Saving and investing for early retirement:
A theoretical analysis

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Abstract

We study optimal consumption and portfolio choice in a framework where investors adjust their labor supply through an irreversible choice of their retirement time. We show that investing for early retirement tends to increase savings and reduce an agent’s effective relative risk aversion, thus increasing her stock market exposure. Contrary to common intuition, an investor might find it optimal to increase the proportion of financial wealth held in stocks as she ages and accumulates assets, even when her income and the investment opportunity set are constant. The model predicts a decrease in risk aversion following strong market gains like those observed in the nineties.

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0. Introduction

Two years ago, when the stock market was soaring, 401(k)'s were swelling and (…) early retirement seemed an attainable goal. All you had to do was invest that big job-hopping pay increase in a market that produced double-digit gains like clockwork, and you could start taking leisurely strolls down easy street at the ripe old age of, say, 55. (Business Week December 31, 2001)

The dramatic rise of the stock market between 1995 and 2000 significantly increased the proportion of workers opting for early retirement (Gustman and Steinmeier, 2002). The above quote from Business Week demonstrates the rationale behind the decision to retire early: a booming stock market raises the amount of funds available for retirement and allows a larger fraction of the population to exit the workforce prematurely.

Indeed, for most individuals, increasing one’s retirement savings seems to be one of the primary motivations behind investing in the stock market. Accordingly, there is an increased need to understand the interactions among optimal retirement, portfolio choice, and savings, especially in light of the growing popularity of 401(k) retirement plans. These plans give individuals a great amount of freedom when determining how to save for retirement. However, such increased flexibility also raises concerns about the extent to which agents’ portfolio and savings decisions are rational. Having a benchmark against which to determine the rationality of people’s choices is crucial for both policy design and in order to form the basis of sound financial advice.

In this paper we develop a theoretical model with which we address some of the interactions among savings, portfolio choice, and retirement in a utility maximizing framework. We assume that agents face a constant investment opportunity set and a constant wage rate while still in the workforce. Their utility exhibits constant relative risk aversion and is nonseparable in leisure and consumption. The major point of departure from preexisting literature is that we model the labor supply choice as an optimal stopping problem: an individual can work for a fixed (nonadjustable) amount of time and earn a constant wage but is free to exit the workforce (forever) at any time she chooses. In other words, we assume that workers can work either full time or retire. As such, individuals face three decision problems: (1) how much to consume, (2) how to invest their savings, and (3) when to retire. The incentive to quit work comes from a discrete jump in their utility due to an increase in leisure once retired. When retired, individuals cannot return to the workforce.¹ We also consider two extensions of the basic framework. In the first extension we disallow the agent from choosing retirement past a pre-specified deadline. In a second extension we disallow her from borrowing against the net present value (NPV) of her human capital (i.e., we require that financial wealth be nonnegative).

The major results that we obtain can be summarized as follows:

First, we show that the agent will enter retirement when she reaches a certain wealth threshold, which we determine explicitly. In this sense, wealth plays a dual role in our model: not only does it determine the resources available for future consumption, but it also controls the “distance” to retirement.

Second, the option to retire early strengthens the incentives to save compared to the case in which early retirement is not allowed. The reason is that saving not only increases

¹This assumption can actually be easily relaxed. For instance, we could assume that retirees can return to the workforce (at a lower wage rate) without affecting any of the major predictions of the model.
consumption in the future but also brings retirement “closer.” Moreover, this incentive is wealth dependent. As the individual approaches the critical wealth threshold to enter retirement, the “option” value of retiring early becomes progressively more important and the saving motive becomes stronger.

Third, the marginal propensity to consume (MPC) out of wealth declines as wealth increases and early retirement becomes more likely. The intuition is simple: an increase in wealth will bring retirement closer, therefore decreasing the length of time the individual remains in the workforce. Conversely, a decline in wealth will postpone retirement. Thus, variations in wealth are somewhat counterbalanced by the behavior of the remaining NPV of income and in turn the effect of a marginal change in wealth on consumption becomes attenuated. Once again this attenuation is strongest for rich individuals who are closer to their goal of early retirement.

Fourth, the optimal portfolio is tilted more towards stocks compared to the case in which early retirement is not allowed. An adverse shock in the stock market will be absorbed by postponing retirement. Thus, the individual is more inclined to take risks as she can always postpone her retirement instead of cutting back her consumption in the event of a declining stock market. Moreover, in order to bring retirement closer, the most effective way is to invest the extra savings in the stock market instead of the bond market.

Fifth, the choice of portfolio over time exhibits some new and interesting patterns. We show that there exist cases in which an agent might optimally increase the percentage of financial wealth that she invests in the stock market as she ages (in expectation), even though her income and the investment opportunity set are constant. This result obtains, because wealth increases over time and hence the option of early retirement becomes more relevant. Accordingly, the tilting of the optimal portfolio towards stocks becomes stronger. Indeed, as we show in a calibration exercise, the model predicts that, prior to retirement, portfolio holdings could increase, especially when the stock market exhibits extraordinary returns as it did in the late 1990s during which time many workers experienced rapid increases in wealth, that allowed them to opt for an earlier retirement date. In fact our model suggests a possible partial rationalization of the (apparently irrational) behavior of individuals who increased their portfolios as the stock market was rising and then liquidated stock as the market collapsed.2

This paper is related to a number of strands in the literature that are surveyed in Ameriks and Zeldes (2001) and Jagannathan and Kocherlakota (1996). The paper closest to ours is that of Bodie et al. (1992) (henceforth BMS). The major difference between BMS and this paper is the different assumption we make about the ability of agents to adjust their labor supply. In BMS, labor can be adjusted in a continuous fashion. However, a significant amount of evidence suggests that labor supply is to a large extent indivisible. For example, in many jobs workers work either full time or they are retired. Moreover, it appears that most people do not return to work after they retire, or if they do, they return to less well-paying jobs or they work only part time. As BMS claim in the conclusion of their paper,

Obviously, the opportunity to vary continuously one’s labor without cost is a far cry from the workings of actual labor markets. A more realistic model would allow

2Some (indirect) evidence to this fact is given in the August 2004 Issue Brief of the Employee Benefit Research Institute (Fig. 2—based on the EBRI/ICI 401(k) Data).
limited flexibility in varying labor and leisure. One current research objective is to analyze the retirement problem as an optimal stopping problem and to evaluate the accompanying portfolio effects.

This is precisely the direction we take here. There are at least two major directions in which our results differ from BMS. First, we show that the optimal retirement decision introduces a nonlinear option-type element in the decision of the individual that is entirely absent if labor is adjusted continuously. Second, the horizon and wealth effects on portfolio and consumption choice in our paper are fundamentally different than those in BMS. For instance, stock holdings in BMS are a constant multiple of the sum of (financial) wealth and human capital. This multiple is not constant in our setup, but instead depends on wealth. Third, the model we present here allows for a clear way to model retirement, which is difficult in the literature that allows for a continuous labor-leisure choice. An important implication is that in our setup, we can calibrate the parameters of the model to observed retirement decisions. In the BMS framework, on the other hand, calibration to microeconomic data is harder because individuals do not seem to adjust their labor supply continuously.

The model is also related to a strand of the literature that studies retirement decisions. A partial listing includes Stock and Wise (1990), Rust (1994), Lazear (1986), Rust and Phelan (1997), and Diamond and Hausman (1984). Most of these models are structural estimations that are solved numerically. Here our goal is different: rather than include all the realistic ramifications that are present in actual retirement systems, we isolate and very closely analyze the new issues introduced by the indivisibility and irreversibility of the labor supply–retirement decision on savings and portfolio choice. Naturally, there is a trade-off between adding realistic considerations and the level of theoretical analysis that we can accomplish with a more complicated model. Other studies in this literature include Sundaresan and Zapatero (1997), who study optimal retirement, but in a framework without disutility of labor, and Bodie et al. (2004), who investigate the effects of habit formation, but without optimal retirement timing.

Some results of this paper share similarities with results that obtain in the literature on consumption and savings in incomplete markets. A highly partial listing includes Viceira (2001), Chan and Viceira (2000), Campbell et al. (2001), Kogan and Uppal (2001), Duffie et al. (1997), Duffie and Zariphopoulou (1993), Koo (1998), and Carroll and Kimball (1996) on the role of incomplete markets and He and Pages (1993) and El Karoui and Jeanblanc-Picqué (1998) on issues related to the inability of individuals to borrow against the NPV of their future income. This literature provides insights on why consumption (as a function of wealth) should be concave, and also offers some implications on portfolio choice. However, while in the incomplete markets literature, the results are driven by the inability of agents to effectively smooth their consumption due to missing markets, in this

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3If we impose a retirement deadline, this multiple also depends on the distance to this deadline.

4Liu and Neis (2002) study a framework similar to BMS, but force an important constraint on the maximal amount of leisure. This, however, omits the issues related to indivisibility and irreversibility, which as we show lead to fundamentally different implications for the resulting portfolios. In sum, the fact that labor supply flexibility is modeled in a more realistic way allows a closer mapping of the results to real-world institutions than is allowed for by a model that exhibits continuous choice between labor and leisure.

5Chan and Viceira (2000) combines insights of both literatures. However, they assume labor-leisure choices that can be adjusted continuously.
paper the results are driven by an option component in an agent’s choices that is related to
the ability of agents to adjust their time of retirement.

Throughout the paper we maintain the assumption that agents receive a constant wage. This is done not only for simplicity, but more importantly because it makes the results more surprising. It is well understood in the literature\(^6\) that allowing for a (positive) correlation between wages and the stock market can generate upward-sloping portfolio holdings over time. What we show is that optimal retirement choice can induce observationally similar effects even when labor income is perfectly riskless. Since the argument and the intuition for this outcome are orthogonal to those in existing models, we prefer to use the simplest possible setup in every other dimension, thereby isolating the effects of optimal early retirement.

