Solutions to Exercises from Kenneth Brown's *Cohomology of Groups*

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I appreciate emails concerning any errors/corrections: **cgerig@berkeley.edu**. Any errors would be due to solely myself, or at least the undergraduate-version of myself when I last looked over this. *Remark made on 1/28/14.*

MAJOR: Applied & Engineering Physics
YEAR: Sophomore Undergraduate

Hey Ken, thanks for taking me under your wing.

Contents

0 Errata to *Cohomology of Groups*

pg62, line 11 $\rightsquigarrow$ missing a paranthesis) at the end.
pg67, line 15 from bottom $\rightsquigarrow$ missing word, should say "as ***an*** abelian group".
pg71, last line of Exercise 4 $\rightsquigarrow$ hint should be on a new line (for whole exercise).
pg85, line 9 from bottom $\rightsquigarrow$ incorrect function, should be $\sum_{g \in C/H} g^{-1} \otimes gm$.
pg114, first line of Exercise 4 $\rightsquigarrow$ misspelled *endomorphism* with an extra r.
pg115, line 5 from bottom $\rightsquigarrow$ missing word, should say "4.4 is ***a*** chain map".
pg141, line 4 from bottom $\rightsquigarrow$ missing a hat $\widehat{\ }$ on the last H^*.
pg149, line 13 $\rightsquigarrow$ the ideal I should be italicized.
pg158, line 20 $\rightsquigarrow$ there should be a space between $SL_2(\mathbb{F}_p)$ and (p odd).
pg160, line 5 from bottom $\rightsquigarrow$ missing the coefficient in cohomology, $\widehat{H}^*(H, M)$.
pg165, line 26 $\rightsquigarrow$ the comma in the homology should be lowered, $H_q(C_{*,p})$.

1 Chapter I: Some Homological Algebra

2.1(a): The set $A = \{g-1 \,|\, g \in G,\, g \neq 1\}$ is linearly independent because $\sum \alpha_g(g-1) = 0 \Rightarrow \sum \alpha_g g = \sum \alpha_g = \sum \alpha_g 1 + \sum 0g$, and since $\mathbb{Z}G$ is a free $\mathbb{Z}$-module, $\sum \alpha_g$ has a unique expression, yielding $\alpha_g = 0\ \forall g \in A$. To show that $I = span(A)$, let $\sum \alpha_g g \in I$ and hence $\sum \alpha_g = 0$. Thus we can write $\sum \alpha_g g = \sum \alpha_g g - 0 = \sum \alpha_g g - \sum \alpha_g = \sum \alpha_g(g-1)$. Since A is a linearly independent set which generates I, it is a basis for I.

2.1(b): Consider the left ideal $A = (\{s-1 \,|\, s \in S\})$ over $\mathbb{Z}G$.
$A \subseteq I$ since $\varepsilon(A) = \sum_s \varepsilon\big((\sum_g r_g g)(s-1)\big) = \sum_s \varepsilon(\sum_g r_g g)\varepsilon(s-1) = y \cdot 0 = 0$.
If $x \in I$ then $x = \sum \alpha_i g_i - \sum \beta_j g_j$ such that $\sum \alpha_i = \sum \beta_j$.

$$x = (\sum \alpha_i g_i - \sum \alpha_i) - (\sum \beta_j g_j - \sum \beta_j) + (\sum \alpha_i - \sum \beta_j)$$

Thus $x = \sum \alpha_i(g_i - 1) - \sum \beta_j(g_j - 1)$ and it suffices to show $g - 1 \in A$ for any $g \in G$ so that $x \in A$ and $I \subseteq A$. Since $G = \langle S \rangle$, we have a representation $x = s_1^{\pm 1} \cdots s_n^{\pm 1}$. By using $ab - 1 = a(b-1) + (a-1)$ and $c^{-1} - 1 = -c^{-1}(c-1)$, the result follows immediately.

2.1(c): Suppose $S \subset G \mid I = (\{s-1 \mid s \in S\})$. Then every element of I is a sum of elements of the form $g - g' \mid g = g's^{\pm 1}$. For $g \in G$, $g - 1 \in I$, so we have a finite sum $g - 1 = \sum_{i=1}^{n-1}(g_i - g_i')$. Since this is a sum of elements in G where G is written multiplicatively, $\exists\, i_0 \mid g_{i_0} = g$, say $i_0 = 1$. Thus $g - 1 = g - g_1' + \sum_{i=2}^{n-1}(g_i - g_i') \Rightarrow g_1' - 1 = \sum_{i=2}^{n-1}(g_i - g_i')$. Another iteration yields $g_2 = g_1'$ and $g = g_1 = g_1' s_1^{\pm 1} = g_2 s_1^{\pm 1} = g_2' s_2^{\pm 1} s_1^{\pm 1}$. At the last iteration, $g_{n-2}' - 1 = g_{n-1} - g_{n-1}' \Rightarrow g_{n-1}' = 1$, $g_{n-2}' = g_{n-1} = g_{n-1}' s_{n-1}^{\pm 1} = 1 s_{n-1}^{\pm 1}$. Through this method we obtain a sequence $g_1, g_2, ..., g_{n-1}, g_n \mid g_i = g_{i+1} s_i^{\pm 1}$ and $g_1 = g$, $g_n = 1$. $\therefore$ g has a representation in terms of elements of S and so $G = \langle S \rangle$ since g was arbitrary.

2.1(d): If G is finitely generated, then by part(b) above, I is finitely generated. For the converse, suppose $I = (a_1, ..., a_n)$ is a left ideal over $\mathbb{Z}G$. Noting from part(a) that $I = \langle g - 1 \rangle_{g \in G}$ as a $\mathbb{Z}$-module, each a_i can be represented as a finite sum $a_i = \sum_j z_j(g_j - 1)$. Since each a_i is generated by finitely many elements, and there are finitely many a_i, I is finitely generated as a left ideal by elements $s - 1$ where $s \in G$. Therefore, we apply part(c) to have $G = \langle s_1, \ldots, s_k \rangle$ and thus G is a finitely generated group.

2.2: Let $G = \langle t \rangle$ with $|G| = n$ and let t be the image of T in $R = \mathbb{Z}[T]/(T^n - 1) \cong \mathbb{Z}G$. The element $T - 1$ is prime in $\mathbb{Z}[T]$ because $\mathbb{Z}[T]/(T-1) \cong \mathbb{Z}$ which is an integral domain (An ideal P of a commutative ring R is prime iff the quotient ring R/P is an integral domain; Proposition 7.4.13[2]). By Proposition 8.3.10[2] (In an integral domain a prime element is always irreducible), $T - 1$ is irreducible in $\mathbb{Z}[T]$. We also could have obtained this result by applying Eisenstein's Criterion with the substitution $T = x + 3$ and using the prime 2. Since $\mathbb{Z}[T]$ is a Unique Factorization Domain, the specific factorization $T^n - 1 = (T-1)(T^{n-1} + T^{n-2} + \cdots + T + 1)$ with the irreducible $T - 1$ factor is unique, considering the latter factor $\sum_{i=0}^{n-1} T^i$ as the expansion of its irreducible factors [note: if n is prime then $\sum_{i=0}^{n-1} T^i = \Phi_n(T)$, a cyclotomic polynomial which is irreducible in $\mathbb{Z}[T]$ by Theorem 13.6.41[2]]. Thus every $f \in R$ is annihilated by $t - 1$ iff it is divisible by $N = \sum_{i=0}^{n-1} t^i$ (and vice versa), and so the desired free resolution of $M = \mathbb{Z} = \mathbb{Z}G/(t-1)$ is:

$$\cdots \to R \xrightarrow{t-1} R \xrightarrow{N} R \xrightarrow{t-1} R \to M \to 0\ .$$

3.1: The right cosets Hg_i are H-orbits of G with the H-action as group multiplication. Since $G = \bigsqcup Hg_i$ where g_i ranges over E, $\mathbb{Z}G = \bigoplus \mathbb{Z}[Hg_i] \cong \bigoplus \mathbb{Z}[H/H_{g_i}]$. G is a free H-set because $hg = g \Rightarrow hgg^{-1} = gg^{-1} \Rightarrow h = 1$, i.e. the isotropy groups H_g are trivial. Therefore, $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module with basis E.

3.2: Let $\langle S \rangle = H \subseteq G$ and consider $\mathbb{Z}[G/H]$. Now $x \in \mathbb{Z}[G/H]$ has the expression $x = \sum z_i(g_i H)$, and there exists an element fixed by H, namely, $x_0 = g_0 H = H$ where $g_0 \in H$. H is annihilated by $I = \mathrm{Ker}\varepsilon = (\{s-1\})$ since $(s-1)H = sH - H = H - H = 0\ \ \forall s \in S$, and so $Ix_0 = 0$. We have $g - 1 \in I$ since $\varepsilon(g-1) = \varepsilon(g) - \varepsilon(1) = 1 - 1 = 0$. Hence $(g-1)x_0 = 0 \Rightarrow gx_0 = x_0\ \forall g \in G \Rightarrow Gx_0 = x_0$. Finally, $GH = H \Rightarrow G \subseteq H$. $\therefore$ $G = H = \langle S \rangle$.

4.1: Orienting each n-cell e^n gives a basis for $C_n(X)$. If X is an arbitrary G-complex, then with

$\sum \eta_i e_i^n \in C_n(X)$, $g \in G$ can reverse the orientation of e^n by inversion (fixing the cell). Thus G need not permute the basis, and hence $C_n(X)$ is not necessarily a permutation module.

4.2: Since X is a free G-complex, it is necessarily a Hausdorff space with no fixed points under the G-action. First, assume G is *finite* and take the set of elements in Gx_0, which are distinct points $g_i x_0$ with $1x_0 = x_0$ for an arbitrary point $x_0 \in X$. Applying the Hausdorff condition, we have open sets U_{gi} containing $g_i x_0$ where each such set is disjoint from U_1 containing x_0. Form the intersection $W = (\bigcap_i g_i^{-1} U_{gi}) \cap U_1$ which contains x_0. Since $g_k W \subseteq U_{gk}$, we have $g_k W \cap W = \varnothing$ for all nonidentity $g_k \in G$, and so W is the desired open neighborhood of x_0 [This result does not follow for arbitrary G since an infinite intersection of open sets need not be open].
Assume the result has been proved for G infinite.
Let $\varphi : X \to X/G$ be the quotient map, which sends the disjoint collection of $g_i W$'s to $\varphi(W)$. Since $\varphi^{-1}\varphi(W) = \coprod_i g_i W$, $g_i W \to \varphi(W)$ is a bijective map (restriction of φ) and thus it is a homeomorphism (φ continuous and open). This covering space is regular because G acts transitively on $\varphi^{-1}(Gx)$ by definition. Elements of G are obviously deck transformations since $Ggx = Gx$, hence $G \subseteq \mathrm{Aut}(X)$. Given $\Gamma \in \mathrm{Aut}(X)$ with $\Gamma(a) = b$, those two points are mapped to the same orbit in X/G (since $\varphi \circ \Gamma = \varphi$), and so $\exists\, g \in G$ sending a to b. By the Lifting Lemma (uniqueness) we have $\Gamma = g$, hence $\mathrm{Aut}(X) \subseteq G \Rightarrow$ G is the group of covering transformations.
If X is contractible then $G \cong \pi_1(X/G)/\pi_1(X) \cong \pi_1(X/G)/0 = \pi_1(X/G)$ and φ is the universal cover of X/G, so X/G is a $K(G,1)$ with universal cover X.
It suffices to show that the G-action is a "properly discontinuous" action on X when G is *infinite*:
Every CW-complex with given characteristic maps $f_{j,n} : (B_j^n, S_j^{n-1}) \to (\bar{\sigma}_j^n, \partial\bar{\sigma}_j^n)$ admits a canonical open cover $\{U_\sigma\}$ indexed by the cells, where U_a and U_b are disjoint open sets for distinct cells of equal dimension (for instance, if X is a simplicial complex we can take $U_\sigma = St(\hat{\sigma})$ which is the open star of the barycenter of σ in the barycentric subdivision of X). More precisely, for one cell $\sigma_k^n \subset X^n$ in each G-orbit of cells define its "barycenter" as $\tilde{\sigma}_k^n \equiv f_{k,n}(b_k)$, where $b_k = 0 \in B^n$ is the origin of the n-disk; for the rest of the cells $\{\sigma_i^n = g_i \sigma_k^n\}$ in each G-orbit define their "barycenters" as $\tilde{\sigma}_i^n \equiv f_{i,n}(b_i)$, where $b_i \in B_i^n$ is chosen so that $\tilde{\sigma}_i^n = g_i \tilde{\sigma}_k^n$. Considering the 0-skeleton X^0, its cells σ_j^0 are by definition open and so the canonical open cover of X^0 is the collection $\{U_{\sigma_j^0} = \sigma_j^0\}$. Proceeding inductively (with U_σ open in X^{n-1}), consider the n-skeleton $X^n = X^{n-1} \bigcup_j \sigma_j^n$ and note that the preimage under $f_{j,n}$ of the open cover of X^{n-1} is an open cover of the unit circle S_j^{n-1}. Take an open set $f_{j,n}^{-1}(U_\sigma)$ in S_j^{n-1} and form the "open sector" $W_{j,\sigma} \subset B_j^n$ which is the union of all line segments emanating from b_j and ending in $f_{j,n}^{-1}(U_\sigma)$, minus b_j and $f_{j,n}^{-1}(U_\sigma)$; each U_σ determines such a $W_{j,\sigma}$. As $f_{j,n}$ is a homeomorphism of $Int(B_j^n)$ with σ_j^n, we have such open sectors $f_{j,n}(W_{j,\sigma})$ in the n-cell. Noting the weak topology on X, the set $U'_\sigma = U_\sigma \bigcup_j f_{j,n}(W_{j,\sigma})$ is open in X^n iff its complement in X^n is closed iff $(X^n - U'_\sigma) \cap \bar{\sigma}_k^i \equiv \bar{\sigma}_k^i - (U'_\sigma \cap \bar{\sigma}_k^i)$ is closed in $\bar{\sigma}_k^i$ for all cells in X^n. For $i < n$, $U'_\sigma \cap \bar{\sigma}^i = U_\sigma \cap \bar{\sigma}^i$ because $f_{j,n}(W_{j,\sigma}) \subset \sigma_j^n$ which is disjoint from the closure of all other cells, and the complement of this intersection in $\bar{\sigma}^i \subset X^{n-1}$ is closed because U_σ is open by inductive hypothesis. Therefore, it suffices to show that $\bar{\sigma}_k^n - (U'_\sigma \cap \bar{\sigma}_k^n)$ is closed in $\bar{\sigma}_k^n$ $\forall\, k$, which is equivalent under topology of cells for $U'_\sigma \cap \bar{\sigma}_k^n$ to be open in $\bar{\sigma}_k^n$. For arbitrary k we have $U'_\sigma \cap \bar{\sigma}_k^n = (U_\sigma \cap \bar{\sigma}_k^n) \bigcup_j (f_{j,n}(W_{j,\sigma}) \cap \bar{\sigma}_k^n) = (U_\sigma \cap \bar{\sigma}_k^n) \bigcup f_{k,n}(W_{k,\sigma})$ and taking the preimage we have $Y = f_{k,n}^{-1}(U'_\sigma \cap \bar{\sigma}_k^n) = f_{k,n}^{-1}(U_\sigma) \cup W_{k,\sigma}$. The n-disk is compact, the CW-complex X^n is Hausdorff, a closed subset of a compact space is compact (Theorem 26.2[6]), the image of a compact set under a continuous map is compact (Theorem 26.5[6]), and every compact subset of a Hausdorff space is closed (Theorem 26.3[6]); thus $f_{k,n}$ is a closed map and hence a quotient map for $\bar{\sigma}_k^n$ (by Theorem 22.1[6]). It suffices to check that $Y \subset B_k^n$ is open, for then $f_{k,n}(Y) = U'_\sigma \cap \bar{\sigma}_k^n$ is open in $\bar{\sigma}_k^n$ by definition of a quotient map. By construction, $Y = \{\mathbf{x} \in B_k^n - b_k \mid r(\mathbf{x}) \in f_{k,n}^{-1}(U_\sigma)\}$ where $r : B_k^n - b_k \to \partial B_k^n \equiv S_k^{n-1}$ is the radial projection $r(\mathbf{x}) = \frac{\mathbf{x} - b_k}{||\mathbf{x} - b_k||}$. As r is continuous and $f_{k,n}^{-1}(U_\sigma)$ is open in ∂B_k^n, we have $Y = r^{-1}(f_{k,n}^{-1}(U_\sigma))$ open in $B_k^n - b_k$ and hence in B_k^n (by Lemma 16.2[6]). Our new collection for X^n is the open sets U'_σ (where $\sigma \subset X^{n-1}$) plus the open n-cells $U'_{\sigma_j^n} = \sigma_j^n$; this is the canonical open cover of X^n and hence completes the induction.
Any point $x \in X$ will lie in an i-cell σ which lies in the open set U'_σ, and we take this as our desired neighborhood of x: since any open set of our constructed cover is bounded by barycenters, and $g \in G$ maps barycenters to barycenters by construction, we have $gU'_\sigma = U'_{g\sigma}$ which is disjoint from U'_σ by construction for all $g \neq 1$.

