B Proofs

Proof of Theorem 2.2. By Shephard's lemma, we know that, to a first order, the productivity shock A_k will change the prices of any industry who purchases inputs, either directly or indirectly, from industry k

$$\frac{\mathrm{d}\log p_i}{\mathrm{d}\log A_k} = -\mathbf{1}(i=k) + \sum_F \tilde{\alpha}_{iF}\frac{\mathrm{d}\log w_F}{\mathrm{d}\log A_k} + \sum_j \tilde{\Omega}_{ij}\frac{\mathrm{d}\log p_j}{\mathrm{d}\log A_k}. \tag{36}$$

Rewrite this in matrix form to get

$$\frac{\mathrm{d}\log p}{\mathrm{d}\log A_k} = (I-\tilde{\Omega})^{-1}(\tilde{\alpha}\frac{\mathrm{d}\log w}{\mathrm{d}\log A_k} - e_k) = \tilde{\Psi}(\tilde{\alpha}\frac{\mathrm{d}\log w}{\mathrm{d}\log A_k} - e_k), \tag{37}$$

where e_k is the kth standard basis vector. Let the household's aggregate consumption good to be the numeraire, so that the household's ideal price index P_c is always equal to one. Then we know that

$$\frac{\mathrm{d}\log P_c}{\mathrm{d}\log A_k} = b'\frac{\mathrm{d}\log p}{\mathrm{d}\log A_k} = 0. \tag{38}$$

Combine this with the previous expression to get

$$-b'\tilde{\Psi}e_k + b'\tilde{\Psi}\tilde{\alpha}\frac{\mathrm{d}\log w}{\mathrm{d}\log A_k} = 0. \tag{39}$$

Note that, $b'\tilde{\Psi} = \tilde{\lambda}$ and $b'\tilde{\Psi}\tilde{\alpha} = \tilde{\Lambda}$. Hence,

$$-\tilde{\lambda}_k + \tilde{\Lambda}'\frac{\mathrm{d}\log w}{\mathrm{d}\log A_k} = 0. \tag{40}$$

Now, note that

$$\Lambda_f = \frac{w_f L_f}{P_c C}. \tag{41}$$

From this, we know that

$$\frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log A_k} = \frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k} + \frac{\mathrm{d}\log L_f}{\mathrm{d}\log A_k} - \frac{\mathrm{d}\log Y}{\mathrm{d}\log A_k}. \tag{42}$$

Substitute this into the previous expression to get

$$-\tilde{\lambda}_k + \tilde{\Lambda}' \frac{\mathrm{d}\log\Lambda}{\mathrm{d}\log A_k} - \tilde{\Lambda}' \frac{\mathrm{d}\log L_f}{\mathrm{d}\log A_k} + \frac{\mathrm{d}\log Y}{\mathrm{d}\log A_k} = 0, \tag{43}$$

where we use the fact that $\sum_f \tilde{\Lambda}_f = 1$. Rearrange this to get the desired result.To get an explicit characterization for $\mathrm{d}\log\Lambda_k/\mathrm{d}\log A_k$ in terms of structural parameters of the model, without loss of generality, assume that each good is produced from a distinct primary factor (this can be achieved by relabelling the input-output matrix). Now note that

$$\lambda_i = b_i + \sum_j \mu_j^{-1}\omega_{ji}\lambda_j, \tag{44}$$

$$w_i = \frac{\alpha_i p_i y_i}{\mu_i G_i(w_i/P_c)}, \tag{45}$$

$$\lambda_i = \frac{p_i y_i}{P_c C}, \tag{46}$$

$$p_i = \frac{1}{A_i}\mathfrak{C}_i(p_1,\ldots,p_N,w_i)\mu_i, \tag{47}$$

$$P_c = \frac{\sum_i p_i c_i}{C} = 1. \tag{48}$$

We can differentiate these to get to our answer. Denote the elasticity of substitution between k and j for the total cost function of industry i by ρ^i_{kl}. Then we can write

$$\begin{aligned}\frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log A_k} &= \mathrm{d}\,b_i + \sum_j \mathrm{d}\,\omega_{ji}\mu_j^{-1}\lambda_j \\ &\quad + \sum_j \omega_{ji}\mu_j^{-1}\frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log A_k} - \sum_j \omega_{ji}\mu_j^{-1}\lambda_j\,\mathrm{d}\log\mu_j,\end{aligned} \tag{49}$$

$$\mathrm{d}\,b_i = b_i \sum_{j\neq i} b_j\left(1-\frac{1}{\rho^c_{ij}}\right)\Big(\mathrm{d}\log p_i - \mathrm{d}\log p_j\Big), \tag{50}$$

$$\mathrm{d}\,\omega_{ij} = \omega_{ij}\sum_{k\neq j}\omega_{ik}\left(1-\frac{1}{\rho^i_{jk}}\right)\Big(\mathrm{d}\log p_j - \mathrm{d}\log p_k\Big) + \omega_{ij}\alpha_i\left(1-\frac{1}{\rho^i_{jL}}\right)\Big(\mathrm{d}\log p_j - \mathrm{d}\log w_i\Big), \tag{51}$$

$$\frac{\mathrm{d}\log w_i}{\mathrm{d}\log A_k} = \sum_j \omega_{ij}\left(1-\frac{1}{\rho^i_{jL}}\right)\Big(\mathrm{d}\log w_i - \mathrm{d}\log p_j\Big) + \mathrm{d}\log p_i + \mathrm{d}\log y_i - \mathrm{d}\log\mu_i, \tag{52}$$

$$\frac{\mathrm{d}\log y_i}{\mathrm{d}\log A_k} = \frac{1}{\lambda_i}\frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log A_k} - \frac{\mathrm{d}\log p_i}{\mathrm{d}\log A_k} + \frac{\mathrm{d}\log P_c C}{\mathrm{d}\log A_k}, \tag{53}$$

The proof for the case with markups is very similar. ■

Proof of Proposition 2.6. Take all derivatives with respect to A_k and normalize $P_c = 1$. In this case, the system reduces to

$$\frac{1}{\lambda_i}\,\mathrm{d}\,\lambda_i = 0, \tag{54}$$

$$\mathrm{d}\log w_i = \mathrm{d}\log p_i + \mathrm{d}\log y_i \tag{55}$$

$$\mathrm{d}\log y_i = -\,\mathrm{d}\log p_i + \mathrm{d}\log Y \tag{56}$$

$$\mathrm{d}\log p_i = -\mathbf{1}(i = k) + \alpha_i\,\mathrm{d}\log w_i + \sum_i \omega_{ij}\,\mathrm{d}\log p_j, \tag{57}$$

$$\mathrm{d}\log Y = \sum_i \frac{w_i l_i}{C}\,\mathrm{d}\log w_i + \sum_i \frac{\pi_i}{C}\left(\mathrm{d}\log\lambda_i + \mathrm{d}\log Y\right). \tag{58}$$

We also know that $b'\,\mathrm{d}\log p = 0$ by normalization. Hence, combining the second and third relationship we get that

$$\mathrm{d}\log w_i = \mathrm{d}\log Y, \tag{59}$$

in other words, all wages move together – there is no pecuniary externality on the factors in response to a shock. Next, we know the pricing equation is

$$\mathrm{d}\log p = (I - \Omega)^{-1}(\alpha\,\mathrm{d}\log Y - e_k). \tag{60}$$

Combine this with $b'\,\mathrm{d}\log p = 0$ to get the desired result that

$$\mathrm{d}\log Y = b'(I - \Omega)^{-1}e_k, \tag{61}$$

following the fact that $b'(I-\Omega)^{-1}\alpha = 1$ — or, without markups, the sum of factor payments would add up to GDP since there are no pure profits. ■

Proof of Proposition 3.1. First, observe that

$$\mathrm{d}\log p_i = -\mathbf{1}(i = k) + \sum_j \omega_{ij}\,\mathrm{d}\log p_j + \alpha_i\,\mathrm{d}\log w. \tag{62}$$

Hence,

$$\mathrm{d}\log p_i = e_i'(I - \Omega)^{-1}\alpha\,\mathrm{d}\log w - e_i'(I - \Omega)^{-1}e_k = \mathrm{d}\log w - \tilde{\psi}_{ik}. \tag{63}$$

Since, $b' \, \mathrm{d} \log p_i = \mathrm{d} \log w - \tilde{\lambda}_k = 0$, this is

$$\mathrm{d} \log p_i = \tilde{\lambda}_k - \tilde{\psi}_{ik}. \tag{64}$$

This completely characterizes the effect on prices, and if the real wage is equal to output, we are done. However, in equilibrium, the real wage is not equal to output. To get the effect on output, we need to take into account how the labor share of income changes. Denote the labor share by Λ_L. Using Theorem 2.2, we have that

$$\mathrm{d} \log Y = \tilde{\lambda}_k - \frac{1}{\Lambda_L} \mathrm{d} \, \Lambda_L. \tag{65}$$

Hence, we need to characterize $\mathrm{d} \, \Lambda_L$. We know that

$$\mathrm{d} \, \Lambda_L = \sum_j \alpha_j \mu_j^{-1} \lambda_j (1 - \theta_j) \left[\mathrm{d} \log w - \mathrm{d} \log p_j - \mathbf{1}(j = k) \right] + \sum_j \alpha_j \mu_j^{-1} \, \mathrm{d} \, \lambda_j. \tag{66}$$

Which simplifies to

$$\mathrm{d} \, \Lambda_L = \sum_j \alpha_j \mu_j^{-1} \lambda_j (1 - \theta_j) \left(\tilde{\psi}_{jk} - \mathbf{1}(j = k) \right) + \sum_j \alpha_j \mu_j^{-1} \, \mathrm{d} \, \lambda_j. \tag{67}$$

