- \lambda_k \Psi_{kL}.
\]

\[\blacksquare\]

C Robustness and Extensions

In this section, we discuss some of the extensions mentioned in the body of the paper. Specifically, we address in more detail how are results extend to situations with elastic factors, capital accumulation/dynamics, fixed costs, entry, and nonlinearities. Proofs for the results are at the end of this section.
C.1 Elastic Factor Supplies

To be able to do counterfactuals, we provide the structural counterparts to these equations in terms of microeconomic elasticities of substitution. These are the formulas that we use to study the impact of monetary policy in models with nominal rigidities (where the endogenous supply of labor is important) in Appendix D.

Assume a nested CES structure similar to the one in Section 3. We have

\[ d \log L_f = \zeta_f d \log \left( \frac{W_f}{P} \right) + \gamma_f d \log Y. \]  

(64)

This implies that

\[ d \log \left( \frac{W_f}{P} \right) = \frac{1}{1 + \zeta_f} d \log \Lambda_f + \frac{1 - \gamma_f}{1 + \zeta_f} d \log Y, \]  

(65)

\[ d \log \left( \frac{W_f}{PY} \right) = \frac{1}{1 + \zeta_f} d \log \Lambda_f - \frac{\gamma_f}{1 + \zeta_f} d \log Y, \]  

(66)

and

\[ d \log L_f = \frac{\zeta_f}{1 + \zeta_f} d \log \Lambda_f + \frac{\gamma_f + \zeta_f}{1 + \zeta_f} d \log Y. \]  

(67)

Changes in factor shares and output solve the following system of equations.\(^{40}\)

\[ d \log \Lambda_f = - \sum_k \lambda_k \Psi_{kf} \Lambda_f d \log \mu_k + \sum_j (\theta_j - 1) \frac{\lambda_j}{\mu_j} \text{Cov}_{\tilde{Q}_j \Theta} \left( \sum_k \Psi_{(k)} d \log A_k - \sum_k \Psi_{(k)} d \log \mu_k, \frac{\Psi(f)}{\Lambda_f} \right) \]

\[ - \sum_j (\theta_j - 1) \frac{\lambda_j}{\mu_j} \text{Cov}_{\tilde{Q}_j \Theta} \left( \sum_g \Psi_{(g)} \frac{1}{1 + \zeta_g} d \log \Lambda_g - \sum_g \Psi_{(g)} \frac{\gamma_g + \zeta_g}{1 + \zeta_g} d \log Y, \frac{\Psi(f)}{\Lambda_f} \right), \]

\[ d \log Y = \frac{1}{\sum_f \tilde{\Lambda}_f \frac{1 - \gamma_f}{1 + \zeta_f}} \left[ \sum_k \tilde{\lambda}_k d \log A_k - \sum_k \tilde{\lambda}_k d \log \mu_k - \sum_f \tilde{\lambda}_f \frac{1}{1 + \zeta_f} d \log \Lambda_f \right]. \]

(68)

\(^{40}\)Note that the first equation can also be written as

\[ d \log \Lambda_f = - \sum_k \lambda_k \Psi_{kf} \Lambda_f d \log \mu_k + \sum_j (\theta_j - 1) \frac{\lambda_j}{\mu_j} \text{Cov}_{\tilde{Q}_j \Theta} \left( \sum_k \Psi_{(k)} d \log A_k - \sum_k \Psi_{(k)} d \log \mu_k, \frac{\Psi(f)}{\Lambda_f} \right) \]

\[ - \sum_j (\theta_j - 1) \frac{\lambda_j}{\mu_j} \text{Cov}_{\tilde{Q}_j \Theta} \left( \sum_g \Psi_{(g)} \frac{1}{1 + \zeta_g} d \log \Lambda_g + \sum_g \Psi_{(g)} \frac{1 - \gamma_g}{1 + \zeta_g} d \log Y, \frac{\Psi(f)}{\Lambda_f} \right). \]
C.2 Capital Accumulation, Adjustment Costs, and Capacity Utilization

In mapping this set-up to the data, there are two ways to interpret this model: either we could interpret final demand as a per-period part of a larger dynamic problem, or we could interpret final demand as an intertemporal consumption function where goods are also indexed by time à la Arrow-Debreu. When we interpret the model intertemporally, then output is the net present-value of consumption streams. When we interpret the model intratemporally, the output function encompasses demand for consumption goods and for investment goods, and treats the two as perfect substitutes. Our results actually generalize to the case where they are not, see Section 6 for more details. The existence of a constant-returns-to-scale aggregate final demand function allows us to unambiguously define real GDP using the corresponding ideal price index.41

In principle, we could also model factor accumulation in the usual Arrow-Debreu manner: treat goods in different time periods as different goods. Then, we could model the process of capital accumulation via intertemporal production functions that transform goods in one period into goods in other periods. This modeling choice would also be well-suited to handle technological frictions to the reallocation of factors such as adjustment costs and variable capacity utilization. Our formulas would apply to these economies without change, but of course, in such a world, the Domar weight of each producer would now be expressed in net-present value terms.

C.3 Fixed Costs

We start by adding fixed costs to the model, and then expand to allow for free entry. To add fixed overhead costs to the model, we need to separate variable cost from total cost. Under these conditions \( \hat{\Omega} \) becomes total-variable-cost based rather than total-cost based. It is the matrix whose \( ij \)th element is

\[
\hat{\Omega}_{ij} = \frac{\partial \log C_i}{\partial \log p_j} = \frac{p_j x_{ij}}{V C_i},
\]

where \( V C_i \) is \( i \)'s total variable cost rather than \( i \)'s total costs. Then we have

\[
d \log Y = \bar{\lambda} ' d \log A - \bar{\lambda} d \log \mu + d H(\bar{\Lambda}, \Lambda).
\]

41We assume the existence of a representative consumer mostly for expositional convenience. Our ex-post reduced-form results could be generalized to cover the generic heterogenous consumers (as long as real GDP is defined using the Laspeyre index), and our ex-ante structural results could be generalized to cover heterogenous consumers with identical homothetic preferences.
where $\Lambda$ is the share of income going to each factor including the payments to the infra-marginal fixed costs, but the cost-based Domar weights $\hat{\lambda}$ and $\hat{\Lambda}$ are constructed using the variable-cost based $\hat{\Omega}$.\footnote{With fixed costs, the interpretation of allocative efficiency as the gap between the passive allocation and the general equilibrium allocation still applies, but the passive allocation is no longer locally the same as general equilibrium when there are no frictions. The reason is that the passive allocation would send resources to the users of the fixed cost, even though the marginal benefit is zero. However, we can change the definition of the passive allocation so that it nets out fixed costs first (and hence becomes equivalent to general equilibrium when there are no frictions). Under this modification, the change in allocative efficiency is given by}

If firms are earning positive economic profits, we might expect that this would induce entry and competition. In this section, we extend our basic results to cover the case where there is free entry. First, we establish that our results can easily be applied to cases where entry is “wasteful.” Typically, entry can be economically meaningful due to several reasons: (1) it increases product variety; (2) it reduces markups; (3) it selects the most productive firms; (4) it counters decreasing returns to scale at the firm level. If we assume that firms have constant-returns-to-scale production functions, are ex-ante identical, markups/wedges are exogenous, and there is no returns to product variety, then entry is entirely socially wasteful.\footnote{A constrained social planner would drive the mass of entrants in each industry to zero.} Under these conditions, our results survive unchanged. Now we discuss how our results could be extended to cover (1), (2), (3), and (4).

