# GIBBS STATES AND THE MERMIN-WAGNER THEOREM 

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## Mixed States and Gibbs States

We are used to thinking about a "quantum state" as a vector $\psi$ in some Hilbert space $\mathcal{H}$, or an equivalence class of such vectors, $\psi \sim e^{i \theta} \psi$. However, this notion of a state is not suited well to describe quantum systems and subsystems. For example, consider a system of two spinless distinguishable particles in $\mathbb{R}^{3}$, described by some unit vector $\psi(\vec{x}, \vec{y}) \in \mathcal{H}=L^{2}\left(\mathbb{R}^{6}\right)$, where $\vec{x}$ is the position of the first particle and $\vec{y}$ is the position of the second particle. If $\psi(\vec{x}, \vec{y})=\psi_{1}(\vec{x}) \psi_{2}(\vec{y})$, a product, then we can say that the state of the first particle is $\psi_{1}$ and the state of the second particle is $\psi_{2}$. That is, we are able to canonically describe the subsystems of this system of two particles.

Now suppose instead that

$$
\psi(\vec{x}, \vec{y})=\psi_{1}(\vec{x}) \psi_{2}(\vec{y})+\phi_{1}(\vec{x}) \phi_{2}(\vec{y})
$$

For this state $\psi$, it isn't clear how to describe the state of the first particle. We could try to say that it is $\psi_{1}(\vec{x})+\phi_{1}(\vec{x})$, but then we could rewrite $\psi$ as

$$
\psi(\vec{x}, \vec{y})=\left(\alpha \psi_{1}(\vec{x})\right)\left(\frac{1}{\alpha} \psi_{2}(\vec{y})\right)+\left(\beta \phi_{1}(\vec{x})\right)\left(\frac{1}{\beta} \phi_{2}(\vec{y})\right)
$$

and our choice for state of the first particle becomes $\alpha \psi_{1}(\vec{x})+\beta \phi_{1}(\vec{x})$, which isn't the same state as before. This is only for a particular system of two particles - in general the way to interpret the state of subsystems becomes even less clear.

In the example above, although not much can be said about the state of each particle, we are still able to take expectation values. Suppose we want to find the expected value of the $j$ 'th component of position the first particle, $x_{j}$. Let $X_{j}^{(1)}$ be the operator of multiplication by this component $x_{j}$, which is a map $\mathbb{R}^{3} \times \mathbb{R}^{3} \rightarrow \mathbb{C}$. The expected value of the $j$ 'th component of the position of the first particle is then

$$
\left\langle\psi, X_{j}^{(1)} \psi\right\rangle=\int_{\mathbb{R}^{6}} x_{j}|\psi(\vec{x}, \vec{y})|^{2} d \vec{x} d \vec{y}
$$

We can similarly take any self-adjoint operator $\mathcal{O}$ on $L^{2}\left(\mathbb{R}^{3}\right)$ for a one particle system, and promote it to an operator in the two particle system $L^{2}\left(\mathbb{R}^{6}\right)$, by acting trivially on the other particle. From this, we can extract expectation values for subsystems of this composite system. This leads us to and expanded notion of "state." Instead of defining the state of a quantum system as a vector in some Hilbert space, we define it by the particular family of expectation values associated to it. This leads us to the notion of density matrices.

Denote by $\mathscr{B}(\mathcal{H})$ the set of bounded linear operators on the Hilbert space $\mathcal{H}$. Observables are typically unbounded linear operators, but expectation values for bounded operators determine expectation values for unbounded operators (see [1]), so we work with $\mathscr{B}(\mathcal{H})$. Recall that the trace of a non-negative self-adjoint operator $A$ is defined as $\operatorname{tr}(A)=\sum_{i}\left\langle e_{i}, A e_{i}\right\rangle$, for an orthonormal basis $\left\{e_{i}\right\}$.
Definition 1. A bounded linear operator $\rho \in \mathscr{B}(\mathcal{H})$ is a density matrix on $\mathcal{H}$ if $\rho$ is self-adjoint, nonnegative, and $\operatorname{tr}(\rho)=1$.

This density matrix will be our notion of a "state" of a system. It is associated to a family of expectation values, as made precise below.
Definition 2. A linear map $\Phi: \mathscr{B}(\mathcal{H}) \rightarrow \mathbb{C}$ is a family of expectation values if the following conditions hold.
(1) $\Phi(\mathrm{id})=1$.
(2) If an operator $A$ is self-adjoint, then $\Phi(A) \in \mathbb{R}$.

[^0](3) If an operator $A$ is self-adjoint and non-negative, then $\Phi(A) \geq 0$.
(4) If $A_{n} \in \mathscr{B}(\mathcal{H})$ is a sequence, and if $\left\|A_{n} \psi-A \psi\right\| \rightarrow 0$ for all $\psi \in \mathcal{H}$, then $\Phi\left(A_{n}\right) \rightarrow \Phi(A)$.