Now, note that

$$\mathrm{d} \, \lambda_i = b_i (1 - \theta_0) \, \mathrm{d} \log p_i + \sum_j \omega_{ji} \mu_j^{-1} (1 - \theta_j) \left(\mathrm{d} \log p_i - \mathrm{d} \log p_j - \mathbf{1}(j = k) \right) + \sum_j \omega_{ji} \mu_j^{-1} \, \mathrm{d} \, \lambda_j. \tag{68}$$

Substitute in prices, and solve this linear system in $\mathrm{d} \, \lambda$ to get

$$\mathrm{d} \, \lambda_m = \sum_i \left[b_i (1 - \theta_0)(\tilde{\lambda}_k - \tilde{\psi}_{ik}) + \sum_j \omega_{ji} \mu_j^{-1} \lambda_j (1 - \theta_j)(\tilde{\psi}_{jk} - \tilde{\psi}_{ik}) + \omega_{ki} \mu_k^{-1} \lambda_k (\theta_k - 1) \right] \psi_{im}. \tag{69}$$

This can be simplified to

$$\begin{aligned} \mathrm{d} \, \lambda_m = & \sum_i b_i (1 - \theta_0) \tilde{\lambda}_k \psi_{im} - \sum_i b_i (1 - \theta_0) \psi_{im} \tilde{\psi}_{ik} \\ & + \sum_{i,j} \omega_{ji} \mu_j^{-1} (1 - \theta_j) \lambda_j \tilde{\psi}_{jk} \psi_{im} - \sum_{i,j} \omega_{ji} \mu_j^{-1} (1 - \theta_j) \lambda_j \tilde{\psi}_{ik} \psi_{im} \end{aligned}$$

$$+ \sum_i \omega_{ki}\mu_k^{-1}\lambda_k(\theta_k - 1)\psi_{im}. \tag{70}$$

Which can be further simplified to

$$\begin{aligned}
\mathrm{d}\,\lambda_m &= \lambda_m\tilde{\lambda}_k(1-\theta_0) - \sum_i b_i(1-\theta_0)\psi_{im}\tilde{\psi}_{ik} \\
&+ \sum_{i,j}\omega_{ji}\mu_j^{-1}(1-\theta_j)\lambda_j\tilde{\psi}_{jk}\psi_{im} - \sum_{i,j}\omega_{ji}\mu_j^{-1}(1-\theta_j)\lambda_j\tilde{\psi}_{ik}\psi_{im} \\
&+ \lambda_k(\theta_k - 1)\left[\psi_{km} - \mathbf{1}(m=k)\right]. \qquad (71)\\
&= (\theta_0 - 1)cov_b^*(\tilde{\Psi}_{(k)}, \Psi_{(m)}) + \sum_j(\theta_j - 1)\lambda_j cov^*_{\mu_j^{-1}\Omega_j}(\tilde{\Psi}_{(k)}, \Psi_{(m)}) \\
&+ \sum_j(\theta_j - 1)\lambda_j\left(\sum_i \omega_{ji}\tilde{\psi}_{ik}\sum_i \mu_j^{-1}\omega_{ji}\psi_{im}\right) \\
&- \sum_{i,j}\omega_{ji}\mu_j^{-1}(\theta_j - 1)\lambda_j\tilde{\psi}_{jk}\psi_{im} + \lambda_k(\theta_k - 1)(\psi_{km} - \mathbf{1}(m=k)), \\
&= (\theta_0 - 1)cov_b^*(\tilde{\Psi}_{(k)}, \Psi_{(m)}) + \sum_j(\theta_j - 1)\lambda_j cov^*_{\mu_j^{-1}\Omega_j}(\tilde{\Psi}_{(k)}, \Psi_{(m)}). \qquad (72)
\end{aligned}$$

Hence,

$$\sum_m \alpha_m\mu_m^{-1}\,\mathrm{d}\,\lambda_m = (\theta_0 - 1)cov_b^*(\tilde{\Psi}_{(k)}, \sum_m \alpha_m\mu_m^{-1}\Psi_{(m)}) + \sum_j(\theta_j - 1)\lambda_j cov^*_{\mu_j^{-1}\Omega_j}(\tilde{\Psi}_{(k)}, \sum_m \alpha_m\mu_m^{-1}\Psi_{(m)}). \tag{73}$$

Putting this altogether, we get

$$\begin{aligned}
\mathrm{d}\,\Lambda_L &= \sum_j \alpha_j\mu_j^{-1}\lambda_j(1-\theta_j)\left(\tilde{\psi}_{jk} - \mathbf{1}(j=k)\right) \\
&+ (\theta_0 - 1)cov_b^*(\tilde{\Psi}_{(k)}, \sum_m \alpha_m\mu_m^{-1}\Psi_{(m)}) + \sum_j(\theta_j - 1)\lambda_j cov^*_{\mu_j^{-1}\Omega_j}(\tilde{\Psi}_{(k)}, \sum_m \alpha_m\mu_m^{-1}\Psi_{(m)}). \qquad (74)
\end{aligned}$$

To finish the proof, note that $\sum_m \mu_m^{-1}\alpha_m\Psi_{im} = \Psi_{iL}$. Then

$$\begin{aligned}
\mathrm{d}\,\Lambda_L &= \sum_j \alpha_j\mu_j^{-1}\lambda_j(1-\theta_j)\left(\tilde{\psi}_{jk} - \mathbf{1}(j=k)\right) \\
&+ (\theta_0 - 1)cov_b^*(\tilde{\Psi}_{(k)}, \sum_m \alpha_m\mu_m^{-1}\Psi_{(m)}) + \sum_j(\theta_j - 1)\lambda_j cov^*_{\mu_j^{-1}\Omega_j}(\tilde{\Psi}_{(k)}, \Psi_{(L)}), \qquad (75)
\end{aligned}$$

$$= (\theta_0 - 1)cov_b^*(\tilde{\Psi}_{(k)}, \sum_m \alpha_m \mu_m^{-1} \Psi_{(m)}) + \sum_j (\theta_j - 1)\lambda_j cov^*_{\mu_j^{-1}[\Omega_j, a_j]}(\tilde{\Psi}_{(k)}, \Psi_{(L)}), \tag{76}$$

where note that $\Psi_{LL} = 1$.

■

Proof of Proposition 3.2. From Theorem 2.2, we know that

$$\mathrm{d}\log Y / \mathrm{d}\log \mu_k = -\tilde{\lambda}_k - \frac{1}{\Lambda_L}\,\mathrm{d}\,\lambda_L. \tag{77}$$

For a quantity m, we have

$$\mathrm{d}\,\lambda_m = \sum_i \left(b_i(1-\theta_0)(\mathrm{d}\log p_i) + \sum_j \Omega_{ji}\mu_j^{-1}(1-\theta_j)\lambda_j[\mathrm{d}\log p_i - \mathrm{d}\log p_j] - \mathbf{1}(j=k)\theta_k \right)\Psi_{im}. \tag{78}$$

Note that

$$\mathrm{d}\log p_i = \mathbf{1}(i=k) + \sum_j \omega_{ij}\,\mathrm{d}\log p_j + \alpha_i\,\mathrm{d}\log w. \tag{79}$$

Hence,

$$\mathrm{d}\log p_i = \mathrm{d}\log w + \tilde{\psi}_{ik}. \tag{80}$$

Since, $b'\,\mathrm{d}\log p_i = \mathrm{d}\log w + \tilde{\lambda}_k = 0$, this is

$$\mathrm{d}\log p_i = -\tilde{\lambda}_k + \tilde{\psi}_{ik}. \tag{81}$$

Substituting this back into (78) and set $m = L$ to get

$$\mathrm{d}\,\lambda_L = \sum_i b_i(1-\theta_0)(\tilde{\psi}_{ik} - \tilde{\lambda}_k)\psi_{iL} + \sum_j \lambda_j(1-\theta_j)\mu_j^{-1}\sum_i \Omega_{ji}[\tilde{\psi}_{ik} - \tilde{\psi}_{jk}]\Psi_{iL} - \theta_k\mu_k^{-1}\lambda_k \sum_i \Omega_{ki}\Psi_{iL}. \tag{82}$$

Rearrange this to get

$$\begin{aligned}\mathrm{d}\,\lambda_L =& (1-\theta_0)Cov_b(\tilde{\Psi}_{(k)}, \Psi_{(L)}) + \sum_j \lambda_j(1-\theta_j)\mu_j^{-1}Cov_{\Omega^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(L)}) \\ &- \mu_k^{-1}(1-\theta_k)\lambda_k \sum_i \Omega_{ki}\Psi_{iL} - \theta_k\mu_k^{-1}\lambda_k \sum_i \Omega_{ki}\Psi_{iL}.\end{aligned}$$

Combine this with (77). ■

Proof of Proposition 3.3. By Shephard's lemma,

$$\mathrm{d}\log p_i = -\mathbf{1}(i=k) + \sum_j \Omega_{ij}\,\mathrm{d}\log p_j + \sum_f \alpha_{if}\,\mathrm{d}\log w_f. \tag{83}$$

Invert this system to get

$$\mathrm{d}\log p_i = -\tilde{\Psi}e_k + \tilde{\Psi}_f\,\mathrm{d}\log w, \tag{84}$$

where $\tilde{\Psi}_f = (I-\Omega)^{-1}\alpha$ is a $N\times K$ matrix of network-adjusted factor intensities by industry.