C.4 Entry

First, we provide a simple case where we can micro-found our industry-level model with constant-returns industry cost functions using a model that has entry and decreasing returns to scale a the firm level. Here, channels (1), (2), and (3) are still shut off, but (4) is operating. Under these conditions, our results for the impact of productivity shocks survive unchanged. Finally, we sketch how a model which potentially allows for all four channels outlined above behaves. Here, our results connect to those of Baqee (2016), who studies network economies with free entry and external economies of scale.
No External Economies

We start with the case where the entry margin has no effect on the marginal productivity of the industry so that mechanisms (1), (2), (3) and (4) are shut off.

Let industry $k$ output be given by

$$y_k = A_k \left( M_k^{-1/\epsilon_k} \int_{M_k} y(i,k)^{\epsilon_k-1} \frac{d}{d i} \right)^{\frac{\epsilon_k}{\epsilon_k - 1}},$$

and suppose the variable cost function of each product $i$ in industry $k$ is given by

$$\frac{1}{A_k z_k(i)} c(w,p)y(i,k),$$

so that firms have constant returns on the margin, and $A_k$ is an industry level TFP. To enter, firms pay a fixed entry cost. After entry, each firm draws an idiosyncratic markup $m_i$ and productivity $z_i$ from some distribution $\Phi(z,m)$. There is also an industry level markup $\mu_k$. The scaling term $M_k^{-1/\epsilon_k}$ is introduced to neutralize love for variety effects and thereby ensure that there are no external economies of scale.

Proposition 6 (No External Economies). Suppose that there is free entry subject to fixed costs, that there are no external economies of scale, and that firms have constant returns to scale. Then

$$\frac{d \log Y}{d \log A_k} = \tilde{\lambda}_k + \frac{d H(\tilde{\Lambda}, \Lambda)}{d \log A_k},$$

and

$$\frac{d \log Y}{d \log \mu_k} = -\tilde{\lambda}_k + \frac{d H(\tilde{\Lambda}, \Lambda)}{d \log \mu_k}.$$

This set up allows us to span the basic framework in, for example, Autor et al. (2017), who argue that the decline in the labor share in the US in recent times is due to an increase in the size of low-labor-share firms.\footnote{The same facts have also been documented by Vincent and Kehrig (2017) and Hartman-Glaser et al. (2016)} They consider a model with entry and where labor intensity falls with the scale of operation because of overhead labor costs. The mechanism Autor et al. (2017) emphasize is an increase in the industry-level elasticity of substitution, whereby more productive firms are able to capture more demand over time.\footnote{Hartman-Glaser et al. (2016) offer a different explanation based on implicit contracts between firms and workers} Our interpretation, consistent with their empirical evidence, is that low-labor share firms charge higher markups, and that these markups are part of the reason why they have a...
low labor share. Our results in Section 5 indicate that these changes in the composition of firms within industries have increased aggregate productivity by improving allocative efficiency.

**Constant External Economies**

While the no-external-economies model described above allows us to accommodate entry, it does so within a very restrictive set up. Now, we sketch a version of the model where, due to decreasing returns to scale at the firm-level, free entry is not socially wasteful. We turn on mechanism (4) but keep (1), (2), and (3) off.

Although individual firms have decreasing returns to scale, industries have constant returns to scale. This simple model can microfound the use of constant-returns-to-scale industry level cost functions, and thereby allows our results to go through unchanged for productivity shocks, as long as we assume that each entrant in industry $k$ has access to a homothetic production function, that all producers charge the same markup, and that overhead costs are paid in units of the industry good.

**Proposition 7 (Constant External Economies).** Suppose that there is free entry subject to fixed entry costs. In addition, suppose that entrants in each industry charge the same markup and have access to the same homothetic production function, and each entrant pays a fixed cost of entry in units of the industry’s input. Then

$$\frac{d \log Y}{d \log A_k} = \tilde{\lambda}_k - \sum_f \tilde{\Lambda}_f \frac{d \log \Lambda_f}{d \log A_k}.$$

This extends our results for productivity shocks to an economy with entry. Unfortunately, the results for markup/wedge shocks do not apply any longer since a change in markups changes the scale of operations of firms, and these changes have associated efficiency changes that cannot be tracked in the same way.

**Increasing External Economies**

Finally, we sketch how our framework would relate to models with richer heterogeneity and entry properties by turning on mechanisms (1), (3), and (4). In particular, we allow for the possibility that entry can induce increasing returns to scale at the industry level.

[46] We could also allow for mechanism (2) with endogenous markups in a model with Cournot competition, or with demand curves with non-constant elasticities. The formula would feature extra terms having to do with the elasticity of the markups to the shocks.
where the industry becomes more productive as more firms enter.\textsuperscript{47}

To deal with the possibility of decreasing returns to scale at the firm-level, we introduce “fictitious” fixed factors, and assume that each entrant in an industry may use a fixed factor. Hence, firm $i$ in industry $k$ has a variable cost function that can be written as

$$
\frac{1}{A_k z_k(i)} c_k(w, p, r_k(i)) y_{ik},
$$

where $A_k$ is industry TFP, $z(i)$ is individual TFP, and $r_k(i)$ is the wage paid to the fixed factor. We also allow for fixed overhead costs on top of the entry costs, which firms pay after observing the realization of their markup and productivity if they decide to be active (see below), thereby creating room for selection effects.

Index producers in industry $k$ by $i$ in such a way that their idiosyncratic productivity $z_k(i)$ is weakly increasing, and suppose that $i$ is distributed according to the distribution $\phi_k(i)$. We assume that the price of the composite industry $k$ good can be written as

$$
p_k = \frac{\mu_k}{A_k} \left( \int_{I_k} \left( \frac{\mu_k(z)}{z_k(i)} c_k(w, p, r_k(z)) \right)^{1-\epsilon_k} \phi_k(i) d i \right)^{\frac{1}{1-\epsilon_k}},
$$

where $i_k$ denotes the cutoff below which firms that have paid the entry cost decide not to be active in order not to pay the fixed overhead cost. Movements in this cutoff drive the strength of selection effects and the external economies of scale arising from love for variety effects.

**Proposition 8 (Increasing External Economies).** Suppose that there is free entry, fixed entry costs, and fixed overhead costs paid after productivity and markup draws conditional on operating, and external economies of scale in the form of love for variety effects. Then

$$
\frac{d \log Y}{d \log A_k} = \lambda_k - \int_f \frac{d \log \Lambda_f}{d \log A_k} \frac{\lambda_j}{\epsilon_j - 1} \frac{p_j(i_j)}{p_j \phi_j(i_j)} \frac{d \log i_j}{d \log A_k},
$$

and

$$
\frac{d \log Y}{d \log \mu_k} = -\lambda_k - \int_f \frac{d \log \Lambda_f}{d \log \mu_k} \frac{\lambda_j}{\epsilon_j - 1} \frac{p_j(i_j)}{p_j \phi_j(i_j)} \frac{d \log i_j}{d \log \mu_k},
$$

where $f$ indexes the set of all factors including the “fictitious” fixed factors.

The model of Baqaee (2016) becomes a special case of this setup. Proposition 8 shows

\textsuperscript{47}Baqaee (2016) shows that, even within the confines of a tightly parameterized model, these forces can significantly alter the propagation and amplification of shocks.
that by incorporating “fictitious” factors, and hence separating profits due to distortions from competitive rents from decreasing returns, we can extend our results to a much more general class of models going even beyond neoclassical production by allowing for increasing returns to scale at the macroeconomic level.

