Math 163: Derived Categories in Algebra and Geometry

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Welcome to Math 163: Derived Categories in Algebra and Geometry. Here's some important information:

- The course webpage is:
  `https://canvas.harvard.edu/courses/85132`
- Office hours are at the following times over Zoom:
  - Friday, 1-3pm ET.
- The text for the course is "Methods of Homological Algebra, 2nd edition," by Sergei Gelfand and Yuri Manin. The book is available free online through Harvard's Springer access.
- Relevant emails are `elmanto@math.harvard.edu` , `forrestflesher@college.harvard.edu`. Email with any questions, comments, or concerns.
- Each student is required to give two presentations, and submit a weekly response paper to presentations.
- You are encouraged to have an iPad or some other writing tablet for sharing work over zoom. If you do not have one, contact HUIT, and they might be able to get one for you (I'm not sure exactly how the process works).
- Feedback on the course is appreciated. Please let us know what you think of the pace, format, and content of the course.

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§1 January 26, 2021

§1.1 Categories

We begin with the definition of a category. In these notes, I will use bold letters for categories.

Definition 1.1 — A **category** $\mathbf{C}$ consists of the following data:

1) A class $\text{Ob}(\mathbf{C})$ of **objects**. We'll typically use X, Y, Z or A, B, C to denote objects.
2) For any two objects X and Y, a set $\text{Hom}_C(X, Y)$ of **morphisms**. We'll typically use f, g, h or other lower case letters to denote morphisms.
3) For any object X, an **identity morphism** id_X in $\text{Hom}_C(X, X)$.
4) For any three objects X, Y, Z, a **composition law**, which is a map

$$\text{Hom}_C(X,Y) \times \text{Hom}_C(Y,Z) \xrightarrow{\circ} \text{Hom}_C(X,Z).$$

And the above must satisfy the following conditions:

a) For any $X, Y \in \text{Ob}(\mathbf{C})$, any $f \in \text{Hom}_C(X,Y)$, then $f \circ \text{id}_X = \text{id}_Y \circ f = f$.
b) The composition law is associative: for any $f \in \text{Hom}_C(X,Y), g \in \text{Hom}_C(Y,Z)$ and $h \in \text{Hom}_C(Z,W)$, then $(h \circ g) \circ f = h \circ (g \circ f)$.

From now on, we will typically condense notation, writing $\text{Hom}(X,Y)$ instead of $\text{Hom}_C(X,Y)$ when clear, $X \in \mathbf{C}$ instead of $X \in \text{Ob}(\mathbf{C})$, and id instead of id_X. I will also often omit parentheses in various situations, such as writing $\text{Ob}\mathbf{C}$.

Note: The above definition is actually a **locally small category**, meaning that $\text{Hom}(X,Y)$ is a *set* (not something bigger) for each pair X, Y. There are categories which are not locally small, such as **Cat**, but we won't deal with those in this course.

Example 1.2 (A monoid as a category)

Say that $\mathbf{C}$ has $\text{Ob}(\mathbf{C}) = \{\star\}$. The only set of morphisms is $\text{Hom}(\star,\star) = \text{End}(\star)$. Elements of $\text{End}(\star)$ have a multiplication rule given by $\circ$, subject to two rules: $f \circ \text{id} = \text{id} \circ f$, and associativity. Those familiar with monoids will recognize this as a monoid.

Example 1.3

A **group** is a monoid such that for any f, there exists and f^{-1} with $f^{-1} \circ f = f \circ f^{-1} = \text{id}$. This leads us to our next definition.

Definition 1.4 — An **invertible morphism** f is a morphism $f : X \to Y$ such that there exists $g : Y \to X$ such that $\text{id}_Y = f \circ g : Y \to Y$ and $\text{id}_X = g \circ f : X \to X$.

Exercise 1.5. Inverses are unique.

Example 1.6

A **groupoid** is a category in which every morphism is invertible. You can also think of a groupoid as a multi-object group.

The above examples have in common the property that the objects form a *set*, and not something bigger (like a collection). A category whose objects and morphisms form sets is called a **small category**. Note especially that locally small and small are different: all small categories are locally small, but not vice-versa. All of our categories will be locally small, but not all of them will be small.

Example 1.7

An example of a category which is not small, is $\mathbf{Vect}_k$, the category of vector spaces over a fixed field k. The objects are vector spaces, and the morphisms are linear transformations. We also have $\mathbf{Vect}_k^{fd} \subseteq \mathbf{Vect}_k$ of finite dimensional vector spaces (technically the objects are finite dimensional vector spaces up to invertible morphisms). The finite dimensional version of this category *is* small.

Example 1.8

More examples of categories which are big are $\mathbf{Set}, \mathbf{Grp}, \mathbf{Ab}, \mathbf{Ring}$. Given the names of these categories, you can work out what the objects and morphisms are, and verify that they are categories.

§1.2 Functors

After categories, the next most important notion in category theory is that of a functor, which we now define. Functors are essentially structure preserving maps between categories, taking objects to objects and morphisms to morphisms.

Definition 1.9 — If $\mathbf{C}$ and $\mathbf{D}$ are categories, then a **functor** $F : \mathbf{C} \to \mathbf{D}$ from $\mathbf{C}$ to $\mathbf{D}$ consists of the following:

i) A map $\mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$, written $X \mapsto F(X)$.

ii) For each $X, Y \in \mathrm{Ob}(\mathbf{C})$, a map $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(F(X), F(Y))$, written $f \mapsto F(f)$, such that
   (a) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, and
   (b) $F(g \circ f) = F(g) \circ F(f)$.

Visually, for $X \xrightarrow{f} Y$, we have $F(X) \xrightarrow{F(f)} F(Y)$.

Example 1.10

Perhaps the simplest example is the identity functor, $\mathbf{C} \to \mathbf{C}$, which takes each object to itself and each morphism to itself.

Example 1.11

For our example of monoids above, a functor is a monoid homomorphism.

Example 1.12

We define the category of pointed topological spaces as follows: the objects are pairs (X, x), where X is a topological space and $x \in X$ is a point, and a morphism from (X, x) to (Y, y) is a continuous map $f : X \to Y$ such that $f(x) = y$.

For each pair (X, x), there is a fundamental group $\pi_1(X, x)$ at x of homotopy classes of loops at x. This gives us a functor from the category of pointed topological spaces to the category of groups.

Example 1.13

a) The **forgetful functor** (or obliviating functor), $\text{oblv} : \mathbf{Grp} \to \mathbf{Set}$, which takes a group to its underlying set. It is called the forgetful functor because it "forgets" the group structure.

b) The free functor, $\text{Free} : \mathbf{Sets} \to \mathbf{Grp}$, which takes a set X to the free group of X.

c) The functor $(\cdot)^{\text{ab}} : \mathbf{Grp} \to \mathbf{AbGrp}$, which takes a group G to $G^{\text{ab}} = G/[G, G]$ (the group modulo its commutator). Notice that G^{ab} is Abelian by construction.

§2 January 28, 2021

§2.1 Natural Transformations

We begin today with the definition of a natural transformation, which describes how to transform one functor into another, giving a sort of "morphism of functors".

Definition 2.1 — If $F, G : \mathbf{C} \to \mathbf{D}$ are functors from category $\mathbf{C}$ to $\mathbf{D}$, then a **natural transformation** is a collection of morphisms for each object $X \in \mathbf{C}$:

$$\{\eta_X \in \text{Hom}_D(F(X), G(X))\}_{X \in \mathbf{C}},$$

such that the following diagram commutes for any morphism $f : X \to Y$:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y). \end{tikzcd}$$

We write the above natural transformation as $\eta : F \Rightarrow G$.

Example 2.2

An example of a functor is the double dual from linear algebra, $\mathrm{id} \Rightarrow (D(D(\cdot)) : \mathbf{Vect}_k \to \mathbf{Vect}_k$.

Natural transformations allow us to define notions of equivalence between categories, since we now have a way to map functors to other functors.

Definition 2.3 — A **natural isomorphism** between categories is a functor $\eta : F \Rightarrow G$ such that η_X is an isomorphism at each object X. An **equivalence of categories** consists of two functors

$$F : \mathbf{C} \to \mathbf{D}$$
$$G : \mathbf{D} \to \mathbf{C},$$

and two natural isomorphisms

$$\eta : \mathrm{id}_C \Rightarrow G \circ F$$
$$\epsilon : F \circ G \Rightarrow \mathrm{id}_D.$$

§2.2 Adjoints

Note: The above definition of equivalence of categories is different from the notion of isomorphism of categories. We now define another notion of comparison between functors which is weaker and often more useful.

Definition 2.4 — Two functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ are called an **adjoint pair** if there exist transformations

$$\eta : \mathrm{id}_C \Rightarrow G \circ F$$
$$\epsilon : F \circ G \Rightarrow \mathrm{id}_D,$$

such that the following two compositions are the identity:

$$F \xRightarrow{\mathrm{id}_F \circ \eta} FGF \xRightarrow{\epsilon \circ \mathrm{id}_F} F$$

$$G \xRightarrow{\eta \circ \mathrm{id}_G} GFG \xRightarrow{\mathrm{id}_G \circ \epsilon} G.$$

The above might be slightly confusing: what does it mean for the composition $F \Rightarrow F$ to be the identity, and what exactly are the maps in between? The confusion likely arises from the fact that here we are actually dealing with two different types of composition of natural transformations: horizontal and vertical. The composition above of the type $A \Rightarrow B \Rightarrow C$ are known as **vertical composition**, so named because can write them in the following way

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left=60, "A"] \arrow[r, "B" description] \arrow[r, bend right=60, "C"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathbf{D}. \end{tikzcd}$$

We get the first composition above by plugging in $A = F, B = FGF, C = F$, and $\alpha = \mathrm{id}_F \circ \eta, \beta = \epsilon \circ \mathrm{id}_F$. The other type of composition in the diagram above is **horizontal composition**, and is probably the source of most confusion. This is a composition of the form

$$\begin{tikzcd}
\mathbf{C} \arrow[r, bend left=50, "A_1", ""{name=U1, below}] \arrow[r, bend right=50, "B_1"', ""{name=D1}] & \mathbf{D} \arrow[r, bend left=50, "A_2", ""{name=U2, below}] \arrow[r, bend right=50, "B_2"', ""{name=D2}] & \mathbf{E}.
\arrow[Rightarrow, from=U1, to=D1, "\alpha"]
\arrow[Rightarrow, from=U2, to=D2, "\beta"]
\end{tikzcd}$$

The horizontal composition is the natural transformation $(\beta \circ \alpha) : A_2A_1 \Rightarrow B_2B_1$. For each object $X \in \mathbf{C}$, we do

$$A_2A_1(X) \to A_2B_1(X) \to B_2B_1(X).$$

The first map sends A_1 to B_1 via α, and the second map sends A_2 to B_2 via β. That is, for $(\beta \circ \alpha)_X$ for some object X, we look at β_Y, and plug in for Y what we get in the image of α_X.

To unpack this in definition definition 2.4 above, we have $(\mathrm{id}_F \circ \eta)_X \in \mathrm{Hom}_D(F(X), FGF(X))$ for some object X. Write $\eta_X : \mathrm{id}(X) \to GF(X)$, where $X \in \mathbf{C}$, and $(\mathrm{id}_F)_Y : F(Y) \to F(Y)$ for $Y \in \mathbf{C}$. Then we want to plug in $GF(X)$ for Y and look at $(\mathrm{id}_F)_{GF(X)} : F(GF(X)) \to F(GF(X))$.

In definition definition 2.4, the map $\eta : \mathrm{id} \Rightarrow G \circ F$ is called the **unit**, and the map $\epsilon : F \circ G \Rightarrow \mathrm{id}$ is called the **counit**. A useful mnemonic device for remembering the order of things is that the unit is one-right-left, and the counit is left-right-one. Left, right, and one refer to F, G and id respectively.

We call F a **left-adjoint** to G, and write $F \dashv G$.

Notice the following: given a morphism $f : F(X) \to Y$, we get a map $\mathrm{Hom}_D(F(X), Y) \to \mathrm{Hom}_C(X, G(Y))$:

$$X \xrightarrow{\eta_X} GF(X) \xrightarrow{f} G(Y).$$

Similarly, given $g : X \to G(Y)$, we get a map $\mathrm{Hom}_X(X, G(Y)) \to \mathrm{Hom}_D(F(X), Y)$:

$$Y \xrightarrow{g} FG(Y) \xrightarrow{\epsilon_Y} Y.$$

This leads us to the following proposition.

Proposition 2.5

For functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$, the following are equivalent:

1. F is left adjoint to G, $F \dashv G$.
2. There are natural isomorphisms $\mathrm{Hom}(F(X), Y) \simeq \mathrm{Hom}(X, G(Y))$.

Example 2.6

Consider the functors Free : **Sets** → **Grp** and oblv : **Grp** → **Sets**. Then Free $\dashv$ oblv, and we have a map $\mathrm{Free}(X) \to G$ if and only if there is a map $X \to \mathrm{oblv}(G)$.

Definition 2.7 — A functor $F : \mathbf{C} \to \mathbf{D}$ is called **essentially surjective** if for all $Y \in \mathbf{D}$ there is $X \in \mathbf{C}$ such that $F(X) \simeq Y$ (note the $\simeq$ and not necessarily an $=$).

A functor $F : \mathbf{C} \to \mathbf{D}$ is **fully faithful** if for all $X, Y \in \mathbf{C}$, then $\mathrm{Hom}_C(X, Y) \simeq \mathrm{Hom}_D(F(X), F(Y))$.

Proposition 2.8

If $F : \mathbf{C} \to \mathbf{D}$ is a functor, then the following are equivalent:

1. There exists G such that G is the inverse of F (F and G form an equivalence of categories).
2. $G \dashv F$ and is an equivalence.
3. F is both essentially surjective and fully faithful.

We'll prove this later.

§2.3 Limits and Colimits

In the following, I is always a small category.

Definition 2.9 — Suppose $F : I \to \mathbf{C}$ is a functor. Then $\underset{I}{\mathrm{colim}} F$ is an object of $\mathbf{C}$ equipped with maps $\phi_i : F(i) \to \underset{I}{\mathrm{colim}} F$ for each $i \in I$ such that

1. For all morphisms $i \to j$, then the following commutes:

$$\begin{tikzcd} F(i) \arrow[r, "\phi_i"] \arrow[d] & \underset{I}{\mathrm{colim}} F \\ F(j) \arrow[ur, "\phi_j"] & \end{tikzcd}$$

2. For all $W \in \mathbf{C}$ with maps $\psi_i : F(i) \to W$, then for all i such that

$$\begin{tikzcd} F(i) \arrow[r, "\psi_i"] \arrow[d] & W \\ F(j) \arrow[ur, "\psi_j"] & \end{tikzcd},$$

commutes, then there exists a *unique* map $\underset{I}{\mathrm{colim}} F \to W$ making the such that the following diagram commutes:

$$\begin{tikzcd}
F(i) \arrow[rr] \arrow[dr, "\phi_i"] \arrow[ddr, "\psi_i"'] & & F(j) \arrow[dl, "\phi_j"'] \arrow[ddl, "\psi_j"] \\
& \underset{I}{\mathrm{colim}} F \arrow[d, "u"] & \\
& W &
\end{tikzcd}$$

Example 2.10 1. For the empty category, can : $\varnothing \to C$ then the colimit is $\underset{\varnothing}{\mathrm{colim}}\mathrm{can}$ is an **initial object** in $\mathbf{C}$, which is an object init such that for all $X \in \mathbf{C}$, there exists a unique map init $\to X$.

2. If I is a discrete category with two objects, then the colimit is called a coproduct (or sum), and is denoted $\mathrm{colim}F = X \sqcup Y$. For sets, this is the disjoint union. For $\mathbf{Mod}_R$, this is the direct sum $\oplus$.

We can rephrase the notion of a colimit in a useful way. If I is a fixed indexing category, then define C^I as $\mathrm{Fun}(I, \mathbf{C})$, the category whose objects are maps $I \to C$ and whose morphisms are natural transformations, called the **functor category**. Then define $\Delta : C \to \mathbf{Fun}(I, \mathbf{C})$, the **diagonal functor**, taking an object X of $\mathbf{C}$ to the constant functor $X \mapsto (i \mapsto X)$.

Lemma 2.11

The colimit $\underset{I}{\mathrm{colim}} : \mathbf{Fun}(I, \mathbf{C}) \to \mathbf{C}$ is left adjoint to the diagonal functor $\Delta : \mathbf{C} \to \mathrm{Fun}(I, \mathbf{C}))$.

Corollary 2.12

If $\mathbf{C}$ has I, J shaped colimits, then $\underset{I\times J}{\mathrm{colim}} \simeq \underset{I}{\mathrm{colim}}\underset{J}{\mathrm{colim}}F \simeq \underset{J}{\mathrm{colim}}\underset{I}{\mathrm{colim}}F$.

Proof. We have

$$\begin{aligned}
\mathrm{Hom}_C(\underset{I}{\mathrm{colim}}\underset{J}{\mathrm{colim}}F, Y) &\simeq \mathrm{Hom}_{C^I}(\underset{J}{\mathrm{colim}}F, \Delta_I(Y)) \\
&\simeq \mathrm{Hom}_{(C^I)^J}(F, \Delta_J(\Delta)I(Y))) \\
&\simeq \mathrm{Hom}_{C^{I\times J}}(F, \Delta_{J\times I}(Y)) \\
&\simeq \mathrm{Hom}_C(\underset{I\times J}{\mathrm{colim}}F, Y).
\end{aligned}$$

The first isomorphism follows from the lemma. For the second step, note that $(C^I)^J$ is $\mathrm{Fun}(J, \mathrm{Fun}(I, C)) = \mathrm{Fun}(J \times I, C)$. It might help to write out the maps explicitly. ☐

A very important type of colimit is called a *filtered* colimit, defined as follows.

Definition 2.13 — Let I be an indexing category. We say that I is **filtered** if

1. $I \neq \varnothing$.
2. I satisfies the **upper bound property**: for all $i, j \in I$, there exists k, and maps $i \to k$ and $j \to k$.
3. I satisfies the **upper bound property for maps**: for all $i, j \in I$ and for all maps
$$\begin{tikzcd} i \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & j, \end{tikzcd} ,$$
there exists $j \to k$ with
$$\begin{tikzcd} i \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & j \arrow[r, "f"] & k \end{tikzcd} ,$$
so that $f \circ u = f \circ v$.

We then have the following theorem, which we'll prove later

Theorem 2.14

The following are equivalent for a category $\mathbf{C}$:

1. $\mathbf{C}$ has colimits.
2. $\mathbf{C}$ has all coproducts, has initial objects, and equalizers.
3. $\mathbf{C}$ has filtered colimits and finite products.

§3 February 2, 2021: CJ Dowd

§3.1 Adjunctions

Today's lecture is given by CJ Dowd. These notes are closely based on his slides.

Recall from last time that a pair of functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{C} \to \mathbf{D}$ are *adjoint* if there exist natural transformations $\eta : \mathrm{id}_{\mathbf{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow \mathrm{id}_{\mathbf{D}}$ such that the compositions

$$F \xRightarrow{\mathrm{id}_F \circ \eta} FGF \xRightarrow{\epsilon \circ \mathrm{id}_F} F$$
$$G \xRightarrow{\eta \circ \mathrm{id}_G} GFG \xRightarrow{\mathrm{id}_G \circ \epsilon} G$$

are both the identity. We can also write this condition as $\epsilon_{FX} \circ F(\eta_X) = \mathrm{id}_{FX}$ and $G(\epsilon_Y) \circ \eta_{GY} = \mathrm{id}_{GY}$ for all $X \in \mathbf{C}$ and $Y \in \mathbf{D}$, which is perhaps more intuitive (and notationally simpler). We will now prove the proposition we stated last time. Recall proposition 2.5:

Proposition 2.5

For functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$, the following are equivalent:

1. F is left adjoint to G, $F \dashv G$.
2. There are natural isomorphisms $\mathrm{Hom}(F(X), Y) \simeq \mathrm{Hom}(X, G(Y))$.

Proof. For (1) $\Rightarrow$ (2), we construct the isomorphism $\varphi : \mathrm{Hom}_{\mathbf{D}}(FX, Y) \to \mathrm{Hom}_{\mathbf{C}}(X, GY)$ and its inverse ψ using the unit and counit. If $f \in \mathrm{Hom}_{\mathbf{D}}(FX, Y)$, define $\varphi(f) \in \mathrm{Hom}_{\mathbf{C}}(X, GY)$ as the composite map

$$\varphi(f) : X \xrightarrow{\eta_X} GF(X) \xrightarrow{G(f)} G(Y).$$

If $g \in \mathrm{Hom}_{\mathbf{C}}(X, GY)$, then define $\psi(g) \in \mathrm{Hom}_{\mathbf{D}}(FX, Y)$ as the composite

$$\psi(g) : F(X) \xrightarrow{F(g)} FG(Y) \xrightarrow{\epsilon_Y} Y.$$

We need to check that these maps are inverses. Let $f \in \mathrm{Hom}_{\mathbf{D}}(FX, Y)$, and consider $(\psi \circ \varphi)(f)$. The outside ψ part of the map is

$$FX \xrightarrow{F(\varphi(f))} FG(Y) \xrightarrow{\epsilon_Y} Y,$$

and expanding the inside φ part this is

$$FX \xrightarrow{F(\eta_X)} FGF(X) \xrightarrow{FG(f)} FG(Y) \xrightarrow{\epsilon_Y} Y.$$

Naturality of ϵ says that $\epsilon_Y \circ FG(f) = \mathrm{id}_{\mathbf{D}} \circ \epsilon_{FX}$. Plugging this in above, the map is

$$FX \xrightarrow{F(\eta_X)} FGF(X) \xrightarrow{\epsilon_{FX}} FX \xrightarrow{f} Y.$$

Applying the adjunction axiom $\epsilon_{FX} \circ F(\eta_X) = \mathrm{id}_{FX}$, this reduces to

$$FX \xrightarrow{f} Y.$$

Thus, $(\psi \circ \varphi)(f) = f$. A similar argument shows that $(\varphi \circ \psi)(g) = g$, and the maps are inverses. It remains to check naturality (the maps must be *natural* isomorphisms). Let $g : Y \to Y'$ be a morphism in $\mathbf{D}$. Checking naturality amounts to checking that the following diagram commutes (where $g \circ -$ is postcomposition):

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}}(FX, Y) \arrow[r, "g\circ -"] \arrow[d, "\varphi"] & \mathrm{Hom}_{\mathbf{D}}(FX, Y') \arrow[d, "\varphi"] \\
\mathrm{Hom}_{\mathbf{C}}(X, GY) \arrow[r, "G(g)\circ -"] & \mathrm{Hom}_{\mathbf{C}}(X, GY')
\end{tikzcd}$$

To check that this is commutative, expand the lower-left triangle $(G(g) \circ -)) \circ \varphi$. To do this, for a map f we first apply φ to get

$$X \xrightarrow{\eta_X} GF(X) \xrightarrow{G(f)} G(Y),$$

and then postcompose with $G(f)$ to get

$$X \xrightarrow{\eta_X} GF(X) \xrightarrow{G(f)} G(Y) \xrightarrow{G(g)} G(Y').$$

For the upper right triangle, firs apply the poscomposition with g to get $(g \circ f)$:

$$FX \xrightarrow{f} Y \xrightarrow{g} Y',$$

and then apply φ to get

$$X \xrightarrow{\eta_X} GF(X) \xrightarrow{G(g\circ f)} G(Y').$$

Since $G(g) \circ G(f) = G(g \circ f)$, we conclude naturality.

For the direction (2) $\Rightarrow$ (1), suppose we have a natural isomorphism $\mathrm{Hom}_{\mathbf{D}}(FX, Y) \simeq \mathrm{Hom}_{\mathbf{C}}(X, GY)$. Then we get a natural isomorphism

$$\alpha \, \mathrm{Hom}_{\mathbf{D}}(FX, FX) \simeq \mathrm{Hom}_{\mathbf{C}}(X, GF(X))$$

for each $X \in \mathbf{C}$. We show that the collection of morphisms $\eta_X := \alpha(\mathrm{id}_{FX}) \in \mathrm{Hom}_{\mathbf{C}}(X, GF(X))$ is the unit of our desired adjuction. First, we show that it is natural. Let $f \in \mathrm{Hom}_{\mathbf{C}}(X, X')$. To show naturality consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}}(FX, FX) \arrow[r, "\sim"] \arrow[d, "F(f)\circ -"] & \mathrm{Hom}_{\mathbf{C}}(X, GF(X)) \arrow[d, "GF(f)\circ -"] \\
\mathrm{Hom}_{\mathbf{D}}(FX, FX') \arrow[r, "\sim"] & \mathrm{Hom}_{\mathbf{C}}(X, GF(X')) \\
\mathrm{Hom}_{\mathbf{D}}(FX', FX') \arrow[r, "\sim"] \arrow[u, "-\circ F(f)"] & \mathrm{Hom}_{\mathbf{C}}(X', GF(X')) \arrow[u, "-\circ f"']
\end{tikzcd}$$

with

$$\begin{tikzcd}
\mathrm{id}_{FX} \arrow[r, mapsto] \arrow[d, mapsto] & \eta_X \arrow[d, mapsto] \\
F(f) \circ \mathrm{id}_{FX} \arrow[r, mapsto] & GF(f) \circ \eta_X \\
\mathrm{id}_{FX'} \arrow[r, mapsto] \arrow[u, mapsto] & \eta_{X'} \arrow[u, mapsto]
\end{tikzcd}$$

Note: $F(f) \circ \mathrm{id}_{FX} = \mathrm{id}_{FX'} \circ F(f)$ and $GF(f) \circ \eta_X = \eta_{X'} \circ f$. The top square of this diagram commutes by naturality of the isomorphism of Hom sets in the second variable. The bottom square commutes by naturality of the isomorphism in the second variable. Since id_{FX} and $\mathrm{id}_{FX'}$ are sent to the same morphism in the left of the diagram, then η_X and η'_X must both be sent to the same map on the right of the diagram. That means that the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_X"] & X' \arrow[d, "\eta_{X'}"] \\
GF(X) \arrow[r, "GF(f)"] & GF(X')
\end{tikzcd}$$

commutes, which is precisely the definition of naturality for η. The argument for η_Y as the image of id_{GY} under $\mathrm{Hom}_{\mathbf{C}}(GY, GY) \simeq \mathrm{Hom}_D(FG(Y), Y)$ is similar.

It remains to show that the unit and counit satisfy the adjunction axioms. That is, that

$$FX \xrightarrow{F(\eta_X)} FGF(X) \xrightarrow{\epsilon(FX)} FX$$

is the identity (and similarly for the other direction). To this end, consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}}(FGF(X), FX) \arrow[r, "\sim"] \arrow[d, "-\circ F(\eta_X)"] & \mathrm{Hom}_{\mathbf{C}}(GF(X), GF(X)) \arrow[d, "-\circ \eta_X"] \\
\mathrm{Hom}_{\mathbf{D}}(FX, FX) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathbf{C}}(X, GF(X)).
\end{tikzcd}$$

with

$$\begin{tikzcd}
\epsilon_{FX} \arrow[r, mapsto] \arrow[d, mapsto] & \mathrm{id}_{GF(X)} \arrow[d, mapsto] \\
\epsilon_{FX} \circ F(\eta_X) \arrow[r, mapsto] & \eta_X.
\end{tikzcd}$$

This diagram commutes by naturality of the isomorphism in the first variable. Considering $\mathrm{id}_{GF(X)}$ in the upper right corner, we get $\epsilon_{FX} \circ F(\eta_X) = \mathrm{id}_{FX}$, as desired. Showing that $G(\epsilon_Y) \circ \eta_{GY}$ is similar. This concludes the proof.

□

Example 3.1 (Free-Forgetful)

Forgetful functors typically have left adjoints, which are known as free functors. For example, the adjunction Free : **Set** $\leftrightarrows$ **Grp** : oblv which we have discussed. Since a map of sets $X \to G$ can uniquely be extended to a group homomorphism $\mathrm{Free}(X) \to G$ via the group law, and from any homomorphism $\mathrm{Free}(X) \to G$ we get a map of sets $X \to G$ by restriction, we have an isomorphism $\mathrm{Hom}_{\mathbf{Grp}}(\mathrm{Free}(X), G) \simeq \mathrm{Hom}_{\mathbf{Set}}(X, G)$.