Now consider a factor L, we have

$$\begin{aligned}
\mathrm{d}\,\Lambda_L =& \sum_i b_i(1-\theta_0)[-\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log w_f]\Psi_{iL},\\
&+ \sum_j (1-\theta_j)\lambda_j\mu_j^{-1}\sum_i \Omega_{ji}[-\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log w_f + \tilde{\Psi}_{jk} - \sum_f \tilde{\Psi}_{jf}\,\mathrm{d}\log w_f]\Psi_{iL}\\
&+ (\theta_k-1)\lambda_k\mu_k^{-1}\sum_i \Omega_{ki}\Psi_{iL}.
\end{aligned}$$

Simplify this to

$$\begin{aligned}
\mathrm{d}\,\Lambda_L =& (\theta_0-1)\left(\sum_i b_i\tilde{\Psi}_{ik}\Psi_{iL} - \sum_i b_i\Psi_{iL}\sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log w_f\right),\\
&+ \sum_j (\theta_j-1)\lambda_j\mu_j^{-1}\left[\sum_i \Omega_{ji}\tilde{\Psi}_{ik}\Psi_{iL} - \sum_i \Omega_{ji}\tilde{\Psi}_{jk}\Psi_{iL}\right]\\
&+ \sum_j (1-\theta_j)\lambda_j\mu_j^{-1}\left[\sum_i \Omega_{ji}\sum_f \left(\tilde{\Psi}_{if} - \tilde{\Psi}_{jf}\right)\mathrm{d}\log w_f\Psi_{iL}\right]\\
&+ (\theta_k-1)\lambda_k\mu_k^{-1}\sum_i \Omega_{ki}\Psi_{iL},\\
=& (\theta_0-1)\left(\sum_i b_i\tilde{\Psi}_{ik}\Psi_{iL} - \sum_i b_i\Psi_{iL}\sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log w_f\right),\\
&+ \sum_j (\theta_j-1)\lambda_j\mu_j^{-1}\left[\sum_i \Omega_{ji}\tilde{\Psi}_{ik}\Psi_{iL} - \left(\sum_i \Omega_{ji}\Psi_{iL}\right)\left(\sum_i \Omega_{ji}\tilde{\Psi}_{ik}\right)\right]\\
&+ \sum_j (1-\theta_j)\lambda_j\mu_j^{-1}\left[\sum_i \Omega_{ji}\sum_f \left(\tilde{\Psi}_{if} - \tilde{\Psi}_{jf}\right)\mathrm{d}\log w_f\Psi_{iL}\right],
\end{aligned}$$

$$
\begin{aligned}
=&(\theta_0-1)\left(\sum_i b_i\tilde{\Psi}_{ik}\Psi_{iL}-\sum_i b_i\Psi_{iL}\sum_f\tilde{\Psi}_{if}\,\mathrm{d}\log w_f\right),\\
&+\sum_j(\theta_j-1)\lambda_j\mu_j^{-1}Cov_{\Omega^{(j)}}(\tilde{\Psi}_{(k)},\Psi_{(L)})\\
&+\sum_j(1-\theta_j)\lambda_j\mu_j^{-1}\sum_f\left[\sum_i\Omega_{ji}\tilde{\Psi}_{if}\Psi_{iL}-\left(\sum_i\Omega_{ji}\Psi_{iL}\right)\left(\sum_i\Omega_{ji}\tilde{\Psi}_{if}\right)\right]\mathrm{d}\log w_f,\\
=&(\theta_0-1)\left(\sum_i b_i\tilde{\Psi}_{ik}\Psi_{iL}-\sum_i b_i\Psi_{iL}\sum_f\tilde{\Psi}_{if}\,\mathrm{d}\log w_f\right),\\
&+\sum_j(\theta_j-1)\lambda_j\mu_j^{-1}Cov_{\Omega^{(j)}}(\tilde{\Psi}_{(k)},\Psi_{(L)})\\
&+\sum_j(1-\theta_j)\lambda_j\mu_j^{-1}\sum_f Cov_{\Omega^{(j)}}(\tilde{\Psi}_{(f)},\Psi_{(L)})\,\mathrm{d}\log w_f,\\
=&(\theta_0-1)\left(\sum_i b_i\tilde{\Psi}_{ik}\Psi_{iL}-\sum_i b_i\Psi_{iL}\sum_f\tilde{\Psi}_{if}\,\mathrm{d}\log w_f\right),\\
&+\sum_j(\theta_j-1)\lambda_j\mu_j^{-1}Cov_{\Omega^{(j)}}(\tilde{\Psi}_{(k)}-\sum_f\tilde{\Psi}_{(f)}\,\mathrm{d}\log w_f,\Psi_{(L)})\\
=&(\theta_0-1)\left(\sum_i b_i\Psi_{iL}\left(\tilde{\Psi}_{ik}-\sum_f\tilde{\Psi}_{if}\,\mathrm{d}\log w_f\right)\right),\\
&+\sum_j(\theta_j-1)\lambda_j\mu_j^{-1}Cov_{\Omega^{(j)}}(\tilde{\Psi}_{(k)}-\sum_f\tilde{\Psi}_{(f)}\,\mathrm{d}\log w_f,\Psi_{(L)})\\
=&(\theta_0-1)Cov_b(\tilde{\Psi}_{(k)}-\sum_f\tilde{\Psi}_{(f)}\,\mathrm{d}\log w_f,\Psi_{(L)})+(\theta_0-1)(\tilde{\lambda}_k-\sum_f\tilde{\Lambda}_f\,\mathrm{d}\log w_f)\lambda_L,\\
&+\sum_j(\theta_j-1)\lambda_j\mu_j^{-1}Cov_{\Omega^{(j)}}(\tilde{\Psi}_{(k)}-\sum_f\tilde{\Psi}_{(f)}\,\mathrm{d}\log w_f,\Psi_{(L)}).
\end{aligned}
$$

Hence, for a productivity shock $\mathrm{d}\log A_k$, letting Λ_L be demand for factor L, and indexing all factors by f, we have

$$
\begin{aligned}
\mathrm{d}\,\lambda_L=&(\theta_0-1)Cov_b\left(\tilde{\Psi}_{(k)}-\sum_f\tilde{\Psi}_{(f)}\,\mathrm{d}\log w_f,\Psi_{(L)}\right)\\
&+\sum_j(\theta_j-1)\mu_j^{-1}\lambda_jCov_{[\Omega_j,\alpha_j]}\left(\tilde{\Psi}_{(k)}-\sum_f\tilde{\Psi}_{(f)}\,\mathrm{d}\log w_f,\Psi_{(L)}\right)
\end{aligned}
$$

$$+ (\theta_0 - 1)\left(\tilde{\lambda}_k - \sum_f \tilde{\lambda}_f \, \mathrm{d}\log w_f\right)\lambda_L. \tag{85}$$

Combine this with the observation that

$$\lambda_L(\mathrm{d}\log w_L + \mathrm{d}\log L_L - \mathrm{d}\log Y) = \mathrm{d}\,\lambda_L. \tag{86}$$

Finally, we know that

$$\mathrm{d}\log Y = \tilde{\lambda}_k + \sum_f \tilde{\lambda}_f \, \mathrm{d}\log \lambda_f. \tag{87}$$

Set $\mathrm{d}\,L_L = 0$, and we have a linear system with $F+1$ equations and $F+1$ unknowns where F is the total number of factors. Substitute

$$\frac{1}{\lambda_f}\,\mathrm{d}\,\lambda_f + \mathrm{d}\log Y = \mathrm{d}\log w_f \tag{88}$$

back into network formula to get

$$\begin{aligned}\Lambda_L \,\mathrm{d}\log \lambda_L =&(\theta_0 - 1)Cov_b\left(\tilde{\Psi}_{(k)} - \sum_f \tilde{\Psi}_{(f)}\,\mathrm{d}\log\lambda_f, \Psi_{(L)}\right)\\ &+ \sum_j (\theta_j - 1)\mu_j^{-1}\lambda_j Cov_{[\Omega_j,\alpha_j]}\left(\tilde{\Psi}_{(k)} - \sum_f \tilde{\Psi}_{(f)}\,\mathrm{d}\log\lambda_f, \Psi_{(L)}\right)\\ &+ (\theta_0 - 1)\left(\tilde{\lambda}_k - \sum_f \tilde{\lambda}_f\,\mathrm{d}\log\lambda_f - \mathrm{d}\log Y\right)\lambda_L.\end{aligned} \tag{89}$$

Use Theorem 2.2 to further simplify this

$$\begin{aligned}\lambda_L \,\mathrm{d}\log \lambda_L =&(\theta_0 - 1)Cov_b\left(\tilde{\Psi}_{(k)} - \sum_f \tilde{\Psi}_{(f)}\,\mathrm{d}\log\lambda_f, \Psi_{(L)}\right)\\ &+ \sum_j (\theta_j - 1)\mu_j^{-1}\lambda_j Cov_{[\Omega_j,\alpha_j]}\left(\tilde{\Psi}_{(k)} - \sum_f \tilde{\Psi}_{(f)}\,\mathrm{d}\log\lambda_f, \Psi_{(L)}\right).\end{aligned} \tag{90}$$

■

Proof of Proposition 3.4. From Theorem 2.2, we know that

$$\mathrm{d}\log Y/\,\mathrm{d}\log\mu_k = -\tilde{\lambda}_k - \frac{1}{\Lambda_L}\,\mathrm{d}\,\lambda_L. \tag{91}$$

For a markup shock, we can differentiate demand for a quantity m to get

$$\mathrm{d}\,\lambda_m = \sum_i \left(b_i(1-\theta_0)(\mathrm{d}\log p_i) + \sum_j \Omega_{ji}\mu_j^{-1}(1-\theta_j)\lambda_j[\mathrm{d}\log p_i - \mathrm{d}\log p_j] - \mathbf{1}(j=k)\theta_k \right)\Psi_{im}. \tag{92}$$

By Shephard's lemma,

$$\mathrm{d}\log p_i = \mathbf{1}(i=k) + \sum_j \Omega_{ij}\,\mathrm{d}\log p_j + \sum_f \alpha_{if}\,\mathrm{d}\log w_f. \tag{93}$$

Invert this system to get

$$\mathrm{d}\log p_i = \tilde{\Psi}e_k + \tilde{\Psi}_f\,\mathrm{d}\log w, \tag{94}$$

where $\tilde{\Psi}_f = (I-\Omega)^{-1}\alpha$ is a $N\times K$ matrix of network-adjusted factor intensities by industry.

