Online Supplementary Appendix to: Rationalizable Partition-Confirmed Equilibrium with Heterogeneous Beliefs

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C The Choice of the Approximation Criterion

To help understand Theorem 2, we now define an alternative notion of ϵ-observational consistency:

Given $\pi \in \Pi$, let $\Pi(D_i(\pi)) := \{\pi' \in \Pi | D_i(\pi') = D_i(\pi)\}$ be the set of strategy profiles that induce the distribution $D_i(\pi)$.

Definition U2(ϵ)$'$ Given a unitary belief model U, version $u_i^k \in U_i$ is strongly ϵ-self-confirming with respect to π^* if $\min_{\tilde{\pi} \in \Pi(D_i(\pi_i^k, \pi_{-i}^*))} \max_{u_{-i} \in \text{supp}(p_i^k)} ||(\pi_i^k, \pi_{-i}(u_{-i})) - \tilde{\pi}|| < \epsilon$.

That is, a version is strongly ϵ-self-confirming with respect to π^* if there exists a single strategy profile of the opponents that (i) induces the same distribution over terminal nodes as π^* and (ii) differs from any strategy profile in the support of the version's belief by up to ϵ.

Definition U3(ϵ)$'$ Given a unitary belief model U, u_i^k is strongly ϵ-observationally consistent if $p_i^k(\tilde{u}_{-i}) > 0$ implies, for each $j \neq i$, $\tilde{u}_j$ is strongly ϵ-self-confirming with respect to $(\pi_i(u_i^k), \pi_{-i}(\tilde{u}_{-i}))$.

Remark 1

(a) There is a $K < \infty$ that does not depend on ϵ such that if the strong ϵ-self-confirming condition holds then the $K\epsilon$-self-confirming condition holds. This is because $||D_i(\pi) - D_i(\pi')|| \leq (\#A)^2 \cdot ||\pi - \pi'||$ for any $\pi, \pi' \in \Pi$ (by Claim 2 in the Appendix of the paper), so

$$||D_i(\pi_i^k, \pi_{-i}) - D_i(\pi_i^k, \pi_{-i}^*)|| = ||D_i(\pi_i^k, \pi_{-i}) - D_i(\tilde{\pi})|| \leq (\#A)^2 \cdot ||(\pi_i^k, \pi_{-i}) - \tilde{\pi}||$$

for any $\tilde{\pi} \in \Pi(D_i(\pi_i^k, \pi_{-i}^*))$.

(b) The ϵ-self-confirming condition allows for ex ante difference of strategies by ϵ, while the strong ϵ-self-confirming condition only allows for an ϵ difference conditional on each information set. To see the difference in an example, consider the game in Example 1 (Figure 1), and suppose now that there is one more player, player 4, who has a singleton action set (we suppress reference to his action in what follows) and the discrete terminal node partition. Let $\pi^* = ((1-\frac{\epsilon}{2})Out_1 + \frac{\epsilon}{2}In_1, U_2, U_3)$, and suppose that version v_4 of player 4 has a belief that assigns a unit mass to $\hat{\pi} = ((1-\frac{\epsilon}{2})Out_1 + \frac{\epsilon}{2}In_1, D_2, D_3)$. Then, v_4 is ϵ-self-confirming with respect to π^* because $D_4(\hat{\pi})$ and $D_4(\pi^*)$ differ only by $\frac{\epsilon}{2}$, as the terminal nodes following player 2's and 3's moves get probability $\frac{\epsilon}{2}$ under either π^* or $\hat{\pi}$. However, v_4 is not strongly ϵ-self-confirming with respect to π^* because $\Pi(D_4(\pi^*)) = \{\pi^*\}$, while $\hat{\pi}$ and π^* differ by 1 at player 2's and 3's information sets. One justification for using the ϵ-self-confirming condition is that in the underlying learning model the agents have large but finite samples, so they will have very few observations at information sets that are unlikely to be reached. In this example, the fact that In_1 is assigned only $\frac{\epsilon}{2}$ probability suggests that the data regarding the play at player 2's and 3's information sets are scarce, so we want to permit a large difference in belief about play here.

Example 6 below shows that if we strengthen the definition of unitary ϵ-RPCE to require strong ϵ-observational consistency then the conclusion of Theorem 2 does not hold. The intuition is that a version v_i can conjecture that she is the only agent who plays a particular action, and that action is the only one that leads to an information set h_j which otherwise would not be reached. Observational consistency in the heterogeneous model allows v_i's belief about play at h_j and the belief held by another player to whom v_i assigns positive probability to be different, as v_i's conjecture assigns measure zero to herself (as in Example 5) and play at a zero probability information set does not affect the distribution over terminal node partitions. However, strong ϵ-observational consistency requires that these two beliefs about play at such information sets be close to each other.