Importantly, allowing for increasing returns makes the “pure” technology effect of a shock endogenous, and this endogenous response of productivity is what the final (and new) term in Proposition 8 accounts for. There are two reasons why technology becomes endogenous: first, an increase in the mass of entrants improves the productivity of firms through its effect on product variety. Second, an increase in the mass of entrants can change the distribution of productivity in each industry, since the change in cutoff value at which firms enter the market changes. Interestingly, we see that these cut-off values for the productivity and their derivatives are sufficient statistics for understanding the first-order impact of the shocks. Analyzing this model in any more detail and expressing these cutoff values and their derivatives as a function of the structural microeconomic parameters of the model is well beyond the scope of this paper, but we pursue it in ongoing work.

C.5 Nonlinear Impact of Shocks

Another limitation of our results is that we neglect nonlinearities. As discussed by Baqaee and Farhi (2017), models with production networks can respond very nonlinearly to productivity shocks. We plan to extend these results to inefficient economies in full generality, but as a first step, we stipulate some conditions under which we can directly leverage these results to inefficient economies. In particular, we show that the amplification of negative shocks due to complementarities emphasized in Baqaee and Farhi (2017) can also work to amplify the negative effects of misallocation.

Consider the quantitative parametric model in Section 5. Let \( \delta_k(i) \), \( \mu_k(i) \), and \( A_k(i) \) denote firm \( i \) in industry \( k \)’s share of industry sales, markups, and productivity. Define

\[
\mu_k = \left( \sum_i \frac{\delta_k(i)}{\mu_k(i)} \right)^{-1}
\]

and

\[
A_k = \mu_k \left( \sum_i \delta_k(i) (\mu_k(i)/A_k(i))^{1-\xi_k} \right)^{\frac{1}{\xi_k}}.
\]

Define the efficiency of each firm \( i \) in industry \( k \) to be \( e_k(i) = 1/\mu_k(i) \). Consider a transformation \( e_k(i) = t_k + (1-t_k)e_k(i) \) which shrinks dispersion in markups relative to its
steady-state value $\mu_k(i) = 1/e_k(i)$. This transformation keeps $\mu_k = 1/e_k$ constant. Define the revenue-based Domar weight of industry $k$ by $\lambda_k$.

**Proposition 9 (Competitive Isomorphism).** Consider an economy where $\mu_k = 1$ for every $k$, then

$$\frac{d \log Y}{d \log t_k} = \lambda_k \frac{d \log A_k}{d \log t_k}$$

(72)

and

$$\frac{d^2 \log Y}{d \log t_k^2} = \lambda_k \frac{d \log A_k}{d \log A_k} \left( \frac{d \log A_k}{d \log t_k} \right)^2 + \lambda_k \frac{d^2 \log A_k}{d \log t_k^2},$$

(73)

with $d \log A_k / d \log t_k \geq 0$.\(^{48}\)

Hence, increases in the dispersion of markups, which keep the harmonic average of markups constant, are isomorphic to negative productivity shocks in a model which is efficient at the industry level. Hence, shocks which increase markup dispersion in an industry can have outsized nonlinear effects on output, if those industries are macro-complementary with other industries in the sense defined by Baqee and Farhi (2017) so that $d \log A_k / d \log t_k < 0$.

This helps flesh out the insight in Jones (2011) that complementarities can interact with distortions to generate large reductions in output, and that these can be quantitatively important enough to explain the large differences in cross-country incomes. Given the examples in Baqee and Farhi (2017), it should be clear how misallocation in a key industry like energy production can significantly reduce output through macro-complementarities. Investigating these nonlinear forces more systematically is an interesting exercise that we leave for future work.

**Proofs of this section**

**Proof of Proposition 6.** The industry level price is then given by

$$p_k = \left( \frac{1}{M_k} \int_{M_k} \frac{\mu_k m_i}{A_k z_i} \Phi(z_i, m_i) d m_i d z_i \right)^{1/\epsilon_k},$$

$$= c(w, p) \frac{\mu_k}{A_k} \frac{m_i}{z_i}^{1-\epsilon_k} \left( \frac{m_i}{z_i} \right)^{\epsilon_k}.$$

\(^{48}\)We typically also have $d^2 \log A_k / d \log t_k^2 \geq 0$. 

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By Shephard’s lemma, we know that,

\[
\frac{d \log p_i}{d \log A_k} = -1(i = k) + \sum_F \tilde{\alpha}_{iF} \frac{d \log w_F}{d \log A_k} + \sum_j \tilde{\Omega}_{ij} \frac{d \log p_j}{d \log A_k},
\]

where \(\tilde{\alpha}_{iF}\) and \(\tilde{\Omega}_{ij}\) are firm-level expenditures on factor \(F\) and industry \(j\) as a share of costs (excluding entry costs). Rewrite this in matrix form to get

\[
\frac{d \log p}{d \log A_k} = (I - \tilde{\Omega})^{-1}(\tilde{\alpha} \frac{d \log w}{d \log A_k} - e_k) = \Psi(\tilde{\alpha} \frac{d \log w}{d \log A_k} - e_k),
\]

where \(e_k\) is the \(k\)th standard basis vector. Let the household’s aggregate consumption good to be the numeraire, so that the household’s ideal price index \(P_c\) is always equal to one. Then we know that

\[
\frac{d \log P_c}{d \log A_k} = b' \frac{d \log p}{d \log A_k} = 0.
\]

Combine this with the previous expression to get

\[
-b' \Psi e_k + b' \Psi \tilde{\alpha} \frac{d \log w}{d \log A_k} = 0.
\]

Note that, \(b' \Psi = \bar{\lambda}\) and \(b' \Psi \tilde{\alpha} = \bar{\Lambda}\). Hence,

\[
-\bar{\lambda}_k + \bar{\Lambda} \frac{d \log w}{d \log A_k} = 0.
\]

Now, note that

\[
\Lambda_f = \frac{w_f L_F}{P_c C}.
\]

From this, we know that

\[
\frac{d \log \Lambda_f}{d \log A_k} = \frac{d \log w_f}{d \log A_k} + \frac{d \log L_f}{d \log A_k} - \frac{d \log Y}{d \log A_k}.
\]

Substitute this into the previous expression to get

\[
-\bar{\lambda}_k + \bar{\Lambda} \frac{d \log \Lambda}{d \log A_k} - \bar{\Lambda} \frac{d \log L_f}{d \log A_k} + \frac{d \log Y}{d \log A_k} = 0,
\]

where we use the fact that \(\sum_f \bar{\Lambda}_f = 1\).

**Proof of Proposition 7.** Let the cost function of entrant \(i\) in industry \(k\) producing \(y_i\) units be
given by
\[ c(w, p)h(y/A), \tag{82} \]
where \( A \) is industry level shock and \( k \) subscripts have been suppressed. Free entry then implies that
\[ \mu c(w, p)h'(y/A) \frac{y}{A} - c(w, p)h(y/A) = f c(w, p), \tag{83} \]
where \( f_k \) is the entry cost in units of the input good. We can simplify this to
\[ \mu h'(y/A) \frac{y}{A} - h(y/A) = f. \tag{84} \]
This equation pins down the efficient scale of operation \( y = Ay'(\bar{f}, \mu). \)

The cost of producing \( q \) units of industry \( k \) output is then given by
\[ nc(w, p)f(y'(\mu, \bar{f})) + nf c(w, p), \tag{85} \]
such that \( ny = q \). Substitute the constraint into this to get
\[ C(w, p)q/A = \frac{q}{y'A} c(w, p)f(y'(\mu, \bar{f})) + \frac{q}{Ay'} f c(w, p). \tag{86} \]
Hence the industry’s cost function is linear \( q/A \) as needed.