The unit for this adjunction is the map which takes the identity $\mathrm{id}_{\mathrm{Free}(X)\to\mathrm{Free}(X)}$ to the map in $\mathrm{Hom}_{\mathbf{Set}}(X, \mathrm{Free}(X))$ which takes X to the corresponding single letter word.

The counit sends the identity $\mathrm{id}_{G\to G}$ to the homomorphism in $\mathrm{Hom}_{\mathbf{Grp}}(\mathrm{Free}(G), G)$ which takes a work "$g_1 \dots g_n$" in G to the element $g_1 \cdots g_n$ in G.

Other free-forgetful adjunctions are similar.

Example 3.2 (Abelianization)

We have an adjunction $(-)^{ab}$: **Grp** $\leftrightarrows$ **Ab** : oblv. For the bijection $\mathrm{Hom}_{\mathbf{Ab}}(G^{ab}, A) \simeq \mathrm{Hom}_{\mathbf{Grp}}(G, A)$, note that any group homomorphism $G \to A$ annihilates the commutator $[G, G]$, and thus descends to an Abelian group homomorphism $G^{ab} \to A$. Similarly an Abelian group homomorphism lifts to a group homomorphism.

The unit $\eta_G \in \mathrm{Hom}_{\mathbf{Grp}}(G, G^{ab})$ is the natural quotient map, and the counit $\epsilon_A \in \mathrm{Hom}_{\mathbf{Ab}}(A^{ab}, A)$ is the isomorphism $A^{ab} \simeq A$.

Example 3.3 (Group Representations)

Let G be a group. There is a functor triv : **Vect** $\to$ **Rep**G sending a vector space to the corresponding trivial representation of G. There is both a left *and* right adjoint to this functor. Suppose V is a G-rep, and let V^G be the fixed space under G. Then

$$\mathrm{Hom}_{\mathbf{Rep}G}(\mathrm{triv}(V), W) \simeq \mathrm{Hom}_{\mathbf{Vect}}(V, W^G).$$

This is because the image of any map of G representations must lie in W^G, which gives a vector space map $V \to W^G$, and any vector space map $V \to W^G$ gives a map of G representations $\mathrm{triv}(V) \to W^G \hookrightarrow W$.

The unit η_V of this adjunction is the vector space isomorphism $V to \mathrm{triv}(V)$, and the counit ϵ_W is the inclusion of G-reps $\mathrm{triv}(W^G) \simeq W^G \hookrightarrow W$.

Also, given a G-rep V, we have a quotient space of coinvariants V_G, which is the largest quotient on which G acts trivially. Then

$$\mathrm{Hom}_{\mathbf{Vect}}(V_G, W) \simeq \mathrm{Hom}_{\mathbf{Rep}G}(V, \mathrm{triv}(W)).$$

This is because any vector space map $V_G \to W$ lifts to a map of G-reps $V \to \mathrm{triv}$ by precomposing with the quotient $V \twoheadrightarrow V_G$, and any map of G-representations $\varphi : V \to \mathrm{triv}(W)$ descends to a map $V_G \to W$, since such a map requires $\varphi(v) = \varphi(gv)$ for all $g \in G$.

The unit η_V is the isomorphism $V \simeq (\mathrm{triv}(V))_G$, and the counit is the natural quotient $W \twoheadrightarrow \mathrm{triv}(\mathrm{W}_\mathrm{G})$.

Example 3.4 (Tensor-Hom)

Let A, B be algebras and let M be an $A - B$ - bimodule (a left A-module and a right B-module). The tensor functor $- \otimes_B M$ sends right B-modules to right A-modules. The internal Hom functor $\underline{\mathrm{Hom}}_A(M, -)$ sends right A-modules to right B-modules, with the B-module structure on a map $\varphi \in \underline{\mathrm{Hom}}_A(M, N)$ given by $\varphi(m) \cdot b = \varphi(mb)$.

We have an adjunction $- \otimes_B M : \mathbf{Mod}_A \leftrightarrows \mathbf{Mod}_B : \underline{\mathrm{Hom}}_B$. The isomorphism

$$\mathrm{Hom}_{\mathbf{Mod}_B}(P \otimes_B M, Q) \simeq \mathrm{Hom}_{\mathbf{Mod}_A}(P, \underline{\mathrm{Hom}}_B(M, Q))$$

is given by

$$((p \otimes_B m) \mapsto q) \mapsto (p \mapsto (m \mapsto q)).$$

For the unit/counit definition, the identity in $\mathrm{Hom}_{\mathbf{Mod}_B}(P \otimes_B M, P \otimes_B M)$ is sent to the unit

$$\eta_P p \mapsto (m \mapsto (p \otimes m)).$$

This is the map which sends $p \in P$ to a function that tensors elements by p.

The counit is given by evaluation: $\epsilon_Q : \mathrm{Hom}_A(M, Q) \otimes_B M \to Q$, with

$$\epsilon_Q(\phi \otimes m) = \phi(m).$$

Example 3.5 (Polynomial Rings)

Let $\mathbf{Ring}_*$ be the category of **pointed rings**. The objects are pairs (R, r), where $r \in R$ is a choice of an element of R, and the morphisms are ring homomorphisms fixing r. (this construction might look familiar to our example of pointed topological spaces and fundamental groups). The forgetful functor $\mathbf{Ring} \to \mathbf{Ring}_*$ has a left adjoint F, which sends a ring R to the pointed ring $(R[x], x)$:

$$F : \mathbf{Ring} \leftrightarrows \mathbf{Ring}_* : \mathrm{oblv},$$

where $R[x]$ is a polynomial ring.

The components of the unit are given by $\eta_R : R \hookrightarrow R[x]$, and the components of the counit are given by evaluation:

$$\begin{aligned} \epsilon_{(R,r)} &: (R[x], x) \to (R, r) \\ &: x \mapsto r. \end{aligned}$$

§3.2 Equivalence of categories

We will now prove the result on equivalences of categories mentioned earlier.

Theorem 3.6

Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. Then the following are equivalent.

1. F is part of an adjoint equivalence of categories, i.e. an equivalence of categories that is also an adjunction.
2. F is part of an equivalence of categories.
3. F is fully faithful and essentially surjective.

Proof. (2) is a special case of (1).

For (2) ⇒ (3), suppose G is adjoint to F. Essential surjectivity follows from the fact that $GF(X) \simeq X$ and $FG(Y) \simeq Y$. Faithfulness of F follows from the fact that the natural transformation

$$\mathrm{Hom}_{\mathbf{C}}(X,Y) \to \mathrm{Hom}_{\mathbf{D}}(FX,FY) \to \mathrm{Hom}_{\mathbf{C}}(GF(X),GF(Y))$$

is an isomorphism. The same argument shows that G is faithful, and thus set-theoretically, F is full.

(3) ⇒ (1). We construct a new category $\mathbf{CD}$. The objects of $\mathbf{CD}$ are triples (C,D,i), where $C \in \mathbf{C}, D \in \mathbf{D}$, and $i : FC \to D$ is an isomorphism. A morphism $(C,D,i) \to (C',D',i')$ is a pair of morphisms (f_c, f_d) such that the following diagram commutes:

$$\begin{tikzcd} FC \arrow[r, "i"] \arrow[d, "F(f_c)"] & D \arrow[d, "f_d"] \\ FC' \arrow[r, "i'"] & D' \end{tikzcd}$$

We have natural forgetful functors:

$$\begin{aligned} P_{\mathbf{C}} &: \mathbf{CD} \to \mathbf{C} \\ &: (C,D,i) \mapsto C \\ &: (f_c, f_d) \mapsto f_c. \\ P_{\mathbf{D}} &: \mathbf{CD} \to \mathbf{D} \\ &: (C,D,i) \mapsto D \\ &: (f_c, f_d) \mapsto f_d. \end{aligned}$$

We also have a natural inclusion functor:

$$\begin{aligned} I_{\mathbf{C}} &: \mathbf{C} \to \mathbf{CD} \\ &: C \mapsto (C, FC, \mathrm{id}_{FC}) \\ &: f_c \mapsto (f_c, F(f_c)), \end{aligned}$$

for any $f_x \in \mathrm{End}(C)$. In particular, $P_{\mathbf{D}} I_C = F$. We show that $P_{\mathbf{D}}$ and $I_{\mathbf{D}}$ are part of an adjoint equivalence.

The composition $P_{\mathbf{C}} I_{\mathbf{C}}$ is the identity functor on $\mathbf{C}$, so the unit $\eta_C = \mathrm{id}_{\mathbf{C}}$. The composition $I_{\mathbf{C}} P_{\mathbf{C}}$ is isomorphic to $\mathrm{id}_{\mathbf{CD}}$ via the natural transformation

$$I_{\mathbf{C}} P_{\mathbf{C}}(C,D,i) = (C, FC, \mathrm{id}_{FC}) \to (C,D,i),$$

given by the morphism $\epsilon_{C,D,i} = (\mathrm{id}_C, i) \in \mathrm{Hom}(I_{\mathbf{C}}P_{\mathbf{C}}(C,D,i),(C,D,i))$. Thus, $P_{\mathbf{C}}$ and $I_{\mathbf{C}}$ are an equivalence of categories $\mathbf{CD} \simeq \mathbf{C}$. Since

$$\epsilon_{(C,FC,i_{FC})} \circ I_{\mathbf{C}}(\mathrm{id}_C) = (\mathrm{id}_C, i) \circ (\mathrm{id}_C, i_{FC}) = (\mathrm{id}_C, \mathrm{id}_{FC}) = \mathrm{id}_{I_{\mathbf{C}}(C)}$$

and

$$P_{\mathbf{C}}(\epsilon_{(C,D,i)}) \circ \mathrm{id}_{P_{\mathbf{C}}(C,D,i)} = \mathrm{id}_C \circ \mathrm{id}_C = \mathrm{id}_{P_{\mathbf{C}}(C,D,i)},$$

then $P_{\mathbf{C}}$ and $I_{\mathbf{C}}$ are actually an *adjoint* equivalence of categories.

We now construct a functor $I_{\mathbf{D}}$ that pairs with $P_{\mathbf{C}}$ to yield an adjoint equivalence $\mathbf{D} \to \mathbf{CD}$. For each $D \in \mathbf{D}$, consider the set S_D of all objects $C \in \mathbf{C}$ such that there exists an isomorphism $i : FC \to D$ (since such isomorphisms are in $\mathrm{Hom}(FC, D)$ and our categories are locally small, S_D is a set). The functor F is essentially surjective, and thus surjective. Using the axiom of choice (Cartesian product of a collection of non-empty sets is non-empty), we can choose an element $C_D \in S_D$ for each D. That is, we can choose a collection $(C_D, D, i_D) \in \mathbf{CD}$, indexed by $D \in \mathbf{D}$.

Define the functor $I_{\mathbf{D}} : \mathbf{D} \to \mathbf{CD}$ by $D \mapsto (C_D, D, i_D)$ on the objects of D. A morphism $f : D \to D'$ uniquely determines a morphism $\tilde{f}_f : F(C_D) \to F(C_{D'})$ making the following diagram commute:

$$\begin{tikzcd}
F(C_D) \arrow[r, "i_D"] \arrow[d, "\tilde{f}_C"] & D \arrow[d, "f"] \\
F(C_{D'}) \arrow[r, "i_{D'}"] & D'.
\end{tikzcd}$$

Since F is fully faithful, there exists a unique $f_C \in \mathrm{Hom}_{\mathbf{C}}(C_D, C_{D'})$ such that $F(f_C) = \tilde{f}_C$ (using the definition of fully faithful). Define $I_{\mathbf{D}}(f) = (f_C, f)$.

From their definitions, we have $P_{\mathbf{D}}I_{\mathbf{D}} = \mathrm{id}_{\mathbf{D}}$. Let $\eta_D = \mathrm{id}_D$. To define the natural transformation $\epsilon : I_{\mathbf{D}}P_{\mathbf{D}} \Rightarrow \mathrm{id}_{\mathbf{CD}}$, suppose $(C, D, i) \in \mathbf{CD}$. Then similarly to above, the full faithfulness of F allows us to uniquely complete the following diagram with an $f_C : C \to C_D$:

$$\begin{tikzcd}
FC \arrow[r, "i"] \arrow[d, "F(f_c)"] & D \arrow[d, "\mathrm{id}_D"] \\
FC_D \arrow[r, "i_D"] & D
\end{tikzcd}$$

Then define $\epsilon_{(C,D,i)} = (f_c, \mathrm{id}_D)$, giving $\epsilon : I_{\mathbf{D}}P_{\mathbf{D}} \Rightarrow \mathrm{id}_{\mathbf{CD}}$, and thus an adjoint equivalence of categories. To see that $I_{\mathbf{D}} : \mathbf{D} \leftrightarrows \mathbf{CD} : P_{\mathbf{D}}$ is in fact an adjoint equivalence, we check the compositions

$$\epsilon_{I_{\mathbf{D}}D} \circ I_{\mathbf{D}}(\mathrm{id}_D) = (\mathrm{id}_{C_D}, \mathrm{id}_D) \circ (\mathrm{id}_{C_D}, \mathrm{id}_D) = \mathrm{id}_{I_{\mathbf{D}},D}$$

$$P_{\mathbf{D}}(\epsilon_{(C,D,i)}) \circ \mathrm{id}_{P_{\mathbf{D}}(C,D,i)} = \mathrm{id}_D \circ \mathrm{id}_D = \mathrm{id}_{P_{\mathbf{D}}(C,D,i)}$$

$$\begin{tikzcd}
 & \mathbf{CD} \arrow[dl, shift left, "P_{\mathbf{C}}"] \arrow[dr, shift right, "I_{\mathbf{D}}"'] & \\
\mathbf{C} \arrow[ur, shift left, "I_{\mathbf{C}}"] \arrow[rr, "F"'] & & \mathbf{D} \arrow[ul, shift right, "P_{\mathbf{D}}"']
\end{tikzcd}$$

This diagram commutes and every pair of functors is an adjoint equivalence. Thus, $F = P_{\mathbf{D}}I_{\mathbf{C}}$ is an adjoint equivalence, concluding the proof. ☐

§4 February 4, 2021: Madison Shirazi

Today's lecture is given by Madison Shirazi.

§4.1 The Yoneda Lemma

We begin with a few definitions. Suppose that $\mathbf{C}$ is a category. The **opposite category**, denoted $\mathbf{C}^{op}$, is the category with the same objects as $\mathbf{C}$, but whose morphisms are reversed, i.e. $\mathrm{Hom}_{\mathbf{C}^{op}}(X, Y) = \mathrm{Hom}_{\mathbf{C}}(Y, X)$. (In general in category theory, we have a principle of *duality*: if one has proven a theorem, one can apply the same theorem to the opposite category, and it is still true.) A functor in the opposite category is called a **contravariant functor**.

A **presheaf** is a functor $F : \mathbf{C}^{op} \to \mathbf{Set}$. The **presheaf category**, $\mathbf{PShv}(\mathbf{C})$ is the category of presheaves, i.e. $\mathrm{Fun}(\mathbf{C}^{op}, \mathbf{Set})$.

Notice that for any $X \in \mathbf{C}$, there is a functor $\mathrm{Hom}(-, X) : \mathbf{C}^{op} \to \mathbf{Set}$ that takes an object Y to $\mathrm{Hom}(Y, X)$, and takes a morphism $f : Y_1 \to Y_2$ to the morphism $\mathrm{Hom}(Y_2, X) \to \mathrm{Hom}(Y_1, X)$, defined by precomposition:

$$\begin{tikzcd}
Y_1 \arrow[rr, "f"] \arrow[dr] & & Y_2 \arrow[dl] \\
& X &
\end{tikzcd}$$

The map $(Y_2 \to X) \in \mathrm{Hom}(Y_2, X)$ is sent to $((Y_2 \to X) \circ f) \in \mathrm{Hom}(Y_1, X)$ (note here: in general $\mathrm{Hom}(-, X)$ is a *functor*, while $\mathrm{Hom}(Y, X)$ is a *set*). We can then define the following important map.

Definition 4.1 — Given a category $\mathbf{C}$, the **Yoneda embedding** is the map

$$Y : \mathbf{C} \to \mathrm{Fun}(\mathbf{C}^{op}, \mathbf{Set})$$
$$Y : X \mapsto \mathrm{Hom}(-, X).$$

A functor is called **representable** if it is in the essential image of Y.

Now, let's look at maps between this Hom-functor and a functor F, which are natural transformations. Let $X \in \mathbf{C}$ and $F \in \mathrm{Fun}(\mathbf{C}^{op}, \mathbf{Set})$. A natural transformation between $Y(X) = \mathrm{Hom}(-, X)$ and F (i.e. an element of $\mathrm{Hom}(Y(X), F)$) is a morphism

$$\eta_Z : \mathrm{Hom}(Z, X) \to F(Z)$$

for each $Z \in \mathbf{C}$. Choosing $X \in \mathbf{C}$, for any natural transformation we can examine $\eta_X : \mathrm{Hom}(X, X) \to F(X)$, and in particular we can always look at the image of the identity $\eta_X(\mathrm{id}_X) \in F(X)$. That is, given $\eta \in \mathrm{Hom}(Y(X), F)$, we can always look at $\eta_X(\mathrm{id}_X)$ so that we have a canonical restriction map

$$\mathrm{Hom}_{\mathrm{Fun}(\mathbf{C}^{op}, \mathbf{Set})}(\mathrm{Hom}(-, X), F) \to F(X)$$
$$\eta \mapsto \eta_X(\mathrm{id}_X).$$

The Yoneda lemma, one of the most important results in category theory, states that this map is actually an isomorphism.

Lemma 4.2 (Yoneda Lemma)
Suppose $X \in \mathbf{C}$, and $F \in \mathrm{Fun}(\mathbf{C}^{op}, \mathbf{Set})$. The the canonical restriction map

$$\mathrm{Hom}_{\mathrm{Fun}(\mathbf{C}^{op},\mathbf{Set})}(Y(X), F) \to F(X)$$

is an isomorphism.

Proof. Let $A \in F(X)$, and define the natural transformation $\epsilon : Y(X) \Rightarrow F$ by

$$\epsilon_Z : \mathrm{Hom}(Z, X) \to F(Z)$$
$$\epsilon_Z : g \mapsto F(g)(A).$$

Note: $\mathrm{Hom}(Z, X) = Y(X)(Z)$, so this definition makes sense. We show first that this map is actually a natural transformation. For a map $h : Z_1 \to Z_2$ in $\mathbf{C}$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}(Z_1, X) \arrow[r] & F(Z_1) \\
\mathrm{Hom}(Z_2, X) \arrow[r] \arrow[u, "h"] & F(Z_2) \arrow[u, "F(h)"']
\end{tikzcd}$$

commutes. This follows from the fact that F is a functor and preserves composition of morphisms.

The composite $F(X) \to \mathrm{Hom}(Y(X), F) \to F(X)$ is the identity, since the image of A under $F(X) \to \mathrm{Hom}(Y(X), F)$ is $\epsilon : Y(X) \Rightarrow F$ as defined above, and then applying the canonical restriction map we get $\epsilon_X(\mathrm{id}_X) = F(\mathrm{id}_X)(A) = \mathrm{id}_{F(X)}(A) = A$.

Now, suppose $\zeta : Y(X) \Rightarrow F$ is a natural transformation with components $\zeta_Z : \mathrm{Hom}(Z, X) \to F(Z)$. If $g \in \mathrm{Hom}(Z, X)$ if a morphism, then since ζ is natural the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(Z, X) \arrow[r, "\zeta_Z"] & F(Z) \\
\mathrm{Hom}(X, X) \arrow[r, "\zeta_X"] \arrow[u, "- \circ g"] & F(X) \arrow[u, "F(g)"']
\end{tikzcd}$$

Starting in the lower left corner with id_X and going up we get $\mathrm{id}_X \circ g = g$, and then going to the right we get $\zeta_Z(g)$. Starting in the lower left with id_X and going right we get $\zeta_X(\mathrm{id}_X)$, and then going up we get $F(g)(\zeta_X(\mathrm{id}_X))$. Then by commutivity, $\zeta_Z(g) = F(g)(\zeta_X(\mathrm{id}_X))$, and we are done. ☐

Corollary 4.3
The Yoneda functor is fully faithful

Proof. Suppose $f : X \to Z$ and $f' : X \to Z$ are distinct members of $\mathrm{Hom}(X, Z)$, where $X, Z \in \mathbf{C}$. To show that Y is faithful, we must show that $Y(f) = f \circ - \neq f' \circ - = Y(f')$. But $f \circ \mathrm{id}_X \neq f' \circ \mathrm{id}_X$, so Y is faithful.

The fullness of the functor follows from lemma 4.2 with $F = \mathrm{Hom}(-, Z)$. ☐

We now describe the relationship between the Yoneda lemma and Cayley's theorem in group theory. Suppose $\mathbf{C}$ is a category with a single object $\{\star\}$ and where every morphism is an isomorphism. Then $G = \mathrm{Hom}_{\mathbf{C}}(\star,\star)$ is a group under composition. In addition, any group can be written this way. A functor $\mathbf{C}^{op} \to \mathbf{Set}$ consists of an $X \in \mathbf{Set}$ and a group homomorphism $G \to \mathrm{Perm}(X)$. That is, X is a G-set. A natural transformation between functors is an equivariant map between G-sets. The functor $\mathrm{Hom}_{\mathbf{C}}(\star,-)$ corresponds to the action of G on itself by right multiplication. Then the Yoneda lemma states that

$$\mathrm{Hom}_{\mathrm{PShv}(\mathbf{C})}(\mathrm{Hom}(-,\star),\mathrm{Hom}(-,\star)) \simeq \mathrm{Hom}(\star,\star) = G$$

This tells us that G is isomorphic to some subgroup of $\mathrm{Perm}(G)$, which is Cayley's theorem.

§4.2 Adjoints and Representable Functors

Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. Define a functor G^{formal} to be

$$\begin{aligned} G^{formal} &: \mathbf{d} \to \mathrm{Fun}(\mathbf{C}^{op}, \mathbf{Set}) \\ &: Y \mapsto (X \mapsto \mathrm{Hom}_{\mathbf{D}}(F(X), Y). \end{aligned}$$

We then have the following useful proposition on existence of adjoint functors.

Proposition 4.4

If F is a functor, then a right adjoint G to F exists if and only if $G^{formal}(Z)$ is representable for every $Z \in \mathbf{D}$.

Proof. Suppose a right adjoint G exists. Then there is a natural isomorphism $\mathrm{Hom}_{\mathbf{D}}(F(X), Z) \simeq \mathrm{Hom}_{\mathbf{c}}(X, G(Z))$, for each $X \in \mathbf{C}$ and $Z \in \mathbf{D}$. Thus,

$$\begin{aligned} G^{formal}(Z)(X) &= \mathrm{Hom}_{\mathbf{D}}(F(X), Z) \\ &\simeq \mathrm{Hom}_{\mathbf{D}}(X, G(Z)) \\ &= \mathrm{Hom}_{\mathbf{C}}(-, G(Z))(X) \\ &= Y(G(Z))(X), \end{aligned}$$

where Y is the Yoneda functor. Thus, G is representable.

Suppose conversely that $G^{formal}(Z)$ is representable. Define $G = P \circ G^{formal}$, where $P : \mathrm{Fun}(\mathbf{C}^{op}, \mathbf{Set})^{repr} \to \mathbf{C}$ is the inverse of Y. Our goal is to show that G is a right adjoint to F, so we must find a natural isomorphism $\mathrm{Hom}_{\mathbf{D}}(F(X), Z) \simeq \mathrm{Hom}_{\mathbf{C}}(X, G(Z))$. We do this as follows:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{C}}(X, G(Z)) &= \mathrm{Hom}_{\mathbf{C}}(X, F \circ G^{formal}(Z)) \\ &\simeq \mathrm{Hom}_{\mathrm{PShv}(\mathbf{C})}(Y(X), G^{formal}(Z) \\ &\simeq G^{formal}(Z)(X) \\ &\simeq \mathrm{Hom}_{\mathbf{D}}(F(X), Z). \end{aligned}$$

□

Using the above proposition, we see that a right adjoint is a functor $G : \mathbf{D} \to \mathbf{C}$ together with a natural isomorphism $Y(G) \simeq G^{formal}$. That is, we have the following diagram

$$\begin{tikzcd} \mathbf{D} \arrow[d, "G^{formal}"'] \arrow[rd, dashed, "G"] & \\ \mathrm{PShv}(\mathbf{C}) & \mathbf{C}. \arrow[l, hook', "Y"] \end{tikzcd}$$

Using the Yoneda lemma, we can see why adjoints are unique: suppose G_1, G_2 are both right adjoint to F. Then $Y(G_1) \simeq G^{formal} \simeq Y(G_2)$, and since Y is fully faithfult hen $G_1 \simeq G_2$.

§4.3 Limits and Functor Categories

Theorem 4.5

Suppose $\mathbf{C}$ and $\mathbf{D}$ are categories, and that I is a small category. If $\mathbf{D}$ has limits of shape I, $\lim_I : \mathrm{Fun}(I, \mathbf{D}) \to \mathbf{D}$, then the composite

$$\mathrm{Fun}(I, \mathrm{Fun}(\mathbf{C}, \mathbf{D})) \simeq \mathrm{Fun}(\mathbf{C}, \mathrm{Fun}(I, \mathbf{D})) \to \mathrm{Fun}(\mathbf{C}, \mathbf{D})$$

is the limit functor.

Proof. Recall that we have an adjunction $\lim_I : F(I, \mathbf{D}) \leftrightarrows D : \Delta$, where Δ is the functor $\Delta : \mathbf{D} \to \mathrm{Fun}(I, \mathbf{D}), X \mapsto (i \mapsto X)$. This functor induces an adjunction

$$\mathrm{Fun}(\mathbf{C}, \mathrm{Fun}(I, \mathbf{D})) \leftrightarrows \mathrm{Fun}(\mathbf{C}, \mathbf{D})).$$

Now, consider the composite

$$\mathrm{Fun}(\mathbf{C}, \mathbf{D}) \to \mathrm{Fun}(\mathbf{C}, \mathrm{Fun}(I, \mathbf{D}) \simeq \mathrm{Fun}(I, \mathrm{Fun}(\mathbf{C}, \mathbf{D})),$$

where the map $\mathrm{Fun}(\mathbf{C}, \mathbf{D}) \to \mathrm{Fun}(\mathbf{C}, \mathrm{Fun}(I, \mathbf{D}))$ is the map induced by $\Delta : \mathbf{D} \to \mathrm{Fun}(I, \mathbf{D})$. This composite coincides with the constant functor

$$\mathrm{Fun}(\mathbf{C}, \mathbf{D}) \to \mathrm{Fun}(I, \mathrm{Fun}(\mathbf{C}, \mathbf{D})).$$

To see this, note that

$$\begin{aligned} \Delta : \mathrm{Fun}(\mathbf{C}, \mathbf{D}) &\to \mathrm{Fun}(I, \mathrm{Fun}(\mathbf{C}, \mathbf{D})) \\ : A &\mapsto (X \mapsto (i \mapsto A(X))), \end{aligned}$$

in the first step of the composite, and then

$$i \mapsto (X \mapsto A(X))$$

when viewed as a functor $I \to \mathrm{Fun}(\mathbf{C}, \mathbf{D})$. □

Exercise 4.6. Show that $\mathrm{Fun}(I, \mathbf{D}) \leftrightarrows \mathbf{D}$ induces the adjunction $\mathrm{Fun}(\mathbf{C}, \mathrm{Fun}(I, \mathbf{D})) \leftrightarrows \mathrm{Fun}(\mathbf{C}, \mathbf{D})$ as stated in the proof above.

Given a presheaf $F : \mathbf{C}^{op} \to \mathbf{Set}$, let $(\star \Rightarrow F)^{op}$ be the category whose objects are pairs (X, Z) with $X \in \mathbf{C}$ and $Z \in F(X)$, and whose morphisms are $X_1 \to X_2 \in \mathbf{C}$. We can then define the functor

$$\begin{aligned} P : (\star \Rightarrow F)^{op} &\to \mathrm{PShv}(\mathbf{C}) \\ : (X, Z) &\mapsto Y(X). \end{aligned}$$

We also have the constant functor

$$\begin{aligned} \Delta : \mathrm{PShv}(\mathbf{C}) &\to \mathrm{Fun}((\star \Rightarrow F)^{op}, \mathrm{PShv}(\mathbf{C})) \\ : G &\mapsto ((X, Z) \mapsto G). \end{aligned}$$

Using these two functors, we can state the following lemma.