Technically, our model extends methods proposed by Karatzas and Wang (2000) (who do not allow for income) to solve optimal consumption problems with discretionary stopping. The extension that we consider in Section 3 uses ideas proposed by Barone-Adesi and Whaley (1987), and in Section 5, we extend the framework in He and Pages (1993) to allow for early retirement.

Finally, three papers that present parallel and independent work on similar issues are Lachance (2003), Choi and Shim (2004), and Dybvig and Liu (2005). Lachance (2003) and Choi and Shim (2004) study a model with a utility function that is separable in leisure and consumption, but that abstracts from a deadline for retirement and/or borrowing constraints.\(^7\) The somewhat easier specification of separable utility does not allow consumption to fall upon retirement as we observe in the data. Technically, these papers solve the problem using dynamic programming rather than convex duality methods, which cannot be easily extended to models with deadlines, borrowing constraints, etc. Our approach overcomes these difficulties. Dybvig and Liu (2005) study a very similar model to that in Section 5 of this paper, with similar techniques. However, they do not consider retirement prior to a deadline as we do. A deadline makes the problem considerably harder (since the critical wealth thresholds become time dependent). Nonetheless, we are able to provide a fairly accurate approximate closed-form solution for this problem in Section 3. One can actually perform simple exercises that demonstrate that in the absence of a retirement deadline, the model-implied distribution of retirement times becomes implausible. Most importantly, compared to the papers above, the present paper goes into significantly greater detail in terms of the economic analysis and implications of the results. In particular, we provide applications (like the analysis of portfolios of agents saving for early retirement in the late 1990s) that demonstrate quite clearly the real-world implications of optimal portfolio choice in the presence of early retirement.

The structure of the paper is as follows: Section 1 contains the model setup. In Section 2 we describe the analytical results if one places no retirement deadline. Section 3 contains an extension to the case in which retirement cannot take place past a deadline, Section 4 contains some calibration exercises, and Section 5 extends the model by imposing borrowing constraints. Section 6 concludes. We present technical details and all proofs in the appendix.

\(^6\)See, e.g., Jagannathan and Kocherlakota (1996) and BMS.

\(^7\)Another model that makes similar assumptions is that of Kingston (2000).
1. Model setup

1.1. Investment opportunity set

The consumer can invest in the money market, where she receives a fixed, strictly positive interest rate \( r > 0 \). We place no limits on the positions that can be taken in the money market. In addition, the consumer can invest in a risky security with a price per share that evolves according to

\[
\frac{dP_t}{P_t} = \mu dt + \sigma dB_t,
\]

where \( \mu > r \) and \( \sigma > 0 \) are known constants and \( B_t \) is a one-dimensional Brownian motion on a complete probability space \((\Omega, F, P)\).\(^{8}\) We define the state-price density process (or stochastic discount factor) as

\[
H(t) = \mathcal{E}(t)Z^*(t), \quad H(0) = 1,
\]

where \( \mathcal{E}(t) \) and \( Z^*(t) \) are given by

\[
\mathcal{E}(t) = e^{-rt},
\]

\[
Z^*(t) = \exp\left\{-\int_0^t \kappa dB_s - \frac{1}{2} \kappa^2 t\right\}, \quad Z^*(0) = 1
\]

and \( \kappa \) is the Sharpe ratio

\[
\kappa = \frac{\mu - r}{\sigma}.
\]

As is standard, these assumptions imply a dynamically complete market (Karatzas and Shreve, 1998, Chapter 1).

1.2. Portfolio and wealth processes

An agent chooses a portfolio process \( \pi_t \) and a consumption process \( c_t > 0 \). These processes are progressively measurable and they satisfy the standard integrability conditions given in Karatzas and Shreve (1998) Chapters 1 and 3. The agent also receives a constant income stream \( y_0 \) while she works and no income stream while in retirement. Retirement is an irreversible decision. We assume until Section 3 that an agent can retire at any time she chooses.

The agent is endowed with an amount of financial wealth \( W_0 \geq -y_0/r \). The process of stockholdings \( \pi_t \) is the dollar amount invested in the risky asset (the “stock market”) at time \( t \). The amount \( W_t - \pi_t \) is therefore invested in the money market. Short selling and borrowing are both allowed. We place no extra restrictions on the (financial) wealth process \( W_t \) until Section 5 of the paper. Additionally, in Section 5 we will impose the restriction \( W_t \geq 0 \). As long as the agent is working, the wealth process evolves according to

\[
dW_t = \pi_t(\mu dt + \sigma dB_t) + (W_t - \pi_t)r dt - (c_t - y_0) dt.
\]

\(^{8}\)We shall denote by \( F = \{F_t\} \) the \( P \)-augmentation of the filtration generated by \( B_t \).
Applying Ito’s Lemma to the product of $H(t)$ and $W(t)$, integrating, and taking expectations, we get for any stochastic time $\tau$ that is finite almost surely
\[
E\left( H(\tau)W(\tau) + \int_0^\tau H(s)[c(s) - y_0] \, ds \right) \leq W_0.
\] (2)

This is the well-known result that in dynamically complete markets one can reduce a dynamic budget constraint of the type in Eq. (1) to a single intertemporal budget constraint of the type in Eq. (2). If the agent is retired, the above two equations continue to hold with $y_0 = 0$.

### 1.3. Leisure, income, and the optimization problem

To obtain closed-form solutions, we assume that the consumer has a utility function of the form
\[
U(l_t, c_t) = \frac{1}{\alpha} \left( \frac{l_t^{1-\alpha} c_t^\gamma}{1 - \gamma^*} \right)^{1-\gamma}, \quad \gamma^* > 0,
\] (3)
where $c_t$ is per-period consumption, $l_t$ is leisure, and $0 < \alpha < 1$. We assume that the consumer is endowed with $\ell$ units of leisure. Leisure can only take two values, $l_1$ or $\ell$: if the consumer is working, $l_t = l_1$; if the consumer is retired $l_t = \ell$. We assume that the wage rate $w$ is constant, so that the income stream is $y_0 = w(\ell - l_1) > 0$. We normalize $l_1 = 1$. Note that this utility is general enough so as to allow consumption and leisure to be either complements ($\gamma^* < 1$) or substitutes ($\gamma^* > 1$). The consumer maximizes expected utility
\[
\max_{c_t, \pi_t, \tau} E \left[ \int_0^\tau e^{-\beta t} U(l_t, c_t) \, dt + e^{-\beta \tau} \int_\tau^\infty e^{-\beta(t-\tau)} U(l_t, c_t) \, dt \right],
\] (4)
where $\beta > 0$ is the agent’s discount factor. The easiest way to proceed is to start backwards by solving the problem
\[
U_2(W_\tau) = \max_{c_t, \pi_t} E \left[ \int_\tau^\infty e^{-\beta(t-\tau)} U(l_t, c_t) \, dt \right],
\]
where $U_2(W_\tau)$ is the value function once the consumer decides to retire and $W_\tau$ is the wealth at retirement. By the principle of dynamic programming we can rewrite (4) as
\[
\max_{c_t, W_\tau, \tau} E \left[ \int_0^\tau e^{-\beta t} U(l_t, c_t) \, dt + e^{-\beta \tau} U_2(W_\tau) \right].
\] (5)

It will be convenient to define the parameter $\gamma$ as
\[
\gamma = 1 - \alpha(1 - \gamma^*)
\]
so we can then reexpress the per-period utility function as
\[
U(l_t, c_t) = l_t^{(1-\alpha)(1-\gamma^*)} c_t^{\frac{1-\gamma}{1-\gamma^*}}.
\]

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9By standard arguments the constant discount factor $\beta$ could also incorporate a constant hazard rate of death, $\lambda$. 
Since we have normalized $l = 1$ prior to retirement, the per-period utility prior to retirement is given by

$$U_1(c) = U(1, c) = \frac{c^{1-\gamma}}{1-\gamma}. \quad (6)$$

Notice that $\gamma > 1$ if and only if $\gamma^* > 1$, and $\gamma < 1$ if and only if $\gamma^* < 1$. Under these assumptions, it follows from standard results (see, e.g., Karatzas and Shreve, 1998, Chapter 3), that once in retirement, the value function becomes

$$U_2(W_t) = (l^{1-\gamma})^{\gamma^*} \left(\frac{1}{\theta}\right)^{\gamma} \frac{W_t^{1-\gamma}}{1-\gamma}, \quad (7)$$

where

$$\theta = \gamma - 1 \left(\frac{r + \frac{\kappa^2}{2\gamma}}{\gamma}\right) + \frac{\beta}{\gamma}.$$

In order to guarantee that the value function is well defined, we assume throughout that $10 \theta > 0$ and $\beta - r < \kappa^2/2$.\(^{11}\) It will be convenient to redefine the continuation value function as

$$U_2(W_t) = K \frac{W_t^{1-\gamma}}{1-\gamma},$$

where

$$K = (l^{1-\gamma})^{\gamma^*} \left(\frac{1}{\theta}\right)^{\gamma}. \quad (8)$$

Since $l > l_1 = 1$, it follows that

$$K^{1/\gamma} > \frac{1}{\theta} \quad \text{if} \quad \gamma < 1 \quad (9)$$

$$K^{1/\gamma} < \frac{1}{\theta} \quad \text{if} \quad \gamma > 1. \quad (10)$$

2. Properties of the solution

Theorem 1 in the appendix presents a formal solution to the problem. The nature of the solution is intuitive: the agent enters retirement if and only if the level of her assets exceeds a critical level $\bar{W}$, which we analyze more closely in Section 2.1. As might be expected, another feature of the solution is that the agent’s marginal utility of consumption equals the stochastic discount factor, both pre- and post-retirement (up to a constant $\lambda^*$, which depends on the wealth of the agent at time 0 and is chosen so that the intertemporal budget

\(^{10}\)Observe that this is guaranteed if $\gamma > 1$.