Finally,
\[ \frac{\partial C(w, p)q/A}{\partial w_j} = \frac{q}{y'A} \frac{\partial c(w, p)}{\partial w_j} f(y'(\mu, \bar{f})) + \frac{q}{Ay'} \frac{\partial c(w, p)}{\partial w_j} = nl_k + nf_k. \tag{87} \]
Therefore, the industry level cost function obeys Shephard’s Lemma and we can replicate
the rest of the proof from Proposition 6.

Proof of Proposition 8.
\[
\begin{align*}
\log p_k &= -1(i = k) \log A_i + \sum_f \left( \int_{z_k}^\infty s_k(z) \hat{\alpha}_{kf}(z) dF(z) \right) \log w_f \\
&\quad + \sum_j \left( \int_{z_k}^\infty s_k(z) \tilde{\alpha}_{kj}(z) dF(z) \right) \log p_j + \left( \int_{z_k}^\infty s_k(z) \hat{\beta}_k(z) d\log r_k(z) dF(z) \right) \\
&\quad - \frac{1}{1 - \epsilon_k} s_k(z) f_k(z_k) d\log z_k.
\end{align*}
\]
where \( s_k(z) \) is firm \( i \)'s share of sales in industry \( k \), and \( \beta_k(z) \) is the cost share of firm \( i \) on its
fictitious fixed factor. We can solve this system to get
\[
d \log p = (I - \tilde{\Omega})^{-1} (-e_i \, d \log A_i + \tilde{\alpha} \, d \log w + \xi - \kappa),
\]
where \( \xi_k = \left( \int_{z_k}^\infty s_k(z) \tilde{\beta}_k(z) \, d \log r_k(z) \, d F(z) \right) \) and \( \kappa_k = \int_{\tilde{z}_k}^{\infty} s_k(\tilde{z}) f_k(\tilde{z}) \, d \log \tilde{z}_k \), and the elements of \( \tilde{\Omega} \) and \( \tilde{\alpha} \) are defined appropriately. Let output be the numeraire so that
\[
b' \, d \log p = 0.
\]
Hence,
\[
- \lambda_i + \tilde{\Lambda}' \, d \log w + \sum_k \tilde{\lambda}_k \xi_k - \sum_k \tilde{\lambda}_k \kappa_k = 0.
\]
For each factor, we know that
\[
d \log w = d \log A + d \log Y.
\]
Substitute this in to get
\[
- \lambda_i + \tilde{\Lambda}' \, d \log A + \tilde{\Lambda}' \, d \log Y + \sum_k \int_{z_k}^{\infty} \tilde{\lambda}_k s_k(z) \tilde{\beta}_k(z) \, d \log A_{r_i(z)} \, d F(z) + \sum_k \int_{\tilde{z}_k}^{\infty} \tilde{\lambda}_k s_k(z) \tilde{\beta}_k(z) \, d \log Y - \sum_k \tilde{\lambda}_k \kappa_k = 0.
\]
Finally, observe that \( \sum_f \tilde{\Lambda}_f + \sum_k \int_{z_k}^{\infty} \tilde{\lambda}_k s_k(z) \tilde{\beta}_k(z) = 1 \).

The proof for markup shocks is very similar.

\[\blacksquare\]

**Proof of Proposition 9.** On the other hand,
\[
\frac{d \log A}{d \tau} = \sum_i s_i \left( 1 - \bar{e}_i \right) = \sum_i s_i (\mu_i - 1) = \sum_i s_i \mu_i - 1 \geq 0.
\]

On the other hand,
\[
\frac{d^2 \log A}{d \tau^2} = (\sigma - 1) Var_s(\mu_i) - \sum_i s_i (\mu_i - 1)^2 = (\sigma - 2) Var_s(\mu_i) - \left( \sum_i s_i \mu_i - 1 \right)^2.
\]

Consider an industry where: all firms \( i \) use the same upstream input bundle with cost \( C \); firms transform this input into a firm-specific variety of output using constant return to returns to scale technology; each firm \( i \) has productivity \( a_i \) and charges a markup \( \mu_i \); the varieties are combined into a composite good by a competitive downstream industry according to a CES production function with elasticity \( \sigma \) on firm \( i \).
We denote the quantity of composite good produced as

\[ Q = \left[ \sum \frac{b_i p_i^{\frac{\sigma-1}{\sigma}}}{a_i^{\frac{\sigma-1}{\sigma}}} \right]^{1-\frac{1}{\sigma}}. \]  

(94)

Firm \( i \) charges a price

\[ p_i = \frac{\mu_i}{a_i} C. \]  

(95)

The resulting demand for firm \( i \)’s variety is

\[ q_i = (\frac{p_i}{P})^{-\sigma} b_i Q, \]  

(96)

where the price index is given by

\[ P = \left[ \sum b_i p_i^{1-\sigma} \right]^{\frac{1}{1-\sigma}}. \]  

(97)

Total profits are given by

\[ \Pi = \sum (p_i - C)(\frac{P_i}{P})^{-\sigma} b_i Q. \]  

(98)

We solve out the price index and profits explicitly and get

\[ P = \left[ \sum b_i \left( \frac{\mu_i}{a_i} \right)^{1-\sigma} \right]^{\frac{1}{1-\sigma}} C, \]  

(99)

\[ \Pi = \sum b_i \left[ \left( \frac{\mu_i}{a_i} \right)^{1-\sigma} \right]^{\frac{\sigma}{1-\sigma}} \left[ \sum b_j \left( \frac{\mu_j}{a_j} \right)^{1-\sigma} \right]^{1-\sigma} b_i CQ. \]  

(100)

For completeness we can also solve for the sales of each firm as a fraction of the sales of the industry

\[ \lambda_i = \frac{p_i q_i}{PQ} = \frac{b_i \left( \frac{\mu_i}{a_i} \right)^{1-\sigma}}{\sum b_j \left( \frac{\mu_j}{a_j} \right)^{1-\sigma}}. \]  

(101)

We want to understand how to aggregate this industry into homogenous industry with productivity \( A \) and markup \( \mu \). These variables must satisfy

\[ P = \frac{\mu}{A} C, \]  

(102)
\[ \Pi = \left( \frac{\mu}{A} - \frac{1}{A} \right) CQ. \] 

(103)

This implies that \( A \) and \( \mu \) are the solutions of the following system of equations

\[ \frac{\mu}{A} = \left[ \sum_i b_i \left( \frac{\mu_i}{a_i} \right)^{1-a} \right]^{1\over 1-a}, \] 

(104)

\[ \left( \frac{\mu}{A} - \frac{1}{A} \right) = \sum_i \left( \frac{\mu_i}{a_i} - \frac{1}{a_i} \right) \left[ \frac{\mu_i}{\sum_i b_i \left( \frac{\mu_j}{a_j} \right)^{1-a}} \right]^{1\over 1-a} b_i. \] 

(105)

The solution is

\[ A = \frac{1}{\left[ \sum_i b_i \left( \frac{\mu_i}{a_i} \right)^{1-a} \right]^{1\over 1-a} - \sum_i \left( 1 - \frac{1}{\mu_i} \right) \frac{\mu_i}{\sum_i b_i \left( \frac{\mu_j}{a_j} \right)^{1-a}} \left[ \frac{\mu_i}{\sum_i b_i \left( \frac{\mu_j}{a_j} \right)^{1-a}} \right]^{1\over 1-a} b_i}, \] 