Proposition. If $\rho$ is a density matrix on $\mathcal{H}$, then $\Phi_{\rho}:=\operatorname{tr}(\rho A)=\operatorname{tr}(A \rho)$ is a family of expectation values. Furthermore, for any family of expectation values $\Phi$, there exists a unique density matrix $\rho$ such that $\Phi(A)=\operatorname{tr}(\rho A)$ for all $A \in \mathscr{B}(\mathcal{H})$.

For a proof of this proposition, the interested reader can see [1]. The first part is a straightforward check of the definition, and the second part is an application of the Riesz representation theorem.

For this notion to make sense, we need to make sure the "density matrix" notion of a state agrees with the "vector in Hilbert space" notion for relevant systems. Let $|\psi\rangle\langle\psi| \in \mathscr{B}(\mathcal{H})$ be the outer product of the vector $|\psi\rangle \in \mathcal{H}$. Since it is an orthogonal projection $|\psi\rangle\langle\psi|$ is bounded, self-adjoint, and non-negative. Choosing an orthonormal basis $\left\{e_{i}\right\}$ with $e_{1}=|\psi\rangle$, then $\operatorname{tr}(|\psi\rangle\langle\psi|)=1$. So $|\psi\rangle\langle\psi|$ is a density matrix. If $A \in \mathscr{B}(\mathcal{H})$ is an operator, then

$$
\operatorname{tr}(|\psi\rangle\langle\psi| A)=\sum_{i}\left\langle e_{i}, \psi\right\rangle\left\langle\psi, A e_{i}\right\rangle=\langle\psi, A \psi\rangle
$$

So for every vector $|\psi\rangle \in \mathcal{H}$, there is a unique $\rho$ associated to it so that the expectation values determined by $\rho$ are the same as for our traditional notion of a state. A state of the form $\rho=|\psi\rangle\langle\psi|$ is called a pure state. If no such $\psi$ exists, $\rho$ is called a mixed state.

We can think of a density matrix $\rho$ as giving a probability distribution over the results of each set of commuting observables; the $\operatorname{tr}(\rho)=1$ condition can be thought of as probabilities adding up to 1 . The density matrix incorporates information both about quantum uncertainties and classical uncertainties, such as lack of knowledge about a system. We can modify the traditional axioms of quantum mechanics to work with density matrices, as follows.

If a system is in state $\rho$, and an observation of $A$ is made and results in a value of $\lambda$, then immediately after the measurement the system will be in the state $\frac{1}{Z} P_{\lambda} \rho P_{\lambda}$, where $P_{\lambda}$ is the operator of orthogonal projection onto the $\lambda$-eigenspace, and $Z=\operatorname{tr}\left(P_{\lambda} \rho P_{\lambda}\right)$ for normalization. If the Hamiltonian of a system is $\hat{H}$, then the time evolution of the state $\rho$ is given by

$$
\frac{d \rho}{d t}=-\frac{1}{i \hbar}[\rho, \hat{H}]
$$

where $[\cdot, \cdot]$ is the commutator $[A, B]=A B-B A$.
With this expanded notion, we are ready to discuss composite systems. Suppose that $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$ are the Hilbert spaces, with inner products $\langle\cdot \mid \cdot\rangle_{1}$ and $\langle\cdot \mid \cdot\rangle_{2}$ respectively. We can define an inner product on the vector space $\mathcal{H}_{1} \otimes \mathcal{H}_{2}$ by $\left\langle u_{1} \otimes u_{2}, v_{1} \otimes v_{2}\right\rangle:=\left\langle u_{1}, v_{1}\right\rangle_{1} \cdot\left\langle u_{2}, v_{2}\right\rangle_{2}$. Then the Hilbert space of the composite system is given by $\mathcal{H}_{1} \hat{\otimes} \mathcal{H}_{1}$, the completion of $\mathcal{H}_{1} \otimes \mathcal{H}_{2}$ with respect to the above inner product. If the two systems $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$ have Hamiltonians $\hat{H}_{1}$ and $\hat{H}_{2}$ respectively, then the Hamiltonian for the noninteracting composite system is $\hat{H}_{1} \otimes I+I \otimes \hat{H}_{2}$. In general the Hamiltonian for a system will be

$$
\hat{H}=\hat{H}_{1} \otimes I+I \otimes \hat{H}_{2}+\hat{H}_{\text {interaction }}
$$