4.3: Given $G = \mathbb{Z} \oplus \mathbb{Z}$ we have the torus T with $\pi_1 T \cong G$ and its universal cover $\rho : \mathbb{R}^2 \to T$. After drawing the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$, pick a square and label that surface L, with the bottom left corner as the basepoint x_0 and the bottom side as the edge e_s and the left side as the edge e_t (so the corners are x_0, sx_0, stx_0, tx_0 going counterclockwise around L, and the top and right sides of L are respectively te_s and se_t). Following Brown's notation [in this section], x_0 generates $C_0(\mathbb{R}^2)$ and e_s, e_t generate $C_1(\mathbb{R}^2)$ with $\partial_1(e_s) = (s-1)x_0$ and $\partial_1(e_t) = (t-1)x_0$. Lastly, L generates $C_2(\mathbb{R}^2)$ with $\partial_2(L) = e_s + se_t - te_s - e_t = (1-t)e_s - (1-s)e_t$. Thus the desired free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ is:

$$0 \to \mathbb{Z}G \xrightarrow{\partial_2} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \ .$$

4.4:

5.1: The homotopy operator h in terms of the $\mathbb{Z}$-basis $g[g_1|\cdots|g_n]$ for F_n is $h(g[g_1|\cdots|g_n]) = [g|g_1|\cdots|g_n]$.

5.2: Using $G = \mathbb{Z}_2 = \{1, s\}$, the elements of the normalized bar resolution $\bar{F}_* = F_*/D_*$ are $[s|s|\cdots|s]$, and each element forms a basis for the corresponding dimension, giving the identification $\bar{F}_n = \mathbb{Z}G$. Denoting $s_i = s \ \ \forall\, i$,

$$d_i[s_1|s_2|\cdots|s_n] = \begin{cases} s[s|\cdots|s] \ , & i = 0 \\ [s_1|\cdots|s_{i-1}|s_i s_{i+1}|\cdots|s_n] = 0 \ , & 0 < i < n \\ [s|\cdots|s] \ , & i = n \end{cases}$$

The middle equation resulted from $s_i s_{i+1} = s^2 = 1$ (so the element lies in D_*). The boundary operator ∂_n then becomes $s-1$ for n odd and $s+1$ for n even.
$\therefore$ the normalized bar resolution is:

$$\cdots \to \mathbb{Z}G \xrightarrow{s-1} \mathbb{Z}G \xrightarrow{s+1} \mathbb{Z}G \xrightarrow{s-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \ .$$

5.3(a): [[*Geometric Realization of a Semi-Simplicial Complex*]]
For each $(n+1)$-tuple $\sigma = (g_0, \ldots, g_n)$, let Δ_σ be a copy of the standard n-simplex with vertices $v_0, \ldots, v_n$. Let $d_i\sigma = (g_0, \ldots, \hat{g}_i, \ldots, g_n)$ and let $\delta_i : \Delta_{d_i\sigma} \to \Delta_\sigma$ be the linear embedding which sends $v_0, \ldots, v_{n-1}$ to $v_0, \ldots, \hat{v}_i, \ldots, v_n$. Consider the disjoint union $X_0 = \coprod_\sigma \Delta_\sigma$ (topologize it as a topological sum) and define the quotient space $X \stackrel{\text{def}}{=} X_0/\!\sim$ using the equivalence relation generated by $(\sigma, \delta_i x) \sim (d_i\sigma, x)$, where we rewrite Δ_σ as $\sigma \times \Delta_\sigma$ for clarity of the relation properties.
We assert that the geometric realization X is a CW-complex with n-skeleton $X^n = (\coprod_{\dim\Delta_\sigma \le n} \Delta_\sigma)/\!\sim$. X^0 is the collection of vertices and hence a 0-skeleton, and so we proceed by induction on n [*sketch*]: In X^n the equivalence relation $\sim$ identifies a point on a boundary $\partial\Delta_\sigma^{(n)}$ with a point in X^{n-1}, and it doesn't touch the interior points of $\Delta_\sigma^{(n)}$. This means that the n-cells are $\{\Delta_\sigma^{(n)}\}$ with the attaching maps induced by $d_i \ \forall\, i$. Refer to Theorem 38.2[4] for the analogous construction with adjunction spaces, providing Hausdorffness of X^n and weak topology w.r.t. $\{X^i\}_{i<n}$. Thus, X is a CW-complex as the union $\bigcup X^i$ with the weak topology.
We define the G-action on the simplices by left multiplication on their associated tuples; it is free since tuples are unique and so the only element which fixes a tuple (and hence a simplex) is the identity element of G. This makes X a G-complex. We deduce that X is contractible because for each simplex, X contains its cone which is contractible, so taking $h\sigma = (1, g_0, \ldots, g_n)$ we can use the straight-line homotopy between $\delta_0 : \Delta_\sigma \to \Delta_{h\sigma}$ and the constant map $\Delta_\sigma \to \Delta_{h\sigma}$ at v_0 (for any point in the domain simplex of this homotopy H which lies on a subsimplex, the homotopy associated to that subsimplex is just the restriction of H, and hence the straight-line homotopy is well defined).
To form the desired isomorphism between the cellular chain complex $C(X)$ and the standard resolution F_*, it suffices to determine the boundary operator on $C(X)$ and see that it provides commutativity of the diagram $F_* \to C(X)$, noting that $C_i(X) \cong F_i$ by the correspondence $\Delta_\sigma \leftrightarrow \sigma$. Now $C(X_0)$ has the boundary $\partial[v_0, \ldots, v_n] = \sum_i (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n]$ and each resulting (n-1)-simplex is the image under a δ_i. Thus $C(X)$ has boundary maps $\partial(\Delta_\sigma) = \sum_i (-1)^i \Delta_{d_i\sigma}$, and these parallel those of F_*, giving commutativity of the diagram.

Note that this provides a solution to ***Exercise I.4.4****: For any group G we can form the contractible*

G-complex X as above, and since $X \to X/G$ is a regular covering map by the result of exercise I.4.2 above, the orbit space X/G is a $K(G,1)$, called the "classifying space."

5.3(b): For the normalized standard resolution $\bar{F}_*$, we simply follow part(a) while making these further identifications in X_0 to collapse degenerate simplices. For each $\sigma = (g_0, \ldots, g_n)$ let $s_i\sigma = (g_0, \ldots, g_i, g_i, \ldots, g_n)$, and when forming the quotient $X_0 \to X$ we also collapse $\Delta_{s_i\sigma}$ to Δ_σ via the linear map L_i which sends $v_0, \ldots, v_{n+1}$ to $v_0, \ldots, v_i, v_i, \ldots, v_n$ (so the only simplices of X are those whose associated tuples have pairwise distinct coordinates). Thus the equivalence relation in part(a) is also generated by $(\sigma, L_i x) \sim (s_i\sigma, x)$. During the inductive process for X^n, if Δ_σ is a degenerate simplex then X^{n-1} will already contain it and so those simplices need not be considered as n-cells. No problems arise when using the homotopy because for tuples of the form $\tau = (1, g_0, \ldots, g_n) \mid g_0 \neq 1$, the cone $\Delta_\tau * v_0 = \Delta_{h\tau} = \Delta_\tau$ has the identity map δ_0 still being nulhomotopic [note: we actually have a deformation retraction since $\Delta_{(1)}$ remains fixed instead of looping around $\Delta_{(1,1)}$ as in part(a)].

6.1: Given a finite CW-complex X with a map $f : X \to X$ such that every open cell satisfies $f(\sigma) \subseteq \bigcup_{\tau \neq \sigma} \tau$ where $\dim\tau \leq \dim\sigma$, we have the condition $f(\sigma) \cap \sigma = \varnothing$ and there are no fixed points. Viewing the open n-cell σ on the chain level in $H_n(X^{(n)}, X^{(n-1)}) = C_n(X)$, $f_\sharp(\sigma)$ does not consist of σ and so the respective matrix has the value 0 at the row-column intersection for σ. Therefore, $tr(f_\sharp, C_n(X)) = 0 \ \forall\, n$ since the diagonal of the matrix for the basis elements is zero. By the Hopf Trace Theorem, $\sum(-1)^i tr(f_i, H_i(X)/torsion) = \sum(-1)^i tr(f_\sharp, C_i(X))$ and so the Lefschetz number $\Lambda(f) = 0$. If X is a homology $(2n-1)$-sphere, then $H_i(X)$ is nontrivial only in dimensions 0 and $2n-1$ (in which case it is isomorphic to $\mathbb{Z}$). Thus, by the Lefschetz Fixed Point Theorem, $(-1)^0 tr(f_0, \mathbb{Z}) + (-1)^{2n-1} tr(f_{2n-1}, \mathbb{Z}) = 1 - d = 0 \ \Rightarrow\ d = 1$ and $f_* : H_{2n-1}(X) \to H_{2n-1}(X)$ is the identity.

6.2: The group action $G \to \mathrm{Homeo}(S^{2n})$ yields a degree map

$$\phi : G \to \mathrm{Aut}(H_{2n}(S^{2n})) \cong \mathrm{Aut}(\mathbb{Z}) = \{\pm 1\} = \mathbb{Z}/2\mathbb{Z}$$

which sends $g \in G$ to the degree $d = \deg(g)$ of its associated homeomorphism $g : S^{2n} \to S^{2n}$ [note: $\deg(g) \cdot \deg(g^{-1}) = \deg(g \cdot g^{-1}) = \deg(id) = 1 \ \Rightarrow\ |d| = 1$].
Consider nontrivial $G \neq \mathbb{Z}/2\mathbb{Z}$ and assert that this ϕ is not injective:
$\mathrm{Ker}\phi = 0 \ \Rightarrow\ 3 \leq |G| = |\mathrm{Ker}\phi| \cdot |\mathrm{Im}\phi| = 1 \cdot |\mathrm{Im}\phi|$. Since $\mathrm{Im}\phi \subseteq \mathbb{Z}/2\mathbb{Z}$, $|\mathrm{Im}\phi| = 1$ *or* 2 . In either case we arrive at a contradiction (since $1, 2 < 3$). $\therefore\ \exists\, g \in \mathrm{Ker}\phi \mid g \neq id \Rightarrow \deg g = 1$. Now assume this action is free, and use the notation $f_i : H_i(S^{2n}) \to H_i(S^{2n})$. By the Lefschetz Fixed Point Theorem, $(-1)^0 tr(f_0) + (-1)^{2n} tr(f_{2n}) = 1 + d = 0 \Rightarrow \deg f = d = -1 \ \forall\, f \neq id$. Our contradiction has now been reached (taking $f = g$ from above).

7.1: Given the finite cyclic group $G = \langle t \rangle$, the free resolution F of $\mathbb{Z}$ over $\mathbb{Z}G$ with period two (chain complex with rotations of S^1), and the bar resolution F', we obtain a commutative diagram where $f : F \to F'$ is the desired augmentation-preserving chain map:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z}G \arrow[r, "t-1"] \arrow[d, "f_3"] & \mathbb{Z}G \arrow[r, "N"] \arrow[d, "f_2"] & \mathbb{Z}G \arrow[r, "t-1"] \arrow[d, "f_1"] & \mathbb{Z}G \arrow[r, "\varepsilon"] \arrow[d, "f_0"] & \mathbb{Z} \arrow[r] \arrow[d, "id_{\mathbb{Z}}"] & 0 \\
\cdots \arrow[r] & F'_3 \arrow[r] & F'_2 \arrow[r] & F'_1 \arrow[r] & F'_0 \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

We define f inductively as $f_{n+1} = k_n f_n \partial$, where k is a contracting homotopy for the augmented complex associated to F', and each map is determined by where it sends the basis element:
$f_0(1) = k_{-1} id_{\mathbb{Z}} \varepsilon(1) = k_{-1} id_{\mathbb{Z}}(1) = k_{-1}(1) = (1) \equiv [\ ]$
$f_1(1) = k_0 f_0 \partial(1) = k_0 f_0(t-1) = k_0\{t f_0(1) - f_0(1)\} = k_0\{t[\ ] - [\ ]\} = [t] - [1]$
$f_2(1) = k_1 f_1 \partial(1) = k_1 f_1(N) = k_1\{\sum_i^{n-1} t^i f_1(1)\} = \sum_i^{n-1}([t^i, t] - [t^i, 1])$
$f_3(1) = k_2 f_2 \partial(1) = k_2 f_2(t-1) = \sum_i^{n-1}([t, t^i, t] + [1, t^i, 1] - [t, t^i, 1] - [1, t^i, t])$

7.2: Here is the axiomatized version, *Lemma 7.4*:
Under the additive category $\mathcal{A}$, let (C, ∂) and (C', ∂') be chain complexes, let r be an integer, and let $(f_i : C_i \to C'_i)_{i \leq r}$ be a class of morphisms such that $\partial'_i f_i = f_{i-1}\partial_i$ for $i \leq r$. If C_i is projective relative to the class $\mathcal{E}$ of exact sequences for $i > r$, and $C'_{i+1} \to C'_i \to C'_{i-1}$ is in $\mathcal{E}$ for $i \geq r$, then $(f_i)_{i \leq r}$ extends to a chain map $f : C \to C'$ and f is unique up to homotopy. (*Theorem 7.5 follows immediately*)

7.3(a): Given an arbitrary category $\mathfrak{C}$, let $A \in \mathrm{Ob}(\mathfrak{C})$ be an object and let $h_A = \mathrm{Hom}_{\mathfrak{C}}(A, -) : \mathfrak{C} \to (Sets)$ be the covariant functor represented by A, with $u_A \in h_A(A)$ as the identity map $A \to A$. Let $T : \mathfrak{C} \to (Sets)$ be an arbitrary covariant functor. Any natural transformation $\varphi : h_A \to T$ yields the commutative diagram (with $f : A \to B$ in $h_A(B)$ arbitrary):

$$\begin{tikzcd} h_A(A) \arrow[r, "h_A(f)"] \arrow[d, "\varphi"] & h_A(B) \arrow[d, "\varphi"] \\ T(A) \arrow[r, "T(f)"] & T(B) \end{tikzcd}$$

For any $v \in T(A)$ suppose we have the natural transformation with $\varphi(u_A) = v$. By commutativity, $T(f)(v) = (\varphi \circ h_A(f))(u_A) = \varphi(f \circ u_A) = \varphi(f)$, and hence the transformation is unique [determined by where it sends the identity]. For existence of the natural transformation $\varphi(f) = T(f)(v)$, we assert that it satisfies the commutative diagram, using arbitrary $g : B \to C$ (and f as above):

$$T(g)[\varphi(f)] = T(g)[T(f)(v)] = T(g \circ f)(v) = \varphi(g \circ f)$$
$$\varphi[h_A(g)(f)] = \varphi(g \circ f) = T(g)[\varphi(f)]$$

Thus, $\mathrm{Hom}_{\mathfrak{F}}(h_A, T) \cong T(A)$ where $\mathfrak{F}$ is the category of functors $\mathfrak{C} \to (Sets)$, and we have finished proving Yoneda's Lemma.

7.3(b): An $\mathfrak{M}$-free functor $F : \mathfrak{C} \to \mathbf{Ab}$ is isomorphic to $\bigoplus_\alpha \mathbb{Z}h_{A_\alpha}$, where $A_\alpha \in \mathfrak{M}$ [$\mathfrak{M}$ is a subclass of $\mathrm{Ob}(\mathfrak{C})$] and $\mathbb{Z}h_A : \mathfrak{C} \to \mathbf{Ab}$ is the composite of h_A and the functor $(Sets) \to \mathbf{Ab}$ which associates to a set the free abelian group it generates. Given the additive category $\mathcal{A}$ whose objects are covariant functors $\mathfrak{C} \to \mathbf{Ab}$ and whose maps are natural transformations of functors, let $\mathcal{E}$ be the class of $\mathfrak{M}$-exact sequences in $\mathcal{A}$. Consider the mapping problem (for all rows in $\mathcal{E}$):

$$\begin{tikzcd} & F \arrow[dl, dotted, "\psi"'] \arrow[d, "\varphi"] \arrow[dr, "0"] & \\ T' \arrow[r, "i"'] & T \arrow[r, "j"'] & T'' \end{tikzcd}$$

By Yoneda's Lemma (part(a) above), each component $\mathbb{Z}h_{A_\alpha}$ of F with any natural transformation in the above diagram is completely determined by the identity u_{A_α}, and so these identities "form a basis" for F. In particular, for the identity u_A we obtain the exact sequence $T'(A) \to T(A) \to T''(A)$ of abelian groups from the above mapping problem since the associated row lies in $\mathcal{E}$ with $A \in \mathfrak{M}$. Thus, for each identity we have $\varphi(u_{A_\alpha}) \in \mathrm{Ker}j = \mathrm{Im}i$, which implies $\exists\ x_\alpha \in T'(A_\alpha) \mid\ i(x_\alpha) = \varphi(u_{A_\alpha})$, and so we form ψ by $\psi(u_{A_\alpha}) = x_\alpha$. This means that F (an $\mathfrak{M}$-free functor) is projective relative to the class $\mathcal{E}$ of $\mathfrak{M}$-exact sequences.