(106)

\[ \mu = \frac{1}{\left[ \sum_i b_i \left( \frac{\mu_i}{a_i} \right)^{1-a} \right]^{1\over 1-a} - \sum_i \left( 1 - \frac{1}{\mu_i} \right) \frac{\mu_i}{\sum_i b_i \left( \frac{\mu_j}{a_j} \right)^{1-a}} \left[ \frac{\mu_i}{\sum_i b_i \left( \frac{\mu_j}{a_j} \right)^{1-a}} \right]^{1\over 1-a} b_i}, \] 

(107)

We can also rewrite this in a useful way as

\[ A = \frac{1}{\left[ \sum_i b_i \left( \frac{\mu_i}{a_i} \right)^{1-a} \right]^{1\over 1-a} - \sum_i \frac{1}{\mu_i} \left( \frac{\mu_i}{\sum_i b_i \left( \frac{\mu_j}{a_j} \right)^{1-a}} \right) \left[ \sum_i \frac{1}{\mu_i} \lambda_i \right]^{1\over 1-a} \frac{1}{\sum_i \frac{1}{\mu_i} \lambda_i}, \] 

(108)

\[ \mu = \frac{1}{\sum_i \frac{1}{\mu_i} \left( \frac{\mu_i}{\sum_i b_i \left( \frac{\mu_j}{a_j} \right)^{1-a}} \right) \left[ \sum_i \frac{1}{\mu_i} \lambda_i \right]^{1\over 1-a} \frac{1}{\sum_i \frac{1}{\mu_i} \lambda_i}}, \] 

(109)

Define the efficiency of each firm \( i \) to be \( e_i = \mu_i^{-1} \). Then, consider a steady state where \( \sum_i s_i \mu_i^{-1} = \sum_i s_i e_i = 1 \). Consider a transformation \( e_i = \tau + (1 - \tau) \bar{e}_i \). This transformation...
keeps $\mu$ constant. On the other hand,

$$
\frac{d \log A}{d \tau} = \sum_i s_i \left(1 - \bar{e}_i \right) = \sum_i s_i (\mu_i - 1) = \sum_i s_i \mu_i - 1 \geq 0.
$$

(110)

On the other hand,

$$
\frac{d^2 \log A}{d \tau^2} = (\sigma - 1) Var_s(\mu_i) - \sum_i s_i (\mu_i - 1)^2 = (\sigma - 2) Var_s(\mu_i) - \left(\sum_i s_i \mu_i - 1\right)^2.
$$

(111)

\section{Nominal Rigidities}

In this Section, we apply our general framework to study the effects of sticky prices in economies with arbitrary production structures.\(^{49}\) In general, sticky prices can be modeled via variable markups: markups which move to ensure that the relevant nominal prices stay constant. This is the point of connection with our framework. We show how to solve for these endogenous markups, and then to trace their impact on the economy.

In this section, we have two goals: first, we show how the existence of nominal rigidities changes the mapping from microeconomic productivity shocks to aggregate output or TFP in economies with distorted steady states; second, we show how monetary policy shocks can be analyzed using our results in economies with distorted steady states, leading to a clean separation the oft-neglected effects of monetary policy shocks on allocative efficiency from their traditional aggregate demand effects. In these applications, the steady-state distortions are the estimated markups discussed above in a given year. The endogenous response of markups to shocks is solved for to ensure that the relevant prices remain fixed.

These exercises are useful demonstrations of how to apply our results more generally in cases where markups are endogenous or variable. Typically, models with sticky prices are linearized around an efficient steady state, which ensures that reallocation terms disappear. We use our framework to study the model’s behavior away from the efficient steady-state using empirically estimated steady-state markups, and with a realistic microeconomic production structure featuring input-output connections, complementarities in production, and substitutability among heterogenous firms within an industry.

\(^{49}\)Starting with Basu (1995), a literature has grown to emphasize the importance of intermediate goods for understanding the business cycle properties of models with sticky prices. See for example Bouakez et al. (2009), Nakamura and Steinsson (2010), Pasten et al. (2016, 2017).
The size and direction of the TFP effects of monetary shocks depend crucially on the correlation pattern between price-stickiness and the level of markups. They can be large and positive if goods with higher markups have stickier prices. Our results draw attention to the fact that the typical loglinearization of New Keynesian models around undistorted steady states is potentially misleading since the allocative efficiency effects can be large away from the efficient steady state.

To model money demand we use the simplest formulation and assume that there is a cash in advance constraint

\[ P_y Y = M, \]

where \( M \) is the instrument of monetary policy.

We index each individual producer by \( i \), and write a firm-level input-output matrix in standard form. To model sticky prices, let \( s \) denote the set of producers with fixed prices, and let \( e_s \) be the \( N \times |s| \) matrix given by \( e_s = [e_i]_{i \in s} \), where \( e_i \) is the \( i \)th standard basis vector. Using the firm-level formulation, we can solve for the change in markups \( \frac{d \log \mu}{\mu} \) that would keep the price of sticky-firms constant, in response to the vector of productivity shocks \( \frac{d \log A}{A} \) and the vector of changes in factor prices \( \frac{d \log w}{w} \):

\[
\frac{d \log \mu}{\mu} = (e_s' \tilde{\Psi} e_s)^{-1} e_s' \tilde{\Psi} (\frac{d \log A}{A} - \tilde{\alpha} \frac{d \log w}{w}).
\]  
(112)

To solve for how the change in factor prices, we use

\[
\frac{d \log w}{w} = \frac{d \log \Lambda}{\Lambda} + \frac{d \log M}{M} - \frac{d \log L}{L}.
\]

Combining these two equations characterizes how output responds to productivity or money shocks in general equilibrium

\[
\frac{d \log Y}{Y} - \tilde{\Lambda}' \frac{d \log L}{L} = \tilde{\lambda}' \frac{d \log A}{A} - \tilde{\lambda}' e_s \frac{d \log \mu}{\mu} + \frac{d H(\tilde{\Lambda}, \Lambda)}{H(\tilde{\Lambda}, \Lambda)},
\]  
(113)

where \( \frac{d \log \mu}{\mu} \) is now determined according to (112).\(^{50}\)

To finish our characterization, we need to make an assumption about labor supply, since equation (113) takes the change in factor supply as given. To fix ideas, we follow convention in the New Keynesian literature, and assume labor is the only factor of production and that the household utility function takes the form:

\[
U(C, L) = \log(C) - \frac{L^{1+1/\nu}}{1 + 1/\nu},
\]

\(^{50}\)See Proposition 12 in the Appendix for a formal statement and proof of these results.
where \( \nu \) is the Frisch elasticity of labor supply. Under these conditions, we can combine equations (113), (17), and (18) to explicitly solve out the labor supply decision.

**Proposition 10** (Nominal Rigidities). Suppose that labor is the only factor of production. Then

\[
d \log Y = \lambda' d \log A - \lambda' e_s d \log \mu - \left( \frac{1}{1 + \nu} \right) \tilde{\Lambda} d \log \Lambda,
\]

with

\[
d \log \mu = (e_s' \Psi e_s)^{-1} e_s' \Psi (d \log A - \tilde{a} \left( d \log M + \left( \frac{1}{1 + \nu} \right) d \log \Lambda \right)),
\]

where \( \Lambda \) is the labor share of income. If the economy has a nested CES form, then

\[
d \log \Lambda = \sum_{k=1}^{N} \left( \sum_{j=0}^{N} (\theta_j - 1) \mu_j^{-1} \lambda_j \text{Cov}_{G \phi} \left( \Psi(k), \frac{\Psi_{(L)}}{\Lambda_L} \right) \right) [d \log A_k - d \log \mu_k] - \sum_{k=1}^{N} \frac{\lambda_k \Psi_{kl}}{\Lambda_L} d \log \mu_k.
\]

Equation (113) gives the change in aggregate TFP in response to either monetary or technology shocks in a New Keynesian type environment and decomposes it into a “pure” change in technology component and change in allocative efficiency component. Proposition 10 characterizes the corresponding changes in output.