\(^{11}\)As we show in the Appendix, this will guarantee that retirement takes place with probability one in this stochastic setup.
constraint is satisfied):

\[ e^{-\beta t} U_C(l_t, c_t) = e^{-\beta t} l_t^{1-2(1-\gamma)} c_t^{-\gamma} = \lambda^* H(t). \]  

(11)

This is just a manifestation of the fact that the market is dynamically complete.\(^{12}\)

Importantly, the marginal utility of consumption is continuous when the agent enters retirement. This is a consequence of a principle in optimal stopping that is known as “smooth pasting,” which implies that the derivative of the value function \( J_W \) is continuous. When smooth pasting is combined with the standard envelope theorem, that is,

\[ U_C(l_t, c_t) = J_W, \]  

(12)

it follows that the marginal utility of consumption \( (U_C) \) is continuous. A consequence of the continuity of the marginal utility of consumption is that consumption itself will jump when the agent enters retirement. This is simply because consumption needs to “counteract” the discrete change in leisure, which enters the marginal utility of consumption in a nonseparable way. The jump is given by

\[ \frac{c_t^+}{c_t^-} = l_t^{(1-2)(1-\gamma)/\gamma} = K^{1/\gamma} \theta. \]  

(13)

Notice that \( \gamma^* > 1 \) will imply a downward jump and \( \gamma^* < 1 \) an upward jump (since \( \tilde{l} > 1 \)). For the empirically relevant case \( (\gamma^* > 1) \), the model predicts a downward jump in consumption, consistent with the data.

In the next three subsections we explore some properties of the solution in more detail. The benchmark model against which we compare our results is a model in which there is a constant labor income stream and no retirement (the worker works forever). This is the natural benchmark for this section, since it keeps all else equal except for the option to retire. The results we obtain in this section allow us to isolate insights related to optimal retirement in a framework in which solutions are not time dependent and therefore are easier to analyze. Fortunately, all of the results continue to hold when we introduce a retirement deadline in Section 3, in which case the natural benchmark model will be one in which the agent is forced to work for a fixed amount of time, which is more natural.

### 2.1. Wealth at retirement

For a constant \( \xi_2 \), where

\[ \xi_2 = \frac{1 - 2(\beta - r)/\kappa^2 - \sqrt{(1 - 2(\beta - r)/\kappa^2)^2 + 8\beta/\kappa^2}}{2}. \]

Theorem 1 gives wealth at retirement as

\[ \overline{W} = \frac{(\xi_2 - 1)K^{(1/\gamma)} \theta}{(1 + \xi_2\gamma/(1 - \gamma)(K^{(1/\gamma)} \theta - 1)) \gamma_0}v_0. \]  

(14)

As Theorem 1 asserts, for wealth levels higher than \( \overline{W} \), it is optimal to enter retirement, whereas for lower wealth levels, it is optimal to remain in the workforce. In the Appendix we show that \( \overline{W} \) is strictly positive, i.e., a consumer will never enter retirement with

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\(^{12}\)See the monograph of Karatzas and Shreve (1998), Chapters 3 and 4.
negative wealth since there is no more income to support post-retirement consumption. It is clear that the critical wealth level $W$ does not depend on the initial wealth of the consumer. The discrete decision between work and retirement is a choice variable that by standard dynamic programming depends only on the current state variable of the system, namely, the current level of wealth. If the current level of wealth is above $W$, retirement is triggered, otherwise it is not. However, it is still true that agents who start life with a higher level of wealth are more likely to hit the retirement threshold in a shorter amount of time. Potentially, an agent might be born with a wealth level sufficiently above $W$ that she can retire immediately.

To understand the forces behind the determination of $W$, we sketch the basic idea behind the derivation of (14). The appendix demonstrates that one can reduce the entire consumption–portfolio–retirement timing problem to a standard optimal stopping problem. After reducing the problem to an optimal stopping problem, one can use well-known intuitions from option valuation. Specifically, retirement can be viewed as an American-type option that allows one to exchange the value of future income with the extra leisure that is brought about by retiring. In this section the option has infinite maturity; in Section 3 its maturity is finite. The key idea behind transforming the problem into a standard optimal timing problem is not to use the level of wealth as the state variable, but instead its marginal value, $J_W$. We can see the advantage of doing so by combining (11) and (12) to obtain

$$J_W = \lambda^* e^{\beta t} H(t).$$

The right-hand side of this equation is exogenous (up to a constant that is chosen so as to satisfy the intertemporal budget constraint). By contrast, the evolution of wealth itself depends on both optimal consumption and portfolio choice, and thus both the drift and the volatility of the wealth process are endogenous.

The next step is to define $Z_t = J_W = \lambda^* e^{\beta t} H(t)$ as a state variable and pose the entire problem as a standard optimal stopping problem in terms of $Z_t$. It is straightforward to show that

$$\frac{dZ_t}{Z_t} = (\beta - r) dt - \kappa dB_t,$$

and thus that $Z_t$ follows a standard geometric Brownian motion with volatility equal to the Sharpe ratio ($\kappa$) and drift equal to the difference between the discount rate and the interest rate ($\beta - r$). To complete the analogy with option pricing, it remains to determine the net payoff from exercising the option of early retirement as a function of $Z_t$. The key difficulty in achieving this is that the cost of forgone income is expressed in monetary terms, while the benefit of extra leisure is in utility terms. The appendix shows that the correct notion of net benefit is the difference in “consumer surplus” enjoyed by an agent who is not working versus someone who is, assuming that the marginal value of wealth is the same for both. In the Appendix we show that this computation leads to the following net payoff upon retirement:

$$Z_t \left[ \frac{1}{\theta (1 - \gamma)} (K^{1/\gamma} - 1) Z_t^{-1/\gamma} - \frac{D_0}{r} \right].$$

The term $Z_t$ outside the square brackets is equal to the marginal value of wealth upon retirement and hence transforms monetary units into marginal utility units. The second
term inside the square brackets \((-y_0/r)\) is negative, capturing the permanent loss of the net present value of income, and can be thought of as the “strike” price of the option.

The first term inside square brackets is always positive and captures the payoff of the option. To analyze this term, it is probably easiest to use the following relation, which we show in the appendix,

\[
\overline{W} = K^{1/\gamma} Z_t^{-1/\gamma}
\]

so that we can rewrite the payoff once the option is exercised as

\[
Z_t \left[ \frac{\gamma (K^{1/\gamma} \theta - 1)}{1 - \gamma} \overline{W} - \frac{y_0}{r} \right].
\]

The first term inside the square brackets now has an intuitive interpretation: it is the monetary equivalent of obtaining the leisure level \(\overline{l}\). Schematically speaking, going into retirement is “as if” the wealth of the agent is increased by \(\frac{\gamma (K^{1/\gamma} \theta - 1)}{1 - \gamma} \overline{W}\) at the fixed cost \(y_0/r\). Summarizing, in order to obtain the optimal retirement time, it suffices to solve the optimal stopping problem

\[
\sup_{\tau} \mathbb{E} \left\{ e^{-\beta \tau} Z_t \left[ \frac{\gamma (K^{1/\gamma} \theta - 1)}{1 - \gamma} \overline{W} - \frac{y_0}{r} \right] \right\}.
\]

There are many direct consequences of this option interpretation. For instance, a standard intuition in optimal stopping is that an increase in the “payoff” of the option upon exercise will increase the opportunity cost of waiting and will make the agent exercise the option earlier. A consequence of Eq. (15) is that an increase in \(y_0\) will reduce the payoff of “immediate exercise” and hence will push the critical retirement wealth upward. Indeed, Eq. (14) demonstrates a linear relation between \(\overline{W}\) and \(y_0\). This homogeneity of degree one shows that one can express the target wealth at retirement in terms of multiples of current income, and suggests the normalization \(y_0 = 1\), which we adopt in all quantitative exercises.

Furthermore, Eq. (15) allows us to perform comparative statics with respect to an agent’s disutility of labor. Assume that \(\gamma^*\) is kept fixed and is larger than one for simplicity.\(^{13}\) Assume now that we increase \(\overline{l}\) so that \(K^{1/\gamma}\) decreases (by Eq. (13) and the fact that \(\overline{l} > 1\)). The value of immediate exercise in Eq. (15) will then increase (since \(\gamma > 1\)) and the agent will decide to enter retirement sooner. This is intuitive: for agents who work longer hours for the same pay, the relative increase in leisure upon retirement \((\overline{l})\) is larger and hence retirement is more attractive, all else equal. Similar comparative statics follow for variations in the relative importance of leisure in the utility function \((\alpha)\). Another standard intuition from option pricing is that increases in the volatility of the underlying state variable (the Sharpe ratio \(\kappa\) in our case) will lead to postponement of exercise (retirement in our case).

Interestingly, there is a direct link between the change in consumption upon retirement and the critical level of wealth, \(\overline{W}\). Note that by combining (13) and (14), we obtain the

\(^{13}\) Similar arguments can be given when \(\gamma^* < 1\).
following relation:

\[
\mathbb{W} = \frac{(\xi_2 - 1)c_{r+}/c_{r-} - y_0}{(1 + \xi_2 \gamma/(1 - \gamma))(c_{r+}/c_{r-} - 1) r}.
\]  

(16)

Thus, assuming \( \gamma > 1 \), lower threshold levels of the critical wealth will be associated with larger decreases in consumption. In the empirical literature this correlation between low levels of wealth at retirement and large decreases in consumption is seen as evidence that workers do not save enough for retirement. Our rational framework suggests the alternative explanation that this correlation is simply the result of preference heterogeneity: agents who value leisure a lot will be willing to absorb larger decreases in their consumption upon retirement (since leisure and consumption enter nonseparably in the utility function) and will have lower levels of retirement-triggering wealth. In option jargon, the payoff of immediate exercise will be too large, as will the opportunity cost of waiting. Hence, low levels of wealth upon entering retirement and large decreases in consumption are merely two manifestations of the same economic force, namely, a stronger preference for leisure.