Lemma 4.7

Suppose $\mathbf{C}$ is a small category, and F and G are presheaves. Then there is a natural isomorphism

$$\operatorname{Hom}_{\mathrm{PShv}(\mathbf{C})}(F,G) \simeq \operatorname{Hom}_{\mathrm{Fun}((\star\Rightarrow F)^{op},\mathrm{PShv}(\mathbf{C}))}(P,\Delta(G)).$$

Exercise 4.8. Prove lemma 4.7.

Corollary 4.9

Every presheaf is a colimit of representables. That is, if F is a presheaf, then

$$F \simeq \operatorname{colim}_{(\star\Rightarrow F)^{op}} P.$$

Corollary 4.10

If $\mathbf{C}$ is a small category and $\mathbf{D}$ is a category with small colimits, then

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{PShv}(\mathbf{C}),\mathbf{D}) \simeq \mathrm{Fun}(\mathbf{C},\mathbf{D}).$$

The category $\mathrm{Fun}^{\mathrm{colim}}$ is the full subcategory of functors which preserve small colimits.

As intuition, this corollary gives the universal property for the formation of presheaf categories:

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "F"] \arrow[d] & \mathbf{D} \\
\mathrm{PShv}(\mathbf{C}) \arrow[ur, dashed, "\hat{F}"'] &
\end{tikzcd}$$

Here, $\hat{F} : \mathrm{PShv}(\mathbf{C}) \to \mathbf{D}$ is defined by using corollary 4.9 to get $G \simeq \operatorname{colim}_{(\star\to G)^{op}} P$ for $G \in \mathrm{PShv}(\mathbf{C})$, and then we define

$$\hat{F}(G) = \operatorname{colim}_{(\star\Rightarrow G)^{op}} F \circ P.$$

Exercise 4.11. Check that the above definition of $\hat{F}$ makes sense and prove corollary 4.10.

§5 February 9, 2021: Philip LaPorte

Today's lecture is given by Philip LaPorte. Our goal will be to abstract properties of categories such as the category of Abelian groups, the category of R-modules, and others, in what are called *Abelian categories*. We'll also discuss how these are related to notions such as kernels, cokernels, and exact sequences.

§5.1 Additive Categories

We begin with the definition of an additive category.

Definition 5.1 — An **additive category** is a category $\mathbf{C}$ with the following three properties:

1. For any $A, B \in \mathbf{C}$, the set $\operatorname{Hom}_{\mathbf{C}}(A, B)$ has the structure of an Abelian group (under composition), and the composition rule must be **bi-additive**, meaning that the composition
$$\operatorname{Hom}(A,B) \times \operatorname{Hom}(B,C) \to \operatorname{Hom}(A,C)$$
is bilinear. This gives a linear map $\operatorname{Hom}(A,B) \otimes \operatorname{Hom}(B,C) \to \operatorname{Hom}(A,B)$.
2. There is a **zero object** $0 \in \operatorname{Ob}(\mathbf{C})$, such that $\operatorname{Hom}(0,0)$ is the trivial group.
3. For any two objects A_1, A_2, there exists an object B and morphisms
$$\begin{tikzcd} A_1 \arrow[r, shift right, "i_1"'] & B \arrow[l, shift right, "\pi_1"'] \arrow[r, shift left, "\pi_2"] & A_2 \arrow[l, shift left, "i_2"] \end{tikzcd}$$
such that the following identities hold:
$$\begin{aligned} \pi_1 i_1 &= \mathrm{id}_{A_1} \\ \pi_2 i_2 &= \mathrm{id}_{A_2} \\ i_1 \pi_2 &= 0 \\ i_2 \pi_1 &= 0 \\ i_1\pi_1 + i_2\pi_2 &= \mathrm{id}_B. \end{aligned}$$

We have some useful and intuitive immediate consequences of the above definition:

(a) For every A, the sets $\operatorname{Hom}(0,A)$ and $\operatorname{Hom}(A,0)$ are the trivial group. This follows from the fact that we have a linear map $\operatorname{Hom}(0,0) \otimes \operatorname{Hom}(0,A) \to \operatorname{Hom}(0,A)$. Since $\operatorname{Hom}(0,0)$ is trivial, then the product on the left is trivial. The map is surjective, since $\mathrm{id} \otimes f \mapsto f \circ \mathrm{id} = f$. Since we have a surjective map from the trivial group to the group $\operatorname{Hom}(0,A)$, then $\operatorname{Hom}(0,A)$ is trivial. Similarly for $\operatorname{Hom}(A,0)$.

(b) Any two zero objects are isomorphic.

(c) We have $B = A_1 \times A_2 = A_1 \sqcup A_2$, i.e. B is both the direct sum and direct product of A_1, A_2. We have that the following diagrams are Cartesian and co-Cartesian, respectively:
$$\begin{tikzcd} B \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & A_1 \arrow[d] \\ A_2 \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} B & A_1 \arrow[l, "i_1"'] \\ A_2 \arrow[u, "i_2"] & 0 \arrow[l] \arrow[u] \end{tikzcd}$$
For the first diagram, suppose that we have
$$A_1 \xleftarrow{p_1} C \xrightarrow{p_2} A_2.$$
Define $\varphi : C \to B$ by $\varphi = i_1 p_1 + i_2 p_2$. We then have that $p_1 = \pi_1 \circ \varphi$ and $p_2 = \pi_2 \circ \varphi$. Furthermore, using the identities from the third property of an additive category

above, if φ' satisfies $p_1 = \pi_1 \circ \varphi'$ and $p_2 = \pi_2 \circ \varphi'$, then $i_1\pi_1\varphi' + i_2\pi_2\varphi' = \mathrm{id}_B\varphi'$, and thus φ is unique, and the first square is Cartesian. The check for the second square is similar.

§5.2 Kernels and Cokernels

Suppose we have a morphism $\varphi : A \to B$. By Yoneda, there is an associated natural transformation of functors $h^A \to h^B$, where $h^A : \mathbf{C}^{op} \to \mathbf{Set}$ is the functor

$$\begin{aligned} h^A &: C \mapsto \mathrm{Hom}(C, A) \\ &: f \mapsto - \circ f. \end{aligned}$$

We can use the fact that the Hom sets now have group structure to define kernels and cokernels. Letting $f : D \to C$, we define a functor $\ker\varphi : \mathbf{C}^{op} \to \mathbf{Ab}$ as follows. Let

$$(\ker\varphi)(C) = \ker(\mathrm{Hom}(C, A) \to \mathrm{Hom}(C, B)).$$

Define $(\ker\varphi)(f)$ as the restriction of $h^A(f) : \mathrm{Hom}(C, A) \to \mathrm{Hom}(D, A)$ to $(\ker\varphi)(C) \subseteq \mathrm{Hom}(C, A)$. To visualize this, consider the following diagram

$$\begin{tikzcd}
(\ker\varphi)(C) = \ker\big(\mathrm{Hom}(C, A) \arrow[r] \arrow[d, "h^A(f)"] & \mathrm{Hom}(C, B)\big) \arrow[d, "h^B(f)"] \\
(ker\varphi)(D) = \ker\big(\mathrm{Hom}(D, A) \arrow[r] & \mathrm{Hom}(D, B)\big).
\end{tikzcd}$$

with a left vertical arrow labelled $(\ker\varphi)(f)$ from $(\ker\varphi)(C)$ to $(ker\varphi)(D)$.

This gives us a natural transformation of functors $\ker\varphi \to h^A \to h^B$. Suppose that $\ker\varphi$ is representable, so that it is equal to h^K for some K. Then our natural transformation is $h^K \to h^A \to h^B$, and we have maps $K \xrightarrow{k} A \xrightarrow{\varphi} B$. We can define the kernel this way, or we can use the following definition.

Definition 5.2 — The **kernel** of a morphism $\varphi : A \to B$ is a pair (K, k) such that $K \xrightarrow{k} A \xrightarrow{\varphi} B$ with $\varphi k = 0$, and such that for any other pair (K', k') with $K' \xrightarrow{k'} A \xrightarrow{\varphi} B$ and $\varphi k' = 0$, there exists a unique morphism $h : K' \to K$ such that $k' \circ u = k$. This is shown in the following diagram:

$$\begin{tikzcd}
K \arrow[r, "k"] & A \arrow[r, "\varphi"] & B \\
K' \arrow[u, dashed, "\exists! h"] \arrow[ur, "k'"'] & &
\end{tikzcd}$$

Alternatively, we can define K as the equalizer:

$$\begin{tikzcd}
K \arrow[r, "k"] & A \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "0"'] & B.
\end{tikzcd}$$

Note that such a pair (K, k) might not exist.

Above we defined the kernel as the object representing the functor $C \mapsto \ker(h^A(C) \to h^B(C))$. A similar definition does not make sense for the cokernel. Instead we define it directly using the universal property.

Definition 5.3 — The **cokernel** of a map $\varphi : A \to B$ is a pair (C, c) such that $A \xrightarrow{\varphi} B \xrightarrow{c} C$ with $c\varphi = 0$, such that for any other pair (C', c') with $A \xrightarrow{\varphi} B \xrightarrow{c'} C'$ and $c'\varphi = 0$, there exists a unique $u : C \to C'$ such that $c' = u \circ c$. This is shown in the following diagram:

$$\begin{tikzcd}
A \arrow[r, "\varphi"] & B \arrow[r, "c"] \arrow[dr, "c'"'] & C \arrow[d, dashed, "u"] \\
& & C'
\end{tikzcd}$$

Alternatively, we can define the cokernel as the coequalizer

$$\begin{tikzcd}
A \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "0"'] & B \arrow[r, "c"] & C.
\end{tikzcd}$$

Again, note that such a pair (C, c) might not exist.

While the kernel and cokernel might not exist, they always exist in what are called *Abelian categories*, which we discuss next. In addition, we will often refer to the kernel or cokernel as simply the object part of the object-morphism pair, which matches with the terminology of kernels you might be familiar with.

Exercise 5.4. Prove that if the kernel exists, it is unique.

Exercise 5.5. Prove that the following definition of cokernel is equivalent to the one above: the cokernel of a morphism $\varphi : A \to B$ is a pair (C, c), with $c : B \to C$, such that for any $X \in \mathrm{Ob}(\mathbf{C})$, the following sequence of groups is exact:

$$0 \to \mathrm{Hom}(C, X) \to \mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X).$$

§5.3 Abelian Categories

An Abelian category is an additive category with an additional property.

Definition 5.6 — An **Abelian category** is an additive category in which the following property is satisfied:

For any morphism $\varphi : A \to B$, there exists a sequence

$$K \xrightarrow{k} A \xrightarrow{i} I \xrightarrow{j} B \xrightarrow{c} C$$

with the following properties:

a) $j \circ i = \varphi$;

b) K is the kernel of φ, $K = \ker(\varphi)$;

c) C is the cokernel of φ, $C = \mathrm{coker}(\varphi)$;

d) I is the cokernel of φ and the kernel of c, $I = \mathrm{coker}(\varphi) = \ker(c)$.

This condition is stronger than simply existence of a kernel and cokernel. Existence of a kernel and cokernel means we have

$$\begin{tikzcd}
K \arrow[r, "k"] & A \arrow[r, "i"] \arrow[dr, "\varphi"'] & I \arrow[r, dotted] \arrow[d, dashed, "h"] \arrow[dr, "\mathrm{id}"] & I' \arrow[r, "j"] & B \arrow[r, "c"] & C \\
 & & B & I \arrow[u, dashed, "u"] \arrow[ur, dashed, "h"'] & &
\end{tikzcd}$$

In an abelian category, the dotted line $I \to I'$ is an isomorphism.

Exercise 5.7. Show that in an Abelian category, any morphism φ with $\ker \varphi = 0$ and $\operatorname{coker} \varphi = 0$ is an isomorphism.

We know several examples of Abelian categories (**Ab**, $\mathbf{Mod}_R$, etc.), so we'll now go over some examples of additive categories which are *not* Abelian.

Example 5.8 (Filtered Abelian Groups)

A **filtered Abelian group** is an Abelian group X with filtration $\cdots F^iX \subset F^{i+1}X \subset \cdots \subset X$. The category **AbF** has as objects filtered Abelian groups, and morphisms defined by

$$\operatorname{Hom}_{\mathbf{AbF}}(X, Y) = \{\varphi : X \to Y | \varphi(F^iX) \subset F^iY \forall i\}.$$

If $\varphi : X \to Y$, write $F^i\varphi : F^iX \to F^iY$ for the restriction of φ on F^iX. You can check that the kernels are $\ker F^i\varphi$, and the cokernels are $F^iY/(F^iY \cap \varphi(X))$. We'll now show that this is not an Abelian category.

Suppose that we have two filtrations, $\{F_1^iX\}, \{F_2^iX\}$, of some Abelian group X, such that $F_1^iX \subseteq F_2^iX$ for all i and $F_1^jX \subsetneq F_2^jX$ for at least one j. Consider $\mathrm{id}_X : X^{(1)} \to X^{(2)}$. This has zero kernel and cokernel in **AbF** (check using the definitions above), but is not an isomorphism.

Example 5.9 (Topological Abelian Groups)

A **topological Abelian group** is an Abelian group with a Hausdorff topology. The category **ABT** has as objects the topological Abelian groups, and as morphisms continuous group homomorphisms. For a morphism $\varphi : A \to B$, $\ker \varphi$ the kernel of φ with the subspace topology, and $\operatorname{coker} \varphi$ is $B/\overline{\varphi(A)}$ with quotient topology, where $\overline{\varphi(A)}$ is a closed subgroup of B. Irrantional winding on the torus ($t \mapsto \alpha t$ for $\alpha \notin \mathbb{Q}$) has dense image, which implies zero cokernel and kernel, but is not an isomorphism (not surjective).

§5.4 Diagram Chasing

We begin with some definitions. In what follows, we will be working in an Abelian category. Let $f : X \to Y$ be a morphism. If for any object $Z \in \mathbf{C}$ and any morphisms $g_1, g_2 : Z \to X$ we have that $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$, then f is called a **monomorphism**. If for any object $Y \in \mathbf{C}$ and any morphisms $g_1, g_2 : X \to Y$, we have that $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$, then f is called an **epimorphism**.

A sequence

$$A \xrightarrow{f} B \xrightarrow{g} \xrightarrow{C}$$

is **exact** at B if $\operatorname{im}(f) = \ker(g)$, or $\operatorname{coim}(g) = \operatorname{coker}(f)$.

Lemma 5.10

Suppose that the right square in the following diagram is Cartesian:

$$\begin{tikzcd}
K' \arrow[r, "k'"] & Z \arrow[r, "f"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\
K \arrow[r, "k"] & X \arrow[r, "f"] & U.
\end{tikzcd}$$

Then if f is an epimorphism, then f' is also an epimorphism. In addition, we have $K = K'$ and $k = g'k'$.

Note: pullbacks and pushouts exist an an Abelian category. In fact, with a couple definitions, we can state a theorem. A monomorphism is **normal** if it is the kernel of some morphism, and an epimorphism is **conormal** if it is the cokernel of some morphism. If every monomorphism in a category is normal, it is called a **normal category**, and if every epimorphism in a category is conormal, it is called a **conormal category**. You can then do the following exercise.

Exercise 5.11. Prove that the following are equivalent for a category $\mathbf{A}$:

1. $\mathbf{A}$ is an Abelian category;
2. $\mathbf{A}$ has kernels, cokernels, finite products, finite coproducts, and is normal and conormal;
3. $\mathbf{A}$ has pullbacks, pushouts, and is normal and conormal.

An **element** of an Abelian category $\mathbf{A}$ is an equivalence class of pairs (X, h), where X is an object and $h : X \to Y$ is a morphism, defined by the equivalence relation $(X, H) \sim (X', h')$ if and only if there exist $Z \in \text{Ob}(\mathbf{A})$ and epimorphisms $u : Z \to X, u' : Z \to X'$ such that $hu = h'u'$. The above lemma helps to check that this is in fact an equivalence relation. We can now state some **diagram chasing rules**. We use = instead of equivalent for elements in the below.

1. A morphism $f : A \to B$ is a monomorphism if and only if $f(x) = 0$ implies $x = 0$ for $x \in A$.
2. A morphism $f : A \to B$ is a monomorphism if and only if $f(x) = f(x')$ implies $x = x'$ for $x, x' \in A$.
3. A morphism $g : B \to C$ is an epimorphism if and only if for all $z \in C$, there exists $y \in B$ such that $g(y) = z$.
4. A morphism $h : R \to S$ is zero if and only if $h(x) = 0$ for all $x \in R$.
5. The sequence $A \xrightarrow{f} B \xrightarrow{g} \xrightarrow{C}$ is exact at B if and only if $g \circ f = 0$ and for all $y \in B$ with $g(y) = 0$, there exists $x \in A$ with $f(x) = y$.
6. Suppose $g : A \to B$ is a morphism, and $x, y \in A$ satisfy $g(x) = g(y)$. Then there exists $z \in A$ such that $g(z) = 0$. This z also satisfies the following properties:
    (a) For any $f : A \to C$ with $f(x) = 0$, then $f(z) = -f(y)$.
    (b) For any $f' : A \to C$ with $f'(y) = 0$, then $f'(z) = f'(x)$.

The first two properties above might look similar to properties of injective maps in familiar categories. The third looks similar to what happens in surjective categories. In the last property, we can think of z as being an analogue to the difference $x - y$.

Using these rules, we can prove the following lemmas.

Lemma 5.12 (Five Lemma)

Consider the following commutative diagram:

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d, "f_1"] & X_2 \arrow[r] \arrow[d, "f_2"] & X_3 \arrow[r] \arrow[d, "f_3"] & X_4 \arrow[r] \arrow[d, "f_4"] & X_5 \arrow[d, "f_5"] \\ Y_1 \arrow[r] & Y_2 \arrow[r] & Y_3 \arrow[r] & Y_4 \arrow[r] & Y_5 \end{tikzcd}$$

Suppose that the rows of this diagram are exact. If f_1 is an epimorphism, f_5 is a monomorphism, and f_2 and f_4 are isomorphisms, then f_3 is an isomorphism.

Lemma 5.13 (Snake Lemma)

Consider the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & X_1 \arrow[r, "g_1"] \arrow[d, "f_1"] & X_2 \arrow[r, "g_2"] \arrow[d, "f_2"] & X_3 \arrow[r] \arrow[d, "f_3"] & 0 \\ 0 \arrow[r] & Y_1 \arrow[r, "h_1"] & Y_2 \arrow[r, "h_2"] & Y_3 \arrow[r] & 0. \end{tikzcd}$$

Suppose that the rows of this diagram are exact. Then the following sequences are exact:

$$0 \to \ker(f_1) \xrightarrow{a_1} \ker(f_2) \xrightarrow{a_2} \ker(f_3),$$
$$\operatorname{coker}(f_1) \xrightarrow{b_1} \operatorname{coker}(f_2) \xrightarrow{b_2} \operatorname{coker}(f_3) \to 0.$$

Here a_1 and a_2 are induced by g_1 and g_2 respectively, and b_1 and b_2 are induced by h_1 and h_2 respectively.

Exercise 5.14. Prove lemma 5.12 and lemma 5.13.

§6 February 11, 2021: Raluca Vlad

§6.1 Projective Modules

Today we discuss modules, which are an example of Abelian categories from last time. Although we will be working with $\mathbf{Mod}_R$, many of the definitions and results generalize to Abelian categories.

Definition 6.1 — A covariant functor $F : \mathbf{Mod} \to \mathbf{Ab}$ is called **exact** if for any short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0,$$

the sequence

$$0 \to F(A) \xrightarrow{F(i)} F(B) \xrightarrow{F(p)} F(C) \to 0$$

is exact, and $F(0) = 0$.

Definition 6.2 — A module M is a **projective module** if the functor $\operatorname{Hom}(M, -)$ is exact. That is, if for any short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0,$$

the sequence

$$0 \to \operatorname{Hom}(M, A) \xrightarrow{-\circ i} \operatorname{Hom}(M, B) \xrightarrow{-\circ p} \operatorname{Hom}(M, C) \to 0$$

is also exact.

Recall that for any module M, then $\operatorname{Hom}(M, -)$ is always left exact. That is, $0 \to \operatorname{Hom}(M, A) \to \operatorname{Hom}(M, B) \to \operatorname{Hom}(M, C)$ is always exact for an exact sequence $0 \to A \to B \to C \to 0$. Thus we can state the above definition equivalently as

Definition 6.3 — A module M is projective if and only if for any surjection $B \twoheadrightarrow C$ then $\operatorname{Hom}(M, B) \twoheadrightarrow \operatorname{Hom}(M, C)$ is also a surjection. That is, for any exact $A \to B \to 0$ and any morphism $\varphi \in \operatorname{Hom}(M, C)$, there is a unique map $\psi \in \operatorname{Hom}(M, B)$, as in the following diagram:

$$\begin{tikzcd}
 & M \arrow[ld, dashed, "\exists\psi"'] \arrow[d, "\varphi"] & \\
B \arrow[r] & C \arrow[r] & 0.
\end{tikzcd}$$

Proposition 6.4

A module M is projective if and only if M is a direct summand of a free module, $M \oplus N = R^I$.

Proof. Suppose that M is projective. Find a surjective map $\pi : R^I \twoheadrightarrow M$. By the definition of projectivity, there exists $f : M \to R^I$ such that the following commutes

$$\begin{tikzcd}
 & M \arrow[ld, dashed, "f"'] \arrow[d, "\mathrm{id}"] & \\
R^I \arrow[r, "\pi"] & M \arrow[r] & 0.
\end{tikzcd}$$

Then $R^I = \operatorname{im}(f) \oplus \ker(\pi) = M \oplus \ker(\pi)$. Thus, if M is projective, then it is a direct summand.

For the other direction, we first show that free modules are projective. That is, for any surjective map $\pi : B \to C$ and any $R^I \to C$, then we must find a map $g : R^I \to B$ such that $\pi \circ g = f$. To this end, define the map on the generators $\{r_i\}$ of R^I by $r_i \mapsto \pi^{-1}(f(r_i))$. This is well defined since π is surjective.

Now, suppose we have M with $M \oplus N = R^I$, for some N, R^I. Let $\pi : B \to C$ be an arbitrary surjection of modules, and suppose we have a map $c : M \to C$. We have a projection map $R^I \xrightarrow{p} M$, and we know that R^I is projective, so we have the map $g : R^I \to B$ defined above. Our desired map for M to be projective is $g\,|_M$, as shown in the following diagram:

$$\begin{tikzcd}
R^I \arrow[dr, "p"] \arrow[dd, "g"'] & \\
 & M \arrow[dl, "g|_M"'] \arrow[d, "c"] \\
B \arrow[r, "\pi"] & C.
\end{tikzcd}$$

□

Note: a module M is projective if and only if $\begin{tikzcd} M \arrow[r, "i"'] & R^I \arrow[l, bend right, "p"'] \end{tikzcd}$. Now, we see that M is a coequalizer, by considering the following diagram

$$\begin{tikzcd}
R^I \arrow[r, shift left, "i\circ p"] \arrow[r, shift right, "\mathrm{id}"'] & R^I \arrow[r] & M.
\end{tikzcd}$$

Conversely, suppose N is a coequalizer

$$\begin{tikzcd}
R^I \arrow[r, shift left, "i\circ p"] \arrow[r, shift right, "\mathrm{id}"'] & R^I \arrow[r, "p"'] & N. \arrow[l, bend right, "i"']
\end{tikzcd}$$

Then N isn't quite projective. We also need the condition that N is a **split coequalizer**, which is a coequalizer with maps s and t such that

$$\begin{tikzcd}
A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B \arrow[l, bend right, "t"'] \arrow[r, "e"'] & C, \arrow[l, bend right, "s"']
\end{tikzcd}$$

with $es = 1_C, se = gt, ft = 1_B$. Then we have that N is projective if and only if it is a split coequalizer:

$$\begin{tikzcd}
R^I \arrow[r, shift left, "i\circ p"] \arrow[r, shift right, "\mathrm{id}"'] & R^I \arrow[l, bend right] \arrow[r, "p"'] & N. \arrow[l, bend right, "i"']
\end{tikzcd}$$

§6.2 Flat Modules

Definition 6.5 — A module M is called a **flat module** if $\otimes M$ is exact. That is, for any exact sequence $0 \to A \to B \to C \to 0$, then

$$0 \to A \otimes M \to B \otimes M \to C \otimes M \to 0$$

is also exact.

Equivalently, a module is flat if for any injective map $A \hookrightarrow B$, then $A \otimes M \to B \otimes M$ is also injective. This follows from the fact that $\otimes M$ is right exact for any M.

We'll now discuss what is called *equational criterion for flatness.*

Definition 6.6 — A relation $\sum r_i m_i = 0$ is called **trivial** if there exist $m \geq 0$, $n_j \in M, j = 1, \ldots, m$, and $a_{ij} \in R, i = 1 \ldots, n, j = 1 \ldots, m$, such that

$$m_i = \sum_j a_{ij} n_j, \forall i, \quad \text{and} \quad \sum_i r_i a_{ij} = 0, \forall j.$$

Proposition 6.7

A module M is flat if and only if every relation in M is trivial.

Proof. Suppose M is flat. Let $\sum r_i m_i = 0$ be a relation in M. Let

$$\begin{aligned} I &= (r_1, \ldots, r_n) \subseteq R \\ K &= \ker(R^n \to I, (a_1, \ldots, a_n) \mapsto \sum a_i r_i). \end{aligned}$$

Then we have an exact sequence $0 \to K \to R^n \to I \to 0$. Since M is flat, we get an exact sequence $0 \to K \otimes M \to R^n \otimes M \to I \otimes M \to 0$. Consider $\sum r_i \otimes m_i$ in $I \otimes M$. Since we have an injection $I \hookrightarrow R$ and $I \otimes M \hookrightarrow R \otimes M$, then $\sum r_i \otimes m_i = 1 \otimes \sum r_i m_i = 0$ in $R \otimes M$ implies that $\sum r_i \otimes m_i = 0$ in $I \otimes M$. Now, look at

$$\sum k_j \otimes n_j \mapsto \sum e_i \otimes m_i \mapsto \sum r_i \otimes m_i = 0.$$

But $k_j = \sum_i a_{ij} e_i$ in R^n, and we are done.

For the other direction, note that it is enough to check that M is flat if $I \to R$ injective implies $I \otimes M \to R \otimes M$ is injective for any finitely generated ideal I. Thus, suppose that $\sum r_i \otimes m_i \mapsto 0$ under the map $I \otimes M \to R \otimes M$. We want to show that $\sum r_i \otimes m_i$ is zero in $I \otimes M$. Since $\sum r_i \otimes m_i = 0$ in $R \otimes M$, then $1 \otimes (\sum r_i m_i) = 0$ in $R \otimes M$, which implies $\sum r_i m_i = 0$ in M. In $I \otimes M$, we can then write

$$\sum_i r_i \otimes m_i = \sum_i r_i \otimes (\sum_j a_{ij} n_j) = \sum_j (\sum_i r_i a_{ij}) \otimes n_j = 0.$$

□

Example 6.8

Here are some examples of flat modules:

- Free modules are flat.
- Direct summands of flat modules are flat.
- Projectives are flat (implied by the above two bullets).

§6.3 Lazard's Theorem

We'll now state and prove Lazard's theorem, which gives an interesting characterization of flat modules, related to the equational criterion of flatness, in the sense that using directed colimits, one cannot introduce nontrivial relations.

Theorem 6.9 (Lazard's Theorem)

A modules M is flat if and only if it is the colimit of a directed system of free finite rank R-modules.

In the above, a colimit of a directed system is a colimit of a functor $F : I \to \mathbf{Mod}$, where $(I, \leq)$ is directed, i.e. for all $i, j \in I$, there exists a k with $i, j \leq k$. The following examples illustrates why we a *directed* colimit and not just a colimit.

Example 6.10

Consider the diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\times 2"] \arrow[d, "\times 2"] \arrow[dr, "\xi"] & \mathbb{Z} \arrow[d, "\phi"] \\
\mathbb{Z} \arrow[r, "\psi"] & \text{colim}
\end{tikzcd}$$

Here, ξ is the map $1 \mapsto (1, 0, 0)$, ϕ is the map $1 \mapsto (1, -1, 0)$, and ψ is the map $1 \mapsto (1, 0, -1)$. The colimit is

$$\text{colim} = \mathbb{Z}^3/((1, 0, -2), (1, -2, 0)).$$

Note that $2(1, -1, -1) = (1, 0, -2) + (1, -2, 0) = 0$, with $(1, -1, -1) \neq 0$. Thus, the colimit has torsion and is therefore not flat.