Typically, New Keynesian models are log-linearized around the efficient steady state, and in those cases, the changes in allocative efficiency are second-order and are therefore neglected.\(^{51}\) In these cases, our results deliver the same conclusion.\(^{52}\) Outside of these special cases, for inefficient steady states, changes in allocative efficiency are not zero, and our results then permit us to isolate these effects.

In their important study, Pasten et al. (2016) characterize the response of output to shocks in a model with Calvo frictions and production networks. They write the input-output matrix at the industry level, and suppose that some fraction \( \delta_i \) of firms in industry \( i \) have flexible prices. Their sharpest analytical result is for the case with log utility in

\(^{51}\)See, for example, Galí (2008).

\(^{52}\)The case of the efficient steady-state is immediate in Proposition 10, since at the efficient steady-state \( \Psi_{(L)} \) is a vector of all ones, and the covariances are all zero. Hence, at the efficient steady state, Proposition 10 implies that the change in allocative efficiency, to a first order, is

\[
-\lambda' d \log \mu - \tilde{\Lambda} d \log \Lambda = - \sum_{k=1}^{N} \lambda_k d \log \mu_k + \sum_{k=1}^{N} \frac{\lambda_k \Psi_{kl}}{\Lambda_L} d \log \mu_k = - \sum_{k=1}^{N} \lambda_k d \log \mu_k + \sum_{k} \lambda_k d \log \mu_k = 0.
\]

Of course, for a monopolistic economy, if the production network is irregular (or asymmetric), then the equilibrium is generically inefficient (due to the heterogeneity in markups implied by double marginalization) even if every producer charges the same markup.
consumption and an infinite Frisch elasticity of labor supply. In this special case, the response of output to shocks takes a very simple form — sticky prices act like shock absorbers to productivity shocks. Using Proposition 10, we can recover their result and shed light on why this happens.

**Proposition 11.** [Pasten et al. 2016] Suppose that utility is log in consumption and the Frisch elasticity of labor supply is infinite. Then Proposition 10 implies

\[ d \log Y = \lambda' \left( I - e_s (e'_s \Psi e_s)^{-1} e'_s \Psi \right) d \log A + \lambda' e_s (e'_s \Psi e_s)^{-1} e'_s 1 d \log M. \]

In the special case where some fraction \( \delta_i \) in industry \( i \) are flexible, then

\[ d \log Y = d \log M - b' (I - \tilde{\Omega})^{-1} \delta (\tilde{\alpha} d \log M - d \log A). \]

Proposition 11 is simple to interpret: for productivity shocks, the impact of a shock is the same as one in a model with flexible prices, but where productivity shocks affect only some fraction \( \delta_i \) of each industry \( i \)'s costs, or in other words, productivity shocks are attenuated by some weight \( \delta_i \) at each industry. The impact of monetary policy shocks on output is given by \( 1 - b' (I - \tilde{\Omega})^{-1} \delta \tilde{\alpha} \), or the total share of value-added which is sticky in the economy.

Crucially, information about elasticities of substitution and changes in allocative efficiency disappear from these calculations. This is due to assumption of infinitely elastic labor supply. In this model, labor supply moves exactly in such a way as to offset changes in allocative efficiency, so that output fluctuations boil down to only how the productivity shocks travel from suppliers into consumer prices. Hence, although output responses can easily be determined without information on elasticities of substitution, allocative efficiency is changing in this environment, and is given by

\[ d \log Y - d \log L = \lambda' d \log A - \lambda' e_s d \log \mu - d \log \Lambda, \]

which is nonzero.\(^{53}\)

We now turn our attention to an application of our results. We calibrate a version of our quantitative model from Section 5, but augmented with a labor-leisure choice, and a Frisch elasticity of labor supply of \( \nu = 1/2 \), which is broadly consistent with the recommendation of Chetty et al. (2011). We create two copies of each firm in our sample,\(^{53}\)

---

\(^{53}\)The fact that changes in allocative efficiency are not required to compute the changes in output is a generic property of infinitely elastic labor supply, and does not depend on parametric assumptions about the production functions, see footnote (39) for more details.
one copy has sticky prices while the other has flexible prices. We then use Proposition 10 to compute the impact of monetary policy shocks and firm-level productivity shocks.

**Monetary Policy Shocks**

In Figure 10, we show the response of output and aggregate TFP for a shock to the money supply. We consider different specifications of the model with different elasticities of substitution. For each specification, consider how output and TFP respond to a 1 log point monetary shock as we vary the ratio of the average markup between the flexible and sticky firms (keeping the average constant).

We find that the movements in aggregate TFP, which are purely caused by changes in allocative efficiency, can become very large if the elasticity of substitution across firms is high, and if average markups are not the same between the flexible and sticky firms. The size and sign of these movements depend crucially on the correlation between price rigidity and markups. These results suggest that empirical work on understanding the correlation between microeconomic price rigidities and the levels of markups could be of great importance.\(^{54}\) They also suggest that the literature on the New Keynesian model, by assuming that the steady-state is efficient, and by assuming away correlations between price rigidities and levels of markups, could potentially be missing important first-order effects.\(^{55}\)

In Figure 11, we plot the output response for the benchmark model and a one sector version of the model with a value-added production function. Comparing the benchmark model to the one-sector model, we recover the famous insight by Basu (1995) that intermediate goods can increase stickiness. If intermediate inputs are sticky, then flexible firms adjust their prices less in response to shocks. The degree of amplification caused by the intermediate-input share is hump-shaped in the fraction of firms \(\delta\) that have sticky prices. In the limit, as all firms become sticky, the intermediate input share becomes irrelevant, and the same occurs when all firms become flexible.\(^{56}\)

\(^{54}\)The sign of this correlation is not ex-ante obvious. In models where the price elasticity of demand is not constant, the pass-through of costs to markups can depend on the level of the markup, so that the desired markups of firms with high markups are less sensitive to changes in costs. In the presence of price-adjustment costs, this means that high markup firms will have stickier prices (see Gopinath and Itskhoki, 2011, 2010; Atkeson and Burstein, 2008; Kimball, 1995). On the other hand, in a Calvo model where the markups are uncorrelated with stickiness on impact, in response to an expansion in the money supply, firms that do not adjust their markups for longer will over time have lower effective markups, inducing a negative correlation between stickiness and markups. Studying these sorts of effects requires a dynamic model however, and we leave this for future work.

\(^{55}\)Typically, second-order effects on allocative efficiency are taken into account only in the computation of welfare, but not in the computation of the equilibrium allocation.