Substituting this back into (92) and set $m = L$ to get

$$\begin{aligned}
\mathrm{d}\,\Lambda_L =& \sum_i b_i(1-\theta_0)[\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log w_f]\Psi_{iL}, \\
&+ \sum_j (1-\theta_j)\lambda_j\mu_j^{-1}\sum_i \Omega_{ji}[\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log w_f - \tilde{\Psi}_{jk} - \sum_f \tilde{\Psi}_{jf}\,\mathrm{d}\log w_f]\Psi_{iL} \\
&- \theta_k\mu_k^{-1}\lambda_k\sum_i \Omega_{ki}\Psi_{iL}.
\end{aligned}$$

Simplify this to

$$\begin{aligned}
\mathrm{d}\,\Lambda_L =& (1-\theta_0)\left(\sum_i b_i\tilde{\Psi}_{ik}\Psi_{iL}\right) + (1-\theta_0)\left(\sum_i b_i\Psi_{iL}\sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log w_f\right), \\
&+ \sum_j (1-\theta_j)\lambda_j\mu_j^{-1}\left[\sum_i \Omega_{ji}\tilde{\Psi}_{ik}\Psi_{iL} - \sum_i \Omega_{ji}\Psi_{iL}\tilde{\Psi}_{jk}\right] \\
&+ \sum_j (1-\theta_j)\lambda_j\mu_j^{-1}\left[\sum_i \Omega_{ji}\sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log w_f\Psi_{iL} - \sum_i \Omega_{ji}\Psi_{iL}\sum_f \tilde{\Psi}_{jf}\,\mathrm{d}\log w_f\right] \\
&- \theta_k\lambda_k\mu_k^{-1}\Psi_{kL},
\end{aligned}$$

$$
\begin{aligned}
=&(1-\theta_0)\left(Cov_b(\tilde{\Psi}_{(k)},\Psi_{(L)})+\tilde{\lambda}_k\Lambda_L\right)\\
&+\sum_j(1-\theta_j)\lambda_j\mu_j^{-1}\left(Cov_{\Omega^{(j)}}(\tilde{\Psi}_{(k)},\Psi_{(L)})-\mathbf{1}(j=k)\Psi_{jL}\right)\\
&+(1-\theta_0)\left(Cov_b\left(\sum_f\tilde{\Psi}_{(f)}\,\mathrm{d}\log w_f,\Psi_{(L)}\right)+\sum_f\tilde{\Lambda}_f\,\mathrm{d}\log w_f\Lambda_L\right)\\
&+\sum_j(1-\theta_j)\lambda_j\mu_j^{-1}\left(Cov_{\Omega^{(j)}}\left(\sum_f\tilde{\Psi}_{(f)}\,\mathrm{d}\log w_f,\Psi_{(L)}\right)-\mathbf{1}(j=k)\Psi_{jL}\right)\\
&-\theta_k\lambda_k\mu_k^{-1}\Psi_{kL},\\
=&(1-\theta_0)\left(Cov_b(\tilde{\Psi}_{(k)}+\sum_f\tilde{\Psi}_{(f)}\,\mathrm{d}\log w_f,\Psi_{(L)})+\tilde{\lambda}_k\Lambda_L\right)\\
&+\sum_j(1-\theta_j)\lambda_j\mu_j^{-1}\left(Cov_{\Omega^{(j)}}(\tilde{\Psi}_{(k)}+\sum_f\tilde{\Psi}_{(f)}\,\mathrm{d}\log w_f,\Psi_{(L)})-\mathbf{1}(j=k)\Psi_{jL}\right)\\
&+(1-\theta_0)\left(\tilde{\lambda}_k+\sum_f\tilde{\Lambda}_f\,\mathrm{d}\log w_f\right)\Lambda_L-\lambda_k\mu_k^{-1}\Psi_{kL},\\
=&(1-\theta_0)\left(Cov_b(\tilde{\Psi}_{(k)}+\sum_f\tilde{\Psi}_{(f)}\,\mathrm{d}\log w_f,\Psi_{(L)})+\tilde{\lambda}_k\Lambda_L\right)\\
&+\sum_j(1-\theta_j)\lambda_j\mu_j^{-1}\left(Cov_{\Omega^{(j)}}(\tilde{\Psi}_{(k)}+\sum_f\tilde{\Psi}_{(f)}\,\mathrm{d}\log w_f,\Psi_{(L)})\right)\\
&-\lambda_k\Psi_{kL}.
\end{aligned}
$$

The final line follows from the fact that $\left(\tilde{\lambda}_k+\sum_f\tilde{\Lambda}_f\,\mathrm{d}\log w_f\right)=b'\,\mathrm{d}\log p=\mathrm{d}\log P_c=0$. Finally, substitute $\mathrm{d}\log w_f=\mathrm{d}\log\Lambda_f+\mathrm{d}\log Y$ into the expression above to get

$$
\begin{aligned}
\mathrm{d}\,\Lambda_L=&(1-\theta_0)\left(Cov_b(\tilde{\Psi}_{(k)}+\sum_f\tilde{\Psi}_{(f)}(\mathrm{d}\log\Lambda_f+\mathrm{d}\log Y),\Psi_{(L)})+\tilde{\lambda}_k\Lambda_L\right)\\
&+\sum_j(1-\theta_j)\lambda_j\mu_j^{-1}\left(Cov_{\Omega^{(j)}}(\tilde{\Psi}_{(k)}+\sum_f\tilde{\Psi}_{(f)}(\mathrm{d}\log\Lambda_f+\mathrm{d}\log Y),\Psi_{(L)})\right)\\
&-\lambda_k\Psi_{kL}.
\end{aligned}
$$

To complete the proof, note that

$$\mathrm{d}\log Y \sum_f \tilde{\Psi}_{(f)} = \mathrm{d}\log Y \mathbf{1}. \tag{95}$$

In other words, this is a vector of all ones multiplied by the scalar $\mathrm{d}\log Y$, and hence it drops out of the covariance operators, since the covariance of a vector of ones with any other vector under any probability distribution is always equal to zero. Hence,

$$\begin{aligned}\mathrm{d}\,\Lambda_L =&(1-\theta_0)\left(Cov_b(\tilde{\Psi}_{(k)} + \sum_f \tilde{\Psi}_{(f)}\,\mathrm{d}\log\Lambda_f, \Psi_{(L)}) + \tilde{\lambda}_k\Lambda_L\right)\\ &+\sum_j (1-\theta_j)\lambda_j\mu_j^{-1}\left(Cov_{\Omega^{(j)}}(\tilde{\Psi}_{(k)} + \sum_f \tilde{\Psi}_{(f)}\,\mathrm{d}\log\Lambda_f, \Psi_{(L)})\right)\\ &-\lambda_k\Psi_{kL}.\end{aligned}$$

This can be combined with Theorem 2.3 to complete the proof. ■

Proof of Example 3.6. We start with the matrices listed in (25) and (26), so that

$$\Gamma = \begin{pmatrix} (1-\theta_0)Cov_b(\widetilde{\Psi}_{(L)}, \Psi_{(L)}) & (1-\theta_0)Cov_b(\widetilde{\Psi}_{(K)}, \Psi_{(L)}) \\ (1-\theta_0)Cov_b(\widetilde{\Psi}_{(K)}, \Psi_{(L)}) & (1-\theta_0)Cov_b(\widetilde{\Psi}_{(K)}, \Psi_{(K)} \end{pmatrix}. \tag{96}$$

$$\delta^{(i)} = \begin{pmatrix} (1-\theta_0)Cov_b(\widetilde{\Psi}_{(i)}, \Psi_{(L)}) \\ (1-\theta_0)Cov_b(\widetilde{\Psi}_{(i)}, \Psi_{(K)} \end{pmatrix}. \tag{97}$$

From the structure of the problem, we can explicitly write the value of Γ as follows:

$$\Gamma = (1-\theta_0)\begin{pmatrix} b_3(b_1\mu_1^{-1} + b_2\mu_2^{-1}) & -b_3(b_1\mu_1^{-1} + b_2\mu_2^{-1}) \\ -b_3(1-b_3)\mu_3^{-1} & b_3(1-b_3)\mu_3^{-1} \end{pmatrix}. \tag{98}$$

First we look at the cases for $i = 1, 2$ (the case where a factor of production is shared). We note that the case is symmetric for $i = 1, 2$ by the structure of the network and the problem. For these values, we have that

$$\delta^{(1)} = (1-\theta_0)\begin{pmatrix} b_1(\mu_1^{-1} - (b_1\mu_1^{-1} + b_2\mu_2^{-1})) \\ -b_1b_3\mu_3^{-1} \end{pmatrix}. \tag{99}$$

Generally (for all cases), in order to solve the system in equation (24), write

$$(\Lambda - \Gamma) = \begin{pmatrix} (b_1\mu_1^{-1} + b_2\mu_2^{-1})(1-(1-\theta_0)b_3) & (1-\theta_0)b_3(\mu_1^{-1}b_1 + b_2\mu_2^{-1}) \\ (1-\theta_0)b_3(1-b_3)\mu_3^{-1} & b_3\mu_3^{-1}(1-(1-\theta_0)(1-b_3)) \end{pmatrix}. \tag{100}$$

Invert this to get

$$(\Lambda - \Gamma)^{-1} = \frac{1}{\det\Gamma}\begin{pmatrix} b_3\mu_3^{-1}(1-(1-\theta_0)(1-b_3)) & -(1-\theta_0)b_3(\mu_1^{-1}b_1 + b_2\mu_2^{-1}) \\ -(1-\theta_0)b_3(1-b_3)\mu_3^{-1} & (b_1\mu_1^{-1} + b_2\mu_2^{-1})(1-(1-\theta_0)b_3) \end{pmatrix}. \tag{101}$$