To prove theorem 6.9, we'll use the following lemma concerning finitely presented modules. A **finitely presented module** is a module M such that there exists an exact sequence $R^{\oplus m} \to R^{\oplus n} \to M \to 0$, for some $m, n \in \mathbb{N}$.

Lemma 6.11

Any module M is a colimit of a directed system of finitely presented modules.

Proof. Let $f : R^I \twoheadrightarrow M$ be a surjection, for some I, and let $K = \ker(f)$. Consider the set of pairs (J, N), where $J \subset I$ is a finite subset and N is a finitely generated submodule of $R^K \cap K$. Define a relation $\leq$ by $(J, N) \leq (J', N')$ if and only if $J \subset J'$ and $N \subset N'$. For each pair $e = (J, N)$, define $M_e = R^J/N$. For any $e \leq e'$, define the map $f_{ee'} : M_e \to M_{e'}$ as the natural map (i.e. $R^J \to R^{J'}/(R^{J'} \cap K)$ induces a map $R^J/(R^J \cap K) \to R^{J'}/(R^{J'} \cap K)$ since $R^J \cap K$ is in the kernel of the map). Now, the $(M_e, f_{ee'})$ form a directed system, and the maps $f_e : M_e \to M = R^J/(K \cap R^J)$ induce an isomorphism $\text{colim}_e M_e \to M$. □

Exercise 6.12. Work out the details of the isomorphism $\text{colim} M_e \to M$ induced by the maps $M_e \to M$.

Proof (of theorem 6.9). First, note that taking directed colimits is exact and commutes with tensor products, so a colimit of a directed system of flat modules is flat. So if M is the colimit of a directed system of free modules, then it is flat (free modules are flat).

For the converse, suppose that M is flat. Let $I = M \times \mathbb{Z}$, and define $f : R^I \twoheadrightarrow M$ as the projection map. Take $E = \{(J, N)\}$, by lemma 6.11, we can write $M = \text{colim}_e M_e$ where $e \in E$. To prove the theorem, it suffices to show that for all $e = (J, N) \in E$, there exists an $e' \geq e$ such that $M_{e'}$ is free (note that the M_e are finite, by definition of J).

This is enough, since then M is a colimit of these e'. To this end, we use the following fact: a module M is flat if and only if for any finitely presented module P and any map $f : P \to M$, there is a finite free F and $h : P \to F$ and $g : F \to M$ such that $f = g \circ h$. This implies that $f_e : R^J/N \to M$ factors as $R^J/N \xrightarrow{h} F \xrightarrow{g} M$, for some finite free F. We now show that $F = M_{e'}$ for some $e' \geq e$. Let $\{b_1, \dots, b_n\}$ be a basis for F. Choose distinct $i_1, \dots, i_n \in I \setminus J$ with $f(r_{i_e}) = g(b_l)$. Let $J' = J \cup \{i_1, \dots, i_n\{$ and define $\varphi : R^{J'} \to F$ by $\varphi : r_i \mapsto h(r_i)$ for $i \in J$ and $\varphi : r_{i_l} \mapsto b_l$. Then let $N' = \ker(\varphi)$, and note that this implies $N' \subset \ker(f)$. We also have that N' is finitely generated, and thus $(J', N') \in E$. This implies that $e \leq e'$, so $F \simeq R^{J'}/N' = M_{e'}$ is free. ☐

Exercise 6.13. In the above proof, verify that $N' \subset \ker(f)$ and that N' is finitely generated.

§6.4 Injective Modules

We now discuss injective modules.

Definition 6.14 — A module M is **injective** if $\mathrm{Hom}(-, M)$ is exact. That is, for any exact sequence $0 \to A \to B \to C \to 0$, then

$$0 \to \mathrm{Hom}(C, M) \to \mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M) \to 0$$

is also exact. Equivalently, a module M is injective if and only if for every injection $A \to B$ and any homomorphism $A \to M$, there is a map $B \to M$ so the following diagram commutes

$$\begin{tikzcd}
M & \\
A \arrow[u] \arrow[r] & B \arrow[ul, dashed]
\end{tikzcd}$$

Example 6.15

A vector space over a field is an injective module.

Example 6.16

The module $\mathbb{Q}$ over $\mathbb{Z}$ is injective. To see this, suppose that $A \to \mathbb{Q}$ is a homomorphism and $A \to B$ is an injection. Look at pairs (A_1, f_1) with $A_1 \subset B$ a subgroup such that $A \subset A_1$, and $f : A_1 \to \mathbb{Q}$ with $f_1 |_A = f$. We say $(A_1, f_1) \geq (A_2, f_2)$ if $A_1 \supseteq A_2$ and $f_1 |_{A_2} = f_2$. Then Zorn's lemma implies that there is a maximal element which is (B, g). To see this, let (A_1, f_1) with $A_1 \subsetneq B$, and take $b \in B \setminus A_1$. Let $A_2 = \langle A_1, b \rangle \subset B$. We want to show there is an $f_2 : A_1 \to \mathbb{Q}$ such that $f_2 |_{A_1} = f_1$. If $nb \notin A_1$ for any nonzero $n \in \mathbb{Z}$, then $f_2(b)$ can be anything. If $nb \in A_1$, then let $n_0 \in \mathbb{Z}$ be the minimal positive integer such that $n_0 b \in A_1$. Define $f_2(b) = f_1(a)/n_0$. We've shown that $A_1 \subsetneq B$ is not maximal, and thus (B, g) is maximal and g is our desired map.

The proof from the example above also works for any divisible Abelian group. In addition, any injective Abelian group is divisible. Since divisible Abelian groups are not finitely generated, then injective Abelian groups are not finitely generated.

§7 February 16, 2021

§7.1 The Ubiquity of Complexes

We begin with basic fact: if K is a field, then any finitely generated K-module (vector space) has a basis. This is why linear algebra works, and is one of the most fundamental results in mathematics. However, if we replace K with a commutative ring, this no longer works. For example, let $R = \mathbb{C}[x, y, z]$, the polynomial ring. Consider the finitely generated R-module $I = (xy, xz)$ (an ideal). This module does not have a basis. If you guessed that $\{xy, xz\}$ is a basis, note that unfortunately $z \cdot (xy) - y \cdot (xz) = 0$, so this is not linearly independent. In the language of this class, we have a sequence

$$0 \to R \to R^{\oplus 2} \to N \to 0,$$

where the maps are given by vector multiplication by $\begin{pmatrix} z \\ -y \end{pmatrix}$ and $\begin{pmatrix} xy & xz \end{pmatrix}$ respectively, and the composite of these maps is zero. Furthermore, the sequence is exact.

The basic idea behind the derived category of modules over a ring is that $0 \to 0 \to R \to R^{\oplus 2}$ is "as good as" the object N itself, and we can think of $0 \to R \to R^{\oplus 2}$ as a graded module where each map composes to zero with the next.

The above example is part of a more general procedure (assume Noetherian). If we have $M \to 0$, we can find a free module F_0 such that $F_0 \twoheadrightarrow M \to 0$. We can then take the kernel K_1 to get an exact sequence $K_1 \to F_0 \twoheadrightarrow M \to 0$. We can then repeat this to find $F_1, K_2, \ldots$ to get

$$\begin{tikzcd}
& & K_1 \arrow[r] & F_0 \arrow[r, two heads] & M \arrow[r] & 0 \\
& K_2 \arrow[r] & F_1 \arrow[u, two heads] \arrow[ru] & & & \\
K_3 \arrow[r] & F_2 \arrow[u, two heads] \arrow[ru] & & & & \\
F_3 \arrow[u, two heads] \arrow[ru] & & & & & ,
\end{tikzcd}$$

where the maps $F_i \to F_{i-1} \to F_0$ compose to zero, and each F_i is free. The kernel K_j is called the j'th syzygy module.

To get to complexes, we discuss some historical derived ideas. Suppose K is a field and we have a map $K^q \xrightarrow{A} K^p$, where A is a matrix. We're interested in solutions of the form $A\vec{x} = 0$. Suppose that $\vec{x}_1, \ldots, \vec{x}_n$ are solutions to $A\vec{x} = 0$. By rank-nullity, $\vec{X} := \{\vec{x}_j\}$ is complete (spans the solution set) if $n \geq q = \operatorname{rank}(A)$. In the language of this class, we say $F^2 \xrightarrow{\vec{X}} K^q \xrightarrow{A} K^p$ is exact if and only if $\ker(A) = \operatorname{im}(X)$ if and only if $\operatorname{rank}(A) + \operatorname{rank}(X) = \operatorname{rank}(K^q)$. In the case that the $\{\vec{x}_j\}$ are linearly independent, we have

$$0 \to F_2 \xrightarrow{\vec{X}} K^q \xrightarrow{A} K^p$$

exact.

From this, Hilbert asked the following question: what happens if A is a matrix with entries in the polynomial ring $K[x_1, \ldots, x_r]$, $r \geq 0$? That is, if

$$\begin{aligned}
\vec{X}_1 &= (p_1(x_1, \ldots, x_r)^{(1)}, \ldots, p_q(x_1, \ldots, x_r)^{(1)}) \\
&\vdots \\
\vec{X}_n &= (p_1(x_1, \ldots, x_r)^{(n)}, \ldots, p_q(x_1, \ldots, x_r)^{(n)}
\end{aligned}$$

is a solution, can we get more solutions by taking linear combinations of the these polynomials? To rephrase this, let $F_1 = K[x_1, \dots, x_r]^{\oplus p}$ and $F_0 = K[x_1, \dots, x_r]^{\oplus q}$, and ask: is it possible to find a number n and a surjection $K[x_1, \dots, x_r]^{\oplus n}$ such that we have the following diagram:

$$\begin{tikzcd}
K \arrow[r] & F_1 \arrow[r, "A"] & F_0 \\
K[x_1, \dots, x_r]^{\oplus n} \arrow[u, two heads] & & .
\end{tikzcd}$$

The answer is yes, by the Hilbert Basis Theorem, which tells us that polynomial rings are Noetherian, and thus Noetherian modules over themselves. By the definition of Noetherian, then any submodule is finitely generated.

Theorem 7.1 (Hilbert Basis Theorem)

Polynomial rings over Noetherian rings are Noetherian.

Of course, the kernel K might not be free, Hilbert then asked if one can bound the minimum length of a resolution. The answer is in the Hilbert syzygy theorem.

Theorem 7.2 (Hilbert Syzygy Theorem)

Any finitely generated $K[x_1, \dots, x_r]$-module has a free resolution

$$0 \to F_j \to F_{j-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

with $j \leq r$.

§8 February 18, 2021

§8.1 Complexes

Today we discuss the fundamentals of (co)chain complexes and go over a few examples.

Definition 8.1 — Let $\mathbf{A}$ be an Abelian category. A **chain complex** in $\mathbf{A}$ is a sequence of objects in $\mathbf{A}$,

$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots,$$

such that $d_n \circ d_{n+1} = 0$. A chain complex is denoted $C_\bullet$.

A **cochain complex** in $\mathbf{A}$ is a sequence

$$\cdots \to C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} C^{n+2} \to \cdots,$$

such that $d_{n+1} \circ d_n = 0$. A cochain complex is denoted $C^\bullet$.

Note: chain complexes and cochain complexes are the same up to reindexing.

There is a category of chain complexes $\operatorname{Kom}(\mathbf{A})$, with chain complexes $C_\bullet$ as objects and morphisms $f_\bullet : C_\bullet \to D_\bullet$ given by collections of maps, so the following commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n+1}"] & \cdots \\
\cdots \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & \cdots.
\end{tikzcd}$$

The most import definition in concerning chains and cochains (and the most important definition in this class) is that of homology (cohomology) of chain complexes (cochain complexes). Note that since $d \circ d = 0$ (sub/superscripts suppressed), and thus $\text{im}(d) \subseteq \ker(d)$. If the $\subseteq$ we and $=$, the sequence would be exact. Homology and cohomology measure the failure of the sequence to be exact.

Definition 8.2 — The n'th **homology group** of a chain complex $C_\bullet$ is

$$H_n(C_\bullet) = \frac{\ker(d_n : C_n \to C_{n-1})}{\text{im}(d_{n+1} : C_{n+1} \to C_n)}.$$

The n'th **cohomology group** of a cochain complex $C^\bullet$ is

$$H^n(C_\bullet) = \frac{\ker(d^n : C^n \to C^{n+1})}{\text{im}(d^{n-1} : C^{n-1} \to C^n)}.$$

We define the n'th **cycle** as $Z_n(C_\bullet) = \ker(d_n)$, and the n'th boundary as $B_n(C_\bullet) = \text{im}(d_{n+1})$, so $H_n = Z_n/B_n$, and similarly for cohomology (defining the **coboundaries** and **cocycles**).

Since these definitions are so important, we'll cover some examples.

Example 8.3

Let M be a finitely generated left module over a ring R. As discussed last time, we have a sequence of free modules:

$$\cdots F_j \to F_{j-1} \to \cdots \to M \to 0.$$

This complex $F_\bullet(M)$ is called a **free resolution** of M. Note that $H_0(F_\bullet(M)) = \ker(M \to 0)/\text{im}(F_0 \twoheadrightarrow M) = M/M = 0$. Similarly

$$\ker(F_j \to F_{j-1}) = \ker(F_j \to K_{j-1}) = \text{im}(F_{j+1} \to K_j) = \text{im}(F_{j+1} \to F_j),$$

so $H_n(F_\bullet(M)) = 0$ for all n. Chain complexes with $H_n(C_\bullet) = 0$ for all n are called **acyclic** complexes.

Example 8.4 (Singular Homology)

Let X be a topological space, and let Δ^n be the standard n-simplex (Δ^0 is a point, Δ^1 is a line segment, Δ^2 is a triangle, Δ^3 is a tetrahedron, etc.). Define a chain complex by letting $C_n(X) = \mathbb{Z}[\text{Hom}_{cts}(\Delta^n, X)]$, where $\text{Hom}_{cts} = \{f : \Delta^n \to X\}$, the set of continuous functions embedding simplices into X. Note that there are two maps $\Delta^0 \to \Delta^1$, there are three maps $\Delta^1 \to \Delta^2$, four maps $\Delta^2 \to \Delta^3$, etc., which place the a simplex in on the edges of a higher dimensional simplex. Now $C_0(X) = \mathbb{Z}[\text{Hom}_{cts}(\Delta^0, X) = X]$, and $C_1(X) = \mathbb{Z}[\text{Hom}_{cts}(\Delta^1, X)]$, and there are two maps $\partial_0, \partial_1 : C_1(X) \to C_0(X)$, given by precomposition with the two maps $\Delta^0 \to \Delta^1$. Similarly, there are three maps $C_2(X) \to C_1(X)$, given by precomposition with the three maps $\partial_0, \partial_1, \partial_2 : \Delta^1 \to \Delta^2$, and so on. To make this into a chain complex, we only want one map from $C_i(X) \to C_{i-1}(X)$. We do this by defining

$$d_n = \sum_{k=0}^{n} (-1)^k \partial_k,$$

where the ∂_k are the maps defined by precomposition with simplex embedding. The maps d_n make $C_\bullet$ into a chain complex. Homology groups of this complex are called **singular homology** groups.

What is the point of doing this? Taking $C_0(X)$ to be the free Abelian group on X, we obtain something more manageable than X itself, but we lose a lot of information. To get back information, we attach the higher C_n. We can think of singular homology as the "correct Abelianization" of X.

Example 8.5 (deRham Cohomology)

Consider $C^\infty(X)$, the set of global C^∞ functions on a smooth manifold X, i.e. the set of infinitely differentiable functions $C \to \mathbb{R}$. Let $\Omega^i(X)$ be the space ($C^\infty(X)$-module) of i-forms. We then have a map

$$C^\infty(X) \to \Omega^1(X) \to \Omega^2(X) \to \cdots.$$

Denote by dx^I the forms $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, where I is an indexing set. Then the boundary maps are defined by

$$d\left(\sum_{|I|=k} f_I dx^I \right) = \sum_{|I|=k} \sum_i \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^I.$$

Cohomology groups of this complex are called **deRham cohomology** groups.

Example 8.6 (Hochschild Complex)

Let K be a commutative base ring (e.g. $\mathbb{Z}$), A an associative K-algebra with a map $K \to A$, and M an A-bimodule, with maps $M \otimes A \to M$ and $A \otimes M \to M$. The Hochschild complex is defined as follows. Note that there are two maps $M \otimes A \to M$, defined by $m \otimes a \mapsto m \cdot a$ and $m \otimes a \mapsto a \cdot m$. Similarly, there are three maps $M \otimes A \otimes A \to M \otimes A$, defined by

$$\begin{aligned} m \otimes a_1 \otimes a_2 &\mapsto m \cdot a_1 \otimes a_2 \\ m \otimes a_1 \otimes a_2 &\mapsto m \otimes a_1 \cdot a_2 \\ m \otimes a_1 \otimes a_2 &\mapsto a_2 m \otimes a_1. \end{aligned}$$

Similarly, there are four maps $M \otimes A \otimes A \otimes A \to M \otimes A \otimes A$, and similarly for higher powers. Define d_{Hoch} as the alternating sum of the maps $M \otimes A^{\otimes n} \to M \otimes A^{\otimes(n-1)}$. Then the Hochschild complex is given by $\mathrm{Hoch}_\bullet^{(K)}(A; M) = (M \otimes A^{\otimes_K^\bullet}, d_{\mathrm{Hoch}})$. We define $HH_\bullet(A; M)$ as the homology of this complex. We have $HH_0(A; A) = A/[A, A]$, where $[A, A]$ is the commutator. To see this, consider $A \otimes A \to A$, and recall the two maps from above are $a \otimes b \mapsto a \cdot b$ and $a \otimes b \mapsto b \cdot a$. Since the map $A \otimes A \to A$ is the alternating sum, then the image is the commutator $[A, A]$. Then consider $A \otimes A \to A \to 0$, we see that $HH_0 = A/[A, A]$.

Furthermore, if R is a commutative K-algebra, then

$$HH_\star(R; R) \simeq HH_\star(\mathcal{M}_n(R); \mathcal{M}_n(R)),$$

where $\mathcal{M}_n(R)$ is the matrix ring over R. This gives a Morita equivalence between R and $\mathcal{M}_n(R)$. Two rings are called **Morita equivalent** if there is an equivalence between their category of modules. The Morita invariance theorem for matrices is related to the generalized trace: $\operatorname{tr} : \mathcal{M}_r(M) \otimes \mathcal{M}_r(A)^{\otimes n} \to M \otimes A^{\otimes n}$.

We'll now cover some basic results about chain complexes.

Theorem 8.7

If $\mathbf{A}$ is an Abelian category, then $\mathrm{Kom}(\mathbf{A})$, the category of chain complexes, is also Abelian.

Proof. The zero element is given by $0 = \cdots \to 0 \to 0 \to \cdots$. We can add chain complexes as

$$C_\bullet \oplus D_\bullet = \cdots \to C_n \oplus D_n \xrightarrow{d_n^C \oplus d_n^D} C_{n-1} \oplus D_{n-1} \to \cdots.$$

A sequence $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is exact if and only if all the rows are exact:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & B_{n+1} \arrow[r] \arrow[d] & C_{n+1} \arrow[r] & 0 \\
0 \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

□

Theorem 8.8 (LES Theorem)

Let $\mathbf{A}$ be an Abelian category. In $\mathrm{Kom}(\mathbf{A})$, if $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is exact, then there exist maps $\partial : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$, such that we get a long exact sequence

$$\cdots \to H_{n+1}(C) \xrightarrow{\partial} H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to \cdots.$$

Exercise 8.9. Prove theorem 8.8. Hint: use the snake lemma: lemma 5.13.

To visualize the above theorem, we can draw the following diagram:

$$\begin{tikzcd}
H_\star(A) \arrow[rr] & & H_\star(B) \arrow[dl] \\
& H_\star(C) \arrow[ul, "-1"] &
\end{tikzcd}$$

§9 February 23, 2021

§9.1 Complexes

We begin by proving the long exact sequence theorem, theorem 8.8 from last time.

Proof. First, consider the following diagram, which is an expansion of the definition for exactness for chain complexes:

$$\begin{tikzcd}
0 \arrow[r, red] & Z_n(A) \arrow[r] \arrow[d] & Z_n(B) \arrow[r] \arrow[d] & Z_n(C) \arrow[d] & \\
0 \arrow[r] & A_n \arrow[r] \arrow[d, "d"] & B_n \arrow[r] \arrow[d, "d"] & C_n \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
& A_{n-1}/dA_n \arrow[r] & B_{n-1}/dB_n \arrow[r] & C_{n-1}/dC_n \arrow[r, blue] & 0,
\end{tikzcd}$$

where the red arrow is injective and the blue arrow is surjective. From this, we can extract the following diagram (using the chain maps):

$$\begin{tikzcd}
& A_n/dA_{n+1} \arrow[r] \arrow[d, "d"] & B_n/dB_{n+1} \arrow[r] \arrow[d, "d"] & C_n/dC_{n+1} \arrow[r, blue] \arrow[d, "d"] & 0 \\
0 \arrow[r, red] & Z_{n-1}(A) \arrow[r] & Z_{n-1}(B) \arrow[r] & Z_{n-1}(C) &
\end{tikzcd}$$

From this, using the definitions of the chain maps, we have

$$\begin{tikzcd}
& H_n(A) \arrow[r] \arrow[d] & H_n(B) \arrow[r] \arrow[d] & H_n(C) \arrow[d] & \\
& A_n/dA_{n+1} \arrow[r] \arrow[d, "d"] & B_n/dB_{n+1} \arrow[r] \arrow[d, "d"] & C_n/dC_{n+1} \arrow[r, blue] \arrow[d, "d"] & 0 \\
0 \arrow[r, red] & Z_{n-1}(A) \arrow[r] \arrow[d] & Z_{n-1}(B) \arrow[r] \arrow[d] & Z_{n-1}(C) \arrow[d] & \\
& H_{n-1}(A) \arrow[r] & H_{n-1}(B) \arrow[r] & H_{n-1}(C) &
\end{tikzcd}$$

To this, we can apply the snake lemma, and get $H_n(C) \to H_{n-1}(A)$ (note that the H_n and H_{n-1} are the kernels and cokernels respectively). Gluing these together for all n, we obtain the long exact sequence as desired. □

Proposition 9.1

Suppose we have

$$\begin{tikzcd}
0 \arrow[r] & A_\bullet \arrow[r] \arrow[d] & B_\bullet \arrow[r] \arrow[d] & C_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'_\bullet \arrow[r] & B'_\bullet \arrow[r] & C'_\bullet \arrow[r] & 0.
\end{tikzcd}$$

Then we get a map of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(C_\bullet) \arrow[r, "\partial"] \arrow[d] & H_{n-1}(A_\bullet) \arrow[r] \arrow[d] & H_{n-1}(B_\bullet) \arrow[r] & \cdots \\
\cdots \arrow[r] & H_n(C'_\bullet) \arrow[r, "\partial"] & H_{n-1}(A'_\bullet) \arrow[r] & H_{n-1}(B'_\bullet) \arrow[r] & \cdots
\end{tikzcd}$$

Exercise 9.2. Prove proposition 9.1

§9.2 Constructions with Complexes (Homotopy)

We begin with some motivation based on ideas from topology. Recall that if X and Y are topological spaces, and $f, g : X \to Y$ maps (continuous), then we say f is **homotopic** to g if there exists a map $H : X \times I \to Y$ such that $H\,|_{X\times 0}= f$ and $H_{X\times 1} = g$. We say that f is **nullhomotopic** if f is homotopic to a constant map (which sends all of X to some point). We say that f is a **homotopy equivalence** if there exists $g : Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y.

As an example, the space $\mathbb{R}^n$ is homotopic to a point. The space $S^1 \times S^1$ (the torus) is *not* homotopic to S^2. We'll now go over some more constructions which will translate into the language of complexes.

Definition 9.3 — Let $f : X \to Y$. The **mapping cylinder** of f is the space $(([0,1] \times X) \sqcup Y)/\sim$, where $\sim$ is given by $(0, x) \sim f(x)$, for $x \in X$. The mapping cylinder is denoted $\mathrm{Cyl}(f)$. That is, we take a cylinder whose ends are X, and glue one of the ends of the cylinder to Y, via the map f. The mapping cylinder is useful in that it factors f as

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, hook] & & Y \\
& \mathrm{Cyl}(f) \arrow[ur, "\simeq"] &
\end{tikzcd},$$

where the map $\mathrm{Cyl}(f) \to Y$ is a homotopy equivalence.

Example 9.4

Consider the pushout of the maps $(+ \sqcup -) \to +$ and $(+ \sqcup -) \to -$, where $+$ and $-$ are points, regarded as topological spaces. The pushout is a point, where we identify $+$ with $-$ (the disjoint union modulo an equivalence relation which identifies the images of both maps). This operation (pushout) is not homotopy invariant, in the sense that we can replace $-$ with something equivalent, and get a different pushout. For example, replace $-$ with the mapping cylinder of $(+ \sqcup -) \to -$, which is an interval I with endpoints $+$ and $-$. The pushout is then the circle S^1, since we identify $+$ and $-$. Thus, taking pushouts is not homotopy invariant.

Example 9.5

Consider the pushout of the maps $S^1 \to \mathrm{pt}$ and $S^1 \to \mathrm{pt}$. The pushout is a point. The mapping cylinder of $S^1 \to \mathrm{pt}$ is a cone (a cylinder with one end attached to a point). Replacing one of the maps $S^1 \to \mathrm{pt}$ with the map $S^1 \to \mathrm{Cyl}(f)$, the pushout becomes a sphere S^2.

The previous two examples lead us to the following construction.

Definition 9.6 — If

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \\ Y & \end{tikzcd}$$

is a diagram of "nice" spaces (think Hausdorff), then the **homotopy pushout** is defined as the pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \arrow[d] \\ \mathrm{Cyl}(f) \arrow[r] & Y \overset{h}{\underset{x}{\bigsqcup}} Z. \end{tikzcd}$$

This homotopy pushout has more desirable properties than the ordinary pushout int the following sense.

Proposition 9.7

Consider

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \\ Y, & \end{tikzcd}$$

If we have

$$\begin{tikzcd} & X \arrow[r] \arrow[dl] \arrow[d] & Z \\ Y \arrow[r, "\sim"] & Y', & \end{tikzcd}$$

where $Y \xrightarrow{\sim} Y'$ is a homotopy equivalence, then $Y \overset{h}{\underset{x}{\bigsqcup}} Z \simeq Y' \overset{h}{\underset{x}{\bigsqcup}} Z$.

Definition 9.8 — The **mapping cone** of $f : X \to Y$ is given by $\mathrm{Cone}(f) = Y \coprod_{x}^{h} \mathrm{pt}$:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\
\mathrm{Cyl}(f) \arrow[r] & \mathrm{Cone}(f).
\end{tikzcd}$$

Now, let's turn back to complexes, and define a notion of homotopy on them. Suppose we have two complexes in an Abelian category $\mathbf{A}$, $C_\bullet$ and $D_\bullet$, and maps $s_n : C_j \to D_{j+1}$. Define the map $f_n = d_{n+1}s_n + s_{n-1}d_n$. This is shown below (note that the s_n maps don't commute, they're just shown for reference):

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[dl, "s_n"] \arrow[d, dashed, "f_n"] & C_{n-1} \arrow[r] \arrow[dl, "s_{n-1}"] \arrow[d, dashed, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Lemma 9.9

The map $f_\bullet : C_\bullet \to D_\bullet$, given by $f_n = d_{n+1}s_n + s_{n-1}d_n$ as above, is a map of chain complexes.

Proof. We have

$$\begin{aligned}
d_n f_n &= d_n(d_{n+1}s_n + s_{n-1}d_n) \\
&= d_n d_{n+1} s_n + d_n s_{n-1} d_n \\
&= d_n s_{n-1} d_n.
\end{aligned}$$

We also have

$$\begin{aligned}
f_{n-1} d_n &= (d_n s_{n-1} + s_{n-2} d_{n-1}) d_n \\
&= d_n s_{n-1} d_n.
\end{aligned}$$

So $d_n f_n = f_{n-1} d_n$ (the d_n on the left side of this are the $D_\bullet$ maps and the d_n on the left are the $C_\bullet$ maps), and thus the f_n form a chain map. □

We can now translate more of our topological notions into the language of complexes.