\(^{56}\)Relatedly, Nakamura and Steinsson (2010) and Pasten et al. (2016) have emphasized that heterogeneity
Figure 10: The elasticity of output and aggregate TFP with respect to monetary policy shocks \((d \log Y / d \log M, d \log TFP / d \log M)\) for Calvo parameter \(\delta = 0.5\). The parameter \(\delta\) is the fraction of each industry with sticky prices. We show the results for the benchmark model, a Cobb-Douglas specification that sets all elasticities equal to one (CD + CD), and a single-industry model with a value-added production function. We then vary the ratio of the markup between the sticky and flexible portions of each sector.
Figure 11: The elasticity of output and aggregate TFP with respect to monetary policy shocks ($d \log Y / d \log M, d \log TFP / d \log M$) as a function of the Calvo parameters $\delta$. The parameter $\delta$ is the fraction of each industry with sticky prices. We show the results for the benchmark model, a Cobb-Douglas specification that sets all elasticities equal to one (CD + CD), and a single-industry model with a value-added production function. The ratio of the markup between the sticky and flexible portions of each sector is 1.
Proofs of this section

Proof of Proposition 12. Order the producers so that the first $s$ producers are the ones with sticky prices. For a vector $x$, denote $x_{(s)} = e_s' x$. From the cash in advance constraint, we know that

$$
d log Y = d log M - d log P_c,$$

$$= d log M - b' d log p,$$

$$= d log M - b' \Psi (\tilde{\alpha} d log w - d log A) + b' \Psi e_s d log \mu,$$

$$= d log M - \tilde{\Lambda}' d log w + \tilde{\lambda}' d log A - \tilde{\lambda} e_s d log \mu,$$

$$= d log M - \tilde{\Lambda}' (d log \Lambda + d log M - d log L) + \tilde{\lambda}' d log A - \tilde{\lambda} e_s d log \mu,$$

$$= d log M - \tilde{\Lambda}' (d log \Lambda - d log L) - d log M + \tilde{\lambda}' d log A - \tilde{\lambda} e_s d log \mu,$$

which implies that

$$d log Y - \tilde{\Lambda}' d log L = \tilde{\lambda}' d log A - \tilde{\lambda}'_{(s)} d log \mu - \tilde{\lambda}' d log \Lambda. \quad (114)$$

To get the markups necessary to keep $p_{(s)}$ sticky, we impose

$$d log p_{(s)} = d log \mu + e_s' \Omega \log p - d log A_{(s)} = 0. \quad (115)$$

This implies

$$d log \mu = -e_s' \Omega d log p - \tilde{\alpha}_{(s)} d log w + d log A_{(s)}. \quad (116)$$

On the other hand, we have

$$d log p = \Psi (\tilde{\alpha} d log w - d log A) + \Psi e_s d log \mu. \quad (117)$$

Substituting this back into the previous expression gives

$$d log \mu = -e_s' \tilde{\Omega} \Psi \tilde{\alpha} d log w - e_s' \tilde{\Omega} \Psi e_s d log \mu + e_s' \tilde{\Omega} \Psi d log A - e_s' \tilde{\alpha} d log w + e_s' d log A. \quad (118)$$

in the frequency of price changes across industries is also quantitatively important. This is mostly because, even in the basic New Keynesian model with a trivial input-output structure, the mapping between the frequency of price changes and the degree of monetary non-neutrality is convex, and so for a given average frequency of price changes, increasing dispersion in the frequency of price changes increases monetary non-neutrality. This is an important dimension of heterogeneity that, for now, we abstract away from in our quantitative examples.
Solve this to get
\[ d \log \mu = (e'_s(I + \tilde{\Omega}\Psi)e_s)^{-1}e'_s(I + \tilde{\Omega}\Psi)(d \log A - \tilde{\alpha} d \log w), \]
\[ = (e'_s\Psi e_s)^{-1}e'_s\Psi (d \log A - \tilde{\alpha} d \log w). \]

To prove Proposition 10, we first prove the following:

**Proposition 12.** Consider an economy with a cash-in-advance constraint, and nominal rigidities. Then,
\[ d \log Y - \tilde{\Lambda}' d \log L = \tilde{\lambda}' d \log A - \tilde{\lambda}' e_s d \log \mu + d H(\tilde{\Lambda}, \Lambda), \]
where
\[ d \log \mu = (e'_s\Psi e_s)^{-1}e'_s\Psi (d \log A - \tilde{\alpha} d \log w), \]
(119)
and
\[ d \log w = d \log \Lambda + d \log M - d \log L. \]

In the special case where some fraction $\delta_i$ in industry $i$ are flexible. Then,
\[ \tilde{\lambda}'(s) d \log \mu = \left(b'(I - \Omega)^{-1} - b'(I - \delta \Omega)^{-1}\delta\right) d \log A - (1 - b'(I - \delta \Omega)^{-1}\delta\alpha) d \log w, \]
(120)
\[ = \left(\tilde{\lambda} - b'(I - \delta \Omega)^{-1}\delta\right) d \log A - (1 - b'(I - \delta \Omega)^{-1}\delta\alpha) d \log w. \]

**Proof of Proposition 10.** The labor-leisure condition and the cash-in-advance condition imply that
\[ L^{1/(v)} = \left(\frac{w}{P_c C}\right) = \left(\frac{w}{M}\right). \]
(121)
Hence,
\[ \frac{1}{v} d \log L = d \log w - d \log M = d \log \Lambda - d \log L + d \log M - d \log M, \]
(122)
or
\[ d \log L = \frac{v}{1 + v} d \log \Lambda. \]
(123)
Therefore,
\[ d \log w = \frac{1}{v + 1} d \log \Lambda + d \log M \]
(124)
To finish, apply Propositions 2 and 3.

**Proof of Proposition 11.** With log utility in consumption and infinite Frisch elasticity of
labor supply we have that

\[ d \log w - d \log P_c = d \log Y, \tag{125} \]

or in other words, substituting in the cash in advance constraint

\[ d \log w = d \log M. \tag{126} \]

Furthermore, the cash in advance constraint implies that

\[ d \log Y = d \log M - d \log P_c, \]

\[ = d \log M - \bar{\lambda}' d \log w + \bar{\lambda}' d \log A - \bar{\lambda}' e_s d \log \mu, \]

\[ = d \log M - d \log M + \bar{\lambda}' d \log A - \bar{\lambda}' e_s d \log \mu, \]

\[ = \bar{\lambda}' d \log A - \bar{\lambda}' e_s d \log \mu, \]

substituting equation (119) from Proposition 12 completes the proof

\[ = \bar{\lambda}' d \log A - \bar{\lambda}(s) (e'_s \Psi e_s)^{-1} e'_s \Psi (d \log A - \bar{\alpha} d \log w), \]

\[ = \bar{\lambda}' d \log A - \bar{\lambda}(s) (e'_s \Psi e_s)^{-1} e'_s \Psi (d \log A - \bar{\alpha} d \log M). \]

In the special case where some fraction \( \delta_i \) in industry \( i \) are flexible. Then,

\[ \bar{\lambda}(s) d \log \mu = \left( b'(I - \bar{\Omega})^{-1} - b'(I - \delta \bar{\Omega})^{-1} \delta \right) d \log A - (1 - b'(I - \delta \Omega)^{-1} \delta \alpha) d \log w, \tag{127} \]

\[ = \left( \bar{\lambda} - b'(I - \delta \bar{\Omega})^{-1} \delta \right) d \log A - (1 - b'(I - \delta \Omega)^{-1} \delta \alpha) d \log w. \]

At the industry level, equation (127) shows that the changes in markups can be interpreted as if some fraction of the firms in each industry change their markup in response to shocks.

\[ \blacksquare \]