The determinant is

$$\begin{aligned}
\det\Gamma =& (b_1\mu_1^{-1} + b_2\mu_2^{-1})(1-(1-\theta_0)b_3)b_3\mu_3^{-1}(1-(1-\theta_0)(1-b_3)) - \\
& (1-\theta_0)^2 b_3^2(1-b_3)\mu_3^{-1}(\mu_1^{-1}b_1 + b_2\mu_2^{-1})), \\
&= \Lambda_L\Lambda_K((1-(1-\theta_0)b_3)(1-(1-\theta_0)(1-b_3)) - (1-\theta_0)^2 b_3(1-b_3)) \\
&= \Lambda_L\Lambda_K(1-(1-\theta_0)b_3 - (1-\theta_0)(1-b_3) + (1-\theta_0)^2 b_3(1-b_3) - (1-\theta_0)^2 b_3(1-b_3)) \\
&= \Lambda_L\Lambda_K(1-(1-\theta_0)(b_3 + 1 - b_3)) \\
&= \Lambda_L\Lambda_K\theta_0.
\end{aligned}$$

Plug this back into (101) and simplify,

$$(\Lambda - \Gamma)^{-1} = \frac{1}{\Lambda_L\Lambda_K\theta_0}\begin{pmatrix} \Lambda_K(1-(1-\theta_0)(1-b_3)) & -(1-\theta_0)b_3\Lambda_L \\ -\Lambda_K(1-\theta_0)(1-b_3) & \Lambda_L(1-(1-\theta_0)b_3) \end{pmatrix}. \tag{102}$$

Returning to the specific case where $i = 1$,

$$(\Lambda - \Gamma)^{-1}\delta^{(1)} = \begin{pmatrix} \frac{(1-(1-\theta_0)(1-b_3))}{\theta_0\Lambda_L} & \frac{(\theta_0-1)b_3}{\Lambda_K\theta_0} \\ -\frac{(1-\theta_0)(1-b_3)}{\theta_0\Lambda_L} & \frac{1-(1-\theta_0)b_3}{\Lambda_K\theta_0} \end{pmatrix}\begin{pmatrix} b_1(\theta_0-1)(\mu_1^{-1} - \Lambda_L) \\ -b_1(\theta_0-1)\Lambda_K \end{pmatrix}. \tag{103}$$

Multiplying the values in (103), and using the identity in (24),

$$\begin{aligned}
\mathrm{d}\log\Lambda =& (\Lambda - \Gamma)^{-1}\delta^{(1)}, \\
&= \frac{b_1(\theta_0-1)}{\theta_0}\begin{pmatrix} (\frac{\mu_1^{-1}}{\Lambda_L} - 1)\theta_0 + b_3(1-\theta_0)\frac{\mu_1^{-1}}{\Lambda_L} + b_3(\theta_0-1) - b_3(\theta_0-1) \\ (\theta_0-1)(1-b_3)(\frac{\mu_1^{-1}}{\Lambda_L} - 1) - (1-(1-\theta_0)b_3) \end{pmatrix}.
\end{aligned}$$

Combine this with (3) gives

$$\begin{aligned}\mathrm{d}\log Y = b_1 - (1-b_3)b_1\frac{\theta_0-1}{\theta_0}\Big(\Big[\frac{\mu_1^{-1}}{\Lambda_L}-1\Big]\theta_0 + b_3(1-\theta_0)\frac{\mu_1^{-1}}{\Lambda_L}\Big)\\ - b_3 b_1\frac{\theta_0-1}{\theta_0}\Big[(\theta_0-1)(1-b_3)\Big(\frac{\mu_1^{-1}}{\Lambda_L}-1\Big) - (1-(1-\theta_0)b_3),\Big].\end{aligned}$$

which further simplifies to

$$\frac{\mathrm{d}\log Y}{\mathrm{d}\log A_1} = b_1 + b_1(\theta_0-1)\Big[1-(1-b_3)\frac{\mu_1^{-1}}{\Lambda_L}\Big]. \tag{104}$$

This gives the desired result for firm 1. Note, as mentioned before, a symmetric result holds for firm 2. In the case of firm 3, we have that

$$\delta^{(3)} = (\theta_0-1)\begin{pmatrix} -b_3(b_1\mu_1^{-1}+b_2\mu_2^{-1}) \\ b_3\mu_3^{-1}(1-b_3)\end{pmatrix}. \tag{105}$$

The results from (102) give the blueprint for solving for the value of $\frac{\mathrm{d}\log Y}{\mathrm{d}\log A_3}$ as well. From this, we can conclude that

$$\frac{d\log Y}{d\log A_3} = b_3 - \begin{pmatrix} b_1+b_2 \\ b_3\end{pmatrix}^T \begin{pmatrix} \frac{(1-(1-\theta_0)(1-b_3))}{\theta_0\Lambda_L} & \frac{(\theta_0-1)b_3}{\Lambda_K\theta_0} \\ -\frac{(1-\theta_0)(1-b_3)}{\theta_0\Lambda_L} & \frac{1-(1-\theta_0)b_3}{\Lambda_K\theta_0}\end{pmatrix}\begin{pmatrix} -b_3(b_1\mu_1^{-1}+b_2\mu_2^{-1}) \\ b_3\mu_3^{-1}(1-b_3)\end{pmatrix}, \tag{106}$$

which further simplifies to

$$= b_3 - \begin{pmatrix} 1-b_3 \\ b_3\end{pmatrix}^T \begin{pmatrix} \frac{b_3(1-\theta_0)}{\theta_0}\Big[1-(1-\theta_0)(1-b_3)\Big] + \frac{(\theta_0-1)^2}{\theta_0}b_3(1-b_3) \\ -\frac{(\theta_0-1)^2}{\theta_0}b_3(1-b_3) + \Big[1-(1-\theta_0)(1-b_3)\Big]\frac{(1-b_3)(\theta_0-1)}{\theta_0}\end{pmatrix}. \tag{107}$$

This simplifies to give the required result

$$\frac{d\log Y}{d\log A_3} = b_3. \tag{108}$$

■

Proof of Proposition 5.1. The industry level price is then given by

$$p_k = \left(\frac{1}{M_k}\int_{M_k}\left(c(w,p)\frac{\mu_k m_i}{A_k z_i}\right)^{1-\varepsilon_k}\Phi(z_i,m_i)\,\mathrm{d}\,m_i\,\mathrm{d}\,z_i\right)^{\frac{1}{1-\varepsilon_k}},$$
$$= c(w,p)\frac{\mu_k}{A_k}E\left(\left(\frac{m_i}{z_i}\right)^{1-\varepsilon_k}\right)^{\frac{1}{1-\varepsilon_k}}.$$

By Shephard's lemma, we know that,

$$\frac{\mathrm{d}\log p_i}{\mathrm{d}\log A_k} = -\mathbf{1}(i=k) + \sum_F \tilde{\alpha}_{iF}\frac{\mathrm{d}\log w_F}{\mathrm{d}\log A_k} + \sum_j \tilde{\Omega}_{ij}\frac{\mathrm{d}\log p_j}{\mathrm{d}\log A_k}, \tag{109}$$

where $\tilde{\alpha}_{iF}$ and $\tilde{\Omega}_{ij}$ are firm-level expenditures on factor F and industry j as a share of costs (excluding entry costs). Rewrite this in matrix form to get

$$\frac{\mathrm{d}\log p}{\mathrm{d}\log A_k} = (I-\tilde{\Omega})^{-1}(\tilde{\alpha}\frac{\mathrm{d}\log w}{\mathrm{d}\log A_k} - e_k) = \tilde{\Psi}(\tilde{\alpha}\frac{\mathrm{d}\log w}{\mathrm{d}\log A_k} - e_k), \tag{110}$$

where e_k is the kth standard basis vector. Let the household's aggregate consumption good to be the numeraire, so that the household's ideal price index P_c is always equal to one. Then we know that

$$\frac{\mathrm{d}\log P_c}{\mathrm{d}\log A_k} = b'\frac{\mathrm{d}\log p}{\mathrm{d}\log A_k} = 0. \tag{111}$$

Combine this with the previous expression to get

$$-b'\tilde{\Psi}e_k + b'\tilde{\Psi}\tilde{\alpha}\frac{\mathrm{d}\log w}{\mathrm{d}\log A_k} = 0. \tag{112}$$

Note that, $b'\tilde{\Psi} = \tilde{\lambda}$ and $b'\tilde{\Psi}\tilde{\alpha} = \tilde{\Lambda}$. Hence,

$$-\tilde{\lambda}_k + \tilde{\Lambda}'\frac{\mathrm{d}\log w}{\mathrm{d}\log A_k} = 0. \tag{113}$$

Now, note that

$$\Lambda_f = \frac{w_f L_F}{P_c C}. \tag{114}$$

From this, we know that

$$\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log A_k} = \frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k} + \frac{\mathrm{d}\log L_f}{\mathrm{d}\log A_k} - \frac{\mathrm{d}\log Y}{\mathrm{d}\log A_k}. \tag{115}$$

Substitute this into the previous expression to get

$$-\tilde{\lambda}_k + \tilde{\Lambda}'\frac{\mathrm{d}\log\Lambda}{\mathrm{d}\log A_k} - \tilde{\Lambda}'\frac{\mathrm{d}\log L_f}{\mathrm{d}\log A_k} + \frac{\mathrm{d}\log Y}{\mathrm{d}\log A_k} = 0, \tag{116}$$

where we use the fact that $\sum_f \tilde{\Lambda}_f = 1$. ■

Proof of Proposition 5.2. Let the cost function of entrant i in industry k producing y_i units be given by

$$c(w,p)h(y_i/A), \tag{117}$$

where A is industry level shock and k subscripts have been suppressed. Free entry then implies that

$$\mu c(w,p)h'(y/A)\frac{y}{A} - c(w,p)h(y/A) = \overline{f}c(w,p), \tag{118}$$

where $\overline{f}_k$ is the entry cost in units of the input good. We can simplify this to

$$\mu h'(y/A)\frac{y}{A} - h(y/A) = \overline{f}. \tag{119}$$

This equation pins down the efficient scale of operation $y = Ay^*(\overline{f},\mu)$.