Definition 9.10 — A map of complexes $f : C_\bullet \to D_\bullet$ is **nullhomotopic** if there exist maps $s_n : C_n \to D_{n+1}$ such that $f = ds + sd$. If such an s exists, it is called a **chain contraction**. Two maps $f, g : C_\bullet \to D_\bullet$ are said to be **chain homotopic** if $f - g$ is nullhomotopic. A map $f : C_\bullet \to D_\bullet$ is a **chain homotopy equivalence** if there exists a $g : D_\bullet \to C_\bullet$ such that $g \circ f$ is chain homotopic to id_C and $f \circ g$ is chain homotopic to id_D.

Exercise 9.11. Consider the singular chain functor $C_\star : \mathbf{Top} \to \mathrm{Kom}(\mathbf{Ab})$, and the deRham complex functor $\Omega : \mathbf{Mnfld} \to \mathrm{Kom}(\mathbf{Ab})$. Show that the notions of homotopy on complexes defined above translate properly to the notions of homotopy from topology.

§10 February 25, 2021

§10.1 Complexes: Homotopy

From now on, our complexes will all be cohomological (increasing indices). Recall that for maps $g, f : K^\bullet \to L^\bullet$, we say g is **homotopic** to f if there exists $s : K^n \to L^{n-1}$ such that $g - f = ds + sd$, and we write $g \sim f$.

Lemma 10.1

If g is homotopic to f (written $f \sim g$), then f and g induce the same maps on cohomology. That is, for $H^n(K^\bullet) \to H^n(L^\bullet)$, then $H^n(f) = H^n(g)$ for all n.

Proof. We have

$$\begin{aligned} H^n(f)(k) &= f^n(\tilde{k}) \\ &= g^n(\tilde{k}) + ds(\tilde{k}) + sd(\tilde{k}) \\ &= H^n(g)(k), \end{aligned}$$

where $\tilde{k} \in K^n$ is a lift of $k \in H^n(K^\bullet)$. □

Corollary 10.2

If f is nullhomotopic then $H^n(f) = 0$.

Definition 10.3 — We say that a cochain complex $C^\bullet$ is **acyclic** if $H^n(C^\bullet) = 0$ for all $n \in \mathbb{Z}$.

Lemma 10.4

The following are equivalent:

(1) $C^\bullet$ is acyclic;

(2) $C^\bullet$ is exact in all degrees;

(3) $0 \to C^\bullet$ has a canonical map which is a quasi-isomorphism (defined below).

Definition 10.5 — A map $f : K^\bullet \to L^\bullet$ is said to be a **quasi-isomorphism** if f induces an isomorphism on cohomology, i.e. $H^n(K^\bullet) \xrightarrow{H^n(f)} H^n(L^\bullet)$ for all n.

Lemma 10.6

If $f : K^\bullet \to L^\bullet$ is a homotopy equivalence, then f is a quasi-isomorphism.

Exercise 10.7. Prove that homotopy equivalence is an equivalence relation, but quasi-isomorphism is not.

Definition 10.8 — Let $C^\bullet = \cdots \to C^{n-1} \to C^n \to C^{n+1} \to \cdots$. We say that $C^\bullet$ is **split** if there exist maps $\cdots \leftarrow C^{n-1} \xleftarrow{s^n} C^n \xleftarrow{s^{n+1}} C^{n+1} \leftarrow \cdots$ such that

$$\begin{tikzcd} C^n \arrow[r, "d^n"] \arrow[d, "="] & C^{n+1} \arrow[d, "="] \arrow[dl, "s^n"'] \\ C^n \arrow[r, "d^n"] & C^{n+1} \end{tikzcd}.$$

We say that $C^\bullet$ is **split exact** if it is split and exact.

Example 10.9

Let $\mathbf{A} = \mathrm{Vect}_K$, where K is a field. Then any complex in $\mathbf{A}$ is split. We can write $C^n = Z^n \oplus (B^n)'$ and $Z^n = B^n \oplus H^n$, where $Z^n = \ker(d^n)$ and $B^n = \mathrm{im}(d^{n-1})$.

Example 10.10

Acyclic complexes need not be split exact. For example, take $\mathbf{A} = \mathbf{Ab}$ ($\mathbb{Z}$-modules). Consider the complex

$$\cdots \to \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot d} \mathbb{Z}/4 \to \mathbb{Z}/4 \to \cdots.$$

This complex is not exact, since $\mathbb{Z}/2 \cdot \mathbb{Z}/4 \ncong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. The complex is quasi-isomorphic to 0, but not homotopy equivalent to 0. Such differences between quasi-isomorphism and homotopy equivalence lead to complications in forming the derived category.

§10.2 Operations on Complexes

Definition 10.11 (Shifted Complex) — Consider a cochain complex $K^\bullet$. The **n-shifted complex**, written $K[n]^\bullet$, is defined as the complex with $(K[n]^\bullet)^i = K^{n+1}$, and $d_{K[n]^\bullet} = (-1)^n d_K$ (here the (-1) represents a shift, not a sign).

For example, consider the following complex:

$$K = K[0] = \cdots \to \overset{-2}{0} \to \overset{-1}{0} \to \overset{0}{M} \to \overset{1}{0} \to \overset{2}{0} \to \cdots$$

Then we have

$$K[1] = \cdots \to \overset{-2}{0} \to \overset{-1}{M} \to \overset{0}{0} \to \overset{1}{0} \to \overset{2}{0} \to \cdots$$

$$K[-1] = \cdots \to \overset{-2}{0} \to \overset{-1}{0} \to \overset{0}{0} \to \overset{1}{M} \to \overset{2}{0} \to \cdots$$

Definition 10.12 (Shift Functor) — The n-shift operation assembles into a functor T^n as follows:

$$\begin{aligned} T^n :& \mathrm{Kom}(\mathbf{A}) \to \mathrm{Kom}(\mathbf{A}) \\ & T^n(K^\bullet) = K[n]^\bullet \\ & T^n(f) : K[n]^\bullet \to L[n]^\bullet. \end{aligned}$$

Definition 10.13 (Cylinder) — Consider $f : K^\bullet \to L^\bullet$. Define the **cylinder** of f as

$$\mathrm{Cyl}(f)^\bullet = K^\bullet \oplus K[1]^\bullet \oplus L^\bullet,$$

where $K[1]^\bullet$ is the shifted complex as defined above. The maps d are

$$\begin{aligned} d : & K^n \oplus K^{n+1} \oplus L^{n+1} \to K^{n+1} \oplus K^{n+2} \oplus L^{n+1} \\ & : (k^n, k^{n+1}, l^n) \mapsto (d(k^n) - k^{n+1}, -d(k^{n+1}), f(k^{n+1}) + d(l^n)). \end{aligned}$$

For example, consider the interval $[0, 1]$ in **Top**, which is two copies of zero dimensional objects (the points 0 and 1) glued to one copy of a one dimensional object (the line $(0, 1)$). We can ask: what is the chain complex version of the interval? Take $\mathbf{A} = \mathbf{Ab} = \mathbf{Mod}_{\mathbb{Z}}$. The interval in complex form is the complex

$$\cdots \to 0 \to \overset{-1}{\mathbb{Z}} \to \overset{0}{\mathbb{Z} \oplus \mathbb{Z}} \to 0 \to \cdots,$$

where the supersets indicate degree. The cohomology is given by

$$H^n = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/\{(n, -n)\} \simeq \mathbb{Z}, & n = 0 \\ 0, & \text{else.} \end{cases}$$

So $C^\bullet(I)$ (the complex of an interval) is the complex $\mathbb{Z}[0]$. This is the cylinder (from topology) on a point, so we expect this (since it's homotopy equivalent to a point). Let's see that this agrees with the notion of a cylinder as above, for $\mathrm{Cyl}(\mathrm{id} : \mathbb{Z} \to \mathbb{Z})$. We have

$$\begin{aligned} \mathrm{Cyl}(\mathrm{id} : \mathbb{Z} \to \mathbb{Z}) &= \mathbb{Z}[0]^\bullet \oplus \mathbb{Z}[1]^\bullet \oplus \mathbb{Z}[0]^\bullet \\ &= \overset{-1}{\mathbb{Z}} \to \overset{0}{\mathbb{Z} \oplus \mathbb{Z}}, \end{aligned}$$

where the maps are $0 \oplus \mathbb{Z} \oplus 0 \to \mathbb{Z} \oplus 0 \oplus \mathbb{Z}$, where $(0, x, 0) \mapsto (-x, 0, x)$.

Definition 10.14 — Suppose $\mathbf{A}$ has a symmetric monoidal structure (a tensor product $\oplus$). Then the shift complex is given by

$$\begin{aligned} (K^\bullet \otimes L^\bullet)^n &= \bigoplus_{i+j=n} K^i \otimes L^j \\ d : & (K^\bullet \otimes L^\bullet)^n \to (K^\bullet \otimes L^\bullet)^{n+1} \\ & : (k^i \otimes l^j) \mapsto (d(k^i) \otimes l^j + (-1)^i k^i \otimes d(l^j)). \end{aligned}$$

The cylinder is given by (for $f : K^\bullet \to L^\bullet$):

$$((K^\bullet \otimes C_\star(I)) \oplus L^\bullet) / \sim .$$

Ignoring the equivalence relation now, the complex is

$$(K^\bullet \otimes C_\star(I)) \oplus L^\bullet = (K^\bullet \oplus K[1]^\bullet) \oplus L^\bullet.$$

Definition 10.15 (Cone) — Let $f : K^\bullet \to L^\bullet$. Then $\mathrm{Cone}(f)^\bullet$ is the complex $K[1]^\bullet \oplus L^\bullet$ with maps

$$\begin{aligned} K^{n+1} \oplus L^n &\to K^{n+2} \oplus L^{n+1} \\ (k^{n+1}, l^n) &\mapsto (-d(k^{n+1}), f(k^{n+1}) + d(l^n)). \end{aligned}$$

Example 10.16

Let $f : M \to N$, a map in $\mathbf{A}$. Then $\mathrm{Cone}(f)^\bullet$ is the complex

$$\cdots \to 0 \to \overset{-1}{M} \xrightarrow{f} \overset{0}{N} \to 0 \to \cdots.$$

§11 March 2, 2021

§11.1 Operations on Complexes

Recall the definition of a cone from last time.

Definition 11.1 — Let $f : K^\bullet \to L^\bullet$. Then the **cone** of f is the complex $\mathrm{Cone}(f)^\bullet = K[1]^\bullet \oplus L^\bullet$, with differential

$$\begin{aligned} d : K^{n+1} \oplus L^n &\to K^{n+2} \oplus L^{n+1} \\ : (k^{n+1}, \ell^n) &\mapsto (-d(k^{n+1}), f(k^{n+1}) + d(\ell^n)). \end{aligned}$$

Example 11.2

Suppose $f : M \to N$ is a map in an Abelian category $\mathbf{A}$. Then $\mathrm{Cone}(f)^\bullet$ is the complex

$$\cdots \to 0 \to \overset{-1}{M} \xrightarrow{f} \overset{0}{N} \to 0 \to \cdots,$$

and has homology

$$\begin{aligned} H^{-1}(\mathrm{Cone}(f)^\bullet) &= \ker(f) \\ H^0(\mathrm{Cone}(f)^\bullet) &= \mathrm{coker}(f). \end{aligned}$$

Lemma 11.3

If $f : M[0] \to N[0]$, then f is an isomorphism if and only if $\mathrm{Cone}(f)^\bullet \underset{q-iso}{\simeq} 0$.

The above lemma highlights a key aspect of the cone: the cone measure to what extent a morphism $f : K^\bullet \to L^\bullet$ is a quasi-isomorphism. This is the way in which we will usually think about cones going forward, since as we will see, we will not usually think of cones as "strict objects."

The following lemma is very important (possibly the "most important lemma in this class"). Before we state the lemma, consider topological spaces. Recall that if we have map $X \to Y$, it can be factored through the cone, $X \hookrightarrow \mathrm{Cyl}(f) \twoheadrightarrow Y$, where the second map is a homotopy equivalence, and we can also collapse the cylinder to a cone $\mathrm{Cyl}(f) \to \mathrm{Cone}(f)$, which then maps to the suspension. The following lemma is a chain complexification of this topological analogy.

Lemma 11.4

Suppose $f : K^\bullet \to L^\bullet$ is a map. Then we can form the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[r] & L^\bullet \arrow[r] \arrow[d] & \mathrm{Cone}(f)^\bullet \arrow[r] \arrow[d, "="] & K[1] \arrow[r] & 0 \\
0 \arrow[r] & K^\bullet \arrow[r] & \mathrm{Cyl}(f)^\bullet \arrow[r] & \mathrm{Cone}(f)^\bullet \arrow[r] & 0, &
\end{tikzcd}$$

where everything is exact. (Apparently, "it is okay if you forget everything else from this class, but you should remember this diagram.") In addition, we have the following:

1. the diagram is functorial in f.
2. $\alpha : L^\bullet \to \mathrm{Cyl}(L^\bullet)$ is a quasi-isomorphism.
3. In the long exact sequence $H^n(K^\bullet) \xrightarrow{\delta} H^n(L^\bullet)$ from the top row, we have $\delta = f^*$, the induced map.

Corollary 11.5

The map $L^\bullet \to K^\bullet$ is a quasi-isomorphism if and only if $\mathrm{Cone}(f)^\bullet$ is acyclic.

Proof. By the bottom row of the diagram in the lemma, we get the exact sequence

$$\cdots \to H^n(K^\bullet) \xrightarrow{f} H^n(\mathrm{Cyl}(f)^\bullet) \to H^n(\mathrm{Cone}(f)^\bullet) \to \cdots,$$

and $H^n(\mathrm{Cyl}(f)^\bullet)$ is isomorphic to $H^n(L^\bullet)$. The map $H^n(K^\bullet) \to H^n(\mathrm{Cyl}(f)^\bullet)$ is identified with f^*. If $H^n(\mathrm{Cone}(f)^\bullet) = 0$, then f^* is an isomorphism. ☐

You can view taking the diagram from lemma 11.4 similarly to taking the kernel and cokernel when working with modules, in the sense that it helps you better understand maps and whether they are isomorphisms.

§12 March 4, 2021: Mark Kong

§12.1 Serre Quotients

Today's lecture is given by Mark Kong

Recall that a **full subcategory** $\mathbf{S}$ of a category $\mathbf{C}$ is a subcategory such that for every pair of objects $X, Y \in \mathbf{S}$, then every morphism $f : X \to Y$ in $\mathbf{C}$ is also in S.

Definition 12.1 — A **Serre subcategory** is a full subcategory $\mathbf{S}$ of an abelian category category $\mathbf{A}$ such that either of the following conditions hold:

(1) If $A \to B \to C$ is exact and if $A, C \in \mathbf{S}$, then also $B \in \mathbf{S}$.

(2) If $0 \to A \to B \to C \to 0$ is exact, then $B \in \mathbf{S}$ if and only if $A, C \in \mathbf{S}$.

If $\mathbf{S}$ is a Serre subcategory, we define the **Serre quotient category** $\mathbf{A}/\mathbf{S}$ as the category whose objects are the objects of $\mathbf{A}$, and whose morphisms are $\operatorname{Hom}_{\mathbf{A}/\mathbf{S}}(X,Y) = \operatorname{colim} \operatorname{Hom}_{\mathbf{A}}(X', Y/Y')$, where the colimit is over $X', Y' \in \mathbf{A}$ such that $Y', X/X' \in \mathbf{S}$, and $X' \subseteq X, Y' \subseteq Y$.

In order to better understand quotient categories, we will study localization (and see that quotient categories can be viewed as a localization).

§12.2 Localization

In general, localization is a means by which to introduce denominators, or inverses, into an object. We begin by recalling localization of commutative rings. Let S be a multiplicatively closed subset of a commutative ring R (closed under multiplication). We would like to form the set of fractions r/s, where $s \in S$. In order to do this, define an equivalence relation on $R \times S$ by $(r_1, s_1) \sim (r_2, s_2)$ if and only if there exists $t \in S$ such that $t(r_1 s_2 - r_2 s_1) = 0$. The localization is denoted $S^{-1}R$.

Alternatively, we can define the localization of a commutative ring via the following universal property (for a commutative ring R): the localization $S^{-1}R$ is a commutative R-algebra, together with a map $h : R \to S^{-1}$ with $h(s)$ invertible in A for all $s \in S$, such that for any other commutative R-algebra A with a map $\varphi : R \to A$ with $\varphi(s)$ invertible in A for all $s \in S$, then there exists a unique $\tilde{\varphi}$ with $\tilde{\varphi} h = \varphi$, i.e. any $R \to A$ sending S to invertible elements factors uniquely through $R \to S^{-1}R \to A$, so following diagram commutes:

$$\begin{tikzcd} R \arrow[rr, "h"] \arrow[dr, "\varphi"] & & S^{-1}R \arrow[dl, dashed, "\tilde{\varphi}"] \\ & A & \end{tikzcd}.$$

More generally we can work in an arbitrary category.

Definition 12.2 — Let $\mathbf{C}$ be a category, and let S be a class (set) of arrows. The set S is called a **left multiplicative system** if it satisfies the following

1. The identity morphism id_X for every $X \in \mathbf{C}$ is in S, and S is closed under composition.
2. For any morphism g in $\operatorname{Hom}_{\mathbf{C}}(X,Y)$ and any $t \in S$, there exists $f \in \operatorname{Hom}_{\mathbf{C}}(Z,W)$ and $s \in S$ such that the following square commutes:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "t"] & Y \arrow[d, dashed, "s"] \\ Z \arrow[r, dashed, "f"] & W. \end{tikzcd}$$

3. If f, g are morphisms $X \to Y$ and $s \in S$ with $sf = sg$, then there exists $t \in S$ with $ft = gt$.

One can similarly define a right multiplicative system. We now define what is known as the calculus of fractions, or localization, of a category.

Definition 12.3 — Let $\mathbf{C}$ be an arbitrary category, and let S be a left multiplicative system. Then we define the (left) **localization** or **left calculus of fractions**, which is a category denoted $S^{-1}\mathbf{C}$, as follows:

- $\mathrm{Ob}(S^{-1}\mathbf{C}) = \mathrm{Ob}(\mathbf{C})$.
- The morphisms $X \to Y$ in $S^{-1}\mathbf{C}$ are pairs (f, s), with $f \in \mathrm{Hom}_{\mathbf{C}}$ and $s \in S$, up to an equivalence relation: the morphisms $(f_1 : X \to Z_1, s_1 : Y \to Z_1)$ and $(f_2 : X \to Z_2, s_2 : Y \to Z_2)$ are equivalent if there exists $(f_3 : X \to Z_3, s_3 : Y \to Z_3)$ and morphisms $u : Z_1 \to Z_3$ and $v : Z_2 \to Z_3$, such that the following diagram commutes:

$$\begin{tikzcd}
 & & Z_3 & & \\
 & Z_1 \arrow[ur, "u"] & & Z_2 \arrow[ul, "v"'] & \\
X \arrow[ur, "f_1"] \arrow[urrr, "f_2"'] & & & & Y \arrow[ul, "s_2"'] \arrow[ulll, "s_1"]
\end{tikzcd}$$

- If $(f : X \to W_1, s : Y \to W_1)$ and $(g : Y \to W_2, t : Z \to W_2)$ are morphisms, then the composition is the pair $(h \circ f : X \to W, u \circ t : Z \to W)$, where h, u are chosen as in the second item in the definition of left multiplicative system, so that we have

$$\begin{tikzcd}
 & & W & & & & & & \\
 & W_1 \arrow[ur, "h"] & & W_2 \arrow[ul, "u"'] & & \Rightarrow & & W & \\
X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \arrow[ur, "g"] & & Z \arrow[ul, "t"'] & & X \arrow[ur, "h\circ f"] & & Z \arrow[ul, "u\circ t"']
\end{tikzcd}$$

- The identity morphism $\mathrm{id} : X \to X$ in $S^{-1}\mathbf{C}$ is the equivalence class of $(\mathrm{id}_x, \mathrm{id}_X)$.

A pair (s, f) with

$$\begin{tikzcd}
 & Z & \\
X \arrow[ur, "s"] & & Y \arrow[ul, "f"']
\end{tikzcd}$$

is called a **roof**. So morphisms in $S^{-1}\mathbf{C}$ are equivalence classes of roofs. There is a little bit to check in the above definition to make sure that we actually get a category. This is proved in the course textbook, and is also stated in the below exercise.

Exercise 12.4. Verify in the above definition that:

(1) The relation $\sim$ defined in the second bullet is an equivalence relation;

(2) The composition rule given in the third bullet is well defined;

(3) That the composition rule in the third bullet is associative.

Lemma 12.5

Let **C** be a category and S a left multiplicative system of morphisms. For any finite collection of morphisms $g : X_i \to Y$ indexed by i, there is pair $s : Y \to Z$ and a family of morphisms $f_i : X_i \to Z$ such that g_i is the equivalence class of (f_i, s) for each i.

Lemma 12.6

Let **C** be a category and S a left multiplicative system of morphisms. If $a, b : X \to Y$ are morphisms given by classes of roofs (f, s) and (g, s) respectively with $f, g : X \to Z$ and $s : Y \to Z$, then $a = b$ if and only if there exists a morphism $\ell : Z \to W$ such that $\ell \circ s \in S$ and $\ell \circ f = \ell \circ g$.

Corollary 12.7

Let **C** be a category and S a left multiplicative system of morphisms. Let $Y \in \mathrm{Ob}(\mathbf{C})$, and let Y/S be the category with objects $(s : Y \to Z) \in S$, and with morphisms being diagrams

$$\begin{tikzcd} & Y \arrow[dl, "s"'] \arrow[dr] & \\ Z \arrow[rr] & & W \end{tikzcd},$$

for arbitrary $Z \to W$. Then the morphisms in $S^{-1}\mathbf{C}$ can be described as

$$\mathrm{Hom}_{S^{-1}\mathbf{C}}(X, Y) = \mathrm{colim}_{(Y \to Z) \in Y/S} \mathrm{Hom}_{\mathbf{C}}(X, Z).$$

Furthermore, taking objects of **C** to objects in $S^{-1}\mathbf{C}$ and morphisms in **C** to equivalence classes in $S^{-1}\mathbf{C}$ defined a functor $Q : \mathbf{C} \to S^{-1}\mathbf{C}$, called the **localization functor**. If $s \in S$, then $Q(s)$ is an isomorphism in $S^{-1}\mathbf{C}$. Any functor $F : \mathbf{C} \to \mathbf{D}$ such that $F(s)$ is invertible for all $s \in S$ can be uniquely factored through the localization, that is, there exists $G : S^{-1}\mathbf{C} \to \mathbf{D}$ such that $F = G \circ Q$.

The last statement in the corollary above shows that the localization we've defined works well with the definition of localization on rings/modules from above (the universal property). Note also, we could have formulated everything above in terms of *right* multiplicative systems instead of left. We'll often work in the case where S is simply a multiplicative system, which is a system of morphisms which is both right multiplicative and left multiplicative.

Theorem 12.8

Let $\mathbf{A}$ be an Abelian category. Then

1. The functor $\mathbf{A} \to S^{-1}\mathbf{A}$ preserves 0 objects;
2. If S is a left multiplicative system, then $S^{-1}\mathbf{A}$ has cokernels and $Q : \mathbf{A} \to S^{-1}\mathbf{A}$ preserves cokernels.
3. If S is a right multiplicative system, then $S^{-1}\mathbf{A}$ has kernels $Q : \mathbf{A} \to S^{-1}\mathbf{A}$ preserves kernels.
4. If S is a left and right multiplicative system, the $Q : \mathbf{A} \to S^{-1}\mathbf{A}$ is exact and $S^{-1}\mathbf{A}$ is Abelian.

§13 March 9, 2021: Philip LaPorte

Today's lecture is given by Philip LaPorte.

§13.1 The Derived Category

Definition 13.1 — Recall that for an Abelian category $\mathbf{A}$, a morphism $f : K^\bullet \to L^\bullet$ is said to be a *quasi-isomorphism* if $H^n(f) : H^n(K^\bullet) \to H^n(L^\bullet)$ is an isomorphism for all n.

Warning: Quais-isomorphisms do not form an equivalence relation. For example, consider the map of chain complexes

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

This is a quasi-isomorphism in the downward direction, but not in the upward direction, since the only upward map is the 0-map.

Note: an arbitrary resolution $Q^\bullet \to X$ of an object X determines a quasi-isomorphism. That is, we have

$$\begin{tikzcd}
\cdots \arrow[r] & \overset{-2}{Q} \arrow[r] \arrow[d] & \overset{-1}{Q} \arrow[r] \arrow[d] & \overset{0}{Q} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & X \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

While quasi-isomorphisms do not form an equivalence relation, they generate an equivalence relation. This is the idea behind derived categories: we aim to study the equivalence relation generated by quasi-isomorphisms. This will allow us to identify an object with its resolutions, which is convenient because many interesting functors can be redefined, e.g. the functor $(M \mapsto M \otimes N) \rightsquigarrow (M \mapsto P^\bullet \otimes M)$, where $P^\bullet$ is a projective resolution of M.

Note: this will force us to consider arbitrary complexes, not just complexes with zero cohomology in nonzero degrees (which we have mostly seen so far).

Definition 13.2 — The **derived category** $D(\mathbf{A})$ of an Abelian category $\mathbf{A}$ is the localization of $\mathrm{Kom}(\mathbf{A})$ at quasi-isomorphisms. That is, there is a canonical functor

$$Q : \mathrm{Kom}(\mathbf{A}) \to D(\mathbf{A})$$

such that

1. $Q(f)$ is an isomorphism whenever f is a quasi-isomorphism;
2. Any functor $\mathrm{Kom}(\mathbf{A}) \to \mathbf{C}$ transforming quasi-isomorphisms to isomorphisms factors uniquely through $D(\mathbf{A})$, i.e. $\mathrm{Kom}(\mathbf{A}) \to D(\mathbf{A}) \to \mathbf{C}$.

Recall, the objects $\mathrm{Ob}(D(\mathbf{A}))$ are $\mathrm{Ob}(\mathrm{Kom}(\mathbf{A}))$, and the morphisms are formal finite sequences of arrows in

$$\mathrm{Mor}(\mathrm{Kom}(\mathbf{A}) \cup \{s^{-1} : Y \to X \mid s : X \to Y \text{ is a q-iso}\})$$

Note: The derived category is no in general Abelian, but is additive.

Definition 13.3 — A complex $K^\bullet$ is a **cyclic complex** if all its differentials are zero. Cyclic complexes form a subcategory $\mathrm{Kom}_0(\mathbf{A}) \subset \mathrm{Kom}(\mathbf{A})$.

For example, the cohomology complex is cyclic, i.e. the cohomology functor takes $\mathrm{Kom}(\mathbf{A})$ to $\mathrm{Kom}_0(\mathbf{A})$, $h : \mathrm{Kom}(\mathbf{A}) \to \mathrm{Kom}_0(\mathbf{A})$, $(K^n, d^n) \mapsto (H^n(K^\bullet), 0)$. This functor also turns quasi-isomorphisms into isomorphisms, so we get an induced functor $k : D(\mathbf{A}) \to \mathrm{Kom}_0(\mathbf{A})$, with $k(s^{-1}) = H^n(s)^{-1}$.

In general, derived categories are difficult to compute, and we'll now compute one of the simpler examples.

§13.2 Computing the Derived Category

First, an Abelian category is called a **semisimple category** if every short exact sequence splits. That is, each exact sequence $0 \to A \to B \to C \to 0$ is isomorphic to the exact sequence $0 \to A \to A \oplus C \to C \to 0$, where the maps are the canonical inclusion and projection maps. As an example, the category $\mathbf{Vect}_k$ is semisimple, but the category $\mathbf{Ab}$ is not.

Proposition 13.4

In a semisimple Abelian category $\mathbf{A}$, the functor $k : D(\mathbf{A}) \to \mathrm{Kom}_0(\mathbf{A})$ is an equivalence of categories.

Proof. Consider the diagram

$$\begin{tikzcd}
\mathrm{Kom}_0(\mathbf{A}) \arrow[r] \arrow[rr, bend left, "\ell"] & \mathrm{Kom}(\mathbf{A}) \arrow[r, "Q"] \arrow[rr, bend left, "h"] & D(\mathbf{A}) \arrow[r, "k"] & \mathrm{Kom}_0(\mathbf{A}).
\end{tikzcd}$$

Note that $k \circ \ell$ is isomorphic to the identity (check with the definition of k).