2.2. Optimal consumption

We concentrate on a consumer with wealth lower than \( \mathbb{W} \), that is, a consumer who has an incentive to continue working. The following proposition characterizes the optimal consumption behavior of the consumer.

**Proposition 1.** Assume that \( W_t < \mathbb{W} \), so that the agent has not retired yet. Let \( c_t \) be the optimal consumption process, and let \( c_t^B \) denote optimal consumption in the benchmark model, in which the consumer has no option of retirement. Then:

(i) Consumption prior to retirement is lower compared to the benchmark case: \( c_t < c_t^B \).

(ii) The marginal propensity to consume out of wealth, \( \partial c_t/\partial W_t \), is a declining function of \( W_t \). By contrast the marginal propensity to consume out of wealth is constant and equal to \( \theta \) in the benchmark case.

The intuition for the first assertion is straightforward: the desire to attain retirement incentivizes the agent to save and accumulate assets compared to the benchmark case. This explains part (i) of the proposition.

Fig. 1 illustrates part (ii) of the proposition. In the standard Merton (1971) framework (with or without an income stream), the marginal propensity to consume out of wealth is fixed at \( \theta \). However, in the present model the marginal propensity to consume approaches \( \theta \) asymptotically as wealth goes to the lowest allowable level, namely, \(-y_0/r \). It declines between \(-y_0/r \) and \( \mathbb{W} \) and then jumps to \( \theta \) when wealth exceeds the retirement threshold, \( \mathbb{W} \), so that the agent enters retirement.

The best way to understand why the marginal propensity to consume is not constant, but rather declining, is to consider the following through experiment. Suppose that we decrease the wealth of an agent by an amount \( x \) prior to retirement, due, e.g., to an unexpected stock market crash. In our framework this will have two effects. First, it will reduce the agent’s total resources and hence will lead to a consumption cutback, as in the standard Merton framework. Second, it will distanciate the agent from the threshold level of wealth that is required to attain retirement. As a result, the agent can now expect to remain in the workforce
longer and thus she will have the opportunity to partially recoup the loss of \( x \) units of wealth by the net present value of the additional income. In short, part of the wealth shock is absorbed by postponing retirement and thus the effect on consumption is moderate.

Naturally, one would expect this effect to be strongest when the distance to the retirement threshold is small (i.e., for wealth levels close to \( W \)). By contrast, when the option of early retirement is completely “out of the money” (for instance, when wealth is close to \(-y_0/r\)), then marginal changes in wealth will have almost no impact on the net present value of future income and thus the marginal propensity to consume will asymptote to \( y_0/r \) as \( W_t \rightarrow -y_0/r \) in Fig. 1.

Of course, if \( W_t > W \), the agent enters retirement and the usual affine relation between consumption and wealth prevails, as is common in Merton-type setups. The marginal propensity to consume is constant at \( y_0/r \) since all of the adjustment to wealth shocks goes through consumption.

It is important to note that the key to these results is not the presence of labor supply flexibility per se, but the irreversibility of the retirement decision along with the indivisibility of labor supply. To substantiate this claim, assume that the agent never retires and that her leisure choice is determined optimally on a continuum at each point in time, so that \( l_t + h_t = l \), where \( h_t \) are the hours devoted to work, and the instantaneous income is \( w h_t \), with \( w \) defined as in Section 1.3. The solution for optimal consumption that one obtains in such a framework\(^\text{14}\) with perfect labor supply flexibility is

\[
c_t = T_1 (W_t + \frac{y_0}{r} C_2)
\]

---

\(\text{14}\)See BMS for a proof. Liu and Neis (2002) impose the constraint \( h_t \geq 0 \) and obtain different results. It is interesting to note that in the framework of Liu and Neis (2002), an individual starts losing labor supply flexibility as she approaches the constraint \( h_t = 0 \). Hence, she effectively becomes more risk averse. In our framework this is true only post-retirement. Pre-retirement, the individual exposes herself to more risk because this is the only way in which she can accelerate retirement. This shows that taking indivisibility and irreversibility into account, the properties of the solution are fundamentally different.
for two appropriate constants \( C_1 \) and \( C_2 \). Notice the simple affine relation between wealth and consumption. These results show an important direction in which the present model sheds some new insights, beyond existing frameworks, into the relations among retirement, consumption, and portfolio choice. In particular, under endogenous retirement, wealth has a dual role. First, as in all consumption and portfolio problems, it controls the amount of resources that are available for future consumption. Second, it controls the distance to the threshold at which retirement is optimal. It is this second channel that is behind the behavior of the marginal propensity to consume that we analyze above.\(^{15}\)

### 2.3. Optimal portfolio

The following proposition gives an expression for the holdings of stock.

**Proposition 2.** Prior to retirement, the holdings of stock are given by

\[
\pi_t = \frac{\kappa}{\sigma \gamma} \left( W_t + \frac{y_0}{r} \right) + \left( \frac{J_w(W_t)}{J_w(W)} \right)^{\xi_2 - 1} \frac{\kappa y_0}{\sigma r} \frac{\xi_2}{\gamma} \left( (\xi_2 - 1) + \frac{1}{\gamma} \right) \left[ \frac{\gamma}{1 - \gamma} \left( \frac{\xi_2 - 1}{1 + \xi_2 \frac{\gamma}{1 - \gamma}} \right) - 1 \right].
\]

The second term in (17) is always

(i) positive, and
(ii) increasing in \( W_t \).

**Post-retirement, the optimal holdings of stock are given by the familiar Merton formula:**

\[
\pi_t = \frac{\kappa}{\sigma \gamma} W_t.
\]

In simple terms, assertion (i) in Proposition 2 implies that the possibility of retirement raises an agent’s appetite to take risk (compared to an infinitely lived Merton investor without the option to retire). The intuition is straightforward: first, the ability to adjust the duration of work effectively hedges the agent against stock market variations. Second, the possibility to attain the extra utility associated with more leisure raises the agent’s willingness to accept more risk.

Fig. 2 provides a graphical illustration of the first assertion, sketching the following three value functions: (a) the value function of a Merton problem wherein the agent has to work forever; (b) the value function of an agent who is already retired; and (c) the value function of the problem involving optimal retirement choice. As we can see, the third value function

---

\(^{15}\)The concavity of the consumption function is also a common result in models that combine non-spanned income and/or borrowing constraints of the form \( W_t \geq 0 \) (e.g., Carroll and Kimball, 1996). A quite important difference between these models and the one we consider here is that in the present model, the effects of concavity are most noticeable for high levels of wealth and not for low wealth levels. In our model the MPC asymptotes to \( \theta \) as \( W_t \rightarrow -y_0/r \), and declines from there to the point where \( W_t = W \). It then jumps back up to \( \theta \), reflecting the loss of the real option associated with remaining in the workforce. By contrast, in models such as Koo (1998) or Duffie et al. (1997) the MPC is above \( \theta \) for low levels of wealth and asymptotes to \( \theta \) as \( W_t \rightarrow \infty \). We discuss this issue further in Section 5, where we introduce borrowing constraints.
looks like an “envelope” of the other two functions which are “more” concave. This implies that relative risk aversion will be lower and hence holdings of stock will be higher in the presence of optimal retirement choice. The value function of the problem involving optimal choice of retirement asymptotes to the first value function as \( \frac{W}{C_0} \approx r \) (i.e., the option of early retirement becomes completely worthless) and it coincides with the second value function when \( W \approx W_{NR} \) (i.e., when the agent enters retirement).

Assertion (ii) in Proposition 2 is driven by a separate intuition. To see why it holds, it is easiest to think of a fictitious asset, namely, a barrier option that could be used to finance retirement. This option pays off when the agent enters retirement. Its payoff is given by

\[
\frac{W}{C_0} W_{NR},
\]

where \( W \) is the target wealth given by Eq. (14) and \( W^{NR} \) is given by the wealth of an agent without the option to retire, keeping the marginal value of a dollar \( (J_W) \) the same across the two agents. It can be shown that \( \bar{W} - W^{NR} \) is always positive and can be expressed as

\[
\bar{W} - W^{NR} = E \int_{\tau}^{\infty} \frac{H_s}{H_0} (c_s^R - c_s^{NR} + y_0) \, ds
\]

\[
= E \int_{\tau}^{\infty} \frac{H_s}{H_0} (c_s^R - c_s^{NR}) \, ds + \frac{y_0}{r},
\]

where \( c_s^R \) is the consumption of a retiree and \( c_s^{NR} \) is the consumption of a worker. In other words, this expression is equal to the difference in the NPV of the consumption streams of a retired versus a nonretired person plus the net present value of forgone income. Hence, \( \bar{W} - W^{NR} \) is the extra wealth that is needed to finance retirement.

The second term in (17) is just the replicating portfolio of such an option. As for most barrier options, the replicating portfolio becomes largest when the option gets closer to becoming exercised, that is, as \( W_{\tau} \rightarrow \bar{W} \); this is why assertion (ii) holds.

It is interesting to relate the above results to BMS. To do so, we start by normalizing the nominal stock holdings by \( W_{\tau} \), so as to obtain nominal holdings of stock as a function of financial wealth. This gives the “portfolio” fraction of stocks \( \phi_{\tau} = \pi_{\tau} / W_{\tau} \),

![Fig. 2. The value functions of three related problems. Financial wealth is denoted as \( W \), \( y_0 \) denotes the constant income stream and \( r \) the constant riskless rate. The wealth threshold that triggers retirement is denoted as \( \bar{W} \).](image-url)
or using (17),

\[
\phi_t = \frac{\kappa}{\gamma} \left(1 + \frac{y_0}{W_t r}\right) \\
+ \left(\frac{J_W(W_t)}{J_W(W)}\right)^{\xi_2-1} \frac{\kappa}{\gamma} \frac{y_0}{W_t r} \xi_2 \left[\left((\xi_2 - 1) + \frac{1}{\gamma}\right) \left[\frac{\gamma}{1 - \gamma(1 + \xi_2\gamma/(1 - \gamma))} - 1\right] \right].
\]

It is interesting to note the dependence of these terms on \(W_t\). By fixing \(y_0/r\) and increasing \(W_t\), one can observe that the first term actually decreases. This is the standard BMS effect: according to BMS, the allocation to stocks depends on the relative ratio of financial wealth to human capital. If an individual is “endowed” with a lot of human capital compared to her financial wealth, it is as if she is endowed with a bond (since labor income is not risky). Hence, she will invest heavily in stocks in order to make sure that a constant fraction of her total resources (financial wealth + human capital) is invested in risky assets. This key intuition of BMS explains why the first term declines as \(W_t\) increases.