We can think of $\hat{H}_{1} \otimes I$ as the energy of the first subsystem, and $I \otimes \hat{H}_{2}$ as the energy of the second subsystem.
These notions of composite system and subsystem make sense. If $A$ and $B$ are bounded operators on $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$ respectively, then there exists a unique bounded operator $A \otimes B$ on $\mathcal{H}_{2} \hat{\otimes} \mathcal{H}_{2}$ with

$$
(A \otimes B)(\phi \otimes \psi)=(A \phi) \otimes(B \psi)
$$

For a density matrix $\rho$ on the composite system $\mathcal{H}_{1} \hat{\otimes} \mathcal{H}_{2}$, there exists a unique density matrix $\rho^{(1)}$ on $\mathcal{H}_{1}$ with

$$
\operatorname{tr}\left(\rho^{(1)} A\right)=\operatorname{tr}(\rho(A \otimes \mathrm{id}))
$$

and similarly for $\mathcal{H}_{2}$. As a special case of this, if $\rho=\rho_{1} \otimes \rho_{2}$, then $\rho^{(1)}=\rho_{1}$ and $\rho^{(2)}=\rho_{2}$. In this case, we say that the states of the two systems are independent. Again for proofs and more background, we refer the reader to [1].

The most important examples of composite systems for our purposes are the microcanonical and canonical ensembles, which are equilibrium ensembles. Equilibrium ensembles should have time independent averages
and from the discussion above on time evolution, a reasonable equilibrium condition is

$$
\frac{d \rho}{d t}=-\frac{1}{i \hbar}[\rho, \hat{H}]=0
$$

For a system with arbitrary $\hat{H}$, we can choose $\rho=\rho(\hat{H})$ to satisfy this condition.
In the microcanonical ensemble, the total energy $E$ is fixed. We can choose a density matrix $\rho(\hat{H})$ to satisfy this for any Hamiltonian $\hat{H}$, by defining

$$
\rho(\hat{H})=\frac{\delta(\hat{H}-E)}{\Omega(E)}
$$

where $\Omega(E)$ is the total number of microstates $\left\{\psi_{\alpha}\right\}$ (the pure states comprising the composite system) with energy $E$. Letting $\{|n\rangle\}$ be the energy eigenbasis, we have

$$
\langle\rho| n|\rho\rangle m=\sum_{\alpha} p_{\alpha}\left\langle n \mid \psi_{\alpha}\right\rangle\left\langle\psi_{\alpha} \mid n\right\rangle= \begin{cases}\frac{1}{\Omega(E)}, & E_{n}=E \text { and } m=n \\ 0, & \text { else. }\end{cases}
$$

This is analogous to the classical statistical mechanical condition of equal equilibrium probabilities. The condition $m=n$ says that the states with energy $E$ are combined with independent random phases.

For the canonical ensemble, we consider a composite system $\mathcal{H}=\mathcal{H}_{1} \hat{\otimes} \mathcal{H}_{2}$ which is a microcanonical ensemble at fixed energy $E$ such that the temperature of the second subsystem $\mathcal{H}_{2}$ is fixed: it is large enough compared to $\mathcal{H}_{1}$ to not affected by changes in its temperature. We call the subsystem $\mathcal{H}_{2}$ the "reservoir." The energy of the composite system is fixed at $E$, but the energy of the subsystems are free to change with fixed temperature. The system $\mathcal{H}_{1}$ is said to be in a canonical equilibrium state, or Gibbs state. The density matrix $\rho_{1}=\rho^{(1)}$ of the system at inverse temperature $\beta=1 /\left(k_{B} T\right)$, is given by

$$
\rho_{1}(\beta)=\frac{e^{-\beta \hat{H}_{1}}}{Z(\beta)}
$$

where $Z(\beta)=\operatorname{tr}\left(e^{-\beta \hat{H}}\right)$ is the partition function.
To see where this comes from, suppose that system 1 has energy $E_{n}$, so the reservoir has energy $E-E_{n}$. Say that there are $v_{n}$ such composite microstates with this reservoir energy, so there are $\sum_{n} v_{n}=v$ total microstates of the composite system. Since the composite system is a microcanonical ensemble and thus each microstate is equally probable, the probability $p_{n}$ of observing system 1 in state $E_{n}$ is

$$
p_{n}=\frac{v_{n}}{v}
$$

Assuming $E \gg E_{n}$, and defining the entropy as $S(E)=\log (v)$, then a Taylor expansion gives

$$
\begin{aligned}
\log \left(p_{n}\right) & =\log \left(v_{n}\right)-\log (v) \\
& =S_{2}\left(E-E_{n}\right)-S_{\mathrm{tot}}(E) \\
& =S_{2}(E)-\beta E_{n}-S_{\mathrm{tot}}(E)
\end{aligned}
$$

Since $S_{2}$ and $S_{\text {tot }}$ don't vary with $n$ (they only depend on the total energy), then

$$
\log \left(p_{n}\right) \propto-\beta E_{n}
$$

and defining $Z(\beta)=\sum_{n} e^{-\beta E_{n}}=\operatorname{tr}\left(e^{-\beta \hat{H}}\right)$ to normalize, we have

$$
p_{n}=\frac{1}{Z(\beta)} e^{-\beta E_{n}}
$$

For the rest of this paper, we will be concerned with Gibbs states nonzero temperature $T>0$, with expectation values given by

$$
\langle A\rangle=\frac{\operatorname{tr}\left(A e^{-\beta \hat{H}}\right)}{Z(\beta)}
$$

For further discussion and background on quantum statistical mechanics, see [4].