Example 6 (Distance between Observations)

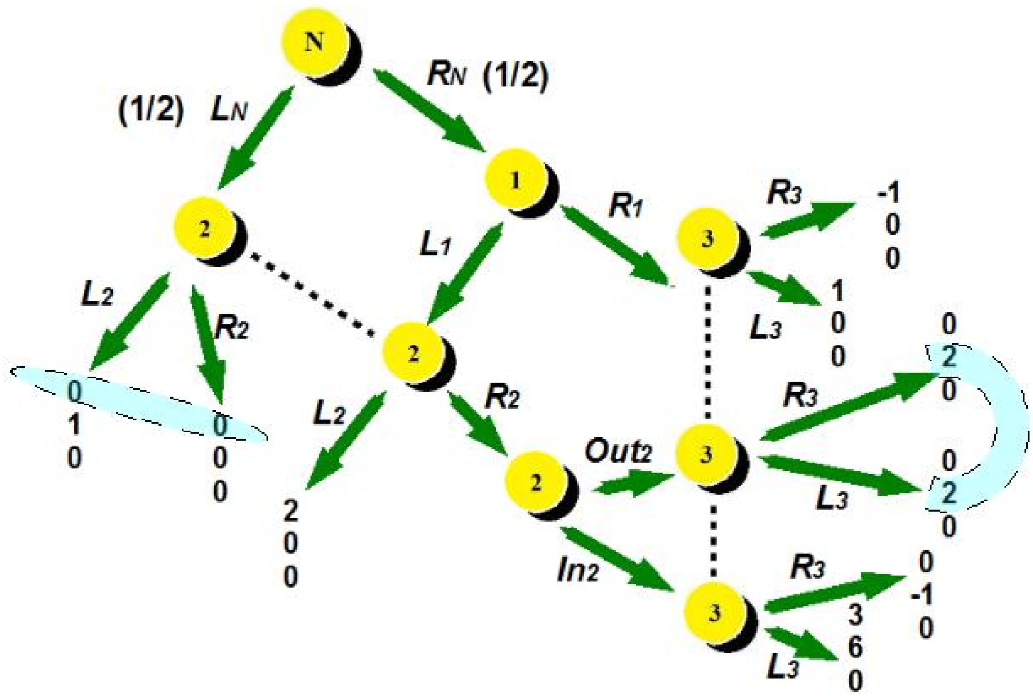

Figure 6

The game in Figure 6 modifies the game in Example 3 by adding decisions for players 2 and 3 after (R_N, L_1, R_2).

The terminal node partitions are such that everyone observes the exact terminal node reached, except that player 1 does not know 2's choice after L_N, and player 2 does not know 3's choice after Out_2.

We first show that $(R_1, (L_2, In_2), L_3)$ is a heterogeneous RPCE. The construction of the belief model is similar to the one for Example 3. Each version of player 1 plays R_1 and believes that all other agents in her player role play L_1. Since in the heterogeneous belief model her conjecture assigns measure zero to the versions who play R_1 (although she herself plays R_1), she is not required (by observational consistency) to think that player 2's observe 3's action and so matches their beliefs with the actual play of player 3.

$$V_1 = \{v_1^1, v_1^2\} \quad \text{with}$$

$$v_1^1 = (R_1, (L_1, (R_2, Out_2), L_3), (v_1^2, v_2^2, v_3^2)),\ v_1^2 = (L_1, (L_1, (R_2, Out_2), R_3), (v_1^2, v_2^2, v_3^3));$$

$$V_2 = \{v_2^1, v_2^2\} \quad \text{with}$$

$$v_2^1 = ((L_2, In_2), (R_1, (L_2, In_2), R_3), (v_1^1, v_2^1, v_3^3)),\ v_2^2 = ((R_2, Out_2), (L_1, (R_2, Out_2), R_3), (v_1^2, v_2^2, v_3^3));$$

$$V_3 = \{v_3^1, v_3^2, v_3^3\} \quad \text{with}\ \ v_3^1 = (L_3, (R_1, (L_2, In_2), L_3), (v_1^1, v_2^1, v_3^1)),$$

$$v_3^2 = (L_3, (L_1, (R_2, Out_2), L_3), (v_1^2, v_2^1, v_3^2));\ v_3^3 = (R_3, (L_1, (R_2, Out_2), R_3), (v_1^2, v_2^1, v_3^3));$$

$$\phi_1(v_1^1) = 1,\ \phi_2(v_2^1) = 1,\ \phi_3(v_3^1) = 1.$$

It is straightforward to check that this belief model satisfies all the conditions for heterogeneous RPCE.