However, in the presence of a retirement option this conclusion is not necessarily true, due to the second term. To see why, compute \(\phi_W\) and evaluate it around \(\overline{W}\) to obtain, after some simplifications,

\[
\phi_W(\overline{W}) = -\frac{1}{\overline{W}} \left(\phi(\overline{W}) - \frac{\kappa}{\sigma} \frac{1}{\gamma}\right) + \frac{1}{\overline{W}} \frac{1}{\phi(\overline{W})} \left(\frac{\kappa}{\gamma}\right)^2 \left(\frac{K^{1/\gamma} (\xi_2 - 1) + \frac{1}{\gamma}}{K^{1/\gamma}}\right) \left(\xi_2\right) \left(1 - \gamma\right) \left[\frac{\gamma}{1 - \gamma(1 + \xi_2\gamma/(1 - \gamma))} - 1\right].
\]

The first term is clearly negative and captures the increase in the denominator of \(\phi = \frac{\gamma}{\gamma}\). The second term is positive and potentially larger than the first term, depending on parameters, and captures the increase in the likelihood that the option of retirement will be exercised. Hence, for values of \(W_t\) close to \(\overline{W}\), it is possible that \(\phi_W > 0\). One can easily construct numerical examples whereby this is indeed the case.

It is noteworthy that this result is driven by the option elements introduced by the irreversibility of the retirement decision and not by labor supply flexibility per se. Indeed, one can show (using the methods in BMS) that allowing an agent to choose labor and leisure freely on a continuum would result in

\[
\pi_t = \frac{1}{\gamma^*} \frac{\kappa}{\sigma} \left(W_t + \frac{y_0}{r} \left(\frac{7}{l - l_1}\right)\right).
\]

This implies that \(\phi\) would have to be decreasing in \(W_t\), despite the presence of labor supply flexibility. The reason for these differences is that in BMS, the amount allocated to stocks as a fraction of total resources (financial wealth + human capital) is a constant. In our framework this fraction depends on wealth. Wealth controls both the resources available for future consumption and the likelihood of “exercising” the real option of retirement.

In summary, not only does the possibility of early retirement increase the incentive to save more, it also increases the incentive of the agent to invest in the stock market because this is the most effective way to attain this goal. Furthermore, this incentive is strengthened as an individual’s wealth approaches the target wealth level that triggers retirement.

3. Retirement before a deadline

None of the claims made so far rely on restricting the time of retirement to lie in a particular interval. The exposition above is facilitated by the infinite horizon setup, which
allows for explicit solutions to the associated optimal stopping problem. However, the trade-off is that in the infinite horizon case, there is no notion of aging, since time plays no explicit role in the solution. Moreover, the “natural” theoretical benchmark for the model in the previous section is one without retirement at all. In this section we are able to extend all the insights of the previous section by comparing the early retirement model to a benchmark model with mandatory retirement at time $T$, which is more natural.

Formally, the only modification that we introduce in this section compared to Section 1 is that Eq. (5) becomes

$$
\max_{c_t, W_t, \tau} \mathbb{E} \left[ \int_{t}^{\tau \wedge T} e^{-\beta(s-t)} U(l_1, c_s) ds + e^{-\beta(\tau \wedge T-t)} U_2(W_{\tau \wedge T}) \right],
$$

where $T$ is the retirement deadline and $\tau \wedge T$ is shorthand notation for $\min(\tau, T)$.

The appendix presents the solution to the above problem in Theorem 2. As might be expected, one needs to use some approximate method to obtain analytical solutions, because now the optimal stopping problem is on a finite horizon. The extended appendix discusses the nature of the approximation and examines its performance against consistent numerical methods to solve the problem. One can easily verify that the formulas for optimal consumption, portfolio, etc. are identical to the respective formulas of Theorem 1 (the sole exception being that the constants are modified by terms that depend on $T - t$). As a result, all of the analysis in Section 2 carries through to this section. This is particularly true for the dependence of consumption, portfolio, etc. on wealth. Here we focus only on the implications of the model for portfolio choice as a function of age. The results for consumption are similar.

By Theorem 2 in the appendix, the optimal holdings of stock as a fraction of financial wealth are given as

$$
\phi_t = \frac{\kappa}{\sigma} \left( 1 + \frac{y_0}{W_t} \frac{1 - e^{-\gamma(T-t)}}{r} \right) + \frac{\kappa y_0}{\sigma W_t} \frac{(1 - e^{-\gamma(T-t)})}{r} \left( \frac{J_W(W_t, T - t)}{J_W(W_{\tau \wedge T}, T - t)} \right)^{\xi_2(T-t)-1} \times \xi_2(T-t) \left( \frac{1}{\gamma} + (\xi_2(T-t) - 1) \right) \left[ \frac{\gamma}{1 - \gamma} \left( 1 + \xi_2(T-t) \right) \frac{\gamma}{1 - \gamma} - 1 \right],
$$

where $W_{T-t}$ is the critical threshold that leads to retirement when an agent has $T - t$ years to mandatory retirement and the terms $\xi_2(T-t)$ are given in the appendix. As in Section 2.3, the first term is the standard BMS term for an investor with $T - t$ years to mandatory retirement. Whether financial wealth increases or the time to mandatory retirement decreases, the first term becomes smaller, which is the standard BMS intuition. Hence, the first term decreases over time (in expectation) because $T - t$ falls, while $W_t$ increases over time (in expectation). The second term captures the replicating portfolio of the early retirement option and is strictly positive. Its relevance is larger (a) the closer the option is to being exercised (in/out of the money) and (b) the more time is left until its expiration ($T - t$). Accordingly, the importance of the second term in (19) should be expected to

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16 An important remark on terminology: the term “finite horizon” refers to the fact that the optimal stopping region becomes a function of the deadline to mandatory retirement. The individual continues to be infinitely lived.

17 The basic idea behind the approximation is to reduce the problem to a standard optimal stopping problem and use the same approximation technique as in Barone-Adesi and Whaley (1987). The most important advantage of this approximation is that it leads to very tractable solutions for all quantities involved.
decrease when $T - t$ is small. However, it should be expected to increase when $W_t$ increases.

This now opens up the possibility of rich interactions between “pure” horizon effects (variations in $T - t$, keeping $W_t$ constant) and “wealth” effects. As an agent ages but is not yet retired, the “pure” horizon effects will tend to decrease the allocation to stocks. However, in expectation wealth increases as well and thus the option to retire early becomes more and more relevant, counteracting the first effect.

We quantitatively illustrate the interplay of these effects in the next section.

4. Quantitative implications

To quantitatively assess the magnitude of the effects described in Section 3 we proceed as follows. First, we fix the values of the variables related to the investment opportunity set to $r = 0.03$, $\mu = 0.1$, and $\sigma = 0.2$. For $\beta$ we choose 0.07 in order to account for both discounting and a constant probability of death. For $\gamma$ we consider a range of values (typically $\gamma = 2, 3, 4$). This leaves one more parameter to be determined, namely $K$. The parameter $K$ controls the shift in the marginal utility of consumption upon entering retirement. It is a well-documented empirical fact that consumption drops considerably upon entering retirement. As such, the most natural way to determine the value of $K$ is to match the agent’s declining consumption upon entering retirement. Aguiar and Hurst (2004) report expenditure drops of 17%, whereas Banks et al. (1998) report changes in log consumption expenditures of almost 0.3 in the five years prior to retirement and thereafter. Since these decreases mainly pertain to food expenditures, which are likely to be inelastic, we also calibrate the model to somewhat larger decreases in consumption.18

In light of (13), we have

$$\frac{c_t^+}{c_t^-} = K^{1/\gamma} \theta \rightarrow K = \left(\frac{c_t^+}{c_t^-}\right)^\gamma \theta^{-\gamma},$$

where $c_{t-}$ is the consumption immediately prior to retirement and $c_{t+}$ is the consumption immediately thereafter. By substituting a post- /pre-retirement ratio of $c_{t+}/c_{t-} = \{0.5, 0.6, 0.7\}$ in the above formula we can determine the respective values of $K$ that will ensure that the retirement decrease is equal to $\{0.5, 0.6, 0.7\}$, respectively. We fix the mandatory retirement age to be $T = 65$ throughout and normalize $y_0$ to be one. The abbreviation “Ref” indicates the solution implied by a model with optimal early retirement (up to time $T$) and “BMS” denotes the solution of a model with mandatory retirement at time $T$, with no option to retire earlier, later, or otherwise adjust their labor supply.

Fig. 3 plots the target wealth that is implied by the model, i.e., the level of wealth required to enter retirement. This figure demonstrates two patterns. First, threshold wealth declines as an agent nears mandatory retirement. This is intuitive, because the option to work is more valuable the longer its “maturity”: as a (working) agent ages, the incentive to keep the option “alive” is reduced and hence the wealth threshold declines. Second, the

18Admittedly, not all of these effects are purely due to nonseparability between leisure and consumption. Home production is undoubtedly a key determinant behind these decreases. It is important to note, however, that our model is not incompatible with such an explanation. As long as (a) the agent can leverage consumption utility with her increased leisure, and (b) time spent on home production is not as painful as work, the present model can be seen as a good reduced-form approximation to a more complicated model that would model home production explicitly.
critical wealth implied by this model varies with the assumptions made about risk aversion, and the disutility of work as implied by a lower $K^{-1/\gamma}$. Risk aversion tends to shift the threshold upwards, whereas lower levels of $K^{-1/\gamma}$ (implying more disutility of labor) bring the threshold down. These are intuitive predictions. An agent who is risk averse wants to avoid the risk of losing the option to work, whereas an agent who cares a lot about leisure will want to enter retirement earlier.