## The Heisenberg Model and the Mermin-Wagner Theorem

In this section we will examine the equilibrium states of the Heisenberg model. Our goal in this section will be to prove the Mermin-Wagner theorem, which states that there is no spontaneous magnetization in the isotropic Heisenberg model in one or two dimensions at nonzero temperature. The term isotropic means that the model reacts the same to an applied magnetic field in any direction. The theorem was first proved by Mermin and Wagner [6], which relied on the work of Hohenberg [2]. Here we follow closely the proof in [7], which is based on that in the original paper by Mermin and Wagner.

Recall the spin operators $\vec{S}=\left(S_{x}, S_{y}, S_{y}\right)$, which satisfy the commutation relations

$$
\begin{aligned}
{\left[S_{i}^{x}, S_{j}^{y}\right] } & =i \hbar S_{i}^{z} \delta_{i j} \\
{\left[S_{i}^{y}, S_{j}^{z}\right] } & =i \hbar S_{i}^{x} \delta_{i j} \\
{\left[S_{i}^{z}, S_{j}^{x}\right] } & =i \hbar S_{i}^{y} \delta_{i j} .
\end{aligned}
$$

We define the spin flip operators $S^{ \pm}=S^{x} \pm i S^{y}$, which satisfy

$$
\begin{aligned}
{\left[S_{i}^{z}, S_{j}^{ \pm}\right] } & = \pm \hbar S_{i}^{ \pm} \delta_{i j} \\
{\left[S_{i}^{+}, S_{j}^{-}\right] } & =2 \hbar S_{i}^{z} \delta_{i j}
\end{aligned}
$$

The Heisenberg model is then defined by the Hamiltonian

$$
\hat{H}=-\sum_{i, j} J_{i j} \vec{S}_{i} \cdot \vec{S}_{j}=-\sum_{i, j} J_{i, j}\left(S_{i}^{+} S_{j}^{-}+S_{i}^{z} S_{j}^{z}\right),
$$

where the sum is over sites in a lattice with $N$ lattice points, and the $J_{i j}$ are coupling constants with $J_{i j}=J_{j i}$ and $J_{i i}=0$ for all $i, j$.

To define the magnetization, we consider this model in a uniform external magnetic field in the $z$-direction $\vec{B}=B_{0} \vec{e}_{z}$. The Hamiltonian in the external magnetic field is given by

$$
\hat{H}=-\sum_{i, j} J_{i j} \vec{S}_{i} \cdot \vec{S}_{j}-b \sum_{i} S_{i}^{z} e^{-i \vec{K} \cdot \vec{R}_{i}}
$$

where the $\vec{R}_{i}$ are position vectors of the lattice points. The $\vec{K}$ allows us to distinguish between different spin structures, such as antiferromagnetic with $\vec{K}=0$, or ferromagnetic with $\vec{K}$ chosen so that $e^{-i \vec{K} \cdot \vec{R}_{i}}=1$ if $\vec{R}_{i}$ refers to one sublattice and $e^{-i \vec{K} \cdot \vec{R}_{i}}=-1$ for the other sublattice. The $b$ is a constant depending on $B_{0}$ (something like $g_{J} \mu_{B} B_{0} / \hbar$, see [7]), which we take to be equal to $B_{0}$. We also assume that the coupling coefficients are short range, meaning

$$
\frac{1}{N} \sum_{i, j}\left|\vec{R}_{i}-\vec{R}_{j}\right|^{2}\left|J_{i j}\right|<\infty
$$

We further assume that the model is in an equilibrium (Gibbs) state, with expectation values given by

$$
\langle A\rangle=\frac{\operatorname{tr}\left(A e^{-\beta \hat{H}}\right)}{Z(\beta)} .
$$

From now on, we work in units with $\hbar=1$.
We define the magnetization by the value

$$
m\left(T, B_{0}, N\right)=\frac{1}{N} \sum_{i} e^{-\vec{K} \cdot \vec{R}_{i}}\left\langle S_{i}^{z}\right\rangle_{T, B_{0}}
$$

Here $\langle\cdot\rangle_{T, B_{0}}$ is the expectation value as given above taken at temperature $T$. The spontaneous magnetization is defined by

$$
m_{s}(T)=\lim _{B_{0} \rightarrow 0} \lim _{N \rightarrow \infty} m\left(T, B_{0}, N\right)
$$

Note that the order in which the limit is taken matters. The quantity $m\left(T, B_{0}\right)=\lim _{N \rightarrow \infty} m\left(T, B_{0}, N\right)$ is called the thermodynamic limit. If $m_{s}(T)=0$, then there is no spontaneous magnetization.