7.3(c): There is a natural chain map in $\mathcal{A}$ from $\mathfrak{M}$-free complexes to $\mathfrak{M}$-acyclic complexes, and it is unique up to homotopy [using the categorial definitions from parts (a) and (b)]. This statement is a result of the combination of part(b) and Exercise 7.2 above, and is precisely the *Acyclic Model Theorem* in a rephrased form ($\mathfrak{M}$ is the set of models).

7.4: Under the category of R-modules, let (C, δ) and $(\bar{C}, \bar{\delta})$ be cochain complexes, let r be an integer, and let $(f_i^\circ : \bar{C}^i \to C^i)_{i \le r}$ be a class of morphisms such that $f_{i-1}^\circ \bar{\delta}_{i-1} = \delta_{i-1} f_i^\circ$ for $i \le r$. If C^i is injective relative to the class $\mathcal{E}^\circ$ of exact sequences for $i > r$ [yielding a cochain complex of injectives], and $\bar{C}^{i-1} \to \bar{C}^i \to \bar{C}^{i+1}$ is in $\mathcal{E}^\circ$ for $i \ge r$ [an acyclic cochain complex], then $(f_i^\circ)_{i \le r}$ extends to a cochain map $f^\circ : \bar{C} \to C$ and f° is unique up to homotopy.
(*The analogous "Theorem 7.5" follows immediately*)

8.1: Obviously the trivial group is one, since $\mathbb{Z}[\{1\}] = \mathbb{Z}$ and $\mathbb{Z}$ is a projective $\mathbb{Z}$-module; so assume G is nontrivial. Give $\mathbb{Z}$ the trivial module structure (so that for $r \in \mathbb{Z}G$, $r \cdot a = \varepsilon(r)a\ \ \forall\ a \in \mathbb{Z}$). Considering the short exact sequence of modules $0 \to I \to \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$, we must find a splitting $\mu : \mathbb{Z} \to \mathbb{Z}G$ for $\mathbb{Z}$ to possibly be $\mathbb{Z}G$-projective. Any such map is determined by where $1 \in \mathbb{Z}$ is sent; say $\mu(1) = x$. Then, for nontrivial $\alpha = \sum r_i g_i$, $\mu(\alpha \cdot 1) = \alpha \cdot \mu(1) = \alpha \cdot x = \alpha x$. But $\mu(\alpha \cdot 1) = \mu(\sum r_i) = \mu(1) \sum r_i = (\sum r_i)x$, so $\alpha x = (\sum r_i)x$. Thus $(\alpha - \sum r_i)x = 0 \Rightarrow \sum r_i g_i = \sum r_i \Rightarrow \sum r_i(g_i - 1) = 0$ [this sum can be viewed

as having no $g_i = 1$, and hence it lies in I]. Restricting our choice of nontrivial α to one which is not an integer, there is some nontrivial r_{i_0} associated to $g_{i_0} \neq 1$ and hence we must have $g_i = 1 \, \forall \, i$ (by freeness of I). But then G is the trivial group, and we are done.

8.2: Assume P is a projective $\mathbb{Z}G$-module and consider the subgroup $H \subseteq G$. Then $\mathcal{F} = P \oplus K$ where $\mathcal{F}$ is a free $\mathbb{Z}G$-module, by Proposition I.8.2[1]. By *restriction of scalars* from $\mathbb{Z}G$ to its subring $\mathbb{Z}H$ [$r \cdot n = f(r)n$ with homomorphism $f : \mathbb{Z}G \to \mathbb{Z}H$ preserving identities], $\mathcal{F}$ has an inherent $\mathbb{Z}H$-module structure. Since P is a direct summand of such a free module, it is a projective $\mathbb{Z}H$-module, by Proposition I.8.2[1]. Alternatively, we can note from Exercise 3.1 above that $\mathbb{Z}G = \bigoplus \mathbb{Z}H$ and so $\mathcal{F}$ is a direct sum of H-modules, hence $\mathbb{Z}H$-free.

8.3(a): Given F as a non-negative acyclic chain complex of projective modules P_n (over an arbitrary ring R), it will be contractible if each short exact sequence $0 \to Z_n \to P_n \xrightarrow{\bar{\partial}_n} Z_{n-1} \to 0$ splits (where $\bar{\partial}$ is induced by the boundary ∂), by Proposition I.0.3[1]. The case $n = 0$ is trivial using the zero map, and so arguing inductively we assume that $\bar{\partial}_{n-1}$ splits. Since $\mathrm{Ker}\bar{\partial}_{n-1} = Z_{n-1}$ is a direct summand of the projective module P_{n-1} and a projective module is a direct summand of a free module by Proposition I.8.2[1], Z_{n-1} is necessarily a direct summand of a free module, and hence is projective by Lemma I.7.2[1]. Therefore, $\bar{\partial}_n$ must have a splitting in the aforementioned short exact sequence by Proposition I.8.2[1].

8.3(b): If R is a principal ideal domain, then submodules of a free R-module are free by Theorem II.7.1[5]. Since a projective module is a direct summand of a free module by Proposition I.8.2[1], it is in particular a submodule and hence is free (over R as a PID). Therefore, submodules of a projective module over a PID are free and hence projective (free modules are projective by Lemma I.7.2[1]). The non-negativity hypothesis of Corollary I.7.7 can be dropped if we then restrict ourselves to PIDs, because we can follow part(a) above but *not* use induction since Z_{n-1} is already projective, being a submodule of the projective chain module P_{n-1} (i.e. we don't need any "starting point" in the resolution to obtain the desired splitting).

8.4: Every permutation module admits the decomposition $\mathbb{Q}X \cong \bigoplus \mathbb{Q}[G/G_x]$ and a direct sum of projective modules is projective iff each summand is projective (by Lemma XVI.3.6[5]). Thus it suffices to show that $\mathbb{Q}[G/G_x]$ is a projective $\mathbb{Q}G$-module, where G is an arbitrary group and G_x is finite. Note that $\mathrm{Hom}_{\mathbb{Q}G}(\mathbb{Q}[G/G_x], -)$ is a left-exact functor (Corollary 10.5.32[2]); it is given by $M \to M^{G_x}$ because any homomorphism φ is determined by $\varphi(G_x)$, and $\varphi(G_x) = \varphi(g \cdot G_x) = g \cdot \varphi(G_x)$ for any $g \in G_x$. Thus, we must show that $\mathrm{Hom}_{\mathbb{Q}G}(\mathbb{Q}[G/G_x], -)$ takes surjective homomorphisms $M \to \bar{M}$ to surjective homomorphisms $M^{G_x} \to \bar{M}^{G_x}$ [because then the functor is exact and then $\mathbb{Q}[G/G_x]$ is projective by definition]. If $\bar{m} \in \bar{M}^{G_x}$, lift $\bar{m}$ to $m \in M$. Then $\frac{1}{|G_x|}\sum_{g \in G_x} gm$ is also a lifting of $\bar{m}$ which lies in M^{G_x} (because $\frac{1}{|G_x|}\sum gm \mapsto \frac{1}{|G_x|}\sum g\bar{m} = \frac{1}{|G_x|} \cdot |G_x|\bar{m} = \bar{m}$), and so $M^{G_x} \to \bar{M}^{G_x}$ is surjective. Thus, $\mathbb{Q}X$ is a projective $\mathbb{Q}G$-module, where X is a G-set and G_x is finite for all $x \in X$.

8.5: If G is finite and k is a field of characteristic zero, consider any short exact sequence of kG-modules of the form $0 \to M' \to M \xrightarrow{\varphi} P \to 0$. Since k-vector spaces are free modules over k, and free modules are projective (by Lemma I.7.2[1]), P is k-projective and hence the sequence [as k-modules] splits by Proposition I.8.2[1]. Choosing a splitting $f : P \to M$ for the underlying sequence of k-vector spaces, we form the homomorphism $\phi : P \to M$ by $x \mapsto \frac{1}{|G|}\sum_{g \in G} gf(g^{-1}x)$.
Since f is a k-module homomorphism, it suffices to show that ϕ is equivariant (compatible with G-action) for it to thus be a kG-module homomorphism:

$$\phi(g_0x) = \tfrac{1}{|G|}\sum gf(g^{-1}g_0x) = \tfrac{1}{|G|}\sum g_0hf(h^{-1}x)$$
$$[\text{where } \; h = g_0^{-1}g]$$
$$g_0\phi(x) = g_0(\tfrac{1}{|G|}\sum gf(g^{-1}g_0x)) = g_0(\tfrac{1}{|G|}\sum hf(h^{-1}x)) = \phi(g_0x)$$
$$[\text{because } \; \textstyle\sum_{g \in G} h = \sum_{g \in G} g, \; \text{ as } \; g_0^{-1} \; \text{ permutes the elements of } \; G]$$

By Proposition 10.5.25[2] it suffices to show that $\varphi\phi = id_P$ for ϕ to thus be a splitting (noting that $\varphi f = id_P$):

$$\varphi(\phi(x)) = \varphi(\tfrac{1}{|G|}\textstyle\sum_{g \in G} gf(g^{-1}x)) = \tfrac{1}{|G|}\sum_{g \in G} \varphi(gf(g^{-1}x)) = \tfrac{1}{|G|}\sum_{g \in G} g\varphi(f(g^{-1}x)) =$$

$$\frac{1}{|G|}\sum_{g\in G} gg^{-1}x = \frac{1}{|G|}\sum_{g\in G} x = \frac{1}{|G|}|G|x = x$$

Since ϕ is a kG-splitting, P is kG-projective by Proposition I.8.2[1].

8.6: Suppose P is an R-module such that $\varphi : P^* \otimes_R P \to \mathrm{Hom}_R(P,P)$ is surjective, where this map is given by $\varphi(u \otimes m)(x) = u(x) \cdot m$. Then in particular we have $id_P = \varphi(\sum f_i \otimes e_i)$, so that $x = id_P(x) = \sum \varphi(f_i \otimes e_i)(x) = \sum f_i(x)e_i$. Thus, P is projective by Proposition I.8.2[1], and it is finitely generated by the e_i elements (noting that the summation is finite for tensor products).

8.7: Assume P is a finitely generated projective R-module and M is any (left) R-module, and take the canonical isomorphism $\varphi : P^* \otimes_R M \to \mathrm{Hom}_R(P,M)$ from Proposition I.8.3[1] which is given by $\varphi(u \otimes m)(x) = u(x) \cdot m$. For any $z \in P^* \otimes_R P$ we define the map $\psi_z : \mathrm{Hom}_R(P,M) \to P^* \otimes_R M$ as $\psi_z(f) = (P^* \otimes f)(z)$, which is a homomorphism since tensors are distributive over sums.

Method 1: View φ^{-1} as a natural transformation $\mathrm{Hom}_R(P,-) \to P^* \otimes_R -$, where $\mathrm{Hom}_R(P,-)$ and $P^* \otimes_R -$ are exact covariant functors from the category **R-mod** to the category **Ab** by Corollary 10.5.41[2] and Corollary 10.5.32[2], noting that P^* is projective by Proposition I.8.3[1] and hence is a flat module by Corollary 10.5.42[2]. By Yoneda's Lemma, φ^{-1} is uniquely determined by $\varphi^{-1}(id_P) = z$, and the proof of the lemma (Exercise I.7.3(a) above) states that $\varphi^{-1}(f) = (P^* \otimes f)(z) = \psi_z(f)$. Thus, the inverse homomorphism φ^{-1} is a map of the form ψ_z (independent of M).

Method 2: By Proposition I.8.2[1] we can choose elements $e_i \in P$ and $f_i \in P^*$ such that for every $x \in P$, $x = \sum f_i(x)e_i$ and $f_i(x) = 0$ for cofinitely many x. Set $z = \sum f_i \otimes e_i$. We have $\psi_z\varphi = id$ because $\psi_z(\varphi(u \otimes m)) = (P^* \otimes u \cdot m)(z) = \sum f_i \otimes u(e_i) \cdot m = \sum f_i \cdot u(e_i) \otimes m = u \otimes m$ [note: $u(x) = u(\sum f_i(x)e_i) = \sum f_i(x) \cdot u(e_i)$ as u is an R-module homomorphism]; we also have $\varphi\psi_z = id$ because $\varphi(\psi_z(f)) = \varphi[(P^* \otimes f)(z)] = \varphi(\sum f_i \otimes f(e_i)) = \sum f_i \cdot f(e_i) = f$ [note: $f(x) = f(\sum f_i(x)e_i) = \sum f_i(x) \cdot f(e_i)$ as f is also an R-module homomorphism]. Thus, the inverse φ^{-1} is a map of the form ψ_z (independent of M).

8.8: Since P is finitely presented, we can form the obvious exact sequence $F_1 \to F_0 \to P \to 0$ with F_0 and F_1 free of finite rank (the generators and relators, respectively). By Theorem 10.5.33[2], $\mathrm{Hom}_R(-,D)$ is a left exact contravariant functor, and so we obtain an exact sequence $0 \to P^* \to F_0^* \to F_1^*$ of right R-modules [notation: $M^* = \mathrm{Hom}_R(M,R)$]. Since P is a flat module, we can tensor this exact sequence with P to obtain the exact sequence $0 \to P^* \otimes_R P \to F_0^* \otimes_R P \to F_1^* \otimes_R P$. Since F_i $[i = 0,1]$ is free, it is necessarily projective (by Lemma I.7.2[1]) and hence $F_i^* \otimes_R P \cong \mathrm{Hom}_R(F_i,P)$ by Proposition I.8.3[1]. Since F_1 is contained in the quotient of F_0 which gives P, $\mathrm{Hom}_R(F_1,P) = 0$ and thus we have the isomorphism $P^* \otimes_R P \cong \mathrm{Hom}_R(F_0,P)$.
Now $\mathrm{Hom}_R(-,P)$ as a functor on the original presentation sequence gives rise to the exact sequence $0 \to \mathrm{Hom}_R(P,P) \to \mathrm{Hom}_R(F_0,P) \to \mathrm{Hom}_R(F_1,P) = 0$, and thus we have the isomorphism $\mathrm{Hom}_R(P,P) \cong \mathrm{Hom}_R(F_0,P)$.
Since the discovered isomorphism $P^* \otimes_R P \xrightarrow{\cong} \mathrm{Hom}_R(P,P)$ is surjective, P is a projective R-module by Exercise I.8.6 above.

2 Chapter II: The Homology of a Group

2.1: For S an arbitrary G-set, $\mathbb{Z}S \cong \bigoplus \mathbb{Z}[G/G_s]$. By the Orbit-Stabilizer Theorem we have a bijection between Gs and G/G_s. When passing to the quotient for the group of co-invariants, the subset $Gs \subseteq S$ is sent to the element $Gs \subseteq S/G$ since s is identified with gs in S/G (giving trivial G-action). Therefore $(\mathbb{Z}S)_G \cong (\bigoplus \mathbb{Z}[G/G_s])_G \cong \bigoplus(\mathbb{Z}[G/G_s])_G \cong \bigoplus \mathbb{Z}[Gs] \cong \mathbb{Z}[S/G]$.

2.2: A weaker hypothesis "*Let X be an arbitrary G-complex without inversions*" suffices. Then $C_*(X) \cong \bigoplus \mathbb{Z}[X_i]$ is a direct sum of permutation modules where X_i is the basis set of i-cells, so by the previous exercise, $C_*(X)_G \cong \bigoplus(\mathbb{Z}[X_i])_G \cong \bigoplus \mathbb{Z}[X_i/G] \cong C_*(X/G)$ which has a $\mathbb{Z}$-basis with one basis element for each G-orbit of cells of X.

2.3(a): With the G-module M and normal subgroup $H \triangleleft G$, $M_H = M/IM$ where I is the augmentation ideal of $\mathbb{Z}H$. The induced G/H-action is given by $gH(m+IM) = gm+IM$. The properties of an action are obviously satisfied, and it is well defined because if g_2 is another coset representative of g_1H, then $g_2 = g_1h$ and $g_2m+IM = g_1hm+IM = g_1hm-g_1m+g_1m+IM = (h-1)g_1m+g_1m+IM = g_1m+IM$.