Now we show that the specified distribution of strategies would not be the result of an ϵ-unitary RPCE in the associated anonymous-matching game with a large number of agents if we replace ϵ-observational consistency with strong ϵ-observational consistency. Intuitively, in the above belief model v_1^1 plays R_1 that leads to player 3's information set and at the same time conjectures that v_2^2 has positive probability. But v_1^1's belief about 3's play (L_3) and v_2^2's belief (R_3) do not coincide. If v_2^2 were to believe that 3 plays L_3 then it would be better for her to play In_2, which would lead 1 to play L_1. In the heterogeneous belief model v_1^1's belief about player 1's play assigns probability exactly equal to zero to R_1, so player 3's node after R_3 gets probability zero even though she herself plays R_1.

Formally, in the unitary model of the associated anonymous-matching game, suppose that there is a positive share of agents of player 1 who play R_1. This means that the share of player 3 who play L_3 must be at least a number close to $\frac{1}{2}$ for large T. With strong ϵ-observational consistency, this means that these versions of player 1 have a conjecture that assigns probability one to a strategy profile close to (R_2, In_2) for large T, as the convex combination of -1 and 6 that assigns to the latter a probability at least a number close to $\frac{1}{2}$ for large T dominates all other payoffs that player 2 can receive in this game. Hence R_1 cannot be a best response.

D The Strength of the Unitary Restriction

Example 7 (The Strength of the Unitary Restriction)