The proof of the Mermin-Wanger theorem uses a result known as the Bogoliubov inequality to put an arbitrarily small upper bound on spontaneous magnetization.

Lemma 3. Define

$$
W_{n}=\frac{1}{Z(\beta)} e^{-\beta E_{n}},
$$

where the $E_{n}$ are energy eigenvalues corresponding to energy eigenstates $|n\rangle$. Then $(\cdot, \cdot)$ given by

$$
(A, B)=\sum_{i, j}^{E_{i} \neq E_{j}}\langle i| A^{\dagger}|j\rangle\langle j| B|i\rangle \frac{W_{j}-W_{i}}{E_{i}-E_{j}}
$$

is a positive semidefinite inner product.
Proof. First, since

$$
\langle i| a_{1} A_{1}+a_{2} A_{2}|j\rangle=a_{1}\langle i| A_{1}|j\rangle+a_{2}\langle i| A_{2}|j\rangle,
$$

then $\left(a_{1} A_{1}+a_{2} A_{2}, B\right)=a_{1}\left(A_{1}, B_{1}\right)+a_{2}\left(A_{2}, B_{2}\right)$.
Next, since

$$
\langle i| A^{\dagger}|j\rangle\langle j| B|i\rangle=\left(\langle i| B^{\dagger}|j\rangle\langle j| A|i\rangle\right)^{*},
$$

and since $\left(W_{j}-W_{i}\right) /\left(E_{i}-E_{j}\right) \in \mathbb{R}$, then $(A, B)=(B, A)^{*}$.
Finally, since $\left(W_{j}-W_{i}\right) /\left(E_{i}-E_{j}\right) \geq 0$, and since

$$
\left.\langle i| A^{\dagger}|j\rangle\langle j| A|i\rangle=|\langle j| A| i\right\rangle\left.\right|^{2},
$$

then $(A, A) \geq 0$. Thus, $(A, B)$ is a positive semidefinite inner product.
Lemma 4 (Boboliubov Inequality). Let $A, C$ be arbitrary operators, and $\hat{H}$ the Hamiltonian from above. Then

$$
\frac{1}{2} \beta\left\langle\left\{A, A^{\dagger}\right\}\right\rangle\left\langle\left[[C, \hat{H}], C^{\dagger}\right]\right\rangle \geq|\langle[C, A]\rangle|^{2} .
$$

Proof. Let $B=\left[C^{\dagger}, H\right]$. Then

$$
\begin{aligned}
(A, B) & =\sum_{i, j}^{E_{i} \neq E_{j}}\langle i| A^{\dagger}|j\rangle\langle j|\left[C^{\dagger}, \hat{H}\right]|i\rangle \frac{W_{j}-W_{i}}{E_{i}-E_{j}} \\
& =\sum_{i, j}\langle i| A^{\dagger}|j\rangle\langle j| C^{\dagger}|i\rangle\left(W_{j}-W_{i}\right) \\
& =\sum_{i, j} W_{j}\langle j| C^{\dagger}|i\rangle\langle i| A^{\dagger}|j\rangle-W_{i}\langle i| A^{\dagger}|j\rangle\langle j| C^{\dagger}|i\rangle \\
& =\sum_{j} W_{j}\langle j| C^{\dagger} A^{\dagger}|j\rangle-\sum_{i} S_{i}\langle i| A^{\dagger} C^{\dagger}|i\rangle \\
& =\left\langle C^{\dagger} A^{\dagger}-A^{\dagger} C^{\dagger}\right\rangle .
\end{aligned}
$$

For the second equality, we have included the terms where $E_{i}=E_{j}$, and in the last equality, we have used the definition of $W_{i}$ from above.

So using $B=\left[C^{\dagger}, H\right]$, we have

$$
(A, B)=\left\langle\left[C^{\dagger}, A^{\dagger}\right]\right\rangle .
$$

Letting $A=B$, we have

$$
0 \leq(B, B)=\left\langle\left[C^{\dagger},[\hat{H}, C]\right]\right\rangle .
$$

Now, using the definition of $W_{i}$, we have

$$
\begin{aligned}
0 & <\frac{W_{j}-W_{i}}{E_{i}-E_{j}} \\
& =\frac{W_{j}+W_{i}}{E_{i}-E_{j}} \tanh \left(\frac{\beta}{2}\left(E_{i}-E_{j}\right)\right) \\
& <\frac{\beta}{2}\left(W_{i}+W_{j}\right) .
\end{aligned}
$$