2.3(b): We can form the group homomorphism $\varphi : M_G \to (M_H)_{G/H}$ using part(a) by $\overline{m} \mapsto \overline{m+IM}$, which is well-defined because $\overline{m} = \overline{gm} \mapsto \overline{gm+IM} = \overline{gH(m+IM)} = \overline{m+IM}$; it is a homomorphism since $\varphi(\overline{m_1}\ \overline{m_2}) = \varphi(\overline{m_1m_2}) = \overline{m_1m_2+IM} = \overline{(m_1+IM)(m_2+IM)} = (\overline{m_1+IM})(\overline{m_2+IM}) = \varphi(\overline{m_1})\varphi(\overline{m_2})$. For the inverse ϕ we use $\overline{m+IM} \mapsto \overline{m}$ which is well-defined since given the equivalent elements $\overline{m+IM}$ and $gm+(h-1)m'+IM$ in $(M_H)_{G/H}$,
$\phi(\overline{gm+(h-1)m'+IM}) = \overline{gm}+\overline{hm'}-\overline{1m'} = \overline{m}+\overline{m'}-\overline{m'} = \overline{m} = \phi(\overline{m+IM})$; it is a homomorphism because $\phi(\overline{m_1+IM}\ \overline{m_2+IM}) = \phi(\overline{m_1m_2+IM}) = \overline{m_1m_2} = \overline{m_1}\ \overline{m_2} = \phi(\overline{m_1+IM})\phi(\overline{m_2+IM})$. Thus, we have the isomorphism $M_G \cong (M_H)_{G/H}$.

2.3(c): Let $\mathbb{Z}[G/H] \otimes_{\mathbb{Z}G} M$ be a G/H-module, where $\mathbb{Z}[G/H]$ is the obvious $(G/H, G)$-bimodule which forms the tensor product and gives it the desired module structure. The map $\mathbb{Z}[G/H] \times M \to M_H$ given by $(a, m) \mapsto a\bar{m}$ is clearly G-balanced, and so by the *universal property of tensor products* (Theorem 10.4.10[2]) there exists the group homomorpism $\varphi : \mathbb{Z}[G/H] \otimes_{\mathbb{Z}G} M \to M_H$ given by $a \otimes m \mapsto a\bar{m}$, and it is clearly a G/H-module homomorphism. There is a well-defined map $\phi : M_H \to \mathbb{Z}[G/H] \otimes_{\mathbb{Z}G} M$ defined as $\bar{m} \mapsto 1 \otimes m$ because of the identity $1H \otimes hm = 1Hh \otimes m = 1H \otimes m$, and it is a G/H-module homomorphism because $\phi(gH \cdot \bar{m}) = \phi(g\bar{m}) = 1H \otimes gm = Hg \otimes m = gH \otimes m = gH1H \otimes m = gH \cdot \phi(\bar{m})$ [noting that $gH = Hg$ since H is normal, and G/H acts on M_H by part(a) above]. Since φ and ϕ are inverses of each other, they are isomorphisms and we obtain $M_H \cong \mathbb{Z}[G/H] \otimes_{\mathbb{Z}G} M$.

3.1: Let $g_1, \dots g_n \in G$ be pairwise-commutative elements and consider $z = \sum(-1)^{\mathrm{sgn}\sigma}[g_{\sigma(1)}|\cdots|g_{\sigma(n)}] \in C_n(G)$, where σ ranges over all permutations of $\{1, \dots, n\}$. *The sign of a permutation is defined here to be the number of swaps between adjacent integers to bring the permuted set back to the identity.* Looking at the boundary ∂z where $\partial = \sum(-1)^i d_i$, a particular d_j with $j \neq 0, n$ will provide elements in $C_{n-1}(G)$ of the form $[\cdots|g_kg_{k'}|\cdots]$. Each of these appears twice because $g_kg_{k'} = g_{k'}g_k$, but the paired elements will have opposite signs and hence will cancel each other. For $j = 0, n$ we have elements of the form $[g_i|\cdots]$ with one g_k missing from each, and each of these elements also appears twice because d_0 will take off g_k from the beginning of some element while d_n will take off g_k from the end of some other element. The paired elements differ by the sign $(-1)^n$ due to the boundary map, and they also differ by the sign $(-1)^{n-1}$ due to the permutation which takes the first slot and sends it to the last slot; since $(-1)^n(-1)^{n-1} = (-1)^{2n-1} = -1$, these paired elements will also cancel each other. Thus, $\partial z = 0$ and z is a cycle in $C_n(G)$.

3.2: Suppose $\mathbb{Z}$ admits a projective resolution of finite length over $\mathbb{Z}G$ where $G = \mathbb{Z}_n$. Then $\exists\, i_0 \mid\ H_iG = 0\ \ \forall\, i > i_0$, and we make note that H_iG is independent of the choice of resolution (see Section II.1[1]). Yet by an earlier calculation II.3.1[1] (using an infinite resolution), $H_iG \cong \mathbb{Z}_n$ for all positive odd integers i. Thus we have arrived at a contradiction.

3.3: If G has torsion, say $\mathbb{Z}_n \subseteq G$, and $\mathbb{Z}$ admits a projective resolution of finite length over $\mathbb{Z}G$,

then since a projective $\mathbb{Z}G$-module is projective as a $\mathbb{Z}H$-module for any subgroup $H \subseteq G$ (by Exercise I.8.2), we would obtain a corresponding finite projective resolution over $\mathbb{Z}[\mathbb{Z}_n]$. But this cannot occur due to the previous exercise, and hence we arrive at a contradiction.

4.1: If Y is a path-connected space and has a contractible regular covering space X with covering group G, then X is its universal cover with free G-action as translation, and $\pi_1 Y = G$ (so $Y \cong X/G$ is a $K(G,1)$-space). The singular chain module $C_n^{sing}(X)$ is a free $\mathbb{Z}$-module with basis the set of singular simplices $\sigma^n : \Delta^n \to X$ (continuous maps of the standard simplex into the space). A G-action on this basis is given by the composition $g\sigma^n : \Delta^n \to X \to X$, and thus $C_n^{sing}(X)$ is a free $\mathbb{Z}G$-module with one basis element for every G-orbit of singular simplices. Denoting the ith face of σ^n as $\sigma^n F_i$ where $F_i = [v_0, \ldots, \hat{v}_i, \ldots, v_n] : \Delta^{n-1} \to \Delta^n$ is the inclusion map, the induced G-action on the simplices maps faces of σ^n to faces of $g\sigma^n$. Thus, the boundary operator is equivariant and the singular [augmented] chain complex $C_*^{sing}(X)$ is a free G-module chain complex. As X is contractible, the complex is exact and so it is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.
Consider the projection $\varphi : C_*^{sing}(X) \to C_*^{sing}(Y)$ given by $\sigma \mapsto q\sigma$, where $q : X \to X/G \cong Y$ is the regular covering map. Noting that $\varphi(g\sigma) = qg\sigma = q\sigma = \varphi(\sigma)$, the projection induces the map $\bar{\varphi} : C_*^{sing}(X)_G \to C_*^{sing}(Y)$ which sends basis elements to basis elements. Since $C_*^{sing}(Y)$ has a $\mathbb{Z}$-basis with one basis element for each G-orbit of singular cells of X as does $C_*^{sing}(X)_G$ [by a property (II.2.3) of the coinvariants functor], $\bar{\varphi}$ is an isomorphism. Therefore, $H_*G \cong H_*Y$ as the homologies of a group are independent of the choice of resolution up to canonical isomorphism.

4.2:

4.3:

5.1: Let Y be an n-dimensional connected CW-complex such that $\pi_i Y = 0$ for $i < n$ ($n \geq 2$), let $\pi = \pi_1 Y$, and let X be the universal cover of Y (so that $\pi_1 X = 0$). Since $\pi_i X \cong \pi_i Y$ for $i > 1$, $\pi_i X$ is trivial for $i < n$ and so by the Hurewicz Theorem $H_i X = 0$ for $0 < i < n$ and the Hurewicz map $h : \pi_n X \to H_n X$ is an isomorphism. In addition, we have a partial free resolution $C_n(X) \to \cdots \to C_0(X) \to \mathbb{Z} \to 0$ whose n^{th} homology group is $Z_n X = H_n X$ (noting that X is n-dimensional). Lemma II.5.1[1] now gives us an exact sequence $0 \to H_{n+1}\pi \to (H_n X)_\pi \to Z_n Y = H_n Y \to H_n \pi \to 0$, where $Y \cong X/\pi$ because every universal cover is regular with covering transformation group $\pi_1 Y$ (by Corollary 81.4[6]). Finally, noting that the coinvariants functor takes isomorphisms to isomorphisms (by right-exactness), the commutative diagram

$$\begin{tikzcd} (\pi_n X)_\pi \arrow[r, "\cong"] \arrow[d, "\cong"] & (H_n X)_\pi \arrow[d] \\ (\pi_n Y)_\pi \arrow[r] & H_n Y \end{tikzcd}$$

yields the desired exact sequence $0 \to H_{n+1}\pi \to (\pi_n Y)_\pi \to H_n Y \to H_n \pi \to 0$.

5.2(a): Let $G = \langle S \ ; \ r_1, r_2, \ldots \rangle = F(S)/R$ where R is the normal closure in $F(S)$ of the words r_i. Consider the abelianization map $r_i r_j \mapsto [r_i] + [r_j]$ from R to the relation module R_{ab} with the denotation $[r_i] = r_i \text{mod}[R, R]$. Any element $x \in R$ has a representation $x = \prod_{i=1}^n (f_i r_i^{\pm 1} f_i^{-1})$ where $f_i \in F = F(S)$. Now F acts by conjugation on R and so induces an F-action on R_{ab}, and R acts trivially on R_{ab} due to the definition of abelianization; it is immediate that we obtain the ($G = F/R$)-action on R_{ab}: $g \cdot [r_i] = [f r_i f^{-1}]$. Subsequently, $[x] = \sum_{i=1}^n (g_i \cdot [r_i])$ and hence R_{ab} is generated as a G-module by the images of the presentation words.

5.2(b): Let $Y = (\bigvee_s S^1) \cup_{r_1} e^2 \cup_{r_2} e^2 \cup \cdots$ be the 2-complex associated to the given presentation of G in part(a), and let $\tilde{Y}$ be its universal cover. Consider the augmented cellular chain complex

$$(*) \qquad 0 \to C_2(\tilde{Y}) \xrightarrow{\partial_2} C_1(\tilde{Y}) \xrightarrow{\partial_1} C_0(\tilde{Y}) \to \mathbb{Z} \to 0$$

Note that $C_0(\tilde{Y}) = \mathbb{Z}G$ and $C_1(\tilde{Y}) = \bigoplus_{|S|} \mathbb{Z}G$ and $C_2(\tilde{Y}) = \bigoplus_{|R|} \mathbb{Z}G$. By Proposition II.5.4[1] we have the exact sequence

$$0 \longrightarrow R_{ab} \longrightarrow C_1(\tilde{Y}) \xrightarrow{\partial_1} C_0(\tilde{Y}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

and hence $R_{ab} = \mathrm{Ker}\partial_1$ is always in the chain complex for $\tilde{Y}$. If Y is a $K(G,1)$ then $\tilde{Y}$ is acyclic and (*) is exact by Proposition I.4.2[1], so $C_2(\tilde{Y}) = R_{ab}$ and hence R_{ab} is a free $\mathbb{Z}G$-module.
Now suppose $\tilde{Y}$ is not acyclic, so that Y is not a $K(G,1)$. Then $H_2\tilde{Y}$ is nontrivial (since $H_i\tilde{Y} = 0$ for $i > 2$ by (*) and $H_1\tilde{Y} = 0$ by the simply-connected property of $\tilde{Y}$) which implies that the boundary map ∂_2 is not injective, and by exactness we can refer to this non-injective map as $C_2(\tilde{Y}) \to \mathrm{Ker}\partial_1 = R_{ab}$. Therefore, there exists a nontrivial $\mathbb{Z}G$-relation amongst the words r_i in $\mathrm{Ker}\partial_2$, so $\{[r_i]\}$ is not $\mathbb{Z}G$-independent in R_{ab} and hence does not generate R_{ab} freely.

5.2(c): Let $G = \langle S; r\rangle$ be an arbitrary one-relator group and write $r = u^n \in F = F(S)$, where $n \geq 1$ is maximal. By a result of Lyndon-Schupp, the image t of u in G has order exactly n, and we let $C = \langle t\rangle = \mathbb{Z}_n$. If $n > 1$ then the relation module R_{ab} is not freely generated by $r \bmod [R,R]$ since this generator is fixed by C, but a result of Lyndon shows that no other relations hold (i.e. the projection $\mathbb{Z}[G/C] \to R_{ab}$ is an isomorphism).
Let Y be a bouquet of circles indexed by S, and let $\tilde{Y}$ be the connected regular covering space of Y corresponding to the normal subgroup R of $F = \pi_1 Y$. Choosing a basepoint $\tilde{v} \in \tilde{Y}$ lying over the vertex of Y, we identify G with the group of covering transformations of $\tilde{Y}$; as explained in [1] on pg15, $\tilde{Y}$ is a (1-dimensional) free G-complex. Since $\tilde{u}$ ends at $t\tilde{v}$, the lifting $\tilde{r}$ is the composite path

$$\begin{tikzcd} t\tilde{v} \arrow[d, "t\tilde{u}"'] & \tilde{v} \arrow[l, "\tilde{u}"'] \\ t^2\tilde{v} \arrow[r, dashed] & t^{n-1}\tilde{v} \arrow[u, "t^{n-1}\tilde{u}"'] \end{tikzcd}$$

Thus the map $S^1 \to \tilde{Y}$ corresponding to $\tilde{r}$ is compatible with the action of C, where C acts on S^1 as a group of rotations (i.e. t^k is multiplication by $e^{2\pi ik/n}$). Consider the 2-complex X obtained by attaching 2-cells to $\tilde{Y}$ along the loops $g\tilde{r}$, where g ranges over a set of representatives for the cosets G/C. Given a 2-cell σ, each $g \in G$ sends $\partial\sigma$ homeomorphically to $\partial\sigma'$ for some 2-cell σ', and so we just pick our favorite extension $g : \sigma \to \sigma'$. This G-action makes X a G-complex since the permutations of 2-cells are determined by the permutations of their boundary loops. Thus, if σ is the 2-cell attached along $\tilde{r}$ then only $C \subset G$ will fix σ, since t^i simply rotates the loop $\partial\sigma$ (i.e. $G_\sigma = C$). Let $\Gamma = (\bigvee_s S^1) \cup_r e^2$ be the standard 2-complex associated to the presentation of G; its universal cover is $\tilde{\Gamma} = \tilde{Y} \bigcup_{g\in G} \sigma_g$ where σ_g is attached along $g\tilde{r}$, and $C_2(\tilde{\Gamma}) = \mathbb{Z}G$. Thus X is the quotient of $\tilde{\Gamma}$ by identifying σ_g with σ_{gt^i} for all i (for each g), and $C_2(X) = \mathbb{Z}[G/C] \cong R_{ab}$. If $n = 1$ then $C = \{1\}$ and $X = \tilde{\Gamma}$. Lyndon's theorem about one-relator groups says that R_{ab} is freely generated by the image of r, provided r is not a power, which is equivalent to Γ being a $K(G,1)$ by part(b) above, and hence equivalent to X being contractible (*X is the Cayley complex associated to the presentation of G*).

5.3(a): With $G = F/R$ and following Kenneth Brown's proof of Theorem II.5.3, let $F = F(S)$, let Y be a bouquet of circles indexed by S, and let $\tilde{Y}$ be the connected regular covering space of Y corresponding to the normal subgroup R of $F = \pi_1 Y$. Choosing a basepoint $\tilde{v} \in \tilde{Y}$ lying over the vertex of Y, we identify G with the group of covering transformations of $\tilde{Y}$. For any $f \in F$ we regard f as a combinatorial path in the CW-complex Y and we denote by $\tilde{f}$ the lifting of f to $\tilde{Y}$ starting at $\tilde{v}$. This path $\tilde{f}$ ends at the vertex $\bar{f}\tilde{v}$, where $\bar{f}$ is the image of f in G. Define the function $d : F \to C_1\tilde{Y}$ by letting df be the sum of the oriented 1-cells which occur in $\tilde{f}$. Since the lifting of f_1f_2 is the path $\tilde{v} \xrightarrow{\tilde{f}_1} \bar{f}_1\tilde{v} \xrightarrow{\bar{f}_1\tilde{f}_2} \bar{f}_1\bar{f}_2\tilde{v}$, we have $d(f_1f_2) = df_1 + \bar{f}_1 df_2$ for all $f_1, f_2 \in F$. Thus, if we regard the G-module $C_1\tilde{Y}$ as an F-module via the canonical homomorphism $q : F \to G$, then d is a derivation [since the F-action is given by restriction of scalars: $f_1 \cdot df_2 = \bar{f}_1 df_2 = q(f_1)df_2$].