The cost of producing q units of industry k output is then given by

$$nc(w,p)f(y^*(\mu,\overline{f})) + n\overline{f}c(w,p), \tag{120}$$

such that $ny = q$. Substitute the constraint into this to get

$$\mathfrak{C}(w,p)\frac{q}{A} = \frac{q}{y^*A}c(w,p)f(y^*(\mu,\overline{f})) + \frac{q}{Ay^*}\overline{f}c(w,p). \tag{121}$$

Hence the industry's cost function is linear q/A as needed.

Finally,

$$\frac{\partial\mathfrak{C}(w,p)q/A}{\partial w_j} = \frac{q}{y^*A}\frac{\partial c(w,p)}{\partial w_j}f(y^*(\mu,\overline{f})) + \frac{q}{Ay^*}\overline{f}\frac{\partial c(w,p)}{\partial w_j} = nl_k + n\overline{f}l_k^f. \tag{122}$$

Therefore, the industry level cost function obeys Shephard's Lemma and we can replicate the rest of the proof from Proposition 5.1 ■

Proof of Proposition 5.3.

$$\begin{aligned}\mathrm{d}\log p_k = &-\mathbf{1}(i=k)\,\mathrm{d}\log A_i + \sum_f \left(\int_{\overline{z}_k}^{\infty} s_k(z)\tilde{\alpha}_{kf}(z)\,\mathrm{d}\,F(z)\right)\mathrm{d}\log w_f \\ &+ \sum_j \left(\int_{\overline{z}_k}^{\infty} s_k(z)\tilde{\omega}_{kj}(z)\,\mathrm{d}\,F(z)\right)\mathrm{d}\log p_j + \left(\int_{\overline{z}_k}^{\infty} s_k(z)\tilde{\beta}_k(z)\,\mathrm{d}\log r_k(z)\,\mathrm{d}\,F(z)\right) \\ &- \frac{1}{1-\varepsilon_k} s_k(\overline{z}) f_k(\overline{z}_k)\,\mathrm{d}\log \overline{z}_k.\end{aligned}$$

where $s_k(z)$ is firm i's share of sales in industry k, and $\beta_k(z)$ is the cost share of firm i on its fictitious fixed factor. We can solve this system to get

$$\mathrm{d}\log p = (I-\tilde{\Omega})^{-1}\left(-e_i\,\mathrm{d}\log A_i + \tilde{\alpha}\,\mathrm{d}\log w + \xi - \kappa\right), \tag{123}$$

where $\xi_k = \left(\int_{\overline{z}_k}^{\infty} s_k(z)\tilde{\beta}_k(z)\,\mathrm{d}\log r_k(z)\,\mathrm{d}\,F(z)\right)$ and $\kappa_k = \frac{1}{1-\varepsilon_k} s_k(\overline{z}) f_k(\overline{z}_k)\,\mathrm{d}\log \overline{z}_k$, and the elements of $\tilde{\Omega}$ and $\tilde{\alpha}$ are defined appropriately. Let output be the numeraire so that

$$b'\,\mathrm{d}\log p = 0. \tag{124}$$

Hence,

$$-\tilde{\lambda}_i + \tilde{\Lambda}'\,\mathrm{d}\log w + \sum_k \tilde{\lambda}_k \xi_k - \sum_k \tilde{\lambda}_k \kappa_k = 0. \tag{125}$$

For each factor, we know that

$$\mathrm{d}\log w = \mathrm{d}\log \Lambda + \mathrm{d}\log Y. \tag{126}$$

Substitute this in to get
$-\tilde{\lambda}_i + \tilde{\Lambda}'\,\mathrm{d}\log \Lambda + \tilde{\Lambda}'\,\mathrm{d}\log Y + \sum_k \int_{\overline{z}_k}^{\infty} \tilde{\lambda}_k s_k(z)\tilde{\beta}_k(z)\,\mathrm{d}\log \Lambda_{r_k(z)}\,\mathrm{d}\,F(z) + \sum_k \int_{\overline{z}_k}^{\infty} \tilde{\lambda}_k s_k(z)\tilde{\beta}_k(z)\,\mathrm{d}\log Y - \sum_k \tilde{\lambda}_k \kappa_k = 0.$
Finally, observe that $\sum_f \tilde{\Lambda}_f + \sum_k \int_{\overline{z}_k}^{\infty} \tilde{\lambda}_k s_k(z)\tilde{\beta}_k(z) = 1$.

The proof for markup shocks is very similar.

■

Proposition B.1. *Consider an economy with a cash-in-advance constraint, and nominal rigidities.*

Then,

$$\mathrm{d}\log Y - \tilde{\Lambda}'\,\mathrm{d}\log L = \tilde{\lambda}'\,\mathrm{d}\log A - \tilde{\lambda}' e_s\,\mathrm{d}\log\mu + \mathrm{d}\,H(\tilde{\Lambda},\Lambda),$$

where

$$\mathrm{d}\log\mu = (e_s'\tilde{\Psi}e_s)^{-1}e_s'\tilde{\Psi}(\mathrm{d}\log A - \tilde{\alpha}\,\mathrm{d}\log w), \tag{127}$$

and

$$\mathrm{d}\log w = \mathrm{d}\log\Lambda + \mathrm{d}\log M - \mathrm{d}\log L.$$

In the special case where some fraction δ_i in industry i are flexible. Then,

$$\begin{aligned}\tilde{\lambda}_{(s)}'\,\mathrm{d}\log\mu &= \left(b'(I-\tilde{\Omega})^{-1} - b'(I-\delta\tilde{\Omega})^{-1}\delta\right)\mathrm{d}\log A - (1 - b'(I-\delta\Omega)^{-1}\delta\alpha)\,\mathrm{d}\log w, \\ &= \left(\tilde{\lambda} - b'(I-\delta\tilde{\Omega})^{-1}\delta\right)\mathrm{d}\log A - (1 - b'(I-\delta\Omega)^{-1}\delta\alpha)\,\mathrm{d}\log w.\end{aligned} \tag{128}$$

Proof of Proposition B.1. Order the producers so that the first s producers are the ones with sticky prices. For a vector x, denote $x_{(s)} = e_s'x$. From the cash in advance constraint, we know that

$$\begin{aligned}\mathrm{d}\log Y &= \mathrm{d}\log M - \mathrm{d}\log P_c, \\ &= \mathrm{d}\log M - b'\,\mathrm{d}\log p, \\ &= \mathrm{d}\log M - b'\tilde{\Psi}\,(\tilde{\alpha}\,\mathrm{d}\log w - \mathrm{d}\log A) + b'\tilde{\Psi}e_s\,\mathrm{d}\log\mu, \\ &= \mathrm{d}\log M - \tilde{\Lambda}'\,\mathrm{d}\log w + \tilde{\lambda}'\,\mathrm{d}\log A - \tilde{\lambda}e_s\,\mathrm{d}\log\mu, \\ &= \mathrm{d}\log M - \tilde{\Lambda}'\,(\mathrm{d}\log\Lambda + \mathrm{d}\log M - \mathrm{d}\log L) + \tilde{\lambda}'\,\mathrm{d}\log A - \tilde{\lambda}e_s\,\mathrm{d}\log\mu, \\ &= \mathrm{d}\log M - \tilde{\Lambda}'\,(\mathrm{d}\log\Lambda - \mathrm{d}\log L) - \mathrm{d}\log M + \tilde{\lambda}'\,\mathrm{d}\log A - \tilde{\lambda}e_s\,\mathrm{d}\log\mu,\end{aligned}$$

which implies that

$$\mathrm{d}\log Y - \tilde{\Lambda}'\,\mathrm{d}\log L = \tilde{\lambda}'\,\mathrm{d}\log A - \tilde{\lambda}_{(s)}'\,\mathrm{d}\log\mu - \tilde{\Lambda}'\,\mathrm{d}\log\Lambda. \tag{129}$$

To get the markups necessary to keep $p_{(s)}$ sticky, we impose

$$\mathrm{d}\log p_{(s)} = \mathrm{d}\log\mu + e_s'\tilde{\Omega}\log p - \mathrm{d}\log A_{(s)} = 0. \tag{130}$$

This implies

$$\mathrm{d}\log\mu = -e_s'\tilde{\Omega}\,\mathrm{d}\log p - \tilde{\alpha}_{(s)}\,\mathrm{d}\log w + \mathrm{d}\log A_{(s)}. \tag{131}$$

On the other hand, we have

$$\mathrm{d}\log p = \tilde{\Psi}\left(\tilde{\alpha}\,\mathrm{d}\log w - \mathrm{d}\log A\right) + \tilde{\Psi} e_s\,\mathrm{d}\log\mu. \tag{132}$$

Substituting this back into the previous expression gives

$$\mathrm{d}\log\mu = -e_s'\tilde{\Omega}\tilde{\Psi}\tilde{\alpha}\,\mathrm{d}\log w - e_s'\tilde{\Omega}\tilde{\Psi}e_s\,\mathrm{d}\log\mu + e_s'\tilde{\Omega}\tilde{\Psi}\,\mathrm{d}\log A - e_s'\tilde{\alpha}\,\mathrm{d}\log w + e_s'\,\mathrm{d}\log A. \tag{133}$$

Solve this to get

$$\begin{aligned}\mathrm{d}\log\mu &= (e_s'(I+\tilde{\Omega}\tilde{\Psi})e_s)^{-1}e_s'(I+\tilde{\Omega}\tilde{\Psi})\left(\mathrm{d}\log A - \mathrm{d}\log w\right),\\ &= (e_s'\tilde{\Psi}e_s)^{-1}e_s'\tilde{\Psi}\left(\mathrm{d}\log A - \mathrm{d}\log w\right).\end{aligned}$$