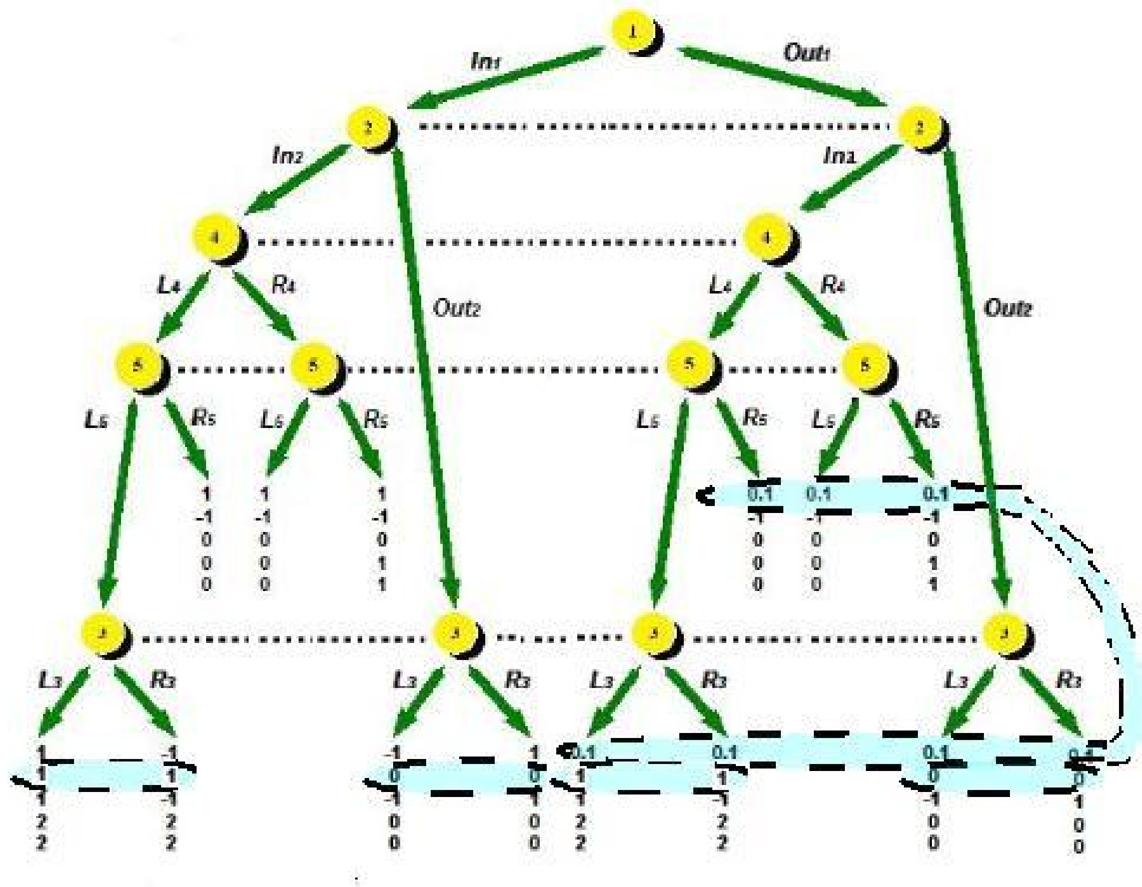

Figure 7

Examples 2 and 3 showed that a pure-strategy heterogeneous RPCE need not be a unitary RPCE. In those examples, there is a version who conjectures that there are other versions in the same player role who play differently. This example shows that even with a restriction that every version (not only actual versions) believes that other versions in the same player role play in the same way as she does, and a restriction that the actual versions' conjectures are correct, a heterogeneous RPCE can be different from a unitary RPCE, because an actual version of one player role can conjecture that different versions in *another* player role play differently.

In particular, we will show that (Out_1, Out_2, R_3) cannot be part of a unitary RPCE, while $(Out_1, Out_2, R_3, R_4, R_5)$ can be a heterogeneous RPCE.

To see that (Out_1, Out_2, R_3) cannot be played in a unitary RPCE, notice that Out_1 can be a best response only when 1 believes that at least one of (In_2, L_4, L_5, R_3) and (Out_2, L_3) is played with positive probability. However, since 3's terminal node partition is discrete and 3's best response is to match 2's action (play L_3 when In_2, and play R_3 when Out_2), observational consistency applied to an actual version of player 1, denoted u_1^k, and the best response condition for player 3 together imply that it cannot be the case that u_1^k believes either L_3 or R_3 is played with probability one, so u_1^k must believe both of L_3 and R_3 are played with positive probability. Since player 3's terminal node partition is discrete, observational consistency applied to u_1^k and the best response condition for player 3 imply that u_1^k believes that In_2 and Out_2 are played with probability $\frac{1}{2}$ for each.

Since player 2's terminal node partition reveals the opponents' play when he plays In_2 with positive probability, u_1^k's belief about players 3, 4, and 5's play must be such that 2 is indifferent between In_2 and Out_2, by observational consistency applied to u_1^k and the best response condition for player 2. But this is possible only when the profile (L_4, L_5) is played with probability exactly equal to $\frac{1}{2}$. Given Out_2, 4 and 5's right-hand information sets lie on the path of play, so u_1^k must expect play there to correspond to a Nash equilibrium in their "subgame" (4 and 5 play a best response to each other given the opponent's play). Thus either (a) (L_4, L_5) is played with probability 1, (b) (L_4, L_5) is played with probability 0, or (c) (L_4, L_5) is played with probability $\frac{1}{9}$. Thus the probability assigned to (L_4, L_5) is not $\frac{1}{2}$, so 2 cannot be indifferent, contradicting observational consistency applied to v_1^k and the best response condition for player 2.

Now we show that $(Out_1, Out_2, R_3, R_4, R_5)$ can be a heterogeneous RPCE. To see this, consider the following belief model:

$$V_1 = \{v_1^1\} \quad \text{with} \;\; v_1^1 = (Out_1, (Out_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, \frac{1}{2}L_3 + \frac{1}{2}R_3, L_4, L_5), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, \frac{1}{2}v_3^3 + \frac{1}{2}v_3^4, v_4^3, v$$

$$V_2 = \{v_2^1, v_2^2, v_2^3\} \quad \text{with} \;\; v_2^1 = (Out_2, (Out_1, Out_2, R_3, R_4, R_5), (v_1^1, v_2^1, v_3^1, v_4^1, v_5^1)),$$

$$v_2^2 = (In_2, (Out_1, In_2, L_3, L_4, L_5), (v_1^1, v_2^2, v_3^2, v_4^2, v_5^2)),$$

$$v_2^3 = (In_2, (Out_1, In_2, R_3, L_4, L_5), (v_1^1, v_2^2, v_3^4, v_4^2, v_5^2));$$

$$V_3 = \{v_3^1, v_3^2, v_3^3, v_3^4\} \quad \text{with} \;\; v_3^1 = (R_3, (Out_1, Out_2, R_3, R_4, R_5), (v_1^1, v_2^1, v_3^1, v_4^1, v_5^1)),$$

$$v_3^2 = (L_3, (Out_1, In_2, L_3, L_4, L_5), (v_1^1, v_2^2, v_3^2, v_4^2, v_5^2));$$

$$v_3^3 = (L_3, (Out_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, L_3, L_4, L_5), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, v_3^3, v_4^4, v_5^4));$$

$$v_3^4 = (R_3, (Out_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, R_3, L_4, L_5), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^3, v_3^4, v_4^5, v_5^5));$$