5.3(b): For any free group F we can apply part(a) above with $R = \{1\}$ to get the desired derivation $d : F \to \Omega$ where $G = F/1 = F$ and $\Omega = C_1\tilde{Y} = \mathbb{Z}F^{(S)}$ which is the free module with basis $(ds)_{s\in S}$ [note: $ds^{-1} = -s^{-1}ds$].
The above note is a result of $d(1) = d(1\cdot 1) = d(1) + 1d(1) = 2d(1) \;\Rightarrow\; d(1) = 0$.
We write the total free derivative df of f as the sum $df = \sum_{s\in S}(\partial f/\partial s)ds$, where $\partial f/\partial s \in \mathbb{Z}F$ is the partial derivative of f with respect to s [the coefficient of ds when df is expressed in terms of the basis (ds)].
It is immediate that $\partial/\partial s : F \to \mathbb{Z}F$ is a derivation because d is a derivation: $ff' \mapsto d(ff') =$

$df + fdf' = \sum[(\partial f/\partial s) + f(\partial f'/\partial s)]ds \mapsto (\partial f/\partial s) + f(\partial f'/\partial s)$. For $t \in S$ we have $dt = \sum(\partial t/\partial s)ds = \sum_{s\neq t} 0ds + 1dt$, and so $\partial t/\partial s = \delta_{s,t}$.

$$\textit{Example}\text{: } S = \{s,t\} \;\Rightarrow\; \partial(ts^{-1}ts^2)/\partial s = \partial ts^{-1}/\partial s + ts^{-1}(\partial ts^2/\partial s) =$$
$$[\partial t/\partial s + t(\partial s^{-1}/\partial s)] + ts^{-1}[\partial t/\partial s + t(\partial ss/\partial s)] =$$
$$0 + t[-s^{-1}(\partial s/\partial s)] + ts^{-1}0 + ts^{-1}t[\partial s/\partial s + s(\partial s/\partial s)] = -ts^{-1}1 + ts^{-1}t[1 + s1] = ts^{-1}ts + ts^{-1}t - ts^{-1}$$

5.3(c): Consider any free group $F = F(S)$ and derivation $d : F \to M$ where M is an F-module. By the representation $f = s_1^{\pm 1} \cdots s_n^{\pm 1}$ and the definition of a derivation $d(gh) = dg + gdh$, we have the equation $df = \sum_{s\in S} w_s ds$ through trivial induction on n, where $w_s = \partial f/\partial s$ by definition of the partial derivative.

5.3(d): Consider θ in the exact sequence $0 \to R_{ab} \xrightarrow{\theta} \mathbb{Z}G^{(S)} \xrightarrow{\partial} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ of G-modules, where $\mathbb{Z}G^{(S)}$ is free with basis $(e_s)_{s\in S}$ and $\partial e_s = \bar{s} - 1$ [bar denotes image in G], and consider $\varphi : R \to \mathbb{Z}G^{(S)}$ given by $r \mapsto \sum_{s\in S} \overline{(\partial r/\partial s)}e_s$ where $\overline{(\partial r/\partial s)}$ is the image of $\partial r/\partial s$ under the canonical map $\mathbb{Z}F \to \mathbb{Z}G$. In order to show that θ is induced by φ we must verify exactness of the above sequence, and so we start by calculating the partial derivatives of the representation $r = s_1^{b_1} \cdots s_n^{b_n} \in R_{ab}$ (where $s_i \neq s_{j\neq i}$). Since $\partial s^b/\partial s = \sum_{j=0}^{b-1} s^j$ for $b > 0$ and $\partial s^b/\partial s = -\sum_{j=1}^{b} s^{-j}$ for $b < 0$ and $\partial(As^b)/\partial s = A(\partial s^b/\partial s)$ with $s \nmid A$, we obtain

$$\partial r/\partial s_i = \begin{cases} s_1^{b_1} \cdots s_{i-1}^{b_{i-1}} \sum_{j=0}^{b_i-1} s_i^j & b_i > 0 \\ -s_1^{b_1} \cdots s_{i-1}^{b_{i-1}} \sum_{j=1}^{b_i} s_i^{-j} & b_i < 0 \end{cases}$$

Injectivity of $\varphi|_{R_{ab}}$ follows immediately from the freeness of $\mathbb{Z}G^{(S)}$ and the fact that any nontrivial r has some nontrivial b_i (hence $\partial r/\partial s_i \neq 0$). It suffices to show that $\partial\varphi(r) = \sum \overline{(\partial r/\partial s)(s-1)} = 0$ for $r \in R_{ab}$. For a particular i, $(\partial r/\partial s_i)(s_i - 1) = s_1^{\pm b_1} \cdots s_{i-1}^{\pm b_{i-1}}(\sum s_i^{1\pm j} - \sum s_i^{\pm j}) = s_1^{\pm b_1} \cdots s_{i-1}^{\pm b_{i-1}}(s_i^{\pm b_i} - 1)$, and $(\partial r/\partial s_i)(s_i - 1) + (\partial r/\partial s_{i+1})(s_{i+1} - 1) = -s_1^{b_1} \cdots s_{i-1}^{b_{i-1}} + s_1^{b_1} \cdots s_i^{b_i} s_{i+1}^{b_{i+1}}$ [suppressing the $\pm$]. Thus, $\sum \overline{(\partial r/\partial s)(s-1)} = -1 + 0 + \cdots + 0 + \bar{s}_1^{b_1} \cdots \bar{s}_n^{b_n} = -1 + \bar{r} = -1 + 1 = 0$, and exactness is satisfied.
If R is the normal closure of a subset $T \subseteq F$, then the projection $\mathbb{Z}G^{(T)} \to R_{ab}$ given by $g \cdot e_t \mapsto g \cdot [t]$ and the above exact sequence provides us with a partial free resolution:

$$\begin{tikzcd}
\mathbb{Z}G^{(T)} \arrow[rr, "\partial_2"] \arrow[dr, two heads] & & \mathbb{Z}G^{(S)} \arrow[r, "\partial_1"] & \mathbb{Z}G \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] & 0 \\
& R_{ab} \arrow[ur, hook, "\theta"'] & & & &
\end{tikzcd}$$

where the matrix of ∂_2 is the "Jacobian matrix" $(\overline{\partial t/\partial s})_{t\in T, s\in S}$.

5.4: *Sketch (via Ken Brown)*: Let $G = F/R$ and use the same notation as in Exercise 5.3(a) above. Consider the following chain map in dimensions ≤ 2 using the bar resolution B_*

$$\begin{tikzcd}
B_2 \arrow[r, "\partial_2"] \arrow[d, "\Gamma_2"] & B_1 \arrow[r, "\partial_1"] \arrow[d, "\Gamma_1"] & B_0 \arrow[r, "\varepsilon"] \arrow[d, "\Gamma_0"] & \mathbb{Z} \arrow[r] & 0 \\
\mathbb{Z}G^{(R)} \arrow[r] & C_1(\tilde{Y}) \arrow[r] & C_0(\tilde{Y}) \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

We have the identification $C_1(\tilde{Y}) = \mathbb{Z}G^{(S)} = I_R$ where I is the augmentation ideal of $\mathbb{Z}F$ (so I_R is a free G-module on the images $\overline{s-1}$), and $\mathbb{Z}G^{(R)}$ is the free G-module with basis $(e_r)_{r\in R}$ which maps onto $R_{ab} = H_1\tilde{Y} \subset C_1(\tilde{Y})$. The specific chain map is given by $\Gamma_2 : [g_1|g_2] \mapsto -\overline{g_1g_2} \cdot r(g_1, g_2)$ and $\Gamma_1 : [g] \mapsto \overline{f(g) - 1}$ and $\Gamma_0 : [\,] \mapsto 1$, where $r(g_1, g_2) \in R$ and $f(g) \in F$ such that $\overline{f(g)} = g \in G$ and $f(g)f(h) = f(gh)r(g, h)$. Applying the coinvariants functor, the group homomorphism $C_2(G) \to R_{ab}$ given by $[g|h] \mapsto r(g, h) \bmod [R, R]$ induces the isomorphism $\varphi : H_2G \to R \cap [F, F]/[F, R]$ by passage to subquotients.
A specific chain map γ in the other direction is given by $\gamma_2 : r \mapsto h\gamma_1(r - 1)$ and $\gamma_1 : \overline{s-1} \mapsto [\bar{s}]$ and $\gamma_0 : 1 \mapsto [\,]$, where $h : B_1 \to B_2$ is the contracting homotopy $h(g \cdot [h]) = [g|h]$. Regarding $C_2(G)$ as an F-module via $f \cdot [g|h] = [\bar{f}g|h]$, the map $D : F \hookrightarrow \mathbb{Z}F \to I \to I_R \xrightarrow{\gamma_1} B_1 \xrightarrow{h} B_2 \to (B_2)_G \equiv C_2(G)$ is a derivation such that $Ds = [1|\bar{s}]$. Moreover, $Df = \sum_{s\in S}[\overline{\partial f/\partial s}|\bar{s}]$, where the symbol $[\cdot|\cdot]$ is $\mathbb{Z}$-bilinear. Then $D|_R : R \to C_2(G)$ is a homomorphism (since R acts trivially on $C_2(G)$) which induces φ^{-1} by passage to subquotients.

Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in F$ such that $r = \prod_{i=1}^n [a_i, b_i] \in R$. The formula $\varphi^{-1}(r \bmod [F,R]) = \sum_{i=1}^n \{[\bar{I}_{i-1}|\bar{a}_i] + [\bar{I}_{i-1}\bar{a}_i|\bar{b}_i] - [\bar{I}_{i-1}\bar{a}_i\bar{b}_i\bar{a}_i^{-1}|\bar{a}_i] - [\bar{I}_{i-1}|\bar{b}_i]\}$, where $I_i = [a_1, b_1] \cdots [a_i, b_i]$, is proven in the universal example where F is the free group on $a_1, \ldots, a_n, b_1, \ldots, b_n$ and R is the normal closure of r. Using the constructed formula for φ^{-1} and the product rule for derivations, the desired formula arises from $Dr = \sum_i I_{i-1} \cdot D[a_i, b_i]$ and $D[a,b] = [1|a] + [a|b] - [aba^{-1}|a] - [aba^{-1}b^{-1}|b]$.

5.5(a): With the presentation $G = \langle s_1, \cdots, s_n \,|\, r_1, \cdots, r_m \rangle$ we associate the 2-complex $Y = (\bigvee_s S^1) \cup_{r_1} e^2 \cdots \cup_{r_m} e^2$ so that $\pi_1 Y \cong G$. By computing the Euler characteristic $\chi(Y)$ two different ways (by Theorem 22.2[4]) we obtain the equation $\sum(-1)^i rk_{\mathbb{Z}}(H_i Y) = \sum(-1)^i c_i$, where $rk_{\mathbb{Z}}$ is the rank and c_i is the number of i-cells. Then $1 - rk_{\mathbb{Z}}(G_{ab}) + rk_{\mathbb{Z}}(H_2 Y) = 1 - n + m$, and so $rk_{\mathbb{Z}}(H_2 Y) = m - n + r$ where $r = rk_{\mathbb{Z}}(G_{ab}) = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes G_{ab})$. Now $H_2 Y = \mathrm{Ker}\partial_2$ is a free abelian group (subgroup of cellular 2-chain group), and by applying Theorem II.5.2[1] we get a surjection $H_2 Y \to H_2 G$ (from the exact sequence in the theorem). Thus $H_2 G$ *can* be generated by $m - n + r$ elements.

5.5(b): Since $G_{ab} = 0$ and the number of generators equals the number of relations (in the finite presentation), $m - n + r = m - m + 0 = 0$. Thus by part(a), $H_2 G$ can be generated by at most 0 elements, and so $H_2 G = 0$.

5.5(c): Given $G_{ab} = 0$, $H_2 G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and n as the number of generators, let m be the number of relations in the presentation of G. Then $H_2 G$ is generated by the two elements $(0,1)$ and $(1,0)$, and $r = 0$, so by part(a), $2 \leq m - n + 0 \Rightarrow m \geq n + 2$. Therefore, any n-generator presentation must involve at least $n + 2$ relations.

5.6(a): From the group extension $1 \to N \hookrightarrow G \twoheadrightarrow Q \to 1$ we have $G/N \cong Q$, and from Hopf's formula we have $H_2 G \cong R \cap [F,F]/[F,R]$ and $H_2 Q \cong S \cap [F,F]/[F,S]$, where $G = F/R$ and $Q = F/S$ with $R \subseteq S \subseteq F$ (so $N \cong S/R$). $(H_1 N)_Q \cong (N_{ab})_Q = N_{ab}/\langle\{(q-1)\cdot n[N,N]\}\rangle = N_{ab}/\langle\{gng^{-1}n^{-1}[N,N]\}\rangle = (N/[N,N])/([G,N]/[N,N]) \cong N/[G,N]$, where the Q-action on N_{ab} is induced by the conjugation action of G on N, and the latter isomorphism follows from the Third Isomorphism Theorem. Now $[G/N, G/N] = \{g_1 N g_2 N g_1^{-1} N g_2^{-1} N\} = \{g_1 g_2 g_1^{-1} g_2^{-1} N\} = [G,G]N/N$, so we have $H_1 Q \cong Q_{ab} = (G/N)/[G/N, G/N] \cong G/(N[G,G])$, where the latter isomorphism follows from the Third Isomorphism Theorem.
Thus, the desired 5-term exact sequence is obtained by showing the exactness of the sequence

$$R \cap [F,F]/[F,R] \xrightarrow{\alpha} S \cap [F,F]/[F,S] \xrightarrow{\beta} N/[G,N] \xrightarrow{\gamma} G/[G,G] \xrightarrow{\delta} G/(N[G,G]) \to 0$$

where γ and δ are induced by the injection and surjection of the group extension. From $\delta : g[G,G] \mapsto g[G,G]N = gN[G,G]$ we have $\mathrm{Ker}\delta = N/[G,G]$ and $\mathrm{Im}\delta = G/(N[G,G])$. From $\gamma : n[G,N] \mapsto n[G,G]$ we have $\mathrm{Ker}\gamma = N \cap [G,G]/[G,N]$ and $\mathrm{Im}\gamma = N/[G,G] = \mathrm{Ker}\delta$. As deduced above, $N/[G,N] = (S/R)/[F/R, S/R] \cong S/(R[F,S])$ and so from $\beta : s[F,S] \mapsto s[F,S]R = sR[F,S]$ we have $\mathrm{Ker}\beta = R \cap [F,F]/[F,S]$ and $\mathrm{Im}\beta = S \cap [F,F]/(R[F,S]) \cong N \cap [G,G]/[G,N] = \mathrm{Ker}\gamma$. Finally, from $\alpha : r[F,R] \mapsto r[F,S]$ we have $\mathrm{Im}\beta = R \cap [F,F]/[F,S] = \mathrm{Ker}\beta$.

5.6(b): Applying part(a) to the group extension $1 \to R \hookrightarrow F \twoheadrightarrow G \to 1$ we obtain the exact sequence

$$\begin{tikzcd}
H_2 F \arrow[r, "\alpha"] \arrow[d, "\cong"] & H_2 G \arrow[r, "\beta"] & (H_1 R)_G \arrow[r, "\gamma"] \arrow[d, "\cong"] & H_1 F \arrow[r, "\delta"] \arrow[d, "\cong"] & H_1 G \arrow[r] \arrow[d, "\cong"] & 0 \\
0 & & R/[F,R] & F/[F,F] & G/[G,G] &
\end{tikzcd}$$

where the first vertical isomorphism follows from Example II.4.1[1] and the second vertical isomorphism arises in the solution to part(a). By the First Isomorphism Theorem and exactness of the sequence, $H_2 G \cong H_2 G/\mathrm{Ker}\beta \cong \mathrm{Im}\beta \cong \mathrm{Ker}\gamma = R \cap [F,F]/[F,R]$.

5.7(a): S^3 is a closed orientable 3-manifold, and it has a group structure under quaternion multplication ($S^3 < \mathbb{H}$ as the elements of norm 1). The finite subgroup G of S^3 provides a multiplication action, and it is free because the only solution in $\mathbb{H}$ to the equation $gx = x$ for nontrivial x is $g = 1$. From a result in the solution to Exercise I.4.2, the G-action is a "properly discontinuous" action and so the

quotient map $\rho : S^3 \to S^3/G$ is a regular covering space.
Following a proof by William Thurston, S^3/G is Hausdorff: considering two points α, β of S^3 in distinct orbits, form respective neighborhoods U_α and U_β such that they are disjoint and neither neighborhood contains any translates of α or β (this can be done by the Hausdorff property of S^3). Taking the union K of these two neighborhoods, $Int(K - \bigcup_{g\neq 1} gK)$ yields a neighborhood of α and a neighborhood of β which project to disjoint neighborhoods of $G\alpha$ and $G\beta$ in S^3/G [note: we needed to refine K because if U_α intersects gU_β, then after projecting to the orbit space, $\rho(gU_\beta) = \rho(U_\beta)$ intersects $\rho(U_\alpha)$].
S^3/G is a closed connected 3-manifold since it has S^3 as a covering space (Theorem 26.5[6] provides the compactness, Theorem 23.5[6] provides the connectedness, and the property of evenly-covered neighborhoods provides the nonboundary and manifold structure).
Since the actions $g : S^3 \to S^3$ are fixed-point free, we have $\deg(g) = (-1)^{3+1} = 1$ by Theorem 21.4[4] and hence G acts by orientation-preserving homeomorphisms [note: *orientation* is in terms of local orientations $\mu_x \in H_3(S^3, S^3 - x) \cong H_3(S^3) \cong \mathbb{Z}$, as defined in [3] on pg234]. Local orientations $\mu_{Gx} = \mu_{\rho(x)}$ of $S^3/G = \rho(S^3)$ are given by the images $\Gamma_x(\mu_x) = \Gamma_{gx}(\mu_{gx})$ under the isomorphisms of local homology groups $\Gamma_x : H_3(S^3, S^3 - x) \to H_3(\rho(S^3), \rho(S^3) - \rho(x))$ which arise from the Excision Theorem and the local-homeomorphism property of covering spaces; the 'local consistency condition' follows in the same respect (where a ball B containing Gx and Gy has as preimage under ρ a union of balls, each of which is homeomorphic to B, and such a homeomorphic ball containing g_1x and g_2y provides the local consistency condition for S^3). Therefore, S^3/G is orientable.
Since S^3 is simply-connected, $\pi_1 S^3 = 0$ and so $G \cong \pi_1(S^3/G)/\pi_1(S^3) \cong \pi_1(S^3/G)$. Applying Poincaré Duality and the Universal Coefficient Theorem we obtain $H_2(S^3/G) \cong H^1(S^3/G) \cong \mathrm{Hom}(H_1(S^3/G), \mathbb{Z}) \cong \mathrm{Hom}(G_{ab}, \mathbb{Z}) = 0$ [noting that G is finite]. A theorem of Hopf (Theorem II.5.2[1]) gives us an exact sequence which includes the surjection $H_2(S^3/G) \to H_2G \to 0$, hence $H_2G = 0$.