Fig. 4 addresses the importance of the real option to retire for portfolio choice. The figure plots the second term in Eq. (19) as a fraction of total stockholdings, $\pi_t$. In other words, it plots the relative importance of stockholdings due to the real option component as a percentage of total stockholdings. This percentage is plotted as a function of two variables, age and wealth. Age varies between 45 and 64 and wealth varies between zero and $x$, where $x$ corresponds to the level of wealth that would make an agent retire (voluntarily) at 64. We normalize wealth levels by $x$ so that the (normalized) wealth levels vary between zero and one. We then plot a panel of figures for different levels of $\gamma$ and $K^{1/\gamma}$. Fig. 4 demonstrates the joint presence of “time to maturity” and “moneyness” effects in the real option to retire. Keeping wealth fixed and varying the time to maturity (i.e., increasing age), the relative importance of the real option to retire declines. Similarly, increasing wealth makes the real option component more relevant, because the real option is more in the money. It is interesting to note that the real option component is large, realizing values as large as 40% for some parameter combinations.
In Fig. 5 we consider the implications of the model for portfolio choice as a function of age. We fix a path of returns that correspond to the realized returns on the CRSP value-weighted index between 1989 and 1999. We then plot the portfolio holdings (defined as total stockholdings normalized by financial wealth) over time for an individual whose wealth in 1989 was just enough to allow her to retire in the end of 1999 at the age of 58.

**Fig. 4.** Stockholdings due to the retirement option normalized by total stockholdings. “Percent of threshold” refers to the level of financial wealth normalized by the threshold wealth that would trigger retirement at age 64. $K^{1/\theta}$ is the ratio of post- to pre-retirement consumption. $\gamma$ is the coefficient of relative risk aversion.

In Fig. 5 we consider the implications of the model for portfolio choice as a function of age. We fix a path of returns that correspond to the realized returns on the CRSP value-weighted index between 1989 and 1999. We then plot the portfolio holdings (defined as total stockholdings normalized by financial wealth) over time for an individual whose wealth in 1989 was just enough to allow her to retire in the end of 1999 at the age of 58.
This is achieved as follows. Assume that in 1989 the investor is 48 years old and has wealth $W_0$. We treat $W_0$ as the unknown variable that we need to solve for. For any given $W_0$, and using both the optimal consumption and portfolio policies and the path of the realized returns between 1989 and 1999, we can determine how much wealth the investor has in 1999, when she is 58 years old. In order to ensure that she retires at that point we know that her wealth must be $W$. Hence, we choose $W_0$ so as to make sure that ten years thereafter (given the optimal policies and the realized path of returns) the wealth has grown to exactly $W$. We repeat the same exercise assuming various combinations of $K^{1/\theta}$ and $\gamma$. In order to be able to compare the results, we also plot the portfolio that would be
implied if the individual had no option of retiring early and we label this later case as “BMS.” Fig. 5 shows that the portfolio of the agent is initially declining and then flat or even increasing over time after 1995. This is in contrast to what would be predicted by ignoring the option to retire early (the “BMS” case). This fundamentally different behavior of the agent’s portfolio over time is due to the extraordinary returns during the latter half of the 1990s, which makes wealth grow faster and hence the real option to retire very important towards the end of the sample. By contrast, if one assumed away the possibility of early retirement, the natural conclusion would be that a run-up in prices would change the composition of the agent’s total resources (financial wealth + human capital) towards financial wealth. For a constant income stream this would therefore mean a decrease in the portfolio chosen.

Fig. 6 demonstrates the above effect more clearly. In this figure we normalize total stockholdings by total resources (human capital + financial wealth). Consistent with our results above, in the BMS case we get a constant equal to \( (\kappa/\sigma)(1/\gamma) \). When we allow for an early retirement option, we observe that the fraction of total resources invested in stock exhibits a stark increase towards the latter half of the 1990s, as the option of early retirement becomes more relevant. The increase in this fraction is small for the first half of the 1990s and large for the latter part of the decade.

Fig. 6 is useful in understanding the behavior of the portfolio holdings in Fig. 5. In the first half of the sample the standard BMS intuition applies. The fraction of total resources invested in the stock market is roughly constant even after taking the option of early retirement into account. Hence, by the standard intuition behind the BMS results, the portfolio of the agent (total stockholdings normalized by financial wealth) declines over time. However, in the latter half of the sample, the increase in the real option to retire is strong enough to counteract the decline in the portfolio implied by standard BMS intuitions.

Figs. 7 and 8 repeat the same exercise as in Figs. 5 and 6, only now for an agent who came close to retirement in 1999, that is, we now assume that her wealth in 1999 was slightly less than sufficient for her to actually retire. To achieve this we just assume that in 1989 she started with slightly less initial wealth than necessary to retire by 1999. It is interesting to note what happens after the stock market crash of 2000. Now, the option of early retirement starts to become irrelevant and the agent’s portfolio declines. The effect of a disappearing option magnifies the decrease in the portfolio. By contrast, in the BMS case the abrupt decrease in the stock market (and hence wealth) would be counterbalanced by a change in the composition between financial wealth and human capital towards human capital. This effect tends to somehow counteract the effects of aging and produces a much more moderate decrease in portfolio holdings.

These figures are meant to demonstrate the fundamentally different economic implications that can result once one takes into account the real option to retire. As such they should be seen as merely an illustrative application. Note, however, that a stronger result can be shown in the context of this exercise. For wealth levels close to the retirement threshold and for our preferred base scenario of \( K^{1/\theta} = 0.7 \) and \( \gamma = 4 \), the portfolio would increase with age in expectation as the agent approaches early retirement. Fig. 9 illustrates this effect. The only difference from Fig. 5 is that in Fig. 9 we perform the counterfactual exercise of assuming that the stock market moved along an “expected path” between 1989 and 1999. To do so, we assume that the increments of the Brownian motion driving returns are zero, so that the expected returns and the realized returns in the stock
As can be seen, the qualitative features of Fig. 5 are preserved. For the base scenario $K^{1/\theta} = 0.7$ and $\gamma = 4$, the fraction invested in stocks is initially decreasing with respect to age, then flat and even slightly increasing (between ages 57 and 58) along the expected path. This increase of the portfolio with age (in expectation) would be impossible in the absence of an early retirement option. Interestingly, the decline in portfolios between ages 48 and 57 is much smaller than what a BMS model would imply. The increase in the importance of the option of early retirement counteracts the pure horizon effect, so that the allocation to stock is almost constant for agents between ages 48 and 57. This may help explain the relatively constant allocations to stock that Ameriks and Zeldes (2001) document empirically.

The present paper is theoretical in nature, and we do not claim to have modeled even a small fraction of all the issues that influence real life retirement, consumption, and
portfolio decisions (e.g., shorting and leverage constraints, transaction costs, undiversifiable income and health shocks, etc.). However, note that the model does produce “sensible” portfolios (for the combination $\gamma = 4$ and $K^{1/\gamma} = 0.7$) as well as variations in portfolio shares between 1995 and 2003. In the bottom right plots of Fig. 7, for instance, the portfolio of the agent grows from 0.58 to 0.62 between 1995 and 1999 and then declines to roughly 0.5 by the beginning of 2003. In comparison, the Employee Benefits Research Institute (EBRI)\textsuperscript{19} reports that the average equity share in a sample of 401(k)s grew steadily from 0.46 to 0.53 between 1995 and 1999 only to fall to 0.4 by the beginning of 2003. The reason the model performs well is that we are considering an agent close to retirement, that is, at a time when the remaining NPV of her income is not a large

\textsuperscript{19}Employee Benefits Research Institute, Issue Brief 272 (Aug 2004), especially Fig. 2.
component of her total wealth in the first place. For young agents the model has similar problems matching the data as BMS, which is to be expected.

5. Borrowing constraint

Thus far we assume that the agent is able to borrow against the value of her future labor income. In this section we impose the additional restriction that it is impossible for the agent to borrow against the value of future income. Formally, we add the requirement that $W_t \geq 0$, for all $t > 0$. To preserve tractability, we assume in this section that the agent is able to go into retirement at any time that she chooses without a deadline. This makes the problem stationary and as a result the optimal consumption and portfolio policies will be given by functions of $W_t$ alone.
The borrowing constraint is never binding post-retirement because the agent receives no income and has constant relative risk aversion. This implies that once the agent is retired, her consumption, her portfolio, and her value function are the same with or without borrowing constraints. In particular, if she enters retirement at time $t$, her expected utility is still $U_2(W_t)$. (20)

The problem the agent now faces is

$$\max_{c_t, W_t, \tau} \mathbb{E} \left[ \int_0^\tau e^{-\beta t} U(l_1, c_t) \, dt + e^{-\beta \tau} U_2(W_\tau) \right]$$

subject to the borrowing constraint

$$W_t \geq 0 \quad \forall t \geq 0,$$ (21)

Fig. 9. Portfolio holdings (defined as total stockholdings normalized by financial wealth) for an individual as a function of her age. We assume that the realized returns in the stock market are equal to the ex-ante expected returns. We plot the portfolio holdings that would be implied by the Bodie et al. (1992) model (denoted by “BMS”) and the model of this paper (denoted by “Ret”). $K^{1/\theta}$ is the ratio of post- to pre-retirement consumption. $\gamma$ is the coefficient of relative risk aversion.
and the budget constraint

\[ dW_t = \pi_t(\mu dt + \sigma dB_t) + (W_t - \pi_t \gamma_0) dt - (c_t - y_0 1(t < \tau)) dt. \] (22)

We present the solution in Theorem 3 in the appendix. We devote the remainder of this section to a comparison of results we obtain in Section 2 with the resulting optimal policies we obtain in Theorem 3.