■

Proof of Proposition 4.1. The labor-leisure condition and the cash-in-advance condition imply that

$$L^{1/(\nu)} = \left(\frac{w}{P_c C}\right) = \left(\frac{w}{M}\right). \tag{134}$$

Hence,

$$\frac{1}{\nu}\,\mathrm{d}\log L = \mathrm{d}\log w - \mathrm{d}\log M = \mathrm{d}\log\Lambda - \mathrm{d}\log L + \mathrm{d}\log M - \mathrm{d}\log M, \tag{135}$$

or

$$\mathrm{d}\log L = \frac{\nu}{1+\nu}\,\mathrm{d}\log\Lambda. \tag{136}$$

Therefore,

$$\mathrm{d}\log w = \frac{1}{\nu+1}\,\mathrm{d}\log\Lambda + \mathrm{d}\log M \tag{137}$$

To finish, apply Propositions 3.1 and 3.2. ■

Proof of Proposition 4.2. With log utility in consumption and infinite Frisch elasticity of labor supply we have that

$$\mathrm{d}\log w - \mathrm{d}\log P_c = \mathrm{d}\log Y, \tag{138}$$

or in other words, substituting in the cash in advance constraint

$$\mathrm{d}\log w = \mathrm{d}\log M. \tag{139}$$

Furthermore, the cash in advance constraint implies that

$$\begin{aligned}
\mathrm{d}\log Y &= \mathrm{d}\log M - \mathrm{d}\log P_c, \\
&= \mathrm{d}\log M - \tilde{\Lambda}'\,\mathrm{d}\log w + \tilde{\lambda}'\,\mathrm{d}\log A - \tilde{\lambda}' e_s\,\mathrm{d}\log \mu, \\
&= \mathrm{d}\log M - \mathrm{d}\log M + \tilde{\lambda}'\,\mathrm{d}\log A - \tilde{\lambda}' e_s\,\mathrm{d}\log \mu, \\
&= \tilde{\lambda}'\,\mathrm{d}\log A - \tilde{\lambda}' e_s\,\mathrm{d}\log \mu,
\end{aligned}$$

substituting equation (127) from Proposition B.1 completes the proof

$$\begin{aligned}
&= \tilde{\lambda}'\,\mathrm{d}\log A - \tilde{\lambda}_{(s)}(e_s'\tilde{\Psi}e_s)^{-1}e_s'\tilde{\Psi}(\mathrm{d}\log A - \tilde{\alpha}\,\mathrm{d}\log w), \\
&= \tilde{\lambda}'\,\mathrm{d}\log A - \tilde{\lambda}_{(s)}(e_s'\tilde{\Psi}e_s)^{-1}e_s'\tilde{\Psi}(\mathrm{d}\log A - \tilde{\alpha}\,\mathrm{d}\log M).
\end{aligned}$$

In the special case where some fraction δ_i in industry i are flexible. Then,

$$\begin{aligned}
\tilde{\lambda}_{(s)}'\,\mathrm{d}\log \mu &= \left(b'(I-\tilde{\Omega})^{-1} - b'(I-\delta\tilde{\Omega})^{-1}\delta\right)\mathrm{d}\log A - (1 - b'(I-\delta\Omega)^{-1}\delta\alpha)\,\mathrm{d}\log w, \\
&= \left(\tilde{\lambda} - b'(I-\delta\tilde{\Omega})^{-1}\delta\right)\mathrm{d}\log A - (1 - b'(I-\delta\Omega)^{-1}\delta\alpha)\,\mathrm{d}\log w.
\end{aligned} \tag{140}$$

At the industry level, equation (140) shows that the changes in markups can be interpreted *as if* some fraction of the firms in each industry change their markup in response to shocks. ■

Proof of Proposition 5.4. On the other hand,

$$\frac{\mathrm{d}\log A}{\mathrm{d}\,\tau} = \sum_i s_i\left(\frac{1-\bar{e}_i}{\bar{e}_i}\right) = \sum_i s_i\,(\mu_i - 1) = \sum_i s_i\mu_i - 1 \geq 0. \tag{141}$$

On the other hand,

$$\frac{\mathrm{d}^2\log A}{\mathrm{d}\,\tau^2} = (\sigma-1)Var_s(\mu_i) - \sum_i s_i\,(\mu_i-1)^2 = (\sigma-2)Var_s(\mu_i) - \left(\sum_i s_i\mu_i - 1\right)^2. \tag{142}$$

Consider an industry where: all firms i use the same upstream input bundle with cost C; firms transform this input into a firm-specific variety of output using constant return to returns to scale technology; each firm i has productivity a_i and charges a markup μ_i; the varieties are combined into a composite good by a competitive downstream industry according to a CES production function with elasticity σ on firm i.

We denote the quantity of composite good produced as

$$Q = \left[\sum b_i^{\frac{1}{\sigma}} p_i^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}. \tag{143}$$

Firm i charges a price

$$p_i = \frac{\mu_i}{a_i} C. \tag{144}$$

The resulting demand for firm i's variety is

$$q_i = (\frac{p_i}{P})^{-\sigma} b_i Q, \tag{145}$$

where the price index is given by

$$P = \left[\sum b_i p_i^{1-\sigma}\right]^{\frac{1}{1-\sigma}}. \tag{146}$$

Total profits are given by

$$\Pi = \sum_i (p_i - C)(\frac{p_i}{P})^{-\sigma} b_i Q. \tag{147}$$

We solve out the price index and profits explicitly and get

$$P = \left[\sum_i b_i \left(\frac{\mu_i}{a_i}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} C, \tag{148}$$

$$\Pi = \sum_i \left(\frac{\mu_i}{a_i} - \frac{1}{a_i}\right) \left[\frac{\frac{\mu_i}{a_i}}{\left[\sum_j b_j \left(\frac{\mu_j}{a_j}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}\right]^{-\sigma} b_i C Q. \tag{149}$$

For completeness we can also solve for the sales of each firm as a fraction of the sales of the industry

$$\lambda_i = \frac{p_i q_i}{PQ} = \frac{b_i \left(\frac{\mu_i}{a_i}\right)^{1-\sigma}}{\sum_j b_j \left(\frac{\mu_j}{a_j}\right)^{1-\sigma}}. \tag{150}$$

We want to understand how to aggregate this industry into homogenous industry

with productivity A and markup μ. These variables must satisfy

$$P = \frac{\mu}{A} C, \tag{151}$$

$$\Pi = \left(\frac{\mu}{A} - \frac{1}{A}\right) CQ. \tag{152}$$

This implies that A and μ are the solutions of the following system of equations

$$\frac{\mu}{A} = \left[\sum_i b_i \left(\frac{\mu_i}{a_i}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}, \tag{153}$$

$$\left(\frac{\mu}{A} - \frac{1}{A}\right) = \sum_i \left(\frac{\mu_i}{a_i} - \frac{1}{a_i}\right) \left[\frac{\frac{\mu_i}{a_i}}{\left[\sum_j b_j \left(\frac{\mu_j}{a_j}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}\right]^{-\sigma} b_i. \tag{154}$$

The solution is

$$A = \frac{1}{\left[\sum_i b_i \left(\frac{\mu_i}{a_i}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} - \sum_i \left(1 - \frac{1}{\mu_i}\right) \left[\frac{\frac{\mu_i}{a_i}}{\left[\sum_j b_j \left(\frac{\mu_j}{a_j}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}\right]^{-\sigma} \frac{\mu_i}{a_i} b_i}, \tag{155}$$

$$\mu = \frac{\left[\sum_i b_i \left(\frac{\mu_i}{a_i}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}{\left[\sum_i b_i \left(\frac{\mu_i}{a_i}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} - \sum_i \left(1 - \frac{1}{\mu_i}\right) \left[\frac{\frac{\mu_i}{a_i}}{\left[\sum_j b_j \left(\frac{\mu_j}{a_j}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}\right]^{-\sigma} \frac{\mu_i}{a_i} b_i}. \tag{156}$$

We can also rewrite this in a useful way as

$$A = \frac{1}{\left[\sum_i b_i \left(\frac{\mu_i}{a_i}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}} \frac{1}{\sum_i \frac{1}{\mu_i} \frac{\left(\frac{\mu_i}{a_i}\right)^{1-\sigma} b_i}{\sum_j b_j \left(\frac{\mu_j}{a_j}\right)^{1-\sigma}}} = \frac{1}{\left[\sum_i b_i \left(\frac{\mu_i}{a_i}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}} \frac{1}{\sum_i \frac{1}{\mu_i} \lambda_i}, \tag{157}$$

$$\mu = \frac{1}{\sum_i \frac{1}{\mu_i} \frac{\left(\frac{\mu_i}{a_i}\right)^{1-\sigma} b_i}{\sum_j b_j \left(\frac{\mu_j}{a_j}\right)^{1-\sigma}}} = \frac{1}{\sum_i \frac{1}{\mu_i} \lambda_i}. \tag{158}$$

Define the efficiency of each firm i to be $e_i = \mu_i^{-1}$. Then, consider a steady state where $\sum_i s_i \mu_i^{-1} = \sum_i s_i \bar{e}_i = 1$. Consider a transformation $e_i = \tau + (1-\tau)\bar{e}_i$. This transformation keeps μ constant. On the other hand,

$$\frac{\mathrm{d} \log A}{\mathrm{d}\, \tau} = \sum_i s_i \left(\frac{1 - \bar{e}_i}{\bar{e}_i} \right) = \sum_i s_i \left(\mu_i - 1 \right) = \sum_i s_i \mu_i - 1 \geq 0. \tag{159}$$