5.7(b): The binary icosahedral group G (of order 120) is the preimage in S^3 which maps onto the alternating group A_5 under $S^3 \to SO(3)$ [up to isomorphism with the group of icosahedral-rotational symmetries] as explained in [3] on pg75. By part(a), $H_2G = 0$. Consider the group extension $1 \to K \to G \to A_5 \to 1$ where K is the central kernel of order 2 (corresponding to G mapping onto A_5). The associated 5-term exact sequence becomes $0 \to H_2(A_5) \to (H_1K)_{A_5} \to 0$ because $H_1G \cong G_{ab} = 0$, and thus we obtain the isomorphism $H_2(A_5) \cong (H_1K)_{A_5}$. The A_5-action on $H_1K \cong K_{ab} = K \cong \mathbb{Z}_2$ is induced by the conjugation G-action on $K \cong \mathbb{Z}_2$ which is the trivial action (since $K \leq Z(G)$), and therefore $H_2(A_5) \cong H_1K \cong \mathbb{Z}_2$.

5.7(c): Consider the abstract group $G = \langle x, y, z \ ; \ x^2 = y^3 = z^5 = xyz \rangle$ which is a finite presentation with the same number of generators as relations. We show that G is perfect ($G = [G, G]$) so that $H_1G \cong G_{ab} = 0$ and $H_2G = 0$ by Exercise 5.5(b) above. Now $G/[G, G]$ is an abelian group with the relations $2\bar{x} = 3\bar{y} = 5\bar{z} = \bar{x} + \bar{y} + \bar{z}$. From this we see that $\bar{x} = \bar{y} + \bar{z}$, so we need not look at $\bar{x}$. Subsequently, $2\bar{y} + 2\bar{z} = 3\bar{y} \Rightarrow \bar{y} = 2\bar{z}$ and so we need not look at $\bar{y}$. Finally, $5\bar{z} = 3(2\bar{z}) = 6\bar{z} \Rightarrow \bar{z} = 0$ and so all generators vanish, i.e. $G/[G, G] = 0$.
Alternatively, the commutator subgroup is $[G, G] \subseteq G$, and to prove the opposite inclusion it suffices to show that x, y, z all lie in $[G, G] \equiv G'$.

$$x^2 = xyz \Rightarrow \overbrace{x = yz} \Rightarrow x^2 = yzyz = y^3 \Rightarrow z^{-1}yzy^{-1} = z^{-2}y \Rightarrow \overbrace{z^2[z^{-1}, y] = y}$$

The two overbraced equations allow us to finish by showing that $z \in G'$.

$$yzyz = y^3 \Rightarrow y^2 = zyz \Rightarrow [y, z] = z^{-1}yz^{-1} \Rightarrow y = z[y, z]z$$

We then have $z^5 = xyz \Rightarrow z^3 = xyz^{-1} = yz \cdot z[y, z]z \cdot z^{-1} = z[y, z]z \cdot z^2[y, z] \Rightarrow z^2 = [y, z]z^3[y, z] = z^3z^{-3} \cdot [y, z]z^3[y, z] \Rightarrow z^{-1} = g[y, z] \Rightarrow \overbrace{z = [y, z]^{-1}g^{-1}} \in G'$ with $g = z^{-3}[y, z]z^3 \in G'$ by normality of the commutator subgroup.

With the cyclic subgroup $C = \langle xyz \rangle$ we have $G/C = A_5$ and hence the group extension $1 \to C \to G \to A_5 \to 1$ $[A_5 = \langle x, y, z \ ; \ x^2 = y^3 = z^5 = xyz = 1 \rangle$ of order 60]. The associated 5-term exact sequence now yields the isomorphism $H_2(A_5) \cong H_1C = C$, noting the trivial A_5-action (since C is generated by a central element) and noting the cyclicity $C_{ab} = C$. Thus, by part(b) we deduce that $|C| = 2$ and hence $|G| = 2 \cdot 60 = 120$. *In fact, $G = SL_2(\mathbb{F}_5)$ is the binary icosahedral group!*

6.1: Given $N \triangleleft G$, let F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and consider the complex F_N. Since a projective $\mathbb{Z}G$-module is also projective as a $\mathbb{Z}H$-module for any subgroup $H \subseteq G$ (by Exercise I.8.2), F is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}N$ and so $H_*(F_N) = H_*N$. By Exercise II.2.3(a), F_N is a complex of G/N-modules and so $H_*(F_N)$ inherits a G/N-action, with $F_N \to F_N$ given by $x \mapsto (gN)x = gx$. For Corollary II.6.3 we have the conjugation action $\alpha : N \to N$ given by $n \mapsto gng^{-1}$ (for $g \in G$), and we have the augmentation-preserving N-chain map $\tau : F \to F$ given by $x \mapsto gx$ [it commutes with the boundary operator ∂ of F since ∂ is equivariant, and it satisfies the condition $\tau(nx) = gnx = gng^{-1}gx = \alpha(n)\tau(x)$]. By Proposition II.6.2[1], if α is conjugation by $g \in N$ then $H_*(\alpha)$ is the identity (hence trivial action), and so $\tau' : F_N \to F_N$ is given by $\tau'(x) = (gN)x = gx$ which agrees with the above map.

6.2: For any finite set A let $\Sigma(A)$ be the group of permutations of A. For $|A| \leq |B|$, choose an injection $i : A \hookrightarrow B$ and consider the injection $\Sigma(A) \hookrightarrow \Sigma(B)$ obtained by extending a permutation on A to be the identity on $B - iA$. In order to show that the induced map $H_*\Sigma(A) \to H_*\Sigma(B)$ is independent of the choice of i, it suffices to show that any two injections i_1 and i_2 give conjugate maps $\bar{i}_1, \bar{i}_2 : \Sigma(A) \hookrightarrow \Sigma(B)$, because the conjugation map $\Sigma(B) \to \Sigma(B)$ induces the identity map on homology $H_*\Sigma(B) \to H_*\Sigma(B)$ by Proposition II.6.2[1]. Let τ be the permutation which takes $i_1(a)$ to $i_2(a)$ for all $a \in A$ and is an arbitrary permutation $(B - i_1A) \to (B - i_2A)$. Then for a permutation $\bar{i}_1(\tilde{\sigma}) = \sigma \in \Sigma(B)$, the permutation $\tau\sigma\tau^{-1}$ is equal to $\bar{i}_2(\tilde{\sigma})$. Thus $\bar{i}_1$ and $\bar{i}_2$ are conjugates of each other by τ, and the result follows.

6.3(a): Given the homomorphism $\alpha : G \to G'$, the n-tuples in $C_n(G)$ are sent to the n-tuples in $C_n(G')$ coordinate-wise via α, where $[gh] \mapsto [\alpha(gh)] = [\alpha(g)\alpha(h)]$. Thus $H_1(\alpha)$ maps $\bar{g}$ to $\overline{\alpha(g)}$, where $\bar{g}$ denotes the homology class of the cycle $[g]$. We also have the explicit isomorphism $H_1G \to G_{ab}$ given by $\bar{g} \mapsto g \bmod [G, G]$. It is immediate that we have the commutative diagram

$$\begin{tikzcd} H_1G \arrow[r, "\cong"] \arrow[d, "H_1(\alpha)"'] & G_{ab} \arrow[d, "\alpha^*"] \\ H_1G' \arrow[r, "\cong"] & G'_{ab} \end{tikzcd}$$

with $\alpha^* : g \bmod [G, G] \mapsto \alpha(g) \bmod [G', G']$, which is precisely the map obtained from α by passage to the quotient. Thus, the isomorphism $H_1(\ ) \cong (\ )_{ab}$ is natural.

6.3(b): Suppose $G = F/R$ and $G' = F'/R'$ with $F = F(S)$ and $F' = F'(S')$ free, and suppose $\alpha : G \to G'$ lifts to $\tilde{\alpha} : F \to F'$. Let Y and $\tilde{Y}$ be associated to the presentation of G as in Exercise II.5.3(a), and similarly for Y' and $\tilde{Y}'$ with G'. Now $\tilde{\alpha}$ yields a map $Y \to Y'$ that sends the combinatorial path $s = S^1_s$ to the combinatorial path $\tilde{\alpha}(s)$. ******Incomplete******

7.1: Consider $G = G_1 *_A G_2$ with $\alpha_k : A \to G_k$ not necessarily injective, and let $\bar{G}_1 = \beta_1(G_1)$, $\bar{G}_2 = \beta_2(G_2)$, $\bar{A} = \beta_1\alpha_1(A) = \beta_2\alpha_2(A)$ be the images of G_1, G_2, A in G [where $\beta_k : G_k \to G$ arises from the amalgamation diagram for G]. Form the amalgam $H = \bar{G}_1 *_{\bar{A}} \bar{G}_2$ and the commutative diagram:

$$\begin{tikzcd} \bar{A} \arrow[r, "i_2"] \arrow[d, "i_1"'] & \bar{G}_2 \arrow[d, "j_2"'] \arrow[ddr, bend left, "\gamma_2"] & \\ \bar{G}_1 \arrow[r, "j_1"'] \arrow[drr, bend right, "\gamma_1"'] & H \arrow[dr, "\varphi"] & \\ & & G \end{tikzcd}$$

where γ_k is the natural inclusion and i_k is the obvious injection [subsequently, we have $\gamma_1 i_1 = \gamma_2 i_2$ and hence the unique map φ from the universal mapping property].

We also have the commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "\alpha_2"] \arrow[d, "\alpha_1"'] & G_2 \arrow[d, "\beta_2"'] \arrow[dr, "r_2"] & \\
G_1 \arrow[r, "\beta_1"'] \arrow[dr, "r_1"'] & G \arrow[dr, "\phi"] & \bar{G}_2 \arrow[d, "j_2"] \\
& \bar{G}_1 \arrow[r, "j_1"'] & H
\end{tikzcd}$$

where r_k is β_k with the codomain restricted to form the inclusion [subsequently, $j_1 r_1 \alpha_1(a) = j_1 \beta_1 \alpha_1(a) = j_1 i_1 \beta_1 \alpha_1(a) = j_2 i_2 \beta_1 \alpha_1(a) = j_2 i_2 \beta_2 \alpha_2(a) = j_2 r_2 \alpha_2(a)$ and hence we have the unique map ϕ from the universal mapping property].
From the diagrams (and dropping subscripts), $\phi\beta = jr$ and $\varphi j = \gamma$.
$\varphi\phi(\beta(g)) = \gamma(r(g)) = \gamma(\beta(g)) = \beta(g)$
$\phi\varphi(j[\beta(g)]) = \phi\gamma[\beta(g)] = \phi(\beta(g)) = j(r(g)) = j(\beta(g))$
Thus $\varphi\phi = id_G$ and $\phi\varphi = id_H$, and so $\varphi = \phi^{-1}$ is an isomorphism ($H \cong G$).
Consequently, any amalgamated free product is isomorphic to one in which the maps $A \to G_k$ are injective.

7.2:

7.3: Consider the Special Linear Group $SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ which is the subgroup of all 2x2 integral matrices with determinant 1. Applying the Mayer-Vietoris sequence for groups and using the fact that $H_*(\mathbb{Z}_k)$ is trivial in positive even dimensions and is isomorphic to $\mathbb{Z}_k$ in positive odd dimensions, we get the exact sequence $0 \to H_{2n}(SL_2(\mathbb{Z})) \xrightarrow{\alpha} \mathbb{Z}_2 \xrightarrow{\beta} \mathbb{Z}_4 \oplus \mathbb{Z}_6 \xrightarrow{\gamma} H_{2n-1}(SL_2(\mathbb{Z})) \to 0$.
Noting that the only nontrivial map $\varphi : \mathbb{Z}_2 = \langle t \rangle \to \mathbb{Z}_4 = \langle s \rangle$ is the canonical embedding defined by $t \mapsto s^2$, we assert that the induced map under H_{2n-1} is the same embedding: Considering the two periodic free resolutions of $\mathbb{Z}$, there exists an augmentation-preserving chain map f between them by Theorem I.7.5[1] and we have a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}[\mathbb{Z}_2] \arrow[r, "1+t"] \arrow[d, "f_{n+1}"] & \mathbb{Z}[\mathbb{Z}_2] \arrow[r, "t-1"] \arrow[d, "f_n"] & \mathbb{Z}[\mathbb{Z}_2] \arrow[d] \\
\mathbb{Z}[\mathbb{Z}_4] \arrow[r, "1+s+s^2+s^3"] & \mathbb{Z}[\mathbb{Z}_4] \arrow[r, "s-1"] & \mathbb{Z}[\mathbb{Z}_4]
\end{tikzcd}$$

The left-side square yields $(1+s+s^2+s^3)f_{n+1}(1) = f_n(1+t) = f_n(1)+\varphi(t)f_n(1) = (1+s^2)f_n(1)$, and by exactness of the bottom row we have $0 = (s-1)(1+s^2)f_n(1) = (s^3-s^2+s-1)f_n(1) \Rightarrow f_n(1) = 1+s$, hence $(1+s+s^2+s^3)f_{n+1}(1) = (1+s^2)(1+s) = 1+s+s^2+s^3 \Rightarrow f_{n+1}(1) = 1$. Then after moving to quotients, the cycle elements (for odd-dimensional homology) are mapped via $\varphi_*(1) = 1+1 = 2$ while the boundary elements are mapped via $\varphi_*(1) = 1$, and the result follows [applies to all odd n, and $f_0(1) = 1$].
In general, *an injection $H \hookrightarrow G = H \times K$ onto a direct summand will pass to an injection under any covariant functor T* because the composition identity $H \hookrightarrow G \twoheadrightarrow H$ yields the identity $T(H) \xrightarrow{j} T(H \times K) \to T(H)$ which implies j is injective [this can be applied to $H = \mathbb{Z}_2$ and $G = \mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$].
Thus we have $\beta(t) = (s_1^2, s_2^3)$ injective and so $H_{2n}(SL_2(\mathbb{Z})) \cong \mathrm{Im}\alpha = \mathrm{Ker}\beta = 0$. From the MV-sequence we see that $H_{2n-1}(SL_2(\mathbb{Z}))$ is a finite abelian group of order dividing $|\mathbb{Z}_4 \oplus \mathbb{Z}_6| = 24 = 2^3 \cdot 3$, hence contains only 2-torsion and 3-torsion by the Primary Decomposition Theorem. Considering the 3-torsion in the MV-sequence, $0 \to 0 \to 0 \oplus \mathbb{Z}_3 \to H_{2n-1}(SL_2(\mathbb{Z}))_{(3)} \to 0$, we have $H_{2n-1}(SL_2(\mathbb{Z}))_{(3)} = \mathbb{Z}_3$ by exactness [this row can be extracted from the MV-sequence because finitely generated abelian groups are direct sums of their Sylow p_i-subgroups (by the Primary Decomposition Theorem) and maps between the abelian groups will send primary components (the Sylow p_i-subgroups) to respective primary components]. For 2-torsion we consider the 2-torsion subgroup and its MV-sequence, and we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\cong"'] & \mathbb{Z}_2 \arrow[r] \arrow[d, "\cong"'] & \mathbb{Z}_4 \oplus \mathbb{Z}_2 \arrow[r] \arrow[d, "\cong"'] & H_{2n-1}(\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_2 = \mathbb{Z}_4) \cong \mathbb{Z}_4 \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d, "\cong"'] & 0 \arrow[d, "\cong"'] \\
0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & \mathbb{Z}_4 \oplus \mathbb{Z}_2 \arrow[r] & H_{2n-1}(SL_2(\mathbb{Z}))_{(2)} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Then by the Five-Lemma, ψ is an isomorphism and so $H_{2n-1}(SL_2(\mathbb{Z}))_{(2)} = \mathbb{Z}_4$. Thus, $H_{2n-1}(SL_2(\mathbb{Z})) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_{12}$.

$$\Rightarrow \quad H_i(SL_2(\mathbb{Z})) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}_{12} & i \text{ odd} \\ 0 & i \text{ even} \end{cases}$$

3 Chapter III: Homology and Cohomology with Coefficients

0.1: Let F be a flat $\mathbb{Z}G$-module and M a G-module which is $\mathbb{Z}$-torsion-free (ie. $\mathbb{Z}$-flat), and consider the tensor product $F \otimes M$ with diagonal G-action. Since $(F \otimes M) \otimes_G - = (F \otimes M \otimes -)_G = F \otimes_G (M \otimes -)$, it suffices to show that $X = F \otimes_G (M \otimes -)$ is an exact functor so that $F \otimes M$ is $\mathbb{Z}G$-flat (by Corollary 10.5.41[2]). But by the same corollary $M \otimes -$ is $\mathbb{Z}$-exact (and a G-module) and $F \otimes_G -$ is $\mathbb{Z}G$-exact, so X is exact and the result follows.