The appendix gives a simple proof as to why wealth at retirement is smaller with borrowing constraints than without (even though quantitatively the effect is negligible). In terms of optimal stockholdings, the presence of borrowing constraints moderates holdings of stock, and decreasingly so as the wealth of the agent increases.

Fig. 10 compares optimal portfolios for four cases formed by those with and without the early retirement option, and those with and without the imposition of borrowing constraints. For the cases in which we allow retirement, we take wealth levels close to retirement but lower than the threshold that would imply retirement. The figure demonstrates that for levels of wealth close to retirement there are only (minor)
quantitative differences between agents with borrowing constrains and agents without. The qualitative properties are the same. Holdings of stock increase with wealth (more than linearly). One can observe that the optimal stockholdings in the presence of early retirement are tilted more towards stocks whether we impose borrowing constraints or not. Similarly, the optimal holdings of stock are smaller when one imposes borrowing constraints (whether one allows for a retirement option or not).

We conclude by summarizing the key insights of this section. Borrowing constraints are relevant for levels of wealth close to zero, where optimal retirement is not an issue. Similarly, the effects of optimal retirement are relevant for high levels of wealth, where borrowing constraints are unlikely to bind in the future. Hence, as long as one examines the effects of the option to retire close to the threshold levels of wealth, borrowing constraints can be safely ignored. However, it is important to note that borrowing constraints can fundamentally affect quantities related to, e.g., the expected time to retirement for a person who starts with wealth close to zero because they will typically imply lower levels of stockholdings and hence a more prolonged time (in expectation) to reach the retirement threshold.

6. Conclusion

In this paper we propose a simple partial equilibrium model of consumer behavior that allows for the joint determination of a consumer’s optimal consumption, portfolio, and time to retirement. The appendix provides essentially closed-form solutions for virtually all quantities of interest. The results can be summarized as follows. The ability to time one’s retirement introduces an option-type character to the optimal retirement decision. This option is most relevant for individuals with a high likelihood of early retirement, that is, individuals with high wealth levels. This option in turn affects both an agent’s incentive to consume out of current wealth and her investment decisions. In general, the possibility of early retirement will lead to portfolios that are more exposed to stock market risk. The marginal propensity to consume out of wealth will be lower as one approaches early retirement, reflecting the increased incentives to reinvest gains in the stock market in order to bring retirement “closer.”

The model makes some intuitive predictions. Here we single out some of the predictions that seem to be particularly interesting. First, the model suggests that during stock market booms, there should be an increase in the number of people that opt for retirement as a larger percentage of the population hits the retirement threshold (some evidence for this may be found in Gustman and Steinmeier (2002) and references therein). Second, the models shows that it is possible that portfolios of aging individuals could exhibit increasing holdings of stock over time, even if there is no variation in the investment opportunity set and the income stream exhibits no correlation with the stock market (or any risk whatsoever). This is interesting in light of the evidence in Ameriks and Zeldes (2001) that portfolios tend to be increasing or hump-shaped with age for the data sets that they consider. Third, according to the model, there should be a discontinuity in the holdings of stock and in consumption upon entering retirement. Ample empirical evidence shows that indeed, this is the case for consumption. (See, e.g., Aguiar and Hurst, 2004 and references therein.) The discontinuity in stockholdings seems to have been less tested an hypothesis. Fourth, the model predicts that all else equal, switching to a more flexible retirement system that links portfolio choice with retirement timing should lead to increased stock
market allocations. This is consistent with the empirical fact that stock market participation increased in the U.S. as 401(k)s were gaining popularity. Fifth, increasing levels of stockholdings during a stock market run-up and liquidations during a stock market fall might not be due to irrational herding; instead, both effects might be due to the behavior of the real option to retire that emerges during the run-up and becomes irrelevant after the fall.

In this paper we try to outline the basic new insights that obtain by the timing of the retirement decision. By no means do we claim that we address all the issues that are likely to be relevant for actual retirement decisions (e.g., health shocks, unspanned income, etc.). Rather, we view the theory developed in this paper as a complement to our understanding of richer, typically numerically solved, models of retirement. Many interesting extensions to this model should be relatively tractable.

A first important extension would be to include features that are realistically present in actual 401(k)-type plans such as tax deferral, employee matching contributions, and tax provisions related to withdrawals. The solutions developed in such a model could be used to determine the optimal saving, retirement, and portfolio decisions of consumers that are contemplating retirement and taking into account tax considerations.

A second extension would be to allow the agent to reenter the workforce (at a lower income rate) once retired. We doubt this would alter the qualitative features of the model, but it is very likely that it would alter the quantitative predictions. It can be reasonably conjectured that the wealth thresholds would be significantly lower in that case, and the portfolios tilted even more towards stocks because of the added flexibility.

A third extension of the model would be to introduce predictability and more elaborate preferences. If one were to introduce predictability, while keeping the market complete (like Wachter, 2002), the methods of this paper can be easily extended. It is also very likely that the model would not loose its tractability if one uses Epstein–Zin utilities in conjunction with the methods recently developed by Schroder and Skiadas (1999).

A fourth extension of the model that we are currently pursuing is to study its general equilibrium implications. This is of particular interest as it would enable one to make some predictions about how the properties of returns are likely to change as worldwide retirement systems begin to offer more freedom to agents in making investment and retirement decisions.

Appendix A. Supplementary data

This version of the appendix contains the statements of the theorems and a sketch of the proof of the main theorem. An extended appendix containing all the proofs is available online at 10.1016/j.jfineco.2005.10.004.

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20The role of labor supply flexibility in a general equilibrium model with continuous labor-leisure choice is considered in Basak (1999). It is very likely that the results we present in this paper could form the basis for a general equilibrium extension. It is well known in the macroeconomics literature that allowing for indivisible labor is quite important if one is to explain the volatility of employment relative to wages. See, for example, Hansen (1985) and Rogerson (1988).
A.1. Theorems and proofs for Section 2

**Theorem 1.** To obtain the solution to the problem we describe in Section 1, define the constants

\[ \xi_2 = \frac{1 - 2 \frac{\theta - r}{\kappa^2} - \sqrt{\left(1 - 2 \frac{\theta - r}{\kappa^2}\right)^2 + 8 \frac{\theta - r}{\kappa^2}}}{2}, \]  

\[ \lambda = \left( \frac{(\xi_2 - 1)\theta}{1 + \xi_2 \frac{\gamma}{1 - \gamma}} \left( K^{1/\gamma} \theta - 1 \right) \frac{r}{K} \right)^{-\gamma} \]  

and

\[ C_2 = \frac{\left[ \frac{\gamma}{1 - \gamma} \left( \xi_2 - 1 \right) \frac{\gamma}{1 - \gamma} \left( 1 + \xi_2 \frac{\gamma}{1 - \gamma} \right) \right] \frac{y_0}{r}}{2 \xi_2 - 1}, \]

assume that

\[ \frac{r}{\theta} \frac{1 - \gamma}{\xi_2 - 1} < 1, \]

and let \( \lambda^* \) be the (unique) solution of

\[ \xi_2 C_2 (\lambda^*)^{\xi_2 - 1} - \frac{1}{\theta} (\lambda^*)^{-1/\gamma} + \frac{y_0}{r} + W_t = 0. \]  

Then

\[ C_2 > 0, \quad \xi_2 < 0, \]

and the optimal policy is given as follows:

(a) If \( W_t < \bar{W} = (\xi_2 - 1)K^{1/\gamma} \theta / (1 + \xi_2 \frac{\gamma}{1 - \gamma}) (K^{1/\gamma} \theta - 1) \frac{y_0}{r}, \)

consumption follows the process:

\[ c_s = \left( \lambda^* e^{\beta(s-t)} \frac{H(s)}{H(t)} \right)^{-1/\gamma} 1[t \leq s < \tau^*], \]

\[ c_s = \lambda^{(1-\gamma)(1-\gamma^*)/\gamma} \left( \lambda^* e^{\beta(s-t)} \frac{H(s)}{H(t)} \right)^{-1/\gamma} 1[s \geq \tau^*], \]

the optimal retirement time is

\[ \tau^* = \inf\{s : W_s = \bar{W}\} \]

\[ = \inf\left\{ s : \lambda^* e^{\beta(s-t)} \frac{H(s)}{H(t)} = \lambda \right\}, \]
and the optimal consumption and stockholdings as a function of $W_t$ are given by

$$c_t = c(W_t) = (\lambda^*(W_t))^{-1/\gamma},$$  \(29\)

$$\pi_t = \pi(W_t) = \frac{\kappa}{\theta} \left( \xi_2 (\xi_2 - 1) C_2 (\lambda^*(W_t))^{\xi_2 - 1} + \frac{11}{\gamma} \theta (\lambda^*(W_t))^{-1/\gamma} \right) .$$  \(30\)

(b) If

$$W_t \geq W = \left( \frac{(\xi_2 - 1) K^{1/\gamma} \theta}{(1 + \xi_2 \frac{\gamma}{1 - \gamma})(K^2 \theta - 1)} \right) y_0 ,$$

the optimal solution is to enter retirement immediately ($\tau^* = t$) and the optimal consumption/portfolio policy is given as in Karatzas and Shreve (1998, Chapter 3).