So we have

$$
\begin{aligned}
(A, A) & <\frac{1}{2} \beta \sum_{i, j}^{E_{i} \neq E_{j}}\langle i| A^{\dagger}|j\rangle\langle j| A|i\rangle\left(W_{i}+W_{j}\right) \\
& \leq \frac{1}{2} \sum_{i, j}\langle i| A^{\dagger}|j\rangle\langle j| A|i\rangle\left(W_{i}+W_{j}\right) \\
& =\frac{1}{2} \beta \sum_{i} W_{i}\left(\langle i| A^{\dagger} A|i\rangle+\langle i| A A^{\dagger}|i\rangle\right) \\
& =\frac{1}{2} \beta\left\langle\left\{A, A^{\dagger}\right\}\right\rangle .
\end{aligned}
$$

We've shown

$$
\begin{align*}
(A, A) & \leq \frac{1}{2} \beta\left\langle\left\{A, A^{\dagger}\right\}\right\rangle \\
(A, B) & =\left\langle\left[C^{\dagger}, A^{\dagger}\right]\right\rangle \\
(B, B) & =\left\langle\left[C^{\dagger},[\hat{H}, C]\right]\right\rangle \tag{1}
\end{align*}
$$

By Plugging these into the Cauchy-Schwarz inequality $(A, A)(B, B) \geq|(A, B)|^{2}$, we have

$$
\begin{equation*}
\left(\frac{1}{2} \beta\left\langle\left\{A, A^{\dagger}\right\}\right\rangle\right)\left(\left\langle\left[C^{\dagger},[\hat{H}, C]\right]\right\rangle\right) \geq|\langle[C, A]\rangle|^{2} \tag{2}
\end{equation*}
$$

which finishes the proof.
Theorem 5 (Mermin-Wagner Theorem). For the isotropic Heisenberg model in one and two dimensions, $M_{s}(T)=0$ for $T>0$.

Proof. We first define the spin operators in momentum space $S^{\alpha}(\vec{k})=\sum_{i} S_{i}^{\alpha} e^{-i k \cdot R_{i}}$, where $\alpha=x, y, z,+,-$, and the $S_{i}^{\alpha}$ are the position space spin operators from above. We will apply the Bogoliubov inequality with

$$
\begin{aligned}
& A=S^{-}(-\vec{k}+\vec{K}) \\
& C=S^{+}(\vec{k})
\end{aligned}
$$

Now, note that

$$
\begin{aligned}
{\left[S^{+}\left(\vec{k}_{1}\right), S^{-}\left(\vec{k}_{2}\right)\right] } & =\left(\sum_{i} S_{i}^{+} e^{-i \vec{k}_{1} \cdot \vec{R}_{i}}\right)\left(\sum_{i} S_{i}^{-} e^{-i \vec{k}_{2} \cdot \vec{R}_{i}}\right)-\left(\sum_{i} S_{i}^{-} e^{-i \vec{k}_{2} \cdot \vec{R}_{i}}\right)\left(\sum_{i} S_{i}^{+} e^{-i \vec{k}_{1} \cdot \vec{R}_{i}}\right) \\
& =\sum_{i}\left(S_{i}^{+} S_{i}^{-}-S_{i}^{-} S_{i}^{+}\right) e^{-i\left(\vec{k}_{1}+\vec{k}_{2}\right) \cdot \vec{R}_{i}} \\
& =\sum_{i} 2 S^{z} e^{-i\left(\vec{k}_{1}+\vec{k}_{2}\right) \cdot \vec{R}_{i}} \\
& =2 S^{z}\left(\vec{k}_{1}+\vec{k}_{2}\right)
\end{aligned}
$$

Similarly, we have

$$
\left[S^{z}\left(\vec{k}_{1}\right), S^{ \pm}\left(\vec{k}_{2}\right)\right]= \pm S^{ \pm}\left(\vec{k}_{1}+\vec{k}_{2}\right)
$$

In order to apply the Bogoliubov inequality (2), we examine each of the terms individually.
(i) We have

$$
\begin{aligned}
\langle[C, A]\rangle & =\left\langle\left[S^{+}(\vec{k}), S^{-}(-\vec{k}+\vec{K})\right]\right\rangle \\
& =2 \hbar\left\langle S^{z}(\vec{K})\right\rangle \\
& =2 \hbar \sum_{i} e^{-i \vec{K} \cdot \vec{R}_{i}}\left\langle S_{i}^{z}\right\rangle
\end{aligned}
$$

where the second equality comes from the above, and the third is the definition of $S^{z}(\vec{K})$. Substituting the definition of magnetization, we have

$$
\langle[C, A]\rangle=2 N m\left(T, B_{0}, N\right)
$$

(ii) Similarly, we evaluate the following sum over $\vec{k}$ :

$$
\begin{aligned}
\sum_{\vec{k}}\left\langle\left\{A, A^{\dagger}\right\}\right\rangle & =\sum_{\vec{k}}\left\langle\left\{S^{-}(-\vec{k}+\vec{K}), S^{+}(\vec{k}-\vec{K})\right\}\right\rangle \\
& =2 N \sum_{i}\left\langle\left(S_{i}^{x}\right)^{2}+\left(S_{i}^{y}\right)^{2}\right\rangle \\
& \leq 2 N \sum_{i}\left\langle\vec{S}_{i}^{2}\right\rangle \\
& \leq 2 N^{2} S(S+1)
\end{aligned}
$$