0.2: Let F be a projective $\mathbb{Z}G$-module and M a $\mathbb{Z}$-free G-module, and consider the tensor product $F \otimes M$ with diagonal G-action. Since $\mathrm{Hom}_G(F \otimes M, -) \cong \mathrm{Hom}(F \otimes M, -)^G \cong \mathrm{Hom}(F, \mathrm{Hom}(M, -))^G \cong \mathrm{Hom}_G(F, \mathrm{Hom}(M, -))$ where the second isomorphism is adjoint associativity (Theorem 10.5.43[2]), it suffices to show that $X = \mathrm{Hom}_G(F, \mathrm{Hom}(M, -))$ is an exact functor so that $F \otimes M$ is $\mathbb{Z}G$-projective (by Corollary 10.5.32[2]). But by the same corollary $\mathrm{Hom}(M, -)$ is $\mathbb{Z}$-exact (and a G-module) and $\mathrm{Hom}_G(F, -)$ is $\mathbb{Z}G$-exact, so X is exact and the result follows.

1.1(a): For a finite group G and a G-module M we have the norm map $\overline{N} : M_G \to M^G$ induced from the map $M \to M$ which is multiplication by the norm element $N = \sum_{g \in G} g$. Noting that $Nm = |G|m$ for both $m \in M_G$ and $m \in M^G$, we see that $|G| \cdot \mathrm{Ker}\overline{N} = 0$ (as $\overline{N}m = Nm/\sim = 0$ by definition of kernel) and $|G| \cdot \mathrm{Coker}\overline{N} = 0$ (as $\mathrm{Coker}\overline{N} = M^G/NM$ and $Nm \bmod NM = 0$).

1.1(b): Suppose M is an induced module ($M = \mathbb{Z}G \otimes A$) where A is an abelian group and G acts by $g \cdot (r \otimes a) = gr \otimes a$. Then $M_G = (\mathbb{Z}G)_G \otimes A = \mathbb{Z} \otimes A$ and $M^G = (\mathbb{Z}G)^G \otimes A = \mathbb{Z} \cdot N \otimes A$, where N is the norm element. The norm map $\overline{N} : M_G \to M^G$ is now given by $z \otimes a \mapsto zN \otimes a$ (for $z \in \mathbb{Z}$) which is clearly a bijection. It is an isomorphism because $\overline{N}[z_1 \otimes a_1 + z_2 \otimes a_2] = Nz_1 \otimes a_1 + Nz_2 \otimes a_2 = z_1 N \otimes a_1 + z_2 N \otimes a_2 = \overline{N}[z_1 \otimes a_1] + \overline{N}[z_2 \otimes a_2]$.

1.1(c): Let M be a projective $\mathbb{Z}G$-module; it is a direct summand of a free module $\mathfrak{F} = \bigoplus_i \mathbb{Z}G$. By application of part(b) above with $A = \bigoplus_i \mathbb{Z}$ we see that $\overline{N}$ is an isomorphism for $\mathfrak{F}$ because $\mathfrak{F} = \bigoplus_i \mathbb{Z}G = \bigoplus_i (\mathbb{Z}G \otimes \mathbb{Z}) = \mathbb{Z}G \otimes (\bigoplus_i \mathbb{Z})$. Now $(M \oplus N)_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} (M \oplus N) \cong (\mathbb{Z} \otimes_{\mathbb{Z}G} M) \oplus (\mathbb{Z} \otimes_{\mathbb{Z}G} N) \cong M_G \oplus N_G$, and $(M \oplus N)^G = M^G \oplus N^G$ under the coordinate-wise G-action (of $\mathfrak{F}$) since $g \cdot (m, n) = (g \cdot m, g \cdot n) = (m, n) \Rightarrow g \cdot m = m\,,\, g \cdot n = n$. Thus $M_G \oplus N_G \cong M^G \oplus N^G$, and since the norm map is bilinear we have $M_G \cong M^G$.

1.2: Using the standard cochain complex, an element of $C^1(G, M)$ is a function $f : G \to M$, and under the coboundary map it is sent to $(\delta f)(g, h) = g \cdot f(h) - f(gh) + f(g)$. The kernel of this map consists of functions which satisfy $f(gh) = f(g) + g \cdot f(h)$, and these are derivations, so $Z^1(G, M) \cong \mathrm{Der}(G, M)$. Since an element of $C^0(G, M)$ is simply $m \in M$, and under the coboundary map it is sent to $(\delta m)(g) = g \cdot m - m$, which is a principal derivation, we have $B^1(G, M) \cong \mathrm{PDer}(G, M)$. Thus, $H^1(G, M) \cong \mathrm{Der}(G, M)/\mathrm{PDer}(G, M)$.
If G acts trivially on M then $\mathrm{PDer}(G, M) = 0$ and $\mathrm{Der}(G, M) = \mathrm{Hom}(G, M)$, so $H^1(G, M) \cong \mathrm{Hom}(G, M) = \mathrm{Hom}(G_{ab}, M) = \mathrm{Hom}(H_1 G, M)$, where the second-to-last equality comes from the fact that any group homomorphism from G to an abelian group factors through the commutator subgroup $[G, G]$ by Proposition 5.4.7[2]. *In particular, $H^1(G) = 0$ for any finite group G.*

1.3: Let A be an abelian group with trivial G-action, and let $F \to \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Then $F \otimes_G A = (F \otimes A)_G \cong \mathbb{Z} \otimes_G (F \otimes A)$, and since the diagonal G-action on $F \otimes A$ is simply the left G-action on F (since G acts trivially on A), we can apply tensor associativity (Theorem 10.4.14[2]) to obtain $\mathbb{Z} \otimes_G (F \otimes A) \cong (\mathbb{Z} \otimes_G F) \otimes A \cong F_G \otimes A$. Thus there is a universal coefficient sequence $0 \to H_n(G) \otimes A \to H_n(G, A) \to \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(G), A) \to 0$ by Proposition I.0.8[1]. Also, $\mathcal{H}om_G(F, A) = \mathcal{H}om(F, A)^G \cong \mathcal{H}om(F_G, A)$, where the last isomorphism arises because $(gu)(m) = g \cdot u(g^{-1}m) = u(g^{-1}m)$ and so we must have $g^{-1}m = m \in F$ for $gu = u$. Thus there is also a universal coefficient sequence $0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(G), A) \to H^n(G, A) \to \mathrm{Hom}(H_n(G), A) \to 0$ by Proposition I.0.8[1].

1.4(a): "Let $f : C' \to C$ be a weak equivalence between arbitrary complexes, and let Q be a non-negative cochain complex of injectives. Then the map $\mathcal{H}om_R(f,Q) : \mathcal{H}om_R(C,Q) \to \mathcal{H}om_R(C',Q)$ is a weak equivalence."
To prove this, note that the mapping cone $C'' = C \oplus \Sigma C'$ of f is acyclic by Proposition I.0.6[1], where $(\Sigma C')_p = C'_{p-1}$ is the 1-fold suspension of C. The mapping cone of $\mathcal{H}om_R(f,Q)$ is $\mathcal{H}om_R(C'',Q)$ because

$$\mathcal{H}om_R(C',Q)_n \oplus \Sigma\mathcal{H}om_R(C,Q)_n = \prod_q \mathrm{Hom}_R(C'_q, Q_{q+n}) \oplus \prod_q \mathrm{Hom}_R(C_q, Q_{q+n-1}) = \\ \prod_q \mathrm{Hom}_R(C'_q \oplus C_{q+1}, Q_{q+n}) = \mathcal{H}om_R(C' \oplus \Sigma^{-1}C, Q)_n = \mathcal{H}om_R(C'',Q)_n$$

noting that $C' \oplus \Sigma^{-1}C = (C'_q \oplus C_{q+1}) = (C'_{q-1} \oplus C_q) = \Sigma C' \oplus C = C''$. Thus it suffices to show that $\mathcal{H}om_R(C'',Q)$ is acyclic (by Proposition I.0.6[1]), i.e., that $H_n(\mathcal{H}om_R(C'',Q)) \equiv [C'',Q]_n = 0 \ \ \forall\, n \in \mathbb{Z}$. By the uniqueness part of the result of Exercise I.7.4, $[C'',Q]_n \equiv [\Sigma^n C'', Q]$ is indeed 0, since any map on Q is zero in negative dimensions (so all extensions off of that zero map are homotopy equivalent).

1.4(b): Let $\varepsilon : F \to \mathbb{Z}$ be a projective resolution and let $\eta : M \to Q$ be an injective resolution. By part(a) above and noting that ε is a weak equivalence (regarded as a chain map with M concentrated in dimension 0), we have a weak equivalence $\mathcal{H}om_R(F,Q) \leftarrow \mathcal{H}om_R(\mathbb{Z},Q)$. Similarly, by Theorem I.8.5[1] we have a weak equivalence $\mathcal{H}om_R(F,M) \to \mathcal{H}om_R(F,Q)$.
In particular, $H^*(G,M) = H^*(Q^G)$ because $H^*(G,M) = H^*(\mathcal{H}om_G(F,M))$ and $H^*(\mathcal{H}om_G(F,M)) \cong H^*(\mathcal{H}om_G(\mathbb{Z},Q)) = H^*(\mathcal{H}om(\mathbb{Z},Q)^G)$, noting that $\mathcal{H}om_{\mathbb{Z}}(\mathbb{Z},Q) \cong Q$.

2.1: Given projective resolutions $F \to M$ and $P \to N$ of arbitrary G-modules M and N, there is an isomorphism of graded modules $\Gamma : F \otimes_G P \overset{\cong}{\to} P \otimes_G F$ given by $f \otimes p \mapsto (-1)^{\mathrm{deg} f \cdot \mathrm{deg} p} p \otimes f$, where we consider diagonal G-action on $F_i \otimes P_j \cong P_j \otimes F_i$ [generally, $\mathrm{deg} x = n$ for $x \in C_n$]. If we show that Γ (a degree 0 map) is a chain map, then it is a homotopy equivalence (hence a weak equivalence) and so $\mathrm{Tor}^G_*(M,N) = H_*(F \otimes_G P) \cong H_*(P \otimes_G F) = \mathrm{Tor}^G_*(N,M)$. Denote by d and d' the boundary operators of F and P, respectively, and denote by D and D' the boundary operators of $F \otimes_G P$ and $P \otimes_G F$, respectively. Then

$$D'\Gamma(f \otimes p) = D'[(-1)^{\mathrm{deg} f \cdot \mathrm{deg} p} p \otimes f] = (-1)^{\mathrm{deg} f \cdot \mathrm{deg} p}(d'p \otimes f + (-1)^{\mathrm{deg} p} p \otimes df) \\ = (-1)^{\mathrm{deg} f \cdot \mathrm{deg} p} d'p \otimes f + (-1)^{\mathrm{deg} p \cdot (\mathrm{deg} f + 1)} p \otimes df$$

and

$$\Gamma D(f \otimes p) = \Gamma[df \otimes p + (-1)^{\mathrm{deg} f} f \otimes d'p] \\ = (-1)^{\mathrm{deg} p \cdot (\mathrm{deg} f - 1)} p \otimes df + (-1)^{\mathrm{deg} f + \mathrm{deg} f \cdot (\mathrm{deg} p - 1)} d'p \otimes f \\ = (-1)^{\mathrm{deg} p \cdot (\mathrm{deg} f + 1)} p \otimes df + (-1)^{\mathrm{deg} f \cdot \mathrm{deg} p} d'p \otimes f$$

Thus $D'\Gamma = \Gamma D$ and so Γ is a chain map.
*(This simultaneously provides a solution to **Exercise I.0.5**)*

3.1: Let P be a projective R-module, and let $\mathfrak{C}$ be a short exact sequence of S-modules which can be regarded as R-modules via restriction of scalars. Since P is projective, $\mathrm{Hom}_R(P,\mathfrak{C})$ is a short exact sequence, and so it suffices to show that the isomorphism of functors $\mathrm{Hom}_S(S \otimes_R P, -) \overset{\cong}{\to} \mathrm{Hom}_R(P,-)$ is natural [because then $\mathrm{Hom}_S(S \otimes_R P, \mathfrak{C})$ is a short exact sequence which implies that $S \otimes_R P$ is a projective S-module]. Given a module homomorphism $\psi : M \to N$, we must check commutativity of the diagram

$$\begin{tikzcd} \mathrm{Hom}_S(S \otimes_R P, M) \arrow[r, "\alpha"] \arrow[d, "\varphi_1"] & \mathrm{Hom}_S(S \otimes_R P, N) \arrow[d, "\varphi_2"] \\ \mathrm{Hom}_R(P,M) \arrow[r, "\beta"] & \mathrm{Hom}_R(P,N) \end{tikzcd}$$

where α and β are given by $f \mapsto \psi \circ f$, and φ_i $(i = 1,2)$ is given by $f \mapsto f \circ i$ under the universal mapping property with $i : P \to S \otimes_R P \ \ , \ \ i(p) = 1 \otimes p$. Now $\varphi_2[\alpha(F)] = \varphi_2[\psi \circ F] = (\psi \circ F) \circ i$ and $\beta[\varphi_1(F)] = \beta[F \circ i] = \psi \circ (F \circ i) = (\psi \circ F) \circ i$. Therefore, $\varphi_2 \alpha = \beta \varphi_1$ and the result follows:
Extension of scalars takes projective R-modules to projective S-modules.

3.2: Let Q be an injective R-module, and let $\mathfrak{C}$ be a short exact sequence of S-modules which can be regarded as R-modules via restriction of scalars. Since Q is injective, $\mathrm{Hom}_R(\mathfrak{C},Q)$ is a short exact sequence,

and so it suffices to show that the isomorphism of functors $\mathrm{Hom}_S(-, \mathrm{Hom}_R(S,Q)) \overset{\cong}{\to} \mathrm{Hom}_R(-,Q)$ is natural [because then $\mathrm{Hom}_S(\mathfrak{C}, \mathrm{Hom}_R(S,Q))$ is a short exact sequence which implies that $\mathrm{Hom}_R(S,Q)$ is an injective S-module]. Given a module homomorphism $\psi : M \to N$, we must check commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Hom}_S(N, \mathrm{Hom}_R(S,Q)) & \xrightarrow{\alpha} & \mathrm{Hom}_S(M, \mathrm{Hom}_R(S,Q)) \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} \\ \mathrm{Hom}_R(N,Q) & \xrightarrow{\beta} & \mathrm{Hom}_R(M,Q) \end{array}$$

where α and β are given by $f \mapsto f \circ \psi$, and φ_i $(i = 1,2)$ is given by $f \mapsto \pi \circ f$ under the universal mapping property with $\pi : \mathrm{Hom}_R(S,Q) \to Q$, $\pi(f) = f(1)$. Now $\varphi_2[\alpha(F)] = \varphi_2[F \circ \psi] = \pi \circ (F \circ \psi)$ and $\beta[\varphi_1(F)] = \beta[\pi \circ F] = (\pi \circ F) \circ \psi = \pi \circ (F \circ \psi)$. Therefore, $\varphi_2 \alpha = \beta \varphi_1$ and the result follows:
Co-extension of scalars takes injective R-modules to injective S-modules.