We must show that $\ell \circ k$ is isomorphism to id. To this end, let $K^\bullet \in \mathrm{Kom}(\mathbf{A})$, and observe that we have two short exact sequences

$$0 \to Z^n \to K^n \xrightarrow{d} B^{n+1} \to 0$$
$$0 \to B^n \to Z^n \to H^n \to 0.$$

Since we are in a semisimple category, then $K^n \simeq Z^n \oplus B^{n+1}$, and $Z^n \simeq B^n \oplus H^n$, and thus $K^n \simeq B^n \oplus H^n \oplus B^{n+1}$, and we can write $d : K^n \to K^{n+1}$ as $d : (b^n, h^n, b^{n+1}) \mapsto (b^{n+1}, 0, 0)$. Now, define quasi-isomorphisms

$$\begin{aligned} f_K &: (K^n, d^n) \to (H^n(K^\bullet), 0) \\ f_K^n &: (b^n, h^n, b^{n+1}) \mapsto h^n \\ g_K &: (H^n(K^\bullet), 0) \to (K^n, d^n) \\ g_K^n &: h^n \mapsto (0, h^n, 0). \end{aligned}$$

Note that f_K and g_K are quasi-isomorphic to each other.

Now, the natural isomorphism from $\ell \circ k$ to $\mathrm{id}_{D(\mathbf{A})}$ is given by the family f_K. The fact that $f_K : (\ell \circ k)(K) \to \mathrm{id}(K)$ is an isomorphism for each K follows from the definitions of k, ℓ, Q, and f_K (similarly for g_K). To check naturality, we can check that for a morphism in $\mathrm{Kom}(\mathbf{A})$, $(K^n, d^n) \to (L^n, d^n)$, the following diagram commutes:

$$\begin{tikzcd} (K^n, d^n) \arrow[r, "f_K"] \arrow[d] & (H^n(K^\bullet), 0) \arrow[d] \\ (L^n, d^n) \arrow[r, "f_L"] & (H^n(L^\bullet), 0), \end{tikzcd}$$

and similarly for g_K. We see that $\ell \circ k$ is naturally isomorphic to the identity. □

As a concrete example of the above, we have that $D(\mathbf{Vect}_k)$ is the category of $\mathbb{Z}$-graded vector spaces.

§13.3 Bounded Derived Category

We are often interested in complexes with finiteness conditions. We define the following complexes:

$$\begin{aligned} \mathrm{Kom}^+(\mathbf{A}) &= \{K^\bullet : K^i = 0 \text{ for } i \le i_0(K^\bullet)\} \\ \mathrm{Kom}^-(\mathbf{A}) &= \{K^\bullet : K^i = 0 \text{ for } i \ge i_0(K^\bullet)\} \\ \mathrm{Kom}^b(\mathbf{A}) &= \mathrm{Kom}^+(\mathbf{A}) \cap \mathrm{Kom}^-(\mathbf{A}). \end{aligned}$$

Question: Should $D^+(\mathbf{A})$ be the localization of $\mathrm{Kom}^+(\mathbf{A})$ at quasi-isomorphisms, or should it be the full subcategory of $D(\mathbf{A})$ of complexes with $H^i(K^\bullet) = 0$ for $i \le i_0$? The answer is that it doesn't matter (although one definition might be better in some cases), since these definitions are equivalent.

Recall the following definition from last time:

Definition 13.5 — A class S of morphisms in $\mathbf{C}$ is (right and left) **localizing** if

(1) S is closed under composition;

(2) If $f \in \mathrm{Mor}(\mathbf{C})$, and $s \in S$ form a span or cospan, then there exist $t \in S$ and $g \in \mathrm{Mor}(\mathbf{C})$ completing the squares for f, s as follows

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "t"] & Z \arrow[d, "s"] & W & Z \arrow[l, "g"] \\ X \arrow[r, "f"] & Y & X \arrow[u, "t"] & Y \arrow[l, "f"] \arrow[u, "s"] \end{tikzcd}$$

(3) Let $f, g : X \to Y$. Then there exists s with $sf = sg$ if and only if there exists t with $ft = gt$.

We define the **homotopy category of chain complexes**, denoted $K(\mathbf{A})$, by

- $\mathrm{Ob}K(\mathbf{A}) = \mathrm{Ob}\mathrm{Kom}(\mathbf{A})$;
- Morphisms in $K(\mathbf{A})$ are homotopy classes of morphisms in $\mathrm{Kom}(\mathbf{A})$.

The class S of quasi-isomorphisms is localizing in $K(\mathbf{A})$, and $D(\mathbf{A})$ is the localization of $K(\mathbf{A})$ at quasi-isomorphisms.

Definition 13.6 — We say that $K^\bullet$ is an H^0-complex if $H^i(K^\bullet) = 0$ for all $i \neq 0$.

Proposition 13.7

The functor $Q : K(\mathbf{A}) \to D(\mathbf{A})$ gives an equivalence of $\mathbf{A}$ with the full subcategory of $D(\mathbf{A})$ formed by H^0-complexes.

Proof. The functor $\mathbf{A} \to K(\mathbf{A}$ sending an object to its corresponding 0-complex is fully faithful, since the only homotopy between morphisms of 0-complexes is the zero-homotopy, so $\mathbf{A}$ is a full subcategory of $K(\mathbf{A})$.

First, we'll prove that for 0-complexes, the canonical mapping

$$a : \mathrm{Hom}_{K(\mathbf{A})}(X, Y) \to \mathrm{Hom}_{D(\mathbf{A})}(Q(X), Q(Y))$$

gives an isomorphism with inverse given by

$$b = H^0 : D(\mathbf{A}) \to A.$$

It is clear that $b \circ a = \mathrm{id}$ on 0-complexes (using definitions).

We check that the same is true for $a \circ b$. Let $\tilde{f} = (f, s)$ be a morphisms of 0-complexes in $D(\mathbf{A})$, represented by

$$\begin{tikzcd}
& Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & & Y,
\end{tikzcd}$$

where s is a quasi-isomorphism. We want to show that $(a \circ b)(\tilde{f}) = \tilde{f}$. We have that $(a \circ b)(\tilde{f}) : X \to Y$ in $D(\mathbf{A})$ is represented by

$$\begin{tikzcd}
& X \arrow[dl, "\mathrm{id}"'] \arrow[dr, "g"] & \\
X & & Y,
\end{tikzcd}$$

where $g = H^0(f) \circ H^0(s)^{-1} : X \to Y$, since $(f, s) \overset{b}{\mapsto} H^0(f) \circ H^0(s)^{-1} \overset{a}{\mapsto} (\xleftarrow{id} \cdot \xrightarrow{g})$. We want to show that

$$\begin{tikzcd}
& X \arrow[dl, "\mathrm{id}"'] \arrow[dr, "g"] & & = & & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & & Y, & & X & & Y.
\end{tikzcd}$$

To this end, we need to find V, r, h so that we have the following diagram:

$$\begin{tikzcd}
 & & V \arrow[dl, "r"'] \arrow[dr, "h"] & & \\
 & Z_1 \arrow[dl, "s"'] \arrow[drrr, "f"] & & Z_2 \arrow[dr, "g"] \arrow[dlll, "\mathrm{id}"'] & \\
X & & & & Y
\end{tikzcd}$$

Define V by $V^i = Z^i$ for $i < 0$, $V^0 = \ker(d_Z^0)$, and $V^i = 0$ for $i > 0$, and with d_V induced by d_Z. Define $r : V \to Z$ to be the natural embedding, and $h : V \to X$ to be the map given by

$$\begin{tikzcd}
V \arrow[d, "h"] & \cdots \arrow[r] & \overset{-2}{Z} \arrow[r] \arrow[d] & \overset{-1}{Z} \arrow[r] \arrow[d] & \ker(d_Z^0) \arrow[r] \arrow[d, "H^0(s)"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
X & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & X \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

It remains to check that r is a quasi-isomorphism, and that the diagram commutes. We've now shown that the functor

$$\begin{tikzcd}
Q: & \mathbf{A} \arrow[rr] \arrow[dr] & & D(\mathbf{A}) \arrow[dl] \\
 & & K(\mathbf{A}) &
\end{tikzcd}$$

is fully faithful. It remains to check that any H^0-complex Z in $D(\mathbf{A})$ is isomorphic to some 0-complex. The desired isomorphism is

$$\begin{tikzcd}
 & V \arrow[dl, "r"] \arrow[dr, "h"] & \\
Z & & H^0(Z).
\end{tikzcd}$$

where both r and h are quasi-isomorphisms. □

Exercise 13.8. Check that r as defined in the above proof is a quasi-isomorphism and that the diagram commutes. In particular, $f \circ r = g \circ h$. Note that $g = H^0(f) \circ H^0(s)^{-1}$, and $h = H^0(s) \circ \varphi$.

As an example, we can define Ext by $\operatorname{Ext}^i_{\mathbf{A}}(X, Y) = \operatorname{Hom}_{D(\mathbf{A})}(X[0], Y[i])$.

§14 March 11, 2021: Fan Zhou

Today's lecture is given by Fan Zhou.

§14.1 Triangulated Categories

We begin with the definition of a triangulated category. First, let $\mathbf{C}$ be an additive category, with an additive autoequivalence $\Sigma : \mathbf{C} \to \mathbf{C}$. A **triangle** is a diagram

$$X \to Y \to Z \to \Sigma X.$$

A morphism of triangles is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'. \end{tikzcd}$$

Note: Before we define triangulated categories, it is important to remember that a triangulated category is a *structure*, not a *property* (i.e., we don't say that a given category is "triangulated," rather we create triangulated categories from existing categories).

From now on, we will denote the autoequivalence Σ by $[1]$, or $[n]$ when applied multiple times (suggestive of a shift of a chain complex), so ΣX will be denoted $X[1]$.

Definition 14.1 — A **triangulated category** consists of an additive category $\mathbf{C}$, a collection of additive functors $\{[n]\}_{n\in\mathbb{N}}$ with $[n]\circ[m]=[n+m]$, and a special set of triangles, called **distinguished triangles**, satisfying the following axioms:

TC1:
- The triangle $X\to X\to 0\to X[1]$ is distinguished.
- Any triangle isomorphic to a distinguished triangle is distinguished.
- Any $f:X\to Y$ can be completed to a distinguished triangle $X\to Y\to \mathrm{Cone}(f)\to X[1]$.

TC2: The triangle $X\xrightarrow{f}Y\xrightarrow{g}Z\xrightarrow{h}X[1]$ is distinguished if and only if $Y\xrightarrow{g}Z\xrightarrow{h}X[1]\xrightarrow{-f[1]}Y[1]$ is distinguished.

TC3: If $X\to Y\to Z\to X[1]$ is distinguished and $X'\to Y'\to Z'\to X'[1]$ is distinguished, then

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & X[1] \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

determines the map $Z\to Z'$.

TC4: This is called the octahedral axiom. (It is somewhat long, and I will use colors.) Let $X,Y,Z\in\mathbf{C}$, with morphisms $f:X\to Y$ and $g:Y\to Z$. Suppose that we have the following three distinguished triangles:

$$X\xrightarrow{f}Y\xrightarrow{p_1}Q_1\xrightarrow{d_1}X[1]$$

$$X\xrightarrow{g\circ f}Z\xrightarrow{p_2}Q_2\xrightarrow{d_2}X[1]$$

$$Y\xrightarrow{g}Z\xrightarrow{p_3}Q_3\xrightarrow{d_3}Y[1].$$

Then there exist morphisms $Q_1\xrightarrow{a}Q_2$ and $Q_2\xrightarrow{b}Q_3$ such that

$$Q_1\xrightarrow{a}Q_2\xrightarrow{b}Q_3\xrightarrow{p_1[1]\circ d_3}Q_1[1]$$

is a distinguished triangle, and such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[rr, "g\circ f"] \arrow[dr, "f"'] & & Z \arrow[rr, "p_3"] \arrow[dr, "p_2"] & & Q_3 \arrow[rr, "p_1[1]\circ d_3"] \arrow[dr, "d_3"] & & Q_1[1] \\
& Y \arrow[ur, "g"'] \arrow[dr, "p_1"] & & Q_2 \arrow[ur, dashed, "b"] \arrow[dr, "d_2"] & & Y[1] \arrow[ur, "p_1[1]"'] & \\
& & Q_1 \arrow[ur, dashed, "a"] \arrow[rr, bend right, "d_1"'] & & X[1] \arrow[ur, "f[1]"'] & &
\end{tikzcd}$$

The colored arrows represent distinguished triangles, and the colored arrow labels denote which distinguished triangle a map comes from. It may also help your intuition to note that the following diagram (a rearrangement of the above) commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & Y \arrow[r, "p_1"] \arrow[d, "g"] & Q_1 \arrow[r, "d_1"] \arrow[d, "a"] & X[1] \arrow[d, "\mathrm{id}"] \\
X \arrow[r, "g\circ f"] \arrow[d, "f"] & Z \arrow[r, "p_2"] \arrow[d, "\mathrm{id}"] & Q_2 \arrow[r, "d_2"] \arrow[d, "b"] & X[1] \arrow[d, "f[1]"] \\
Y \arrow[r, "g"] & Z \arrow[r, "p_3"] & Q_3 \arrow[r, "d_3"] & Y[1]
\end{tikzcd}$$

Looking at this diagram, we can see why distinguished triangles are often called **exact triangles** (look at the rows, and the second to last column).

From axiom TC2, we have the following "double helix" diagram:

$$\begin{tikzcd}
X[2] \arrow[rr, "f[2]"] & & Y[2] \\
& Z[1] \arrow[ul, "-h[1]"'] & \\
X[1] \arrow[rr, "-f[1]"] & & Y[1] \arrow[ul, "-g[1]"'] \\
& Z \arrow[ul, "h"'] & \\
X \arrow[rr, "f"] & & Y \arrow[ul, "g"']
\end{tikzcd}$$

We'll use this in the proof of the following.

Proposition 14.2

Given a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$, then for all $A \in \mathbf{C}$, the following sequences are exact:

$$\cdots \operatorname{Hom}(A, X[n]) \xrightarrow{f[n]_*} \operatorname{Hom}(A, Y[n]) \xrightarrow{g[n]_*} \operatorname{Hom}(A, Z[n]) \xrightarrow{h[n]_*} \operatorname{Hom}(A, X[n+1]) \cdots$$

$$\cdots \operatorname{Hom}(Z[n], A) \xrightarrow{g[n]^*} \operatorname{Hom}(Y[n], A) \xrightarrow{f[n]^*} \operatorname{Hom}(X[n], A) \xrightarrow{h[n-1]^*} \operatorname{Hom}(Z[n-1], A) \cdots$$

Proof. Using the helix construction from above, it suffices to check exactness of the diagram at $\operatorname{Hom}(A, Y) = \operatorname{Hom}(A, Y[0])$. First, note that $gf = 0$, using axiom TC3 applies as follows:

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, dashed, "u"] & X[1] \arrow[d] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & X[1].
\end{tikzcd}$$

The only possible u is $u = 0$, and the since the diagram commutes, $gf = 0$. Then $g_* f_* = 0$ as well.

It remains to check that if $v : A \to Y$ satisfies $gv = 0$, then $v = fu$ for some $u : A \to X$. First, use TC3 and TC2 to get $u[1]$:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & A[1] \arrow[r, "-\mathrm{id}"] \arrow[d, dashed, "u[1]"] & A[1] \arrow[d, "v[1]"] \\
Y \arrow[r, "g"] & Y \arrow[r, "h"] & X[1] \arrow[r, "-f[1]"] & Y[1].
\end{tikzcd}$$

Then do this again to get g:

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}"] \arrow[d, "v"] & A \arrow[r, "u"] \arrow[d] & 0 \arrow[r] \arrow[d] & A[1] \arrow[d, "u[1]"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & X[1].
\end{tikzcd}$$

This finishes the proof for the first diagram. The proof for the second is similar. ☐

Corollary 14.3 (a) If $X \to Y \to Z \to X[1]$ and $X' \to Y' \to Z' \to X'[1]$ are distinguished triangles with isomorphisms θ and ϕ such that

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\theta"] & Y \arrow[r] \arrow[d, "\phi"] & Z \arrow[r] \arrow[d, dashed, "\psi"] & X[1] \arrow[d, "\theta[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1],
\end{tikzcd}$$

then ψ is an isomorphism as well.

(b) The completion of $X \to Y$ to an exact triangle from TC2 is unique up to isomorphism.

Proof. We sketch the proof of (a), and note that (b) follows immediately.

Use Yoneda lemma to note that we only need to check ψ is an isomorphism after applying Hom to everything. Using the previous lemma, apply $\operatorname{Hom}(Z',\cdot)$ to the entire diagram, then apply the five-lemma to get that $\psi^* : \operatorname{Hom}(Z',Z) \to \operatorname{Hom}(Z',Z')$ is an isomorphism. ☐

Corollary 14.4

Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} X'[1]$ are distinguished triangles, and suppose $\phi : Y \to Y'$ satisfies $g'\phi f = 0$. Then ϕ completes to a morphism of triangles:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dashed, "\theta"] & Y \arrow[r, "g"] \arrow[d, "\phi"] & Z \arrow[r, "h"] \arrow[d, dashed, "\psi"] & X[1] \arrow[d, dashed, "\theta[1]"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1],
\end{tikzcd}$$

Furthermore, if $\operatorname{Hom}(X, Z'[-1]) = 0$, then the morphism is unique.

Proof. Apply $\operatorname{Hom}(X,\cdot)$ to the lower distinguished triangle to get

$$\cdots \longrightarrow \operatorname{Hom}(X,Z'[-1]) \xrightarrow{-h'[1]_*} \operatorname{Hom}(X,X') \xrightarrow{f'_*} \operatorname{Hom}(X,Y') \longrightarrow \operatorname{Hom}(X,Z') \longrightarrow \cdots.$$

A morphism $X \to X'$ such that

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dashed] & Y \arrow[d, "\phi"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

commutes is the preimage $(f'_*)^{-1}(\phi \circ f)$. This preimage is unique up to an element of the kernel $\ker(f'_*) = \operatorname{im}(-h'[1]_*)$. This gives θ making the square commute, which by TC2 gives a morphism of triangles. ☐

§15 March 18, 2021: Fan Zhou

Today's lecture is also given by Fan Zhou.

§15.1 Homotopy Category of Chain Complexes is Triangulated

Recall the definitions of Cone and Cyl. For a map of complexes $f : A^\bullet \to B^\bullet$, then $\operatorname{Cone}(f)^\bullet$ is the complex $A[1]^\bullet \oplus B^\bullet$, with $d^n_{\operatorname{Cone}(f)^\bullet}(a,b) = (-d(a), f(a) + d(b))$. And $\operatorname{Cyl}(f)^\bullet = A^\bullet \oplus A[1]^\bullet \oplus B$, with $d_{\operatorname{Cyl}(f)^\bullet}(a,a',b) = (d(a) - a', -d(a'), f(a') + d(b))$.

Recall that for $f : A^\bullet \to B^\bullet$, we can write the following diagram

$$\begin{tikzcd}
 & \operatorname{Cyl}(f)^\bullet \arrow[dr, "\pi_{23}"] & & \\
A^\bullet \arrow[ur, "\iota_1"] \arrow[r] & B^\bullet \arrow[u, "\iota_3"'] \arrow[r] & \operatorname{Cone}(f)^\bullet \arrow[r] & A[1]^\bullet.
\end{tikzcd}$$

Here [1] is the shift operator on complexes. Now, recall the category $K(\mathbf{A})$, the category whose objects are chain complexes and whose maps are chain homotopy equivalence classes of maps in Kom($\mathbf{A}$). We say that a triangle $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$ is a distinguished triangle if it is isomorphic to a triangle of the form $A_1^\bullet \xrightarrow{u_1} B_1^\bullet \to \mathrm{Cone}(u_1) \to A_1[1]$. We show now that with these triangles, the category $K(\mathbf{A})$ (and $K^\pm(\mathbf{A})$) is triangulated. That is, if there exists a commutative (up to chain homotopy equivalence) diagram

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"] & A[1] \\
A_1 \arrow[r, "u_1"] & B_1 \arrow[r, "v_1"] & \mathrm{Cone}(u_1) \arrow[r, "\delta"] & A_1[1].
\end{tikzcd}$$

From now on, we'll continue to drop the dot notation for chain complexes, and write $A^\bullet = A$.

Theorem 15.1

The categories $K(\mathbf{A})$ and $K^\pm(\mathbf{A})$ are triangulated categories.

Proof. **TC1:** Recall that the cone of the identity Cone(id) is split exact, and thus isomorphic to 0 in $K(\mathbf{A})$. Thus, we see that $X \to X \to 0 \to X[1]$ is indeed a distinguished triangle by

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, "0"] & X[1] \arrow[d] \\
X \arrow[r] & X \arrow[r] & \mathrm{Cone}(\mathrm{id}) \arrow[r] & X[1]
\end{tikzcd}$$

By definition, any triangle isomorphic to a distinguished triangle is distinguished, and any $f : X \to Y$ can be completed to $X \to Y \to \mathrm{Cone}(f) \to X[1]$ (by the diagram above).

TC2: Given $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ distinguished, we want to show that $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is distinguished. We have $\mathrm{Cone}(g) \simeq Y[1] \oplus X[1] \oplus Y[1]$ (since $X \to Y \to Z \to X[1]$ is distinguished and so $Z = X[1] \oplus Y$) and

$$d_{\mathrm{Cone}(g)} = \begin{pmatrix} -d_Y & 0 & \mathrm{id}_X \\ 0 & -d_X & f[1] \\ 0 & 0 & d_Y \end{pmatrix}.$$

Define $\theta : X[1] \to \mathrm{Cone}(g)$ by $x \mapsto (-f(x), x, 0)$. We claim now that

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "\mathrm{id}"] & Y \arrow[r, "h"] \arrow[d, "\mathrm{id}"] & Z \arrow[r, "-f{[1]}"] \arrow[d, "\theta"] & Y[1] \arrow[d, "Y{[1]}"] \\
Y \arrow[r] & Z \arrow[r, "s"] & \mathrm{Cone}(g) \arrow[r] & Y[1]
\end{tikzcd}$$

is an isomorphism. The difficult part is to show that s is homotopic to $\theta \circ h$. The homotopy is given by $\eta^n : Z^n \to \mathrm{Cone}(g)^{n-1} = Y[1] \oplus X[1] \oplus Y$ (here again $Z = X[1] \oplus Y$), with $\eta^n : (x^{n+1}, y^n) \mapsto (y^n, 0, 0)$. It remains to check that this morphism of triangles is an isomorphism. We now check that $\theta : X[1] \to \mathrm{Cone}(g)$ is an invertible isomorphism. We have $\pi_2 \circ \theta = \mathrm{id}_{X[1]}$ (where π_2 is the projection onto the second component), and $\theta \circ \pi_2 \simeq \mathrm{id}_{Y[1] \oplus X[1] \oplus Y}$ with $(y', x, y) \mapsto (-f(x), x, 0)$, via the homotopy $(y', x, y) \mapsto (y, 0, 0)$.

TC3: Suppose we have distinguished triangles with

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r, "u'"] & Y' \arrow[r] & Z' \arrow[r] & X'[1]. \end{tikzcd}$$

We can assume $Z = \mathrm{Cone}(u)$ and $Z' = \mathrm{Cone}(u')$, and we can take the map $Z \to Z'$ to be $f[1] \oplus g$.

TC4: The proof of this is long, and is found in Gelfand and Manin.

☐

§15.2 The Derived Category is Triangulated

Recall our definition of a localizing class from definition 13.5. We make the following definition

Definition 15.2 — A localizing class S is **compatible** if

- $s \in S$ if and only if $\Sigma s \in S$ (where Σ is the shift operator);
- If $f, g \in S$, then there exists a morphism $h \in S$ which completes the morphism of triangles in TC3:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f\in S"] & Y \arrow[r] \arrow[d, "g\in S"] & Z \arrow[r] \arrow[d, dashed, "\exists h\in S"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]. \end{tikzcd}$$

We can then state our main theorem.

Theorem 15.3

Let $\mathbf{C}$ be a triangulated category, and S a compatible localizing class. Define a triangle to be distinguished in $S^{-1}\mathbf{C}$ if it is isomorphic to the image of a distinguished triangle in $\mathbf{C}$ under $\mathbf{C} \to S^{-1}\mathbf{C}$. With these distinguished triangles, $S^{-1}\mathbf{C}$ is a triangulated category.

*Proof.***TC1:** The distinguishedness of $X \to X \to X \to X[1]$ in $S^{-1}\mathbf{C}$ is inherited from its distinguishedness in $\mathbf{C}$. That any triangle isomorphic to a distinguished triangle is distinguished is inherent in the definition. It remains to show that any morphism of roofs can be completed to a distinguished triangle of roofs.

For a morphism $X \xrightarrow{fs^{-1}} Y$ in $S^{-1}\mathbf{C}$, we denote the roof by

$$\begin{tikzcd} & \widehat{XY} \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

We construct the triangle as follows. We can complete $\widehat{XY} \xrightarrow{f} Y$ to a triangle $\widehat{XY} \xrightarrow{f} Y \to \mathrm{Cone}(f) \xrightarrow{g} \widehat{XY}[1]$, and we can complete $X \xrightarrow{fs^{-1}} Y$ to a triangle

$X \xrightarrow{fs^{-1}} Y \to \mathrm{Cone}(f) \xrightarrow{s[1]\circ g} X[1]$. These are isomorphic, since

$$\begin{tikzcd}
\widehat{XY} \arrow[r] \arrow[d, "s"] & Y \arrow[r] \arrow[d, "="] & \mathrm{Cone}(f) \arrow[r] \arrow[d, "="] & \widehat{XY}[1] \arrow[d, "{s[1]}"] \\
X \arrow[r] & Y \arrow[r] & \mathrm{Cone}(f) \arrow[r] & X[1].
\end{tikzcd}$$

with s and $s[1]$ both invertible. Thus, the triangle is distinguished.

TC2: This axiom is inherited from $\mathbf{C}$.