**E  Relabelling**

As mentioned previously, our results can be applied to any nested CES economy, with any arbitrary pattern of nested substitutabilities and complementarities among intermediate inputs and factors. For concreteness, we describe the relabeling for one specific example.
Let the household’s consumption aggregator be

\[
\frac{C}{\hat{C}} = \left( \sum_k b_k \left( \frac{c_k}{\hat{c}_k} \right)^{\frac{\alpha_k}{\sigma_k}} \right)^{\frac{\sigma_k}{\sigma_k - 1}}. \tag{128}
\]

Suppose each producer \( k \) produces using a CES aggregator of value-added \( VA \) and intermediate inputs \( X \):

\[
\frac{y_k}{\hat{y}_k} = A_k \left( \frac{VA_k}{V\hat{A}_k} \right)^{\frac{\eta_k - 1}{\eta_k}} + (1 - \alpha_k) \left( \frac{X_k}{\hat{X}_k} \right)^{\frac{\eta_k - 1}{\eta_k}}. \tag{129}
\]

Value-added consists of different factors

\[
\frac{VA_k}{V\hat{A}_k} = \sum_{j=1}^{F} \nu_{kj} \left( \frac{l_{kj}}{l_{kj}^{\frac{\eta_k - 1}{\eta_k}}} \right)^{\frac{\eta_k}{\eta_k - 1}}.
\]

where \( l_{kj} \) is factor of type \( j \) used by \( k \). Intermediate input consists of inputs from other producers

\[
\frac{X_k}{\hat{X}_k} = \sum_{j=1}^{N} \omega_{kj} \left( \frac{x_{kj}}{x_{jk}} \right)^{\frac{\epsilon_k}{\epsilon_k - 1}}. \tag{129}
\]

Denote the matrix of \( \nu_{kj} \) using \( V \) and the matrix of \( \omega_{kj} \) as \( \Omega \).

We rewrite this economy in the standard form we require, and put hats on the new expenditure shares. The new input-output matrix \( \hat{\Omega} \) has dimension \((1+3N+F) \times (1+3N+F)\). The first industry is the household’s consumption aggregator, the next \( N \) industries are the original industries, the next \( N \) industries produce the value-added of the original industries, the next \( N \) industries produce the intermediate inputs of the original industries, and the final \( F \) industries correspond to the factors.

Under the relabeling, we have

\[
\hat{b} = (1, 0, \ldots, 0)',
\]

\[
\hat{\Omega} = \begin{pmatrix}
    b & 0 & 0 & 0 & 0 \\
    0 & 0 & \text{diag}(a) & \text{diag}(1-a) & 0 \\
    0 & 0 & 0 & 0 & V \\
    \Omega & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
with
\[ \hat{\theta} = (\sigma, \theta, v, \epsilon, 1), \]  
and
\[ \hat{\mu} = (1, \mu, 1, \ldots, 1). \]

F Aggregation of cost-based Domar weights

In this Appendix we show that recovering cost-based Domar weights from aggregated data is, in principle, not possible. We also show how the Basu-Fernald decomposition can detect changes in misallocation even in acyclic economies (where the equilibrium is efficient, and there is no possibility of reallocation of resources). The vertical economy in Figure 1a also shows the failure of the aggregation property implied by Hulten’s theorem. The easiest way to see this is to consider aggregating the input-output table for the economy in Figure 1a. For simplicity, suppose that markups are the same everywhere so that \( \mu_i = \mu \) for all \( i \). Since there is no possibility of reallocation in this economy, and since markups are uniform, this is our best chance of deriving an aggregation result, but even in this simplest example, such a result does not exist. Suppose that we aggregate the whole economy \( S = \{1, \ldots, N\} \). Then, in aggregate, the economy consists of a single industry that uses labor and inputs from itself to produce. In this case, the input-output matrix is a scalar, and equal to the intermediate input share of the economy

\[ \Omega_{S S} = \frac{1 - \frac{1}{\mu^{N-1}}}{1 - \frac{1}{\mu}}, \]  

and the aggregate markup for the economy is given by \( \mu \). Therefore, \( \tilde{\lambda}_S \) constructed using aggregate data is

\[ \tilde{\lambda}_S = 1'(I - \mu \Omega)^{-1} = \frac{\mu^{N-1} - \frac{1}{\mu}}{1 - \frac{1}{\mu}}. \]

However, we know from the example that

\[ \frac{d \log Y}{d \log A} = \sum_{i \in S} \tilde{\lambda}_i = N \neq \tilde{\lambda}_S = \frac{\mu^{N-1} - \frac{1}{\mu}}{1 - \frac{1}{\mu}}, \]

except in the limiting case without distortions \( \mu \to 1 \). Therefore, even in this simplest case, with homogenous markups and no reallocation, aggregated input-output data cannot be used to compute the impact of an aggregated shock.
To compare our decomposition with that of Basu and Fernald (2002), consider the acyclic economy in Figure 12 with two factors $L_1$ and $L_2$. Now, we know that

$$
\frac{d \log Y}{d \log A_2} = \tilde{\lambda}_2 = (1 - \alpha_1).
$$

(135)

The Basu-Fernald decomposition for this economy gives

$$
\frac{d \log Y}{d \log A_2} = \lambda_2 + R_M + \bar{\mu} R_M = \frac{1 - \alpha_1}{\mu_1} + \bar{\mu} R_M,
$$

(136)

where the first term (in this case, the sales-share of 2) is the “pure” technology effect, and $R_M$ is the reallocation of intermediate inputs, even though in this economy, there is no capacity for reallocating resources and the equilibrium is efficient.

![Figure 12: Acyclic economy where the solid arrows represent the flow of goods. The flow of profits and wages from firms to households has been suppressed in the diagram. The two factors in this economy are $L_1$ and $L_2$.](image)

### G Extra Examples

**Example G.1.** Next, consider the minimal example with two elasticities of substitution, which demonstrates the principle that changes in misallocation are driven by how each node switches its demand across its supply chain in response to a shock. To this end, we apply Proposition 2 to the economy depicted in Figure 13.

$$
\frac{d \log Y}{d \log A_3} = \tilde{\lambda}_3 - \frac{1}{\Lambda L} \left( (\theta_0 - 1) \left[ b_1 (\omega_{13} \mu_1^{-1} (\omega_{13} \mu_5^{-1} + \omega_{14} \mu_4^{-1})) - \omega_{13} b_1 \Lambda L \right] 
+ (\theta_1 - 1) \mu_1^{-1} \lambda_1 \left[ \omega_{13} \mu_3^{-1} - \omega_{13} \left( \omega_{13} \mu_5^{-1} + \omega_{14} \mu_4^{-1} \right) \right] \right).
$$
Figure 13: An economy with two elasticities of substitution.

The term multiplying \((\theta_0 - 1)\) captures how the household will shift their demand across 1 and 2 in response to the productivity shock, and the relative degrees of misallocation in 1 and 2’s supply chains. The term multiplying \((\theta_1 - 1)\) takes into account how 1 will shift its demand across 3 and 4 and the relative amount of misallocation of labor between 3 and 4. Not surprisingly, if instead we shock industry 1, then only the household’s elasticity of substitution matters, since industry 1 will not shift its demand across its inputs in response to the shock to industry 2:

\[
\frac{d \log Y}{d \log A_1} = b_1 - \frac{1}{\Lambda_L} (\theta_0 - 1) \left[ b_1 \mu_1^{-1} (\omega_{13} \mu_3^{-1} + \omega_{14} \mu_4^{-1}) - b_1 \Lambda_L \right].
\]

This illustrates the general principle in Proposition 2 that an elasticity of substitution \(\theta_j\) matters only if \(j\) is somewhere downstream from \(k\).