On the other hand,

$$\frac{\mathrm{d}^2 \log A}{\mathrm{d}\, \tau^2} = (\sigma - 1) Var_s(\mu_i) - \sum_i s_i \left(\mu_i - 1 \right)^2 = (\sigma - 2) Var_s(\mu_i) - \left(\sum_i s_i \mu_i - 1 \right)^2. \tag{160}$$

■

C Relabelling

As mentioned previously, our results can be applied to any nested CES economy, with any arbitrary pattern of nested substitutabilities and complementarities among intermediate inputs and factors. For concreteness, we describe the relabeling for one specific example. Let the household's consumption aggregator be

$$\frac{C}{\overline{C}} = \left(\sum_k b_k \left(\frac{c_k}{\overline{c}_k} \right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}. \tag{161}$$

Suppose each producer k produces using a CES aggregator of value-added VA and intermediate inputs X:

$$\frac{y_k}{\overline{y}_k} = A_k \left(\alpha_k \left(\frac{VA_k}{\overline{VA}_k} \right)^{\frac{\theta_k - 1}{\theta_k}} + (1 - \alpha_k) \left(\frac{X_k}{\overline{X}_k} \right)^{\frac{\theta_k - 1}{\theta_k}} \right)^{\frac{\theta_k}{\theta_k - 1}}.$$

Value-added consists of different factors

$$\frac{VA_k}{\overline{VA}_k} = \left(\sum_{j=1}^{F} \nu_{kj} \left(\frac{l_{kj}}{\bar{l}_{kj}} \right)^{\frac{\eta_k - 1}{\eta_k}} \right)^{\frac{\eta_k}{\eta_k - 1}},$$

where l_{kj} is factor of type j used by k. Intermediate input consists of inputs from other producers

$$\frac{X_k}{\overline{X}_k} = \left(\sum_{j=1}^{N} \omega_{kj} \left(\frac{x_{kj}}{\bar{x}_{jk}} \right)^{\frac{\varepsilon_k - 1}{\varepsilon_k}} \right)^{\frac{\varepsilon_k}{\varepsilon_k - 1}}.$$

Denote the matrix of ν_{kj} using $\mathcal{V}$ and the matrix of ω_{kj} as Ω.

We rewrite this economy in the standard form we require, and put hats on the new expenditure shares. The new input-output matrix $\hat{\Omega}$ has dimension $(1+3N+F)\times(1+3N+F)$. The first industry is the household's consumption aggregator, the next N industries are the original industries, the next N industries produce the value-added of the original industries, the next N industries produce the intermediate inputs of the original industries, and the final F industries correspond to the factors.

Under the relabeling, we have

$$\hat{b} = (1, 0, \ldots, 0)',$$

$$\hat{\Omega} = \begin{pmatrix} \boldsymbol{b} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & diag(a) & diag(1-a) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{V} \\ \Omega & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{162}$$

with

$$\hat{\theta} = (\sigma, \boldsymbol{\theta}, \boldsymbol{\nu}, \boldsymbol{\varepsilon}, \mathbf{1}), \tag{163}$$

and

$$\hat{\mu} = (1, \boldsymbol{\mu}, 1, \ldots, 1). \tag{164}$$

D Aggregation of cost-based Domar weights

In this Appendix we show that recovering cost-based Domar weights from aggregated data is, in principle, not possible. We also show how the Basu-Fernald decomposition can detect changes in misallocation even in acyclic economies (where the equilibrium is efficient, and there is no possibility of reallocation of resources). Example 2.1 also shows the failure of the aggregation property implied by Hulten's theorem. The easiest way to see this is to consider aggregating the input-output table for the economy in Example 2.1. For simplicity, suppose that markups are the same everywhere so that $\mu_i = \mu$ for all i. Since there is no possibility of reallocation in this economy, and since markups are uniform, this is our best chance of deriving an aggregation result, but even in this simplest example, such a result does not exist. Suppose that we aggregate the whole economy $S = \{1, \ldots, N\}$. Then, in aggregate, the economy consists of a single industry that uses labor and inputs from itself to produce. In this case, the input-output matrix is a scalar, and equal to the intermediate input share of the economy

$$\Omega_{SS} = \frac{1 - \frac{1}{\mu^{N-1}}}{1 - \frac{1}{\mu^N}}, \tag{165}$$

and the aggregate markup for the economy is given by μ. Therefore, $\tilde{\lambda}_S$ constructed using aggregate data is

$$\tilde{\lambda}_S = 1'(I - \mu\Omega)^{-1} = \frac{\mu^{N-1} - \frac{1}{\mu}}{1 - \frac{1}{\mu}}. \tag{166}$$

However, we know from the example that

$$\frac{\mathrm{d}\log Y}{\mathrm{d}\log A} = \sum_{i \in S} \tilde{\lambda}_i = N \neq \tilde{\lambda}_S = \frac{\mu^{N-1} - \frac{1}{\mu}}{1 - \frac{1}{\mu}}, \tag{167}$$

except in the limiting case without distortions $\mu \to 1$. Therefore, even in this simplest case, with homogenous markups and no reallocation, aggregated input-output data cannot be used to compute the impact of an aggregated shock.

To compare our decomposition with that of Basu and Fernald (2002), consider the acyclic economy in Figure 13 with two factors L_1 and L_2. Now, we know that

$$\frac{\mathrm{d}\log Y}{\mathrm{d}\log A_2} = \tilde{\lambda}_2 = (1 - \alpha_1). \tag{168}$$

The Basu-Fernald decomposition for this economy gives

$$\frac{\mathrm{d}\log Y}{\mathrm{d}\log A_2} = \lambda_2 + R_M + \overline{\mu}^v R_M = \frac{1-\alpha_1}{\mu_1} + \overline{\mu}^v R_M, \tag{169}$$

where the first term (in this case, the sales-share of 2) is the "pure" technology effect, and R_M is the reallocation of intermediate inputs, even though in this economy, there is no capacity for reallocating resources and the equilibrium is efficient.

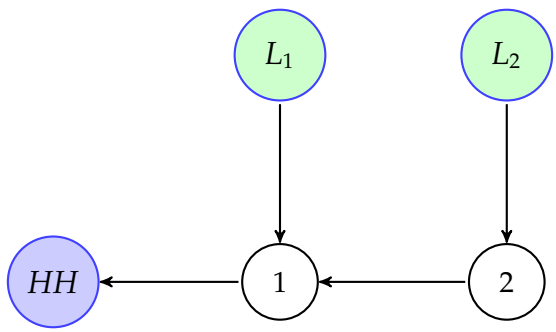

Figure 13: Acyclic economy where the solid arrows represent the flow of goods. The flow of profits and wages from firms to households has been suppressed in the diagram. The two factors in this economy are L_1 and L_2.

E Extra Examples

Example E.1. Next, consider the minimal example with two elasticities of substitution, which demonstrates the principle that changes in misallocation are driven by how each node switches its demand across its supply chain in response to a shock. To this end, we apply Proposition 3.1 to the economy depicted in Figure 14.

$$\begin{aligned}\frac{\mathrm{d}\log Y}{\mathrm{d}\log A_3} =& \tilde{\lambda}_3 - \frac{1}{\Lambda_L}\Big((\theta_0 - 1)\left[b_1(\omega_{13}\mu_1^{-1}(\omega_{13}\mu_3^{-1} + \omega_{14}\mu_4^{-1})) - \omega_{13}b_1\Lambda_L\right] \\ &+(\theta_1 - 1)\mu_1^{-1}\lambda_1\left[\omega_{13}\mu_3^{-1} - \omega_{13}\left(\omega_{13}\mu_3^{-1} + \omega_{14}\mu_4^{-1}\right)\right]\Big).\end{aligned}$$

The term multiplying $(\theta_0 - 1)$ captures how the household will shift their demand across 1 and 2 in response to the productivity shock, and the relative degrees of misallocation in 1 and 2's supply chains. The term multiplying $(\theta_1 - 1)$ takes into account how 1 will shift its demand across 3 and 4 and the relative amount of misallocation of labor between 3 and

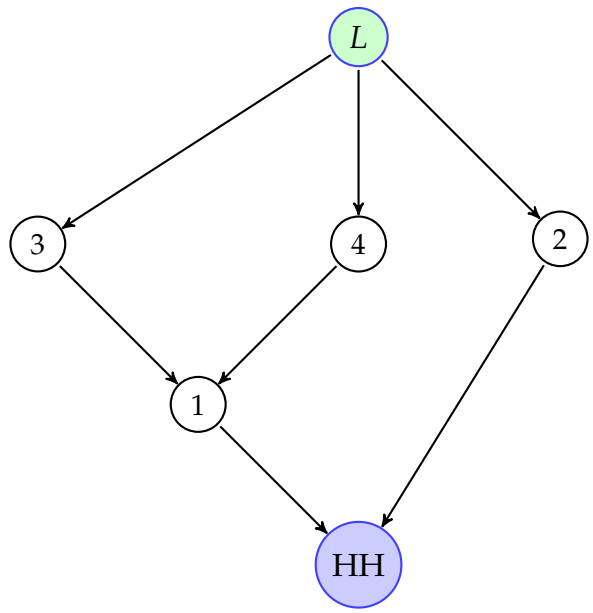

Figure 14: An economy with two elasticities of substitution.

4. Not surprisingly, if instead we shock industry 1, then only the household's elasticity of substitution matters, since industry 1 will not shift its demand across its inputs in response to the shock to industry 2:

$$\frac{\mathrm{d}\log Y}{\mathrm{d}\log A_1} = b_1 - \frac{1}{\Lambda_L}(\theta_0 - 1)\left[b_1\mu_1^{-1}(\omega_{13}\mu_3^{-1} + \omega_{14}\mu_4^{-1}) - b_1\Lambda_L\right].$$

This illustrates the general principle in Proposition 3.1 that an elasticity of substitution θ_j matters only if j is somewhere downstream from k.