TC3: We can represent the distinguished triangles is $S^{-1}\mathbf{C}$ as images of those in $\mathbf{C}$. The morphisms in $S^{-1}\mathbf{C}$ are represented by roofs. That is, we write

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\theta s^{-1}"] & Y \arrow[r, "g"] \arrow[d, "\phi t^{-1}"] & Z \arrow[r, "h"] & X[1] \arrow[d, "{\theta s^{-1}[1]}"] \\
x' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1]
\end{tikzcd}$$

as

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \arrow[rr, "g"] & & Z \arrow[r, "h"] & X[1] & \\
& \widehat{XX'} \arrow[ul, "s"'] \arrow[dl, "\theta"] & & \widehat{YY'} \arrow[ul, "t"'] \arrow[dl, "Y'"] & & & \widehat{XX'}[1] \arrow[ul, "{s[1]}"'] \arrow[dl, "{\theta[1]}"] \\
X' \arrow[rr] & & Y' \arrow[rr] & & Z' \arrow[r] & X'[1] &
\end{tikzcd}$$

(*)

We want to show there is a roof $\widehat{ZZ'}$ that fills in the gap above. Now, we can complete the following square with some $\tilde{X}, \tilde{f}, \tilde{s}$, by

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{s}"] & \widehat{YY'} \arrow[d, "t"] \\
\widehat{XX'} \arrow[r, "fs"] & Y
\end{tikzcd}$$

Now, consider

$$\begin{tikzcd}
& & \tilde{X} \arrow[dl, "\tilde{s}"'] \arrow[dr, "\mathrm{id}"] & & \\
& \widehat{XX'} \arrow[dl, "s"'] \arrow[drrr, "\theta"] & & \tilde{X} \arrow[dlll, "s\tilde{s}"] \arrow[dr, "\theta\tilde{s}"] & \\
X & & & & X'
\end{tikzcd}$$

You can check that this diagram commutes. Then this tells us that we can replace

$$\begin{tikzcd}
& \widehat{XX'} \arrow[dl, "s"'] \arrow[dr, "\theta"] & & & & \tilde{X} \arrow[dl, "s\tilde{s}"'] \arrow[dr, "\theta\tilde{s}"] & \\
X & & X' & \mapsto & X & & X'
\end{tikzcd}$$

Thus, by construction, we can replace the square with $\widehat{XX'}$ in eq. (*) by

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
\tilde{X} \arrow[u, "s\tilde{s}"] \arrow[r, "\tilde{f}"] & \widehat{YY'}, \arrow[u, "t"']
\end{tikzcd}$$

which commutes by construction. We see that eq. (*) becomes

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \arrow[rr, "g"] & & Z \arrow[r, "h"] & X[1] & \\
& \tilde{X} \arrow[ul, "s\tilde{s}"] \arrow[rr, "\tilde{f}"] \arrow[dl, "\theta\tilde{s}"] & & \widehat{YY'} \arrow[ul, "t"'] \arrow[dl, "Y'"] & & & \widehat{XX'}[1] \arrow[ul, "s[1]"] \arrow[dl, "\theta[1]"] \\
X' \arrow[rr] & & Y' \arrow[rr] & & Z' \arrow[r] & X'[1] &
\end{tikzcd}$$

(**)

Note that we now have a map $\tilde{X} \to \widehat{YY'}$. Now we do this again. Consider the part of the new diagram eq. (**)

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "\theta\tilde{s}"] & \widehat{YY'} \arrow[d, "\phi"] \\
X' \arrow[r, "f'"] & Y'.
\end{tikzcd}$$

There exists $\tilde{\tilde{X}}$ and $\tilde{\tilde{s}}$ such that

$$\begin{tikzcd}
\tilde{\tilde{X}} \arrow[r, "\tilde{\tilde{s}}"] & \tilde{X} \arrow[r, shift left, "f'\theta\tilde{s}"] \arrow[r, shift right, "\phi\tilde{f}"'] & Y'
\end{tikzcd}$$

Replace this into eq. (**) to get

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \arrow[rr, "g"] & & Z \arrow[r, "h"] & X[1] & \\
& \tilde{\tilde{X}} \arrow[ul, "s\tilde{s}\tilde{\tilde{s}}"] \arrow[rr, "\tilde{f}\tilde{\tilde{s}}"] \arrow[dl, "\theta\tilde{s}\tilde{\tilde{s}}"] & & \widehat{YY'} \arrow[ul, "t"'] \arrow[dl, "Y'"] & & & \widehat{XX'}[1] \arrow[ul, "s[1]"] \arrow[dl, "\theta[1]"] \\
X' \arrow[rr] & & Y' \arrow[rr] & & Z' \arrow[r] & X'[1] &
\end{tikzcd}$$

Now, we can complete $\tilde{\tilde{X}} \xrightarrow{\tilde{f}\tilde{\tilde{s}}} \widehat{YY'}$ to a distinguished triangle in **C**. Then TC3 in **C** gives

$$\begin{tikzcd}
\tilde{\tilde{X}} \arrow[r] \arrow[d, "\theta\tilde{s}\tilde{\tilde{s}}"] & \widehat{YY'} \arrow[r] \arrow[d, "\phi"] & \widehat{ZZ'} \arrow[r] \arrow[d, dashed, "\exists\psi"] & \tilde{\tilde{X}}[1] \arrow[d, "(\theta\tilde{s}\tilde{\tilde{S}})[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1].
\end{tikzcd}$$

and also in **C**

$$\begin{tikzcd}
X \arrow[r] & Y \arrow[r] & X \arrow[r] & X[1] \\
\tilde{\tilde{X}} \arrow[r] \arrow[u, "s\tilde{s}\tilde{\tilde{s}}"] & \widehat{YY'} \arrow[r] \arrow[u, "t"] & \widehat{ZZ'} \arrow[r] \arrow[u, dashed, "\exists r"] & \tilde{\tilde{X}}. \arrow[u, "(s\tilde{s}\tilde{\tilde{s}})[1]"]
\end{tikzcd}$$

Now, our desired roof for Z and Z' is

$$\begin{tikzcd} & \widetilde{ZZ'} \arrow[dl, "r"'] \arrow[dr, "\psi"] & \\ Z & & Z' \end{tikzcd}$$

This completes the proof for TC3.

TC4: This proof is long, and is in Gelfand and Manin.

□

The rest of today's lecture is given by Elden Elmanto.

§15.3 Stable Infinity Categories

We mentioned earlier that triangulated categories aren't necessarily the right way to view things. To explain this, pretend you are an early human. You start counting, and you decide that the numbers you want are $\mathbb{N}$. You could have instead started with **Fin**. If you start with $\mathbb{N}$, you get $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, etc., and develop algebra and analysis. If instead you start with **Fin**, you get the sphere spectrum $\mathbf{S}$ and you end up with higher algebra. The notion of category goes to higher category, and $\mathrm{Hom}(X, Y)$ becomes $\mathrm{Maps}(X, Y)$, which is a "space." Then, here's a definition.

Definition 15.4 (Lurie) — Let $\mathbf{C}$ be a higher category. Then we say that $\mathbf{C}$ is **stable** if

1) $\mathbf{C}$ is pointed (has a 0 object);
2) $\mathbf{C}$ has finite limits and colimits;
3) A square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

is a pushout if and only if it is a pullback.

Theorem 15.5

If $\mathbf{C}$ is a stable ∞-category, then $h(\mathbf{C})$, with objects the same as $\mathbf{C}$ and $\mathrm{Hom}(X, Y) = \pi_0 \mathrm{Maps}(X, Y)$, is triangulated.

The key idea is that if $f : X \to Y$ is a map in $\mathbf{C}$, then a triangle is obtained through the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & W \arrow[r] & X[1] \end{tikzcd}$$

§16 March 23, 2021: Madison Shirazi

§16.1 Tor in Z-Modules

We begin by considering Tor in $\mathbb{Z}$-modules. Recall that the operation of tensoring is right exact, but not left exact. That is, for an exact sequence $0 \to A \to B \to C \to 0$, then the sequence $A \otimes M \to B \otimes M \to C \otimes M \to 0$ is an exact sequence, but $0 \to A \otimes M \to B \otimes M \to C \otimes M$ is not necessarily exact. For example, one can tensor the exact sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2BZ \to 0$ by the module $\mathbb{Z}/2\mathbb{Z}$ to get a sequence which is not exact on the left. The Tor functor is a way of extending this sequence to the left to get an exact sequence.

Definition 16.1 — Given A, B Abelian groups, choose a free resolution of A:

$$0 \to \mathbb{Z}^m \to \mathbb{Z}^n \to A \to 0,$$

which is an exact sequence. Then delete A to get a sequence which is not necessarily exact:

$$0 \to \mathbb{Z}^m \to \mathbb{Z}^n \to 0.$$

Then tensor the sequence with B, and define $\operatorname{Tor}_1(A, B)$ as the module which makes the following tensored sequence exact:

$$0 \to \operatorname{Tor}_1(A, B) \to B^m \to B^n \to 0.$$

(Recall that $\mathbb{Z}^n \otimes B = B^n$.)

We'll go over some examples of computing Tor for Abelian groups. Note that Tor preserves direct sums: $\operatorname{Tor}(A \oplus B, C) = \operatorname{Tor}(A, C) \oplus \operatorname{Tor}(B, C)$.

Example 16.2

- We compute $\operatorname{Tor}(\mathbb{Z}, G)$. Take the resolution $0 \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0$. Deleting $\mathbb{Z}$ gives $0 \to 0 \to \mathbb{Z} \to 0$, and tensoring with G gives $0 \to \operatorname{Tor}(\mathbb{Z}, G) \to 0 \to G$ for the definition of Tor. Thus, we have $\operatorname{Tor}(\mathbb{Z}, G) = 0$, the module which makes this sequence exact.

- We compute $\operatorname{Tor}(\mathbb{Z}/n\mathbb{Z}, G)$. Take the resolution $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$. Deleting $\mathbb{Z}/n\mathbb{Z}$ and tensoring with G gives $0 \to \operatorname{Tor}(\mathbb{Z}/n\mathbb{Z}, G) \to G \xrightarrow{\times n} G \to 0$. The module that makes this sequence exact is the elements of order n, i.e. the set of n-torsion elements. Note that for $G = \mathbb{Z}/m\mathbb{Z}$, we have $\operatorname{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \operatorname{Tor}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$. That is, the n-torsion elements in $\mathbb{Z}/m\mathbb{Z}$ are equal to the m-torsion elements in $\mathbb{Z}/n\mathbb{Z}$.

We'll now go over some properties of Tor for $\mathbb{Z}$-modules, which will motivate the properties of Tor (and derived functors) in general.

Proposition 16.3 (1) Tor is well defined, i.e. it doesn't depend on the resolution;

(2) Tor is a functor in both arguments: $f : B \to C$ induces a canonical map $\mathrm{Tor}(A, B) \to \mathrm{Tor}(A, C)$;

(3) Tor is symmetric

(4) Tor induces a long exact sequence: for $0 \to A \to B \to C \to 0$ exact, then there is a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}(A, M) \arrow[r] & \mathrm{Tor}(B, M) \arrow[r] & \mathrm{Tor}(C, M) \arrow[dlll, out=-30, in=150] \\
A \otimes M \arrow[r] & B \otimes M \arrow[r] & C \otimes M \arrow[r] & 0
\end{tikzcd}$$

Proof. (1) Choose two resolutions $0 \to \mathbb{Z}^{m_1} \to \mathbb{Z}^{n_1} \to A \to 0$ and $0 \to \mathbb{Z}^{m_2} \to \mathbb{Z}^{n_2} \to A \to 0$. Consider the following diagram, obtained by lifting id_A to two the projective resolutions in either direction:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^{m_1} \arrow[r] \arrow[d, shift right] & \mathbb{Z}^{n_1} \arrow[r] \arrow[d, shift right] \arrow[dl, "\exists s"] & A \arrow[r] \arrow[d, leftrightarrow, "="] & 0 \\
0 \arrow[r] & \mathbb{Z}^{m_2} \arrow[r] \arrow[u, shift right] & \mathbb{Z}^{n_2} \arrow[r] \arrow[u, shift right] & A \arrow[r] & 0.
\end{tikzcd}$$

The maps, after tensoring with id_B, induce mutually inverse isomorphisms between the homology complexes. Moreover, any two maps between projective resolutions (or more generally between a complex of projectives and an exact sequence), are homotopic. Thus the isomorphisms on homology are independent of the choices of maps.

(2) The diagram from part (1) also gives functorality.

(3) Take $0 \to \mathbb{Z}^m \to \mathbb{Z}^n \to A \to 0$ and $0 \to \mathbb{Z}^s \to \mathbb{Z}^t \to B \to 0$. Consider the following diagram:

$$\begin{tikzcd}
 & & & \mathrm{Tor}(B, A) \arrow[d, red] & \\
 & \mathbb{Z}^m \otimes \mathbb{Z}^s \arrow[r] \arrow[d] & \mathbb{Z}^n \otimes \mathbb{Z}^s \arrow[r, red] \arrow[d, red] & A \otimes \mathbb{Z}^s \arrow[r] \arrow[d] & 0 \\
 & \mathbb{Z}^m \otimes \mathbb{Z}^t \arrow[r, red] \arrow[d, red] & \mathbb{Z}^n \otimes \mathbb{Z}^t \arrow[r] \arrow[d] & A \otimes \mathbb{Z}^t \arrow[r] \arrow[d] & 0 \\
\mathrm{Tor}(A, B) \arrow[r, red] & \mathbb{Z}^m \otimes B \arrow[r] \arrow[d] & \mathbb{Z}^n \otimes B \arrow[r] \arrow[d] & A \otimes B \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By following the arrows and lifting along the red path, we get the desired equivalence.

(4) This will follow from the general result we'll prove later.

□

§16.2 Tor in R-modules

Definition 16.4 (Tor) — For a ring R, and R-modules A, B, define Tor as follows:

1) Take a resolution of A:
$$\cdots \to R^{n_3} \to R^{n_2} \to R^{n_1} \to R^{n_0} \to A \to 0,$$
which is an exact sequence.

2) Tensor the deleted resolution (the resolution above with A taken out) with B, to get
$$A \overset{L}{\otimes} B := \cdots \to B^{n_2} \to B^{n_1} \to B^{n_0} \to 0,$$
which is not in general exact.

3) Define $\mathrm{Tor}_i^R(A, B)$, the i'th component of Tor, by
$$\mathrm{Tor}_i^R(A, B) := \frac{\ker(B^{n_i} \to B^{n_{i-1}})}{\mathrm{im}(B^{n_{i+1}} \to B^{n_i})} =: H^i(A \overset{L}{\otimes} B)$$

We can ask: which of the properties from the $\mathbb{Z}$-module case does this Tor still satisfies. It turns out, many of the properties still hold: this Tor is well defined (doesn't depend on choice of resolution), functorial (it is a functor), symmetric ($\mathrm{Tor}_i^R(A, B) = \mathrm{Tor}_i^R(B, A)$), and gives a long exact sequence: given $0 \to A \to B \to C \to 0$ exact, the following is also exact:

$$\begin{tikzcd}
\cdots \mathrm{Tor}_2^R(A, M) \arrow[r] & \mathrm{Tor}_2^R(B, M) \arrow[r] & \mathrm{Tor}_2^R(C, M) \arrow[dll, rounded corners] \\
\mathrm{Tor}_1^R(A, M) \arrow[r] & \mathrm{Tor}_1^R(B, M) \arrow[r] & \mathrm{Tor}_1^R(C, M) \arrow[dll, rounded corners] \\
A \otimes M \arrow[r] & B \otimes M \arrow[r] & C \otimes M \arrow[r] & 0
\end{tikzcd}$$

We'll now go over a few examples in R-modules.

Example 16.5 (1) Let $R = k[x]/(x^2)$, where k is a field. This is two dimensional as a k-vector space (with basis $\{1, x\}$. Let $M := R/(x)$. We compute $\mathrm{Tor}_i^R(M, M)$. A resolution of M is

$$\begin{tikzcd}
\cdots R \arrow[r] & R \arrow[r] & R \arrow[r] & M \arrow[r] & 0 \\
(1 \arrow[r, mapsto] & x), (1 \arrow[r, mapsto] & x), (1 \arrow[r, mapsto] & 1)
\end{tikzcd}$$

Tensoring the deleted sequence with M, we get $\cdots \to M \overset{0}{\to}\overset{0}{\to} M \overset{0}{\to} M \overset{0}{\to} 0$. Since all the maps are zero, $\mathrm{Tor}_i = M$ for each i, and there is no finite resolution.

(2) Define $R = k[x, y]$, for a field k, and let $k_{00} = R/(x, y)$, considered as a module over R. There is a resolution $0 \to R \to R \oplus R \to R \to k_{00} \to 0$, where the map $R \to R \oplus R$ is $1 \mapsto (y, -x)$, the map $R \oplus R \to R$ is $(1, 0) \mapsto x$, $(0, 1) \mapsto y$.

Tensoring with k_{00} to compute $\operatorname{Tor}_i^R(k_{00}, k_{00})$, we get

$$0 \to k_{00} \xrightarrow{0} k_{00}^2 \xrightarrow{0} k_{00} \to 0.$$

Since all the maps are zero, $\operatorname{Tor}_0 = k_{00}$, $\operatorname{Tor}_1 = k_{00}$ and $\operatorname{Tor}_2 = k_{00}$.

§16.3 Derived Functors

We begin by studying "classical" derived functors. The definition is analogous to the above definition for modules

Definition 16.6 — Let $T : \mathbf{A} \to \mathbf{C}$ be a (additive) functor between two Abelian categories $\mathbf{A}, \mathbf{C}$, and let $A \in \mathbf{A}$. Take a projective resolution of A (assume we can do this), to get

$$P := \cdots \to P_3 \to P_2 \to P_1 \to P_0 \to A \to 0.$$

Form the deleted sequence

$$P_A := \cdots \to P_3 \to P_2 \to P_1 \to P_0 \to 0.$$

Apply the functor to the sequence to get

$$TP_A := \cdots TP_2 \to TP_1 \to TP_0 \to 0.$$

Then define the **classical left derived functors** $(L_iT)(A) := H_i(TP_A)$, for each i. These are functors $\mathbf{A} \to \mathbf{C}$ for each i. We define the **classical right derived functors** analogously.

We'll define the left derived functor LT so that $L_iT == H^i(LT)$. In particular, we'll define it so that we don't have to first take a resolution.

Given an exact triple $0 \to A \to B \to C \to 0$, if T is left exact, then we get a long exact sequence

$$\begin{tikzcd} \cdots R^iF(A) \arrow[r] & R^iF(B) \arrow[r] & R^iF(C) \arrow[dll, out=0, in=180] \\ R^{i+1}F(A) \arrow[r] & \cdots \end{tikzcd}$$

We get a similar sequence with descending indices is T is right exact.

Definition 16.7 — A functor on a triangulated category is called *exact* if it maps distinguished triangles to distinguished triangles.

Definition 16.8 — Given a functor $F : \mathbf{A} \to \mathbf{B}$ which is right exact, define $K^+(F) : K^+(\mathbf{A}) \to K^+(\mathbf{B})$ as the canonical induced functor. (Applying F to a complex pointwise leaves homotopic morphisms homotopic.)

Proposition 16.9

Let F, and $K^+(F)$ as above.

(1) $K^+(F)$ transforms quasi-isomorphisms into quasi-isomorphisms, so it induces a functor $D^+(F) : D^+(\mathbf{A}) \to D^+(\mathbf{B})$.

(2) $D^+(F)$ is exact, i.e. it maps distinguished triangles to distinguished triangles.

Proof. A map $f : K^\bullet \to L^\bullet$ induces a canonical isomorphism $F(\mathrm{Cone}(f) \to \mathrm{Cone}(F(f))$, since $F(\mathrm{Cone}(f)^i) = F(K[1]^i \oplus L^i)$ and $\mathrm{Cone}(F(f))^i = F(K[1])^i \oplus F(L)^i$. Let $K' = F(K)$ and $L' = F(L)$, so $\mathrm{Cone}(F(f))^i = K'[1]^i \oplus (L')^i$. Now, $f : K^\bullet \to L^\bullet$ is a quasi-isomorphism if and only if $\mathrm{Cone}(f)$ is acyclic. Now $K \to L \to \mathrm{Cone}(f) \to K[1]$ induces a long exact sequence

$$\cdots H_{i-1}(\mathrm{Cone}(f)) \to H_i(K) \xrightarrow{f^*} H_i(L) \to H_i(\mathrm{Cone}(f)) \cdots.$$

Then for a quasi-isomorphism f, $C(f)$ is acyclic, and by exactness $F(\mathrm{Cone}(f)) \simeq \mathrm{Cone}(F(f))$ is acyclic, and thus $F(f)$ is a quasi-isomorphism. That is, $K^+(F)$ mapso quasi-isomorphisms to quasi-isomorphisms. □

Exercise 16.10. Fill in the details in the above proof.

§17 March 25, 2021: Madison Shirazi

§17.1 Adapted Classes of Objects

Recall that our idea to construct the derived functor was to apply F elementwise to get a map of complexes, but we don't want to apply F to every object of every complex. We introduce *adapted classes* to make this precise. First, note that we define a **subobject** of an object C as an isomorphism class of monomorphisms $A \hookrightarrow C$. Two monomorphisms $i : A \hookrightarrow C$ and $k : B \hookrightarrow C$ are isomorphic if there exists an isomorphism $k : A \to B$ such that $i = jk$. A given map $i : A \to C$ a subobject inclusion if and only if for all X, $\mathrm{Hom}(C, X) \to \mathrm{Hom}(A, X)$ is a surjection.

Definition 17.1 — A class of objects $R \subset \mathrm{Ob}(\mathbf{A})$ is said to be **adapted** to a right exact (resp. left exact) functor F if

(1) It is stable under finite direct sums;

(2) F maps any acyclic complex from $\mathrm{Kom}^+(R)$ (resp. $\mathrm{Kom}^-(R)$) to an acyclic complex.

(3) Any object in $\mathbf{A}$ is a quotient (resp. subobject) of an object from R.

Proposition 17.2

Let R be a class of objects adapted to a right exact functor $F : \mathbf{A} \to \mathbf{B}$, and S_R a class of quasi-isomorphisms in $K^+(R)$. Then S_R is a localizing class of morphisms in $K^+(R)$, and the canonical functor $K^+(R)[S_R^{-1}] \to D^+(\mathbf{A})$ is an equivalence of categories.

Proof. We know that the localization of $K^+(A)$ at quasi-isomorphisms is canonically isomorphic to $D^+(\mathbf{A})$. Since R is closed under direct sums, $K^+(R)[S_R^{-1}]$, then $K^+(R)[S_R^{-1}] \to D^+(\mathbf{A})$. A necessary condition for S_R to be a localizing class of morphisms is that for any $s : X' \to X$, with $s \in S$, $X \in \mathrm{Ob}(\mathbf{B})$, there exists $f : X'' \to X'$ such that $sf \in S$ and $X'' \in \mathrm{Ob}(\mathbf{B})$. ☐

Exercise 17.3. Fill in the details in the above proof. See section III.6.4 in the course text.

Example 17.4

Let $\mathbf{A} = \mathbf{Ab}$, and R the class of finitely generated free Abelian groups, then $\mathbb{Z}/n\mathbb{Z} \in D^+(\mathbf{A})$. We can obtain $\mathbb{Z}/n\mathbb{Z}$ just from $K^+(R)$, using $\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$ and taking a resolution.

We now define RF on objects of $K^+(R)[S_R^{-1}]$ term by term:

$$RF(K^\bullet)^i = F(K^i)$$

for $K^\bullet \in \mathrm{Ob}(K^+(R))$, and $F : \mathbf{A} \to \mathbf{B}$. Since quasi-isomorphisms in $K^+(R)$ are mapped to quasi-isomorphisms, $RF : K^+(R)[S_R^{-1}] \to D^+(\mathbf{B})$. From the previous proposition, we have a natural embedding (equivalence) $K^+(R)[S_R^{-1}] \to D^+(\mathbf{A})$, so choose an inverse and we get $\Phi : D^+(\mathbf{A}) \to K^+(R)[S_R^{-1}]$. This allows us to finally define the derived functor on the derived category:

$$RF(K^\bullet) := RF(\Phi(K^\bullet)).$$

This doesn't depend on Φ, which is clear. It is also doesn't depend on the choice of R, which is less clear: we state below in a formal definition(/proposition) of the derived functor, which can be used to verify independence.

Definition 17.5 — The derived functor of an additive left exact functor $F : \mathbf{A} \to \mathbf{B}$ is a pair

$$(RF : D^+(\mathbf{A}) \to D^+(\mathbf{B}), \epsilon_F : Q_B \circ K^+(F) \to RF \circ Q_A) :$$

$$\begin{tikzcd}
 & D^+(\mathbf{A}) \arrow[dr, "R_F"] & \\
K^+(\mathbf{A}) \arrow[ur, "Q_A"] \arrow[dr, "K^+(F)"'] & & D^+(B) \, , \\
 & K^+(\mathbf{B}) \arrow[ur, "Q_B"'] \arrow[uu, Rightarrow, "\epsilon_F"] &
\end{tikzcd}$$

satisfying the following property: for any exact functor $G : D^+(\mathbf{A}) \to D^+(\mathbf{B})$, and any $\epsilon : Q_B \circ K^+(F) \to G \circ Q_A$, there exists a unique morphism of functors $\eta : RF \to G$ such that the following commutes:

$$\begin{tikzcd}
 & Q_B \circ K^+(F) \arrow[dl, "\epsilon_F"] \arrow[dr, "\epsilon"] & \\
RF \circ Q_A \arrow[rr, "\eta \circ Q_A"] & & G \circ Q_A .
\end{tikzcd}$$

Suppose we have two derived functors (RF, ϵ) and $(\tilde{RF}, \tilde{\epsilon})$. Then we can see that the derived functor is unique by gluing together the diagrams

$$\begin{tikzcd}
 & Q_B \circ K^+(F) \arrow[dl, "\epsilon_F"'] \arrow[dr, "\tilde{\epsilon}_F"] & \\
RF \circ Q_A \arrow[rr, "\eta \circ Q_A"] & & \tilde{RF} \circ Q_A.
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Q_B \circ K^+(F) \arrow[dl, "\tilde{\epsilon}_F"'] \arrow[dr, "\epsilon_F"] & \\
\tilde{RF} \circ Q_A \arrow[rr, "\eta' \circ Q_A"] & & RF \circ Q_A.
\end{tikzcd}$$

Exercise 17.6. Show that the definition of the derived functor above matches with the definition in terms of adapted classes.

§18 March 30, 2021

Today I missed class, and I would like to thank Mark Kong for these notes.

§18.1 Sheaves

Let X be a topological space. The category $\mathrm{Opens}(X) = \mathrm{Op}(X)$ has as objects open subsets of X, and as morphisms injections.

Definition 18.1 — Fix $\mathbf{A}$ an Abelian category (assume this category is “reasonable” in the sense that it’s $\mathbf{Ab}$, or more generally $\mathbf{LMod}_R$ for an associative unital ring R). The **category of A-valued presheaves** is the functor category

$$\mathrm{Fun}(\mathrm{Op}(X)^{op}, \mathbf{A}) = \{\mathcal{F} : \mathrm{Op}(X)^{op} \to \mathbf{A}\}.$$

This category is denoted $\mathrm{PSh}(X; \mathbf{A})$. Within $\mathrm{PSh}(X; \mathbf{A})$, there is a subcategory of **sheaves**, which is the category of all $\mathcal{F} : \mathrm{Op}(X)^{op} \to \mathbf{A}$ such that for all $U \in \mathrm{Op}(X)$ and $\{U_i \to U\}_{i \in I}$ with $\bigcup\limits_{i \in I} U_i = U$, the following is an equalizer:

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j).$$

Unpacking this, the sheaf condition on an open cover $\{U_i\}$ of an open subset $U \subseteq X$ says that for any choices of elements $(s_i \in \mathcal{F}(U_i))_{i \in I}$ with $s_i |_{U_i \cap U_j} = s_j |_{U_j \cap U_i}$ for all i, j, we can glue the s_i together in the sense that there exists a unique $s \in F(u)$ such that $s |_{U_i} = s_i \in \mathcal{F}(U_i)$.

Example 18.2

Fix an object $A \in \mathbf{A}$. Consider the constant presheaf where all objects map to A and all morphisms map to the identity. This is not a sheaf. First reason why: $\varnothing$ is an open cover of $\varnothing \subseteq X$ (don’t confuse this with the open cover $\{\varnothing\}$). Indeed, the hypotheses are vacuously satisfied. The objects involved in the equalizer condition are then the empty product, which is 0 (that is, the 0 object in $\mathbf{A}$). Thus, $F(\varnothing)$ must be the equalizer of $0 \rightrightarrows 0$, which is 0. However, by the definition of the constant presheaf, $F(\varnothing) = A$, and A usually isn’t 0.

Can we fix this by sending $\mathcal{F}(\varnothing) = 0$? Again the answer is no. Let $X = X_0 \coprod X_1$ be a disjoint union (so this is a disconnected space). Then $\{X_0, X_1\}$ forms an open cover of X. The equalizer condition implies $\mathcal{F}(X) \simeq \mathcal{F}(X_0) \times \mathcal{F}(X_1)$ (it's the equalizer of two maps from $A \times A$ to 0), but $A \not\simeq A \times A$ usually.

Example 18.3

Let $p \in X$ and let $A \in \mathbf{A}$ be an object. The **skyscraper sheaf** (or **Dirac-δ sheaf**) is the functor

$$(i_{p^*}A)(U) := \begin{cases} A & p \in U \\ 0 & p \notin U \end{cases}.$$

This is a sheaf.

Example 18.4

Let X be a manifold. Then the **sheaf of ∞-differentiable functions** is

$$\mathcal{O}_X^{\text{diff}} : U \mapsto \{f : U \to C^\infty(U)\}.$$

We could also have replaced "manifold" with "$\mathbb{C}$-manifold" and "C^∞" with "analytic/holomorphic."

Exercise 18.5. Verify that the skyscraper sheaf and sheaf of ∞-differentiable functions are actually sheaves. For the skyscraper sheaf, you can do this sectionwise/componentwise.

§18.2 Sheafification

We have an injection $\text{Sh}(X, \mathbf{A}) \hookrightarrow \text{PSh}$. Abstractly, there must exist a left adjoint, by the adjoint functor theorem.

We will now give a construction of the sheafification. The construction we will use will be one that generalizes to higher categories; the other methods of sheafification are historical mistakes and should be forgotten.

We will use L_{Zar} to denote the left adjoint. The construction proceeds in four steps.