(iii) The last term, $\left\langle\left[[C, \hat{H}], C^{\dagger}\right]\right\rangle$ is more tedious. We first substitute the form of the Hamiltonian $H$ and take the commutator with $S_{p}$, to get

$$
\left[S_{p}^{+}, \hat{H}\right]=-\sum_{i} J_{i p}\left(2 S_{i}^{+} S_{p}^{z}-S_{i}^{z} S_{p}^{+}-S_{p}^{+} S_{i}^{z}\right)+B_{0} S_{p}^{+} e^{-i \vec{K} \cdot \vec{R}_{p}}
$$

We now take the commutator of this with $S_{q}^{-}$, to get

$$
\left[\left[S_{p}^{+}, \hat{H}\right], S_{q}^{-}\right]=2 \sum_{i} J_{i q} \delta_{p q}\left(S_{i}^{+} S_{q}^{-}+2 S_{i}^{z} S_{q}^{z}\right)-2 J_{p q}\left(S_{p}^{+} S_{q}^{-}+2 S_{p}^{z} S_{q}^{z}\right)+2 B_{0} \delta_{p q} S_{q}^{z} e^{-i \vec{K} \vec{R}_{q}}
$$

We then substitute the definition of $C$, to get

$$
\begin{align*}
\left\langle\left[[C, \hat{H}], C^{\dagger}\right]\right\rangle & =\sum_{p, q} e^{-i \vec{k} \cdot\left(\vec{R}_{p}-\vec{R}_{q}\right)}\left\langle\left[\left[S_{p}^{+}, \hat{H}\right], S_{q}^{-}\right]\right\rangle \\
& =2 B_{0} \sum_{q}\left\langle S_{q}^{z}\right\rangle e^{-i \vec{K} \cdot \vec{R}_{q}}+2 \sum_{p, q} J_{p q}\left(1-e^{-i \vec{k} \cdot\left(\vec{R}_{p}-\vec{R}_{q}\right)}\right)\left\langle S_{p}^{+} S_{q}^{-}+2 S_{p}^{z} S_{q}^{z}\right\rangle \tag{3}
\end{align*}
$$

We'll now obtain an upper bound for this value. Recall that this value must be nonnegative by the inner product in equation (1), and that this does not change if $C=S^{+}(\vec{k})$ is replaced by $\bar{C}=S^{+}(-\vec{k})$. This tells us that the right hand side of (3) above is still positive if $\vec{k}$ is replaced by $-\vec{k}$. Thus, we add the $\vec{k}$ and $-\vec{k}$ replaced expressions to get an upper bound

$$
\left\langle\left[[C, \hat{H}], C^{\dagger}\right]\right\rangle \leq 4 B_{0} \sum_{q}\left\langle S_{q}^{z}\right\rangle e^{-i \vec{K} \cdot \vec{R}_{q}}+4 \sum_{p, q} J_{p q}\left(1-\cos \left(\vec{k} \cdot\left(\vec{R}_{p}-\vec{R}_{q}\right)\right)\right)\left\langle\vec{S}_{p} \cdot \vec{S}_{q}+S_{p}^{z} S_{q}^{z}\right\rangle .
$$

Since this is nonnegative, we can apply the triangle inequality to get

$$
\begin{aligned}
\left\langle\left[[C, \hat{H}], C^{\dagger}\right]\right\rangle & \leq 4 B_{0} N\left|\left\langle S_{q}^{z}\right\rangle\right|+4 \sum_{p, q}\left|J_{p q}\right|\left|1-\cos \left(\vec{k} \cdot\left(\vec{R}_{p}-\vec{R}_{q}\right)\right)\right| \cdot\left(\left|\left\langle\vec{S}_{p} \cdot \vec{S}_{q}\right\rangle\right|+\left|\left\langle S_{p}^{z} S_{q}^{z}\right\rangle\right|\right) \\
& \leq 4 B_{0} N\left|\left\langle S_{q}^{z}\right\rangle\right|+8 \hbar^{2} S(S+1) \cdot\left(\sum_{p, q}\left|J_{p q}\right|\left|1-\cos \left(\vec{k} \cdot\left(\vec{R}_{p}-\vec{R}_{q}\right)\right)\right|\right)
\end{aligned}
$$

Substituting the definition of magnetization and simplifying, we have

$$
\left\langle\left[[C, \hat{H}], C^{\dagger}\right]\right\rangle \leq 4\left|B_{0} m\left(T, B_{0}, N\right)\right|+8 S(S+1) \sum_{p, q}\left|J_{p q}\right| \frac{1}{2} k^{2}\left|\vec{R}_{p}-\vec{R}_{q}\right|^{2}
$$