$$V_4 = \{v_4^1, v_4^2, v_4^3, v_4^4, v_4^5\} \quad \text{with} \;\; v_4^1 = (R_4, (Out_1, Out_2, R_3, R_4, R_5), (v_1^1, v_2^1, v_3^1, v_4^1, v_5^1)),$$

$$v_4^2 = (L_4, (Out_1, In_2, L_3, L_4, L_5), (v_1^1, v_2^2, v_3^2, v_4^2, v_5^2)),$$

$$v_4^3 = (L_4, (Out_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, \frac{1}{2}L_3 + \frac{1}{2}R_3, L_4, L_5), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, \frac{1}{2}v_3^3 + \frac{1}{2}v_3^4, v_4^3, v_5^3));$$

$$v_4^4 = (L_4, (Out_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, L_3, L_4, L_5), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, v_3^3, v_4^4, v_5^4));$$

$$v_4^5 = (L_4, (Out_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, R_3, L_4, L_5), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, v_3^4, v_4^5, v_5^5));$$

$$V_5 = \{v_5^1, v_5^2, v_5^3, v_5^4, v_5^5\} \quad \text{with} \;\; v_5^1 = (R_5, (Out_1, Out_2, R_3, R_4, R_5), (v_1^1, v_2^1, v_3^1, v_4^1, v_5^1)),$$

$$v_5^2 = (L_5, (Out_1, In_2, L_3, L_4, L_5), (v_1^1, v_2^2, v_3^2, v_4^2, v_5^2));$$

$$v_5^3 = (L_5, (Out_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, \frac{1}{2}L_3 + \frac{1}{2}R_3, L_4, L_5), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, \frac{1}{2}v_3^3 + \frac{1}{2}v_3^4, v_4^3, v_5^3));$$

$$v_5^4 = (L_5, (Out_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, L_3, L_4, L_5), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, v_3^3, v_4^4, v_5^4));$$

$$v_5^5 = (L_5, (Out_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, R_3, L_4, L_5), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, v_3^4, v_4^5, v_5^5));$$

$$\phi_1(v_1^1) = 1,\ \phi_2(v_2^1) = 1,\ \phi_3(v_3^1) = 1,\ \phi_4(v_4^1) = 1,\ \phi_5(v_5^1) = 1.$$

The key here is that v_1^1, the actual version of player 1, can conjecture that there are two actual versions (v_2^1 and v_2^2) in the role of player 2, each of whom conjectures that they are the only one in that role, e.g. that all player 2's reason and play as they do. Out_1 can be a best response only when player 2's action corresponds to a mixed strategy whose support is the strategies played by v_2^1 and v_2^2, but such a mixed strategy would violate the best response condition in the unitary belief model.

E Heterogeneous Beliefs under Common Knowledge of Observation Structure

Here we show that common knowledge of the observation structure on its own does not rule out heterogeneous beliefs in Example 4.

Consider requiring conditions 1 and 3 in the definition of heterogeneous RPCE and a weaker version of condition 2 where we require optimality only at the on-path information sets (all $h \in H(\pi_i^k, \pi_{-i})$ for all π_{-i} is in the support of γ_i^k). This concept would correspond to relaxing the unitary assumption of partition-confirmed equilibrium defined in FK. With this definition, player 2 is not assumed to know the payoff function of player 1, so he can believe that 3 and 4 play (R_3, R_4). Specifically, it is easy to check by inspection that all the above conditions are satisfied in the following belief model:

$$V_1 = \{v_1^1\} \quad \text{with} \ \ v_1^1 = (In_1, (In_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, L_3, L_4), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, v_3^1, v_4^1));$$

$$V_2 = \{v_2^1, v_2^2\} \quad \text{with} \ \ v_2^1 = (In_2, (In_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, L_3, L_4), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, v_3^1, v_4^1)),$$

$$v_2^2 = (Out_2, (In_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, R_3, R_4), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, v_3^1, v_4^1);$$

$$V_3 = \{v_3^1\} \quad \text{with} \ \ v_3^1 = (L_3, (In_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, L_3, L_4), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, v_3^1, v_4^1)),$$

$$V_4 = \{v_4^1\} \quad \text{with} \;\; v_4^1 = (L_4, (In_1, \frac{1}{2}In_2 + \frac{1}{2}Out_2, L_3, L_4), (v_1^1, \frac{1}{2}v_2^1 + \frac{1}{2}v_2^2, v_3^1, v_4^1)),$$

$$\phi_1(v_1^1) = 1, \; \phi_2(v_2^1) = \phi_2(v_2^2) = \frac{1}{2}, \; \phi_3(v_3^1) = 1, \phi_4(v_4^1) = 1.$$