§18.2.1 Step 1

Let $\mathcal{U} = \{U_1 \to U\}$ be a cover of U in $\text{Op}(X)$. If $\mathcal{F}$ were a sheaf, then

$$\mathcal{F}(U) \simeq \text{Eq}(\prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \cap U_j)).$$

Since we don't know that F is a sheaf, instead set

$$H^0(\mathcal{U}, \mathcal{F}) := \text{Eq}(\prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \cap U_j)),$$

which is

$$\{(s_i)_{i \in I} \in \prod \mathcal{F}(U_i) : s_i \mid_{U_i \cap U_j} = s_j \mid_{U_i \cap U_j}\}.$$

This depends on the cover $\mathcal{U}$, but we want an equalizer condition to hold for all covers, so we assemble all the covers of U into a category

$$\text{Cov}_U = \{\mathcal{U}(= \{U_i \to U\}_{i \in I})\};$$

the morphisms are maps $\mathcal{U}(=\{U_i \to U\}_{i\in I} \to \mathcal{V}(=\{V_j \to U\}_{j\in J})$ such that there exists some set function $\alpha : I \to J$ with inclusions $U_i \to V_{\alpha(i)}$.

This is the category of all covers. These morphisms are often called **refinements** (one says $\mathcal{U}$ is a refinement of $\mathcal{V}$ in the above notation). We then have a functor

$$\mathrm{Cov}_U^{op} \to \mathbf{A}$$

given by $\mathcal{U} \mapsto H^0(\mathcal{U};\mathcal{F})$.

§18.2.2 Step 2

The only natural thing you can do is to set

$$\mathcal{F}^+(U) \coloneqq \operatorname*{colim}_{\mathcal{U}\in\mathrm{Cov}_U^{op}} H^0(\mathcal{U};\mathcal{F}).$$

Warning: This isn't a sheaf yet. This is called the "+-construction."

Lemma 18.6 (Key Lemma)
Cov_U^{op} is filtered.

Then $U \mapsto \mathcal{F}^+(U)$ is a functor (as can be verified using the universal property).

Observe: Given an element $s \in \mathcal{F}^+(U) = \operatorname*{colim}_{\mathrm{Cov}_U} H^0(\mathcal{U},\mathcal{F})$, because we are taking a filtered colimit, s is defined on some cover $\mathcal{U}$.

(The lemma we are using is that compact objects factor through filtered colimits. In **Set**, every element of a colimit is an element of one of the original objects, but this isn't true in a general Abelian category, e.g. **Ab**).

New apply Yoneda on an open set U. We get a functor $\mathrm{Op}(X) \to \mathrm{PSh}(X;\mathbf{A})$ by sending $U \mapsto Y_U(V) = \mathrm{Hom}(V,U)$.

(How is the Yoneda functor defined if $\mathbf{A}$ is not **Sets**? If $\mathbf{A} = \mathbf{Ab}$, then Y_U is "free Abelian homs," meaning the group of formal linear combinations of homs. If $\mathbf{A} = \mathrm{LMod}_R$, then Y_U is the free left R-module on the set of homs.)

The following diagram will be relevant:

$$\begin{tikzcd}
\coprod Y_{U_i} \arrow[r, dashed] \arrow[d, dashed] & \mathcal{F} \arrow[d, "\theta"] \\
Y_U \arrow[r] & \mathcal{F}^+
\end{tikzcd}$$

§18.2.3 Step 3

Although $\mathcal{F}^+$ isn't a sheaf, we claim $\mathcal{F}^+$ has "improved" $\mathcal{F}$ in some sense. To make this precise, we make the following definition:

Definition 18.7 — $\mathcal{F}$ is **separated** if

$$\mathcal{F}(U) \hookrightarrow \prod_{i\in I} \mathcal{F}(U_I)$$

for all covers $\{U_i \hookrightarrow U\}$ (note the hooks in the arrows!)

If $\mathcal{F}$ lands in $\mathbf{A}$, an Abelian category, then $\mathcal{F}$ is separated if and only if "covers detect zero," that is, for all covers $\{U_i \hookrightarrow U\}$, $s \in \mathcal{F}(U)$ is 0 if and only if $s\,|_{U_i} = 0$ for all i.

This is the sense in which $\mathcal{F}^+$ improves $\mathcal{F}$:

Lemma 18.8

$\mathcal{F}^+$ is separated.

Proof. We prove this in the case when $\mathcal{F}$ lands in an Abelian category. In this case, pick $S \in \mathcal{F}^+(U)$ with $\{U_i \to U\} = \mathcal{U}$ such that $s\,|_{U_i} \in \mathcal{F}^+(U_i) = 0$ for all $i \in I$. We want to show $s = 0$.

Let θ denote the map $\mathcal{F} \to \mathcal{F}^+$ from the commutative square in step 2. By the observation earlier, for each U_i there exists a fixed cover $\{V_{ij} \to U_i\}$ such that $0 = s\,|_{U_i}|_{V_{ij}} = \theta(s_{ij})$ for some $s_{ij} \in \mathcal{F}(V_{ij})$. Up to further refinement, $s_{ij} = 0$ for all i, j.

Granted this, s is 0 in $H^0(\{V_{ij} \to U\})$, and hence s is 0 in $\mathcal{F}^+(U)$.

The content of the observation is that if s is zero then it must have come from 0 in a finite stage. That is, if $A = \operatorname*{colim}_{i\in I} A_i$ and I is filtered and $s = 0$ in A, then there exists $i \in I$, $s_i \mapsto s$ and $s_i = 0$ in A_i. The proof of this in the case $\mathbf{A} = \mathbf{Ab}$ is left as an exercise. □

§18.2.4 Step 4

Lemma 18.9

If $\mathcal{F}$ is separated, then $\mathcal{F}^+$ is a sheaf and $\mathcal{F} \to \mathcal{F}^+$ is injective.

Separatedness is quite close to the sheaf condition: the separated condition says $\mathcal{F}(U) \hookrightarrow \prod \mathcal{F}(U_i)$, which is indeed implied by the sheaf condition.

Proof. This is the cleanest way in which filteredness enters the picture. Let's prove that $\mathcal{F} \to \mathcal{F}^+$ is injective first. For any cover $\mathcal{U}$,

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[dr] & H^0(\mathcal{U};\mathcal{F}) \arrow[d] \\
 & \prod_{i\in I} \mathcal{F}(U_i)
\end{tikzcd},$$

and the diagonal arrow is an injection since $\mathcal{F}$ is separated. Thus, $\mathcal{F}(U) \to H^0(\mathcal{U};\mathcal{F})$ is injective.

But now $\mathcal{F}(U) \xrightarrow{\theta} \mathcal{F}^+(U) = \operatorname*{colim}_{\mathrm{Cov}_u^{op}} H^0(\mathcal{U},\mathcal{F})$ is a colimit of injections, and a filtered colimit of injections is an injection.

To see that $\mathcal{F}^+$ is a sheaf, we need to check the gluing condition: if $s_i \in \mathcal{F}^+(U_i)$ are chosen such that $s_i\,|_{U_i \cap U_j} = s_j\,|_{U_j \cap J_i}$ for all i, j, then the S_i glue to a unique section (i.e. element) of $\mathcal{F}^+(U)$. Uniqueness is guaranteed by the first claim. □

We'll finish the proof in the next class. It is a diagram chase. As a hint, it uses the injectivity we just proved. For now, take the lemma for granted.

Theorem 18.10

Let $\mathcal{F} \in \mathrm{PSh}(X, \mathbf{A}), \mathbf{A} \in \{\mathbf{Ab}, \mathbf{LMod}_R, \mathbf{Sets}\}$. Then

- $\mathcal{F}^{++}$ is a sheaf;
- $L_{\mathrm{Zar}}\mathcal{F} = \mathcal{F}^{++}$;
- The map $\mathcal{F} \to \mathcal{F}^{++}$ is the sheafification map.

Proof. The first claim is clear (apply the lemma to $\mathcal{F}^+$).

There's a universal property that if $\mathcal{G}$ is a sheaf, then any map $\mathcal{F} \to \mathcal{G}$ factors through $\mathcal{F}^{++}$. The proof is using the universal property of the colimit we used to define $\mathcal{F}^{++}$. ☐

§19 April 1, 2021

§19.1 Sheafification

Recall from last time the "+-concstruction," which is $\mathcal{F}^+(\mathcal{U}) := \operatorname*{colim}_{\mathrm{Cov}_U^{op}}(H^0(\mathcal{U}); \mathcal{F})$, where $\mathcal{U} = \{U_i \to U\}$ and $H^0(\mathcal{U}; \mathcal{F}) := \mathrm{Eq}(\prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j))$. If $\mathcal{F}$ takes values in $\mathbf{A}$, this is $\ker(\prod \mathcal{F}(U_i) \xrightarrow{r_1 - r_2} \prod \mathcal{F}(U_i \cap U_j)$. So far we've been vague about $\mathbf{A}$. We will make this more clear today.

Recall from last time that $\mathcal{F}^+$ is a separated presheaf. We started the proof of the following lemma, which we will now finish.

Lemma 19.1

If $\mathcal{F}$ is separated, then $\mathcal{F} \to \mathcal{F}^+$ is injective and $\mathcal{F}^+$ is a sheaf.

Proof. We saw injectivity last time. We show that $\mathcal{F}^+$ is a sheaf. Let $\mathcal{I} = \{U_i \to U\}$ be a cover of U, and let $s_i \in \prod_{i \in I} \mathcal{F}^+(U_i)$ be compatible, meaning $s_i\,|_{U_i \cap U_j} = s_j\,|_{U_i \cap U_j}$ for all i, j. Our goal is to produce an $s \in \mathcal{F} + (U)$ so that $s\,|_{U_i} = s_i$ for all i. Once we have such an s, uniqueness will be guaranteed by the separatedness of $\mathcal{F}$ (which implies separatedness of $\mathcal{F}^+$.

To produce such an s, for each i, choose a cover $\{V_{i_j} \to U_i\}_{i,j}$ of U_i such that $s_i\,|_{V_{i_j}} = \theta(t_{i_j})$, where $t_{i_j} \in \mathcal{F}(V_{i_j})$ and $\theta : \mathcal{F} \to \mathcal{F}^+$. Then let $\mathcal{V} = \{V_{i_j} \to U\}_{i_j}$. We claim now that $(t_{i_j})_{i_j}$ defined an element of $H^+(\mathcal{V}; \mathcal{F})$, and hence an element of $\mathcal{F}^+(U)$. To show this, we need to show that $t_{i_j}\,|_{V_{i_j} \cap V_{i'_{j'}}} = t_{i'_{j'}}\,|_{V_{i_j} \cap V_{i'_{j'}}}$ for all i, j, i', j'. By the part of the lemma we showed last time, $\mathcal{F} \to \mathcal{F}^+$ is injective (note this uses separability). So using the assumed sheaf condition on the s_i and applying θ, we have $\theta : t_{i_j} \to s_i$, and the t_{i_j} inherit the sheaf property from s_i. Thus we have $t_{i_j}\,|_{V_{i_j} \cap V_{i'_{j'}}} = t_{i'_{j'}}\,|_{V_{i_j} \cap V_{i'_{j'}}}$, and so these define an element t of $\mathcal{F}^+(U)$. Then check that $t\,|_{U_i} = s_i$, and we are done. ☐

Theorem 19.2

The functor $L : \mathrm{Shv}(X; \mathbf{A}) \to \mathrm{PShv}(X; \mathbf{A})$ has left adjoint $L_{\mathrm{Zar}} : \mathrm{PShv}(X; \mathbf{A}) \to \mathrm{Shv}(X; \mathbf{A})$, and L_{Zar} is computed as $L_{\mathrm{Zar}}\mathcal{F} = \mathcal{F}^{++}$ (which is a sheaf).

You may have sen sheafification before in "one-step." You can do this, but it is useful to use colimits. Our "+" formulation relies on two properties of $\mathbf{A}$, namely that it has filtered colimits, and that filtered colimits preserve monomorphisms. To make $\mathrm{Shv}(X; \mathbf{A})$ to be an Abelian category, these two conditions are sufficient.

Proof. Take $\mathcal{F} \to \mathcal{F}^+ \to \mathcal{F}^{++}$, where $\mathcal{F}^+$ is separable and $\mathcal{F}^{++}$ is a sheaf. Check the universal property

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[dr] & \mathcal{F}^{++} \arrow[d, dashed] \\
 & \mathcal{G}
\end{tikzcd}$$

$\square$

Our next goal will be to answer the question: when is $\mathrm{Shv}(X;\mathbf{A})$ an Abelian category? First, we introduce the notion of *stalks*.

Definition 19.3 (Stalk) — Fix $x \in X$. The **stalk**, $\mathcal{F}_x$ of $\mathcal{F}$ at x is defined as the filtered colimit (direct limit)

$$\mathcal{F}_x = \mathrm{colim}_{U \ni x} \mathcal{F}(U).$$

Intuitively, a stalk takes you from a big neighborhood to a little neighborhood around a point, i.e. it is "concentrating at a point."

Lemma 19.4

If $\mathcal{F}$ is a presheaf, then $\mathcal{F}_x \simeq (L_{\mathrm{Zar}}\mathcal{F})_x$. That is, sheafification does not change stalks.

The basic idea of the proof is that colimits commute and filtered colimits commute with finite limits, in particular equalizers.

Example 19.5

Let $A \in \mathbf{A}$. Then $\underline{A} : U \mapsto A$ is not a sheaf. What is $L_{\mathrm{Zar}}(\underline{A})$?

The above lemma implies that $(L_{\mathrm{Zar}}(\underline{A})_x = (\underline{A})_x = A$. Observe that if U is a connected topological space, then since every continuous map from a connected to a discrete space is constant, we have $A = \mathrm{Hom}_{\mathrm{cts}}(U, A)$. We know that in general, $(L_{\mathrm{Zar}}\underline{A})(U) \neq A$; for example if $U = U_1 \sqcup U_2$, then $(L_{\mathrm{Zar}}\underline{A}) = A(U_1) \times A(U_2) = A \times A$ (if each U_i is connected).

We can guess then that $L_{\mathrm{Zar}}\underline{A}(U) \simeq \mathrm{Hom}_{\mathrm{cts}}(U, A)$. This works at every stalk, and it is in fact actually true, since continuous maps can be glued.

In general, it is very difficult to compute using the + construction. In practice, the best way to compute things is to guess, using stalks.

We still want to know when Shv is Abelian. We'll now prove a result to this end.

Lemma 19.6

Let $\mathbf{A}$ be an Abelian category, and let $\mathbf{B} \subseteq \mathbf{A}$ be a full subcategory, and let $L : \mathbf{A} \to \mathbf{B}$ be left-adjoint to inclusion. If L is left exact, then $\mathbf{B}$ is Abelian.

Proof. Take $X \xrightarrow{f} Y$ a monomorphism in $\mathbf{B}$. Then $\ker f = \ker_{\mathbf{B}}(f) = \ker_{\mathbf{A}}(f)$ computes in $\mathbf{A}$: the kernel is a limit, and $\mathbf{B} \hookrightarrow \mathbf{A}$ is a right adjoint, and right adjoints preserve limits. This implies (check) that a kernel in $\mathbf{A}$ is a kernel in $\mathbf{B}$: i.e. $\hookrightarrow$ "creates limits." Now, $\mathrm{coker}(f) = L(\mathrm{coker}(f))$, but since coker is a colimit, there is no reason for it to be preserved in the other direction, by the inclusion $\mathbf{B} \hookrightarrow \mathbf{A}$. To check the last axiom of Abelian categories, i.e. that $\mathrm{coker}(\ker(f) \to X) \simeq \ker(Y \to \mathrm{coker}(f))$, note that $L(\mathrm{coker}(f) \to X)) = \mathrm{coker}_{\mathbf{B}}(\ker(f) \to X)$. Compare this to $\ker(Y \to L(\mathrm{coker}(f))) = \ker(Y \to \mathrm{coker}_{\mathbf{B}}(f))$. We want to show that $L(\mathrm{coker}(\ker(F) \to X)) \simeq \ker(Y \to L(\mathrm{coker}(f)))$. By assumption, L is left exact, so if $0 \to A \to B \to C \to 0$ is exact then so is $0 \to LA \to LB \to LC$, and L preserves kernels, and so $\ker(Y \to L(\mathrm{coker}(f))) = L(\ker(Y \to \mathrm{coker}(f)))$. Then since

$\mathrm{coker}(\ker(f) \to X) \simeq \ker(Y \to \mathrm{coker}(f))$ is an isomorphism in A, then $L(\mathrm{coker}(\ker(f) \to X))) \simeq L(\ker(Y \to \mathrm{coker}(f))) \simeq \ker(Y \to L(\mathrm{coker}(f)))$, by left-exactness. ☐

Now, if $L_{\mathrm{Zar}}(x; \mathbf{A}) \to \mathrm{Shv}(X; \mathbf{A})$ is left exact, we could say that $\mathrm{Shv}(X; \mathbf{A})$ is Abelian. What does left-exactness mean? We want L_{Zar} to preserve kernels. Since L_{Zar} is a filtered colimit, it doesn't in general preserve kernels. We therefore make the following definition.

Definition 19.7 — A **Grothendieck Abelian category** is an Abelian category in which $\underset{\overrightarrow{i}}{\mathrm{colim}}$ are exact. That is, if $\{0 \to A_i \to B_i \to C_i\}_{i \in I}$ exact, then $0 \to \mathrm{colim} A_i \to \mathrm{colim} B_i \to \mathrm{colim} C_i$ exact.

Such a category $\mathbf{A}$ has a **generator** in the following sense: there exists an object $U \in \mathbf{A}$ such that for any M and any $0 \to N \hookrightarrow M$, there is a map $U \to M$ which does not factor through N,

$$\begin{tikzcd}
& & U \arrow[dl, dashed, "\nexists"'] \arrow[d, "\exists"] \\
\forall(0 \arrow[r] & N \arrow[r, hook] & M)
\end{tikzcd}$$

Theorem 19.8

Whenever $\mathbf{A}$ is a Grothendieck Abelian category, then $\mathrm{Shv}(X; \mathbf{A})$ is Abelian.

Proof. $L_{\mathrm{Zar}} = \mathcal{F}^{++}$, which is left exact. ☐

Recall that we constructed $D(\mathrm{Shv}(X; \mathbf{A}) = \mathrm{Kom}(\mathrm{Shv}(X; \mathbf{A}))[\text{q-iso}^{-1}]$, and that this is a triangulated category. This is the derived category of sheaves. We have a functor $\Gamma : \mathrm{Shv}(X; \mathbf{A}) \to \mathbf{A}$ with $\gamma(\mathcal{F}) := \mathcal{F}(X)$ (global sections), which is left exact, so we can take $R\Gamma(\mathcal{F})$, and then $H^i(X; \mathcal{F}) := H^i(R\Gamma(\mathcal{F}))$.

Lemma 19.9

$\Gamma(X; \mathbf{A}) \to \mathbf{A}$ is a right adjoint functor.

Proof. Its left adjoint is $\mathbf{A} \to \mathrm{Shv}(X; \mathbf{A})$, with $A \mapsto L_{\mathrm{Zar}}(\underline{\underline{A}})$. ☐

§20 April 6, 2021

§20.1 Computing Cohomology

In this section, X is a topological space, $\mathcal{U} = (U_i)_{i \in I}$ an open cover, and $\mathcal{F}$ a (pre)sheaf on X. Furthermore, suppose there is an ordering $<$ chosen on I. We begin by defining the Čech complex/cohomology.

Definition 20.1 (Čech complex) — The **Čech complex** $\check{C}^\bullet(\mathcal{U}; \mathcal{F})$ is defined as follows. The links in the complex are the Abelian groups

$$\check{C}^n = \prod_{i_0 < i_1 < \cdots < i_n} \mathcal{F}(U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_n}).$$

For an element $\alpha \in \prod_{i_0 \in I} \mathcal{F}(U_{i_0})$, there are two natural maps into $\prod_{i_0 < i_1} \mathcal{F}(U_{i_0} \cap U_{i_1})$, which are $\alpha \mapsto (d^0\alpha)_{i_0<i_1} = \alpha_{i_0}|_{U_{i_0}\cap U_{i_1}}$, and $\alpha \mapsto (d^1\alpha)_{i_0<i_1} = \alpha_{i_1}|_{U_{i_0}\cap U_{i_1}}$, where the restrictions arise from the sheaf. Similarly, for any n, there are $n+1$ natural maps, $\alpha \mapsto (d^k\alpha)_{i_0<\cdots<i_{n+1}} = \alpha_{i_0<\cdots<\hat{i}_k<\cdots<i_{n+1}}|_{U_{i_0}\cap\cdots\cap U_{i_{n+1}}}$, for $\alpha \in \prod_{i_0<\cdots<i_n} \mathcal{F}(U_{i_0}\cap\cdots\cap U_{i_n})$. The differential is then given by the alternating sum

$$d\alpha_{i_0<\cdots<i_{n+1}} = \sum_{j=0}^{n+1}(-1)^k d^k\alpha = \sum_{j=0}^{n+1}(-1)^j \alpha_{i_0<\cdots<\hat{i}_j<\cdots<i_{n+1}}|_{U_{i_0}\cap\cdots\cap U_{i_{n+1}}}.$$

We notate the ith cohomology group of this complex by $\check{H}^j(\mathcal{U};\mathcal{F}) = H^j(\check{C}(\mathcal{U};\mathcal{F}))$. In general, this will depend on the space, the cover, and the presheaf, but we will sometimes write $\check{H}(X;\mathcal{F})$.

Lemma 20.2

If $\mathcal{F}$ is a sheaf, then $\check{H}^0(X;\mathcal{F}) \simeq \mathcal{F}(X)$, the global section.

Proof. This follows immediately from the sheaf condition, and the definition of the first boundary map as $\alpha_{i_0}|_{U_{i_0}\cap U_{i_1}} - \alpha_{i_1}|_{U_{i_0}\cap U_{i_1}}$. ☐

Example 20.3

Let $X = S^1$, and $\mathcal{F} = L_{\text{Zar}}\underline{\underline{\mathbb{Z}}}$. Then

$$\check{H}^j(X;\mathcal{F}) = \begin{cases} \mathbb{Z} & j=0 \\ \mathbb{Z} & j=1 \\ 0 & j>2 \end{cases}.$$

For $j=0$, this follows from the fact that continuous maps from X to $\mathbb{Z}$ are isomorphic to $\mathbb{Z}$, i.e. $\mathbb{Z}^X \simeq \mathbb{Z}$. We can cover X with two open sets (we can't cover with a single set), so the higher degree terms are zero. To justify the $j=1$ term, call the two covering sets A and B (they can be viewed as overlapping half-circles). In degree 0, we have $\mathcal{F}(A)\times\mathcal{F}(B) \simeq \mathbb{Z}\times\mathbb{Z}$. For their intersection, we have $\mathcal{F}(A\cap B) \simeq \mathbb{Z}\times\mathbb{Z}$, since the intersection is two arcs. The differential gives $\mathcal{F}(A)\times\mathcal{F}(B) \to \mathcal{F}(A\cap B)$ as

$$\begin{aligned} \mathbb{Z}\times\mathbb{Z} &\to \mathbb{Z}\times\mathbb{Z} \\ (a,b) &\mapsto (a-b, a-b). \end{aligned}$$

The image is the diagonal, which is isomorphic to $\mathbb{Z}$, so $\check{H}^1 = \ker/\operatorname{im} \simeq \mathbb{Z}\times\mathbb{Z}/\mathbb{Z} \simeq \mathbb{Z}$.

Example 20.4

Consider $\mathbb{Z}P^1 \simeq S^2$, the complex projective space. For an open $U \subset \mathbb{C}P^1$, we have $U \to \Omega^1_U$, the set of complex differential forms. For example, $\mathbb{A}^1_{\mathbb{C}} \mapsto \Omega_{\mathbb{A}^1_{\mathbb{C}}} \simeq \mathbb{C}[x]dx$, and $\mathbb{A}^1_{\mathbb{C}\smallsetminus\{0\}} \mapsto \mathbb{C}[x,1/x]dx$. We can cover S^2 with two copies of $\mathbb{A}^1_{\mathbb{C}}$: these represent "punctured spheres," one punctured at the bottom, and one punctured at the top. Their intersection is the set $\mathbb{C}\smallsetminus\{0\}$. Then $C^0 = \mathbb{C}[x]dx\times\mathbb{C}[y]dy$, and $C^1 = \mathbb{C}[x,1/x]$, where we have made a preference for x over y, which is arbitrary. The elements of interest in C^0 are

$$\begin{aligned} x \mapsto x \quad & y \mapsto 1/x \\ dx \mapsto dx \quad & dy \mapsto dy (= -1/x^2 dx). \end{aligned}$$

The kernel of d is $\langle f(x)dx, g(y)dy \rangle$, where $f(x)$ and $g(x)$ have $f(x) = -\frac{1}{x^2} g(1/x)$. But this cannot occur unless $f = g = 0$, so we have $\check{H}^0(\mathcal{U}; \Omega) = 0$. But

$$\check{H}^1 = \frac{\mathbb{C}[x, 1/x]dx}{\left(f(x) + \frac{1}{x^2} g(1/x)dx\right)} = \mathbb{C}\{x^{-1}dx = d\log x\},$$

which is a one dimensional $\mathbb{C}$-vector space.

Example 20.5

Let X be a space, and $x \in X$, and let $i_{x,\star}A$ be a skyscraper. Then

$$H^j(X; i_{x,\star}A) = \begin{cases} A & j = 0 \\ 0 & \text{else} \end{cases}.$$

To see this, take a nice cover of X, $\{U_i \to U\}$, so that $U_0 \ni x$ and $U_i \not\ni x$ for $i \neq 0$.

Now, take $\mathbf{A} = \mathbf{Ab}$.

Lemma 20.6

If X is a topological space, then $\mathrm{Shv}(X; \mathbf{A})$ has enough injectives.

Proof. Let A be an Abelian group. Recall that any A has an **injective hull** (see here)

$$0 \to A \to I(A),$$

where $I(A)$ is injective. In fact, I is functorial in A. Once we know this, consider the presheaf $I_{\mathcal{F}} : U \mapsto \prod_{x \in U} I(\mathcal{F}_x)$. We have the following sublemma

Lemma 20.7

$I_{\mathcal{F}}$ is a sheaf.

$I_{\mathcal{F}}$ is a product of skyscrapers $I_{\mathcal{F}} = \prod_{x \in X} i_{x,\star}(\mathcal{F}_x)$, where the equality is in presheaves.

Now, we have a map $\mathcal{F} \to I_{\mathcal{F}}$ given by $s \mapsto \prod_{x \in U} I(s_x)$. Since this map is injective, it suffices to show that $I_{\mathcal{F}}$ is injective.

Suppose we have $0 \to \mathcal{G}_0 \to \mathcal{G}_1$ a monomorphism in Shv, and $\mathcal{G}_0 \to I_{\mathcal{F}}$, we want to show that the following map exists

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G}_0 \arrow[r] \arrow[d] & \mathcal{G}_1 \arrow[dl, dashed] \\ & I_{\mathcal{F}} & \end{tikzcd}$$

We have

$$\begin{aligned}\mathrm{Hom}(\mathcal{G}, I_{\mathcal{F}}) &\simeq \mathrm{Hom}(\mathcal{G}, \prod_{x\in X} i_{x,\star}\mathcal{F}_x) \\ &\simeq \prod_{x\in X} \mathrm{Hom}(\mathcal{G}, i_{x,\star}\mathcal{F}_x) \\ &\simeq \prod_{x\in X} \mathrm{Hom}(\mathcal{G}_x, \mathcal{F}_x)\end{aligned}$$

So to map $\mathcal{G}_1 \to I_{\mathcal{F}}$ is the same as maps $\{\mathcal{G}_{1_x} \to I_{\mathcal{F}_x}\}_{x\in X}$. But by construction, since $I(\mathcal{F}_x)$ is injective, then there exists a map as in the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G}_{0_x} \arrow[r] \arrow[d] & \mathcal{G}_{1_x} \arrow[dl, dashed] \\ & I_{\mathcal{F}_x} & \end{tikzcd}$$

This implies that $I_{\mathcal{F}}$ is an injective sheaf. ☐

The key ideas are as follows:

- To prove that $\mathrm{Shv}(X; \mathbf{A})$ is Abelian, we needed that L_{Zar} is left exact, which is guaranteed as soon as filtered colimits are left exact, which is guaranteed in a Grothendieck Abelian category.
- In a Grothendieck Abelian category, there's a second condition on existence of generators, which implies any Grothendieck Abelian category has enough injectives.
- At this point, we know that sheaf cohomology in a Grothendieck Abelian category exists.

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