Now by definition, since the interactions are short range, the value

$$
Q=\frac{1}{N} \sum_{i, j}\left|\vec{R}_{i}-\vec{R}_{j}\right|^{2}\left|J_{i j}\right|<\infty
$$

exists. We can use this to simplify the above expression, and we have

$$
\left\langle\left[[C, \hat{H}], C^{\dagger}\right]\right\rangle \leq 4\left|B_{0} m\left(T, B_{0}, N\right)\right|+4 N k^{2} \hbar^{2} Q S(S+1)
$$

Now, we substitute the above inequalities into the Bogoliubov inequality (2) to get

$$
\beta S(S+1) \geq \frac{m\left(T, B_{0}, N\right)^{2}}{N^{2}} \sum_{\vec{k}} \frac{1}{\left|B_{0} m\left(T, B_{0}, N\right)\right|+k^{2} \hbar^{2} N Q S(S+1)},
$$

again, the sum is over vectors $\vec{k}$ in the first Brillouin zone (see [5]). This is the key inequality for the theorem, and to proceed, we convert the sum in this inequality into an integral, as

$$
\beta S(S+1) \geq m\left(T, B_{0}, N\right)^{2} \frac{1}{N^{2}} \int_{\vec{k}} \frac{1}{(2 \pi)^{d}} \frac{1}{\left|B_{0} m\left(T, B_{0}, N\right)\right|+k^{2} N Q S(S+1)}
$$

where the integration is over the vectors $\vec{k}$ in the first Brillouin zone, and we suppose our system sits inside a $d$-dimensional volume $V_{d}$ (this is the "dimension" that we are concerned with in the theorem). Suppose it contains $N_{d}=n^{d}$ spins, and suppose that in the thermodynamic limit $N \rightarrow \infty$, then $V_{d} / N_{d} \rightarrow v_{d}$, some constant value. Since

$$
\frac{1}{\left|B_{0} m\left(T, B_{0}, N\right)\right|+k^{2} N Q S(S+1)} \geq 0
$$

then instead of integrating over the entire first Brillouin zone, we can integrate over a sphere contained entirely inside the first Brillouin zone. Let $k_{0}$ be the distance from the nearest Bragg plane to the origin, so the integration will be over a sphere of radius $k_{0}$. We have the inequality

$$
S(S+1) \geq \frac{m\left(T, B_{0}\right)^{2} v_{d} \Omega_{d}}{\beta(2 \pi)^{d}} \int_{0}^{k_{0}} \frac{k^{d-1} d k}{\left|B_{0} m\left(T, B_{0}\right)\right|+k^{2} Q S(S+1)}
$$

where $\Omega_{d}$ is the surface area of the $d$-dimensional unit sphere.
For $d=1$ and $d=2$, we can evaluate this integral, and obtain
(a) $(d=1)$. Evaluating the integral in this case gives

$$
S(S+1) \geq \frac{m\left(T, B_{0}\right)^{2} v_{1}}{2 \pi \beta} \frac{\arctan \left(k_{0} \sqrt{\frac{Q \hbar^{2} S(S+1)}{\left|B_{0} m\left(t, B_{0}\right)\right|}}\right)}{\sqrt{Q S(S+1)\left|B_{0} m\left(T, B_{0}\right)\right|}}
$$

For small $B_{0}$, we have

$$
\left|m\left(T, B_{0}\right)\right| \leq \text { const. } \frac{B_{0}^{1 / 3}}{T^{2 / 3}}
$$

(b) $(d=2)$ : Evaluating the integral for $d=2$, we obtain the inequality

$$
S(S+1) \geq \frac{m\left(B_{0}, T\right)^{2} v_{2}}{2 \pi \beta\left(g_{J} \mu_{B}\right)^{2}} \frac{\log \left(\frac{Q S(S+1) k_{0}^{2}+\left|B_{0} m\left(B_{0}, T\right)\right|}{\left|B_{0} m\left(T, B_{0}\right)\right|}\right)}{2 Q S(S+1)}
$$

For small $B_{0}$, we have

$$
\left|m\left(T, B_{0}\right)\right| \leq \text { const. } \frac{1}{T^{1 / 2}\left|\log \left(B_{0}\right)\right|^{1 / 2}}
$$

In either of the above cases, for $d=1,2$ and $T \neq 0$, we have

$$
\lim _{B_{0} \rightarrow 0} m\left(T, B_{0}\right)=0
$$

which finises the proof the theorem.
Some remarks on the theorem:
(1) The theorem only holds for the isotropic Heisenberg model.
(2) The proof rules out only spontaneous magnetization, and does not exclude the possibility of other phase transitions.
(3) The theorem can also be proved for other models, such as the Hubbard model, X-Y model, or Kondo-lattice model.
(4) For a more general discussion of the absence of continuous symmetry breaker in 2D lattice systems, the reader is referred to [3].

## References

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