The remainder of this section is to provide a sketch for the proof of Theorem 1. Throughout this section we fix $t = 0$ without loss of generality. For a concave, strictly increasing and continuously differentiable function $U : (0, \infty) \to \mathbb{R}$, we can define the inverse $I(\cdot)$ of $U(\cdot)$. Let $U$ be given by

$$U(x) = \min_{y > 0} [U(x) + xy] = U(I(y)) - yI(y), \quad 0 < y < \infty .$$

It is easy to verify that $\tilde{U}(\cdot)$ is strictly decreasing and convex, and satisfies

$$\tilde{U}'(y) = -I(y), \quad 0 < y < \infty ,$$

$$U(x) = \min_{y > 0} [\tilde{U}(y) + xy] = \tilde{U}(U'(x)) + xU'(x), \quad 0 < x < \infty .$$  \(31\)

To start, we fix a stopping time $\tau$ and define

$$V_\tau(W_0) = \max_{c_1, \pi_1} E \left[ \int_0^\tau e^{-\beta t} U_1(c_t) dt + e^{-\beta t} U_2(W_\tau) \right] ,$$  \(32\)

where $U_1$ and $U_2$ are as defined in Eqs. (6) and (7). The following result is a generalization of the equivalent result in Karatzas and Wang (2000) to allow for income.

**Lemma 1.** Let

$$\tilde{J}(\lambda; \tau) = E \left[ \int_0^\tau [e^{-\beta t} \tilde{U}_1(\lambda e^{\beta t} H(t)) + \lambda H(t)y_0] dt + e^{-\beta \tau} \tilde{U}_2(\lambda e^{\beta \tau} H(\tau)) \right] .$$

For any $\tau$ that is finite almost surely, there exists $\lambda^*$ such that

$$V_{\tau}(W_0) = \inf_{\lambda > 0} [\tilde{J}(\lambda; \tau) + \lambda W_0] = \tilde{J}(\lambda^*; \tau) + \lambda^* W_0$$

and the optimal solution to (32) entails

$$W_\tau = I_2(\lambda^* e^{\beta \tau} H_\tau) ,$$  \(33\)

$$c_t = I_1(\lambda^* e^{\beta t} H(t)) 1\{ t < \tau \}$$  \(34\)
with $I_1$ and $I_2$ defined similarly to Eq. (31). Moreover, the value function $V(W_0)$ of the problem outlined in Section 2 satisfies

$$V(W_0) = \sup_{\tau} V_\tau(W_0) = \sup_{\tau} \inf_{\lambda > 0} [\tilde{J}(\lambda; \tau) + \lambda W_0] = \sup_{\tau} [\tilde{J}(\lambda^*; \tau) + \lambda^* W_0].$$

Karatzas and Wang (2000) show that one can reduce the entire joint portfolio-consumption-stopping problem into a pure optimal stopping problem by examining whether the inequality

$$V(W_0) = \sup_{\tau} \inf_{\lambda > 0} [\tilde{J}(\lambda; \tau) + \lambda W_0] \leq \inf_{\lambda > 0} \sup_{\tau} [\tilde{J}(\lambda; \tau) + \lambda W_0] = \inf_{\lambda} [\tilde{V}(\lambda) + \lambda W_0]$$

(35)

becomes an equality, with $\tilde{V}(\lambda)$ defined as

$$\tilde{V}(\lambda) = \sup_{\tau} \tilde{J}(\lambda; \tau) = \sup_{\tau} \mathbb{E} \left[ \int_0^\tau [e^{-\beta t} \tilde{U}_1(\lambda e^{\beta t} H(t)) + \lambda H(t) y_0] dt + e^{-\beta \tau} \tilde{U}_2(\lambda e^{\beta \tau} H(\tau)) \right].$$

(36)

Inequality (35) follows from a standard result in convex duality (see, e.g., Rockafellar, 1997). Reversing the order of maximization and minimization in (35) makes the problem significantly more tractable, since $\tilde{V}(\lambda)$ is the value of a standard optimal stopping problem, for which one can apply well known results. In particular, the parametric assumptions that we make in Section 1.3 allow us to solve this optimal stopping problem explicitly. This forms a substantial part of the proof and we present it in the extended appendix.

The cost of reversing the order of the minimization and the maximization, however, is that it will only give us an upper bound to the value function. The rest of the proof here is therefore devoted to showing that the inequality in (35) is actually an equality.

**Remark 1.** The option pricing interpretation given in Section 2.1 is based on a slight rewriting of equation (36). To see this, note that

$$\tilde{V}(\lambda) = \sup_{\tau} \tilde{J}(\lambda; \tau) = \sup_{\tau} \mathbb{E} \left[ \int_0^\tau [e^{-\beta t} \tilde{U}_1(\lambda e^{\beta t} H(t)) + \lambda H(t) y_0] dt + e^{-\beta \tau} \tilde{U}_2(\lambda e^{\beta \tau} H(\tau)) \right]$$

$$= \mathbb{E} \left[ \int_0^\infty [e^{-\beta t} \tilde{U}_1(\lambda e^{\beta t} H(t)) + \lambda H(t) y_0] dt \right]$$

$$+ \sup_{\tau} \mathbb{E} \left[ e^{-\beta \tau} \tilde{U}_2(\lambda e^{\beta \tau} H(\tau)) - \int_\tau^\infty [e^{-\beta t} \tilde{U}_1(\lambda e^{\beta t} H(t)) + \lambda H(t) y_0] dt \right].$$

The extended appendix shows that the third line can be further rewritten as

$$\sup_{\tau} \mathbb{E} \left\{ e^{-\beta \tau} Z_\tau \left[ \frac{1}{\theta} \frac{\gamma}{1 - \gamma} (K^{1/\gamma} - 1) Z^{-1/\gamma}_\tau - \frac{y_0}{r} \right] \right\},$$

which is precisely the option we analyze in Section 2.1.

The following result is a straightforward extension of a result in Karatzas and Wang (2000) and is given without proof.
Lemma 2. Let \( \tilde{\tau} \) be the optimal stopping rule associated with \( \lambda \) in Eq. (36). Then

\[
\tilde{V}'(\lambda) = -E \left[ \int_{0}^{\tilde{\tau}} H(t) (I_1(e^{\lambda t} - y_0) dt + H(\tilde{\tau}) I_2(e^{\lambda \tilde{\tau} / e^{\lambda t}}) \right], \quad \lambda \in (0, \infty).
\]

The final steps towards proving Theorem 1 use this observation in order to replace the inequality in (35) with an equality sign. In particular, the extended appendix shows how to use Lemma 2 to demonstrate that the policies we propose in the statement of the theorem are feasible and their associated payoff provides an upper bound to the value function. We then conclude that they are the optimal policies.

A.2. Proofs for Sections 2.2 and 2.3

See extended appendix.

A.3. Theorems and proofs for Section 3

The statement of Theorem 2 is almost identical to the statement of Theorem 1 with the main exception that all the constants now depend on \( T / C_0 \). In the following statement of the theorem we isolate the results related to the optimal portfolio and leave the precise statement of the entire theorem along with its proof for the extended appendix.

Theorem 2. The optimal portfolio is given as

\[
\pi_t = \pi(W_t) = \frac{\kappa}{\sigma} \left( \xi_2(T-t) (\lambda^*(W_t)) \xi_2(T-t) - 1 \right) C_2(T-t) (\lambda^*(W_t))^{\xi_2(T-t) - 1} + \frac{1}{\gamma \theta(T-t)} (\lambda^*(W_t))^{1/\gamma} \right],
\]

where \( \lambda^*(W_t) \) is the unique solution of

\[
\xi_2(T-t) C_2(T-t) (\lambda^*)^{\xi_2(T-t) - 1} - \frac{1}{\theta(T-t)} (\lambda^*)^{-1/\gamma} + \frac{y_0(1 - e^{-\theta(T-t)})}{\gamma} + W_t = 0
\]

and \( C_2(T-t), \xi_2(T-t), \theta(T-t) \) are constants that depend on \( T - t \), which we give explicitly in the extended appendix.

Since Theorem 2 involves a finite-horizon optimal stopping problem, we use an approximation (along the lines of Barone-Adesi and Whaley, 1987) in order to solve it. The extended appendix discusses the quality of this approximate solution and finds that it is very accurate.

A.4. Theorems and proofs for Section 5

Theorem 3. Under technical conditions given in the extended appendix, there exist appropriate constants \( C_1, C_2, Z_L, Z_H, \xi_1, \) and \( \xi_2 \) and a positive decreasing process \( X^*_s \) with \( X^*_T = 1 \) so that the optimal policy triple \( (\hat{\tau}_s, W^*_s, \tilde{\tau}) \) is

(a) If \( W_t < \bar{W} = K^{1/\gamma} Z_L^{-1/\gamma}, \)

\[
\hat{\tau}_s = \left( \lambda^* e^{\theta(t-s)} X^*_s \frac{H(s)}{H(t)} \right)^{-1/\gamma} \{s < \tilde{\tau},
\]

\[\text{ARTICLE IN PRESS}\]

\[ \hat{W}_T = \overline{W}, \]

\[ \hat{r} = \inf \{ s : W_s = \overline{W} \} = \inf \left\{ s : \lambda^* e^{(s-t)X_s} \frac{H(s)}{H(t)} = Z_L \right\}, \]

and \( \lambda^* \) is given by

\[ \xi_1 C_1(\lambda^*)^{\gamma - 1} + \xi_2 C_2(\lambda^*)^{\gamma - 1} - \frac{1}{\theta} (\lambda^*)^{-1/\gamma} + \frac{Y_0}{r} + W_t = 0. \]  

(38)

Using the notation \( \lambda^*(W_t) \) to make the dependence of \( \lambda^* \) on \( W_t \) explicit, the optimal consumption and portfolio policy is given by

\[ c_t = c(W_t) = (\lambda^*(W_t))^{-1/\gamma}, \]

\[ \pi_t = \pi(W_t) = -\frac{\kappa \lambda^*(W_t)}{\sigma \lambda_{W_t}^*(W_t)}, \]

where \( \lambda_{W_t}^* \) denotes the first derivative of \( \lambda^*(W_t) \) with respect to \( W_t \).

(b) If \( W_T > \overline{W} = K^{1/\gamma}Z_L^{-1/\gamma} \), the optimal solution is to enter retirement immediately (\( \hat{r} = t \)) and the optimal consumption policy is given as in the standard Merton (1971) infinite horizon problem.

We include the proof and the precise assumptions behind this theorem in the extended appendix to this paper.

References

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