3.3: Given S which is flat as a right R-module, let Q be an injective S-module, let $\mathfrak{C}$ be a short exact sequence of S-modules, and consider $\mathrm{Hom}_R(\mathfrak{C}, Q)$ where $\mathfrak{C}$ and Q are regarded as R-modules via restriction of scalars. Since S is R-flat, $S \otimes_R \mathfrak{C}$ is a short exact sequence, and since Q is S-injective, $\mathrm{Hom}_S(S \otimes_R \mathfrak{C}, Q)$ is a short exact sequence. It suffices to show that the isomorphism of functors $\mathrm{Hom}_S(S \otimes_R -, Q) \overset{\cong}{\to} \mathrm{Hom}_R(-,Q)$ is natural [because then $\mathrm{Hom}_R(\mathfrak{C}, Q)$ is a short exact sequence which implies that Q is an injective R-module]. Given a module homomorphism $\psi : M \to N$, we must check commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Hom}_S(S \otimes_R N, Q) & \xrightarrow{\alpha} & \mathrm{Hom}_S(S \otimes_R M, Q) \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} \\ \mathrm{Hom}_R(N,Q) & \xrightarrow{\beta} & \mathrm{Hom}_R(N,Q) \end{array}$$

where α is given by $f \mapsto f \circ (S \otimes_R \psi)$, β is given by $f \mapsto f \circ \psi$, φ_1 is given by $f \mapsto f \circ i_N$ for the natural map $i_N : N \to S \otimes_R N$, and φ_2 is given similarly by $f \mapsto f \circ i_M$. Now $\varphi_2[\alpha(F)] = \varphi_2[F \circ (S \otimes_R \psi)] = (F \circ (S \otimes_R \psi)) \circ i_M = F \circ ((S \otimes_R \psi) \circ i_M)$ and $\beta[\varphi_1(F)] = \beta[F \circ i_N] = (F \circ i_N) \circ \psi = F \circ (i_N \circ \psi)$. Also, $i_N[\psi(m)] = 1 \otimes \psi(m) = (S \otimes_R \psi)(1 \otimes m) = (S \otimes_R \psi)[i_M(m)]$. Therefore, $\varphi_2 \alpha = \beta \varphi_1$ and the result follows:
Restriction of scalars takes injective S-modules to injective R-modules if S is a flat right R-module.

3.4: Given S which is projective as a left R-module, let P be a projective S-module, let $\mathfrak{C}$ be a short exact sequence of S-modules, and consider $\mathrm{Hom}_R(P, \mathfrak{C})$ where $\mathfrak{C}$ and P are regarded as R-modules via restriction of scalars. Since S is R-projective, $\mathrm{Hom}_R(S, \mathfrak{C})$ is a short exact sequence, and since P is S-projective, $\mathrm{Hom}_S(P, \mathrm{Hom}_R(S, \mathfrak{C}))$ is a short exact sequence. It suffices to show that the isomorphism of functors $\mathrm{Hom}_S(P, \mathrm{Hom}_R(S, -)) \overset{\cong}{\to} \mathrm{Hom}_R(P, -)$ is natural [because then $\mathrm{Hom}_R(P, \mathfrak{C})$ is a short exact sequence which implies that P is a projective R-module]. Given a module homomorphism $\psi : M \to N$, we must check commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Hom}_S(P, \mathrm{Hom}_R(S,M)) & \xrightarrow{\alpha} & \mathrm{Hom}_S(P, \mathrm{Hom}_R(S,N)) \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} \\ \mathrm{Hom}_R(P,M) & \xrightarrow{\beta} & \mathrm{Hom}_R(P,N) \end{array}$$

where α is given by $f \mapsto \phi \circ f$ for $\phi(g) = \psi \circ g$ $(g : S \to M)$, β is given by $f \mapsto \psi \circ f$, φ_1 is given by $f \mapsto \pi_M \circ f$ under the universal mapping property with $\pi : \mathrm{Hom}_R(P,M) \to M$, $\pi_M(f) = f(1)$, and φ_2 is given similarly by $f \mapsto \pi_N \circ f$. Now $\varphi_2[\alpha(F)] = \varphi_2[\phi \circ F] = \pi_N \circ (\phi \circ F) = (\pi_N \circ \phi) \circ F$ and $\beta[\varphi_1(F)] = \beta[\pi_M \circ F] = \psi \circ (\pi_M \circ F) = (\psi \circ \pi_M) \circ F$. Also, $\pi_N[\phi(f)] = \pi_N[\psi \circ f] = (\psi \circ f)(1) = \psi[f(1)] = \psi[\pi_M(f)]$. Therefore, $\varphi_2 \alpha = \beta \varphi_1$ and the result follows:
Restriction of scalars takes projective S-modules to projective R-modules if S is a projective left R-module.

4.1: Let $R = \mathbb{Z}/n\mathbb{Z}$, and note that the ideals of R are the ideals $I_x = (x) \bmod(n)$ for $x|n \in \mathbb{Z}$ by the 4th Isomorphism Theorem. It suffices to show that every map $\varphi : I \to R$ extends to a map $R \to R$ so that R is self-injective by Baer's Criterion (Proposition III.4.1[1]). Given I_x and $\varphi(x \bmod(n)) = r \bmod(n)$, we can write $n = yx$ so that $\varphi(yx \bmod(n)) = 0$. But $\varphi(yx \bmod(n)) = y\varphi(x \bmod(n)) \bmod(n) = yr \bmod(n)$

and thus $yr = mn = myx \;\Rightarrow\; r = xm$. Then, since $\varphi(x \bmod(n)) = [x \bmod(n)][m \bmod(n)]$ we can extend φ to the domain R by setting $\varphi(1 \bmod(n)) = m \bmod(n)$.

(a): Let A be an abelian group such that $nA = 0$, and let $C \subseteq A$ be a cyclic subgroup of order $|C| = n$. We can regard A as an R-module because $x^n = x^0 \in R$ and R acts on A by $x^i \cdot a = ia$. As C is a [self-injective] subgroup of A, we have an inclusion $C \hookrightarrow A$ of an injective R-module into an R-module. By definition of "injective module" (pg.782-783 of [5], statement XX.4.I1), every exact sequence of modules $0 \to Q \to M \to M' \to 0$ splits for injective Q, hence Q is a direct summand of M. Therefore, C is a direct summand of A.

(b): Given A as above (i.e. an arbitrary abelian group of finite exponent), we regard A as an R-module. Since n is minimal (to annihilate A) there exists an element of order n and hence a cyclic subgroup $C \cong R$ of order n in A. If we can show that $A = A' \oplus A''$ for A' of smaller exponent then by induction on n we have that A' is a direct sum of cyclic groups; thus it remains to show that A'' is a direct sum of modules (each isomorphic to R) and is a direct summand of A. An application of Zorn's Lemma on the set of direct sums with summands in A isomorphic to R (noting that the set is nonempty because it contains C) provides the maximal element A''; we can use this lemma because an upper bound for any chain would be the direct sum of those elements (the direct sums) in that chain. A ring is Noetherian iff every ideal is finitely generated (by Theorem 15.1.2[2]); thus $\mathbb{Z}$ is Noetherian (being a Principal Ideal Domain). Now $R = \mathbb{Z}_n$ is also Noetherian, because a quotient of a Noetherian ring by an ideal is Noetherian (by Proposition 15.1.1[2]). It is a fact that a ring is Noetherian iff an arbitrary direct sum of injective modules (over that ring) is injective. Thus $A'' = \bigoplus_i R$ is R-injective, so A'' is a direct summand of A, and A is a direct sum of cyclic groups.
This result is known as Prufer's Theorem for abelian groups.

5.1: For any H-module M consider the G-module $\mathrm{Ind}_H^G M = \bigoplus_{g \in G/H} gM$ where this equality follows from Proposition III.5.1[1]. The summand gM is a gHg^{-1}-module and hence gHg^{-1} is the isotropy group of this summand in $\mathrm{Ind}_H^G M$. By Proposition III.5.3[1], $\mathrm{Ind}_H^G M \cong \mathrm{Ind}_{gHg^{-1}}^G gM$.
In particular, by Proposition III.5.6[1] we have the K-isomorphism

$$\bigoplus_{g \in E} \mathrm{Ind}_{K \cap gHg^{-1}}^K \mathrm{Res}_{K \cap gHg^{-1}}^{gHg^{-1}} gM \cong \bigoplus_{gg' \in E} \mathrm{Ind}_{K \cap gg'H(gg')^{-1}}^K \mathrm{Res}_{K \cap gg'H(gg')^{-1}}^{gg'H(gg')^{-1}} gg'M$$

Thus the K-module $\mathrm{Ind}_{K \cap gHg^{-1}}^K \mathrm{Res}_{K \cap gHg^{-1}}^{gHg^{-1}} gM$ depends up to isomorphism only on the class of $g \in E$ in $K \backslash G / H$.

5.2(a): For any H-module M and G-module N consider the tensor product $N \otimes \mathrm{Ind}_H^G M$ which has the diagonal G-action. By Proposition III.5.1[1] and the fact that tensor products commute with direct sums, we have $N \otimes \mathrm{Ind}_H^G M \cong N \otimes (\bigoplus_{g \in G/H} gM) \cong \bigoplus_{g \in G/H} (N \otimes gM)$ which has $N \otimes M$ as a direct summand in the underlying abelian group. Treating this as an H-module $\mathrm{Res}_H^G N \otimes M$ with a diagonal action so that H is its isotropy group, we have $N \otimes \mathrm{Ind}_H^G M \cong \mathrm{Ind}_H^G(\mathrm{Res}_H^G N \otimes M)$ by Proposition III.5.3[1].
In particular, for $M = \mathbb{Z}$ we have $N \otimes \mathbb{Z}[G/H] \cong \mathrm{Ind}_H^G \mathrm{Res}_H^G N$.

5.2(b): For any H-module M and G-module N consider $U = \mathrm{Hom}(\mathrm{Ind}_H^G M, N)$ which has the "diagonal" G-action given by $(gu)(m) = g \cdot u(g^{-1}m)$. By Proposition III.5.1[1] and the fact that the Hom-functor "commutes" with direct sums/products, we have $U \cong \mathrm{Hom}(\bigoplus gM, N) \cong \prod \mathrm{Hom}(gM, N)$, where the indices on the sum/product symbols are implicitly the coset representatives in G/H. Thus U admits a direct product decomposition $(\pi_g : U \twoheadrightarrow \mathrm{Hom}(gM, N))$. Using the denotation $\pi_g(u) = u_g$, we have $[(\pi_g g_0)(u)](m) = [\pi_g(g_0 u)](m) = g_0 \cdot u_g(g_0^{-1}m) = g_0 \cdot u_{g_0^{-1}g}(m) = g_0 \cdot [\pi_{g_0^{-1}g}(u)](m)$, and so $\pi_g g_0 \sim \pi_{g_0^{-1}g}$, with $m \in \mathrm{Ind}_H^G M$. This decomposition has $U \twoheadrightarrow \mathrm{Hom}(M, N)$ as one of the surjections in the underlying abelian group. Treating this surjection as an H-module $\pi_1 : U \twoheadrightarrow \mathrm{Hom}(M, \mathrm{Res}_H^G N)$ so that H is its isotropy group, we have $\mathrm{Hom}(\mathrm{Ind}_H^G M, N) \cong \mathrm{Coind}_H^G \mathrm{Hom}(M, \mathrm{Res}_H^G N)$ by Proposition III.5.8[1].
An analogous proof will provide $\mathrm{Hom}(N, \mathrm{Coind}_H^G M) \cong \mathrm{Coind}_H^G \mathrm{Hom}(\mathrm{Res}_H^G N, M)$.

5.3: Let F be a projective G-module and M a $\mathbb{Z}$-free G-module, and consider the tensor product $F \otimes M$ with diagonal G-action. By Corollary III.5.7[1], $\mathbb{Z}G \otimes M$ is a free G-module since M is free as a $\mathbb{Z}$-module.

Since F is projective, it is a direct summand of a free G-module $\mathfrak{F} = \bigoplus_i \mathbb{Z}G$, so that $\mathfrak{F} = F \oplus K$. As the tensor product commutes with arbitrary direct sums (Corollary XVI.2.2[5]), $\mathfrak{F} \otimes M = \bigoplus_i (\mathbb{Z}G \otimes M)$ and hence is a free G-module. Finally, $\mathfrak{F} \otimes M = (F \oplus K) \otimes M = (F \otimes M) \oplus (K \otimes M)$ and so $F \otimes M$ is a direct summand of $\mathfrak{F} \otimes M$, hence G-projective.
Note that this gives a new proof of the result of ***Exercise III.0.2*** *above.*

5.4(a): If $|G : H| = \infty$ then there are infinitely many distinct coset representatives in G/H. Now $\mathrm{Ind}_H^G M = \bigoplus_{g \in G/H} gM$ by Proposition III.5.1[1], and it has a transitive G-action which permutes the summands. Consider an arbitrary [nontrivial] element $x = \sum_{i=1}^N g_i m_i$ with all $m_i \neq 0$ (this also refers to the sum over all coset representatives with cofinitely many $m_i = 0$). We may take the summand entry $g_1 m_1$ of x and a summand $g'M$ which doesn't appear in the representation of x (i.e. $g' \neq g_j$ for $1 \leq j \leq N$), and there then exists $g'' \in G$ such that $g'' \cdot g_1 m_1 = g'm$ because the G-action is transitive on the summands. Thus x is not fixed by G (since $g'' \cdot x \neq x$), and so $(\mathrm{Ind}_H^G M)^G = 0$.

Note that if the index is finite, then this result does not hold. For instance, take $2\mathbb{Z} \subset \mathbb{Z} = \langle x \rangle$ which has index $|\mathbb{Z} : 2\mathbb{Z}| = |\mathbb{Z}_2| = 2$. Then $\mathrm{Ind}_{2\mathbb{Z}}^{\mathbb{Z}} M = M \oplus xM$ where x is the coset representative of $x(2\mathbb{Z})$ which generates $\mathbb{Z}_2$. Since $x^{2i} \cdot (m_1, xm_2) = (x^{2i} m_1, x(x^{2i} m_2))$, we must have $(\mathrm{Ind}_{2\mathbb{Z}}^{\mathbb{Z}} M)^{\mathbb{Z}} \subseteq M^{2\mathbb{Z}} \oplus xM^{2\mathbb{Z}}$. Since $x^{2j+1} \cdot (\bar{m}_1, x\bar{m}_2) = x \cdot (\bar{m}_1, x\bar{m}_2) = (\bar{m}_2, x\bar{m}_1)$, we must have $\bar{m}_1 = \bar{m}_2$ and hence $(\mathrm{Ind}_{2\mathbb{Z}}^{\mathbb{Z}} M)^{\mathbb{Z}} = \{(\bar{m}, x\bar{m}) \mid \bar{m} \in M^{2\mathbb{Z}}\}$. For $M = \mathbb{Z}_4$ with $2\mathbb{Z}$-action defined as $x^{2i} \cdot m = (-1)^i m$, the largest submodule on which $2\mathbb{Z}$ acts trivially [so that $-m = m$] is $\mathbb{Z}_2$. Thus $(\mathrm{Ind}_{2\mathbb{Z}}^{\mathbb{Z}} \mathbb{Z}_4)^{\mathbb{Z}} \neq 0$ is our desired example.

5.4(b):

5.4(c): Assume statement (i), so that there is a finitely generated subgroup $G' \subseteq G$ such that $|G' : G' \cap gHg^{-1}| = \infty$ for all $g \in G$ (with $H \subseteq G$). Using the analogue of Proposition III.5.6[1] for coinduction and passing to G'-coinvariants, we obtain $(\mathrm{Res}_{G'}^G \mathrm{Coind}_H^G M)_{G'} \cong (\bigoplus_{g \in E} \mathrm{Coind}_{G' \cap gHg^{-1}}^{G'} \mathrm{Res}_{G' \cap gHg^{-1}}^{gHg^{-1}} gM)_{G'} \cong \bigoplus_{g \in E} (\mathrm{Coind}_{G' \cap gHg^{-1}}^{G'} M')_{G'}$ where this latter isomorphism follows from commutativity of the tensor product with direct sums (and E is the set of representatives for the double cosets KgH). It is a fact that if G is finitely generated and $|G : H| = \infty$ then $(\mathrm{Coind}_H^G M)_G = 0$ for any H-module M [part(b) above], so by statement (i) we have $(\mathrm{Res}_{G'}^G \mathrm{Coind}_H^G M)_{G'} = \bigoplus 0 = 0$ for any H-module M. Noting that restriction of scalars gives the action $g' \cdot x = \varphi(g')x$ where φ is a map $G' \to G$, we have $(\mathrm{Coind}_H^G M)_G \subseteq (\mathrm{Res}_{G'}^G \mathrm{Coind}_H^G M)_{G'}$. Thus $(\mathrm{Coind}_H^G M)_G = 0$ and (i) implies (ii).
Assume statement (ii), so that $(\mathrm{Coind}_H^G M)_G = 0$ for all H-modules M. Then in particular (for $M = \mathbb{Z}$) there is only one element of $(\mathrm{Coind}_H^G \mathbb{Z})_G$, and that element must be zero, so (ii) implies (iii).
Assume statement (iii), so that the element of $(\mathrm{Coind}_H^G \mathbb{Z})_G$ represented by the augmentation map $\varepsilon \in \mathrm{Coind}_H^G \mathbb{Z}$ is zero. We first note that [in general] if $\bar{n}_0 = 0 \in N_G$ for some $n_0 \in N$ then it is also zero in $N_{G'}$ for some finitely generated subgroup $G' \subseteq G$; this is because $N_G = N/\langle gn - n \rangle$ and so n_0 can be written as a finite $\mathbb{Z}$-linear combination of elements of the form $gn - n$, which implies that we can take G' to be the subgroup generated by those specific g's. In particular, $\bar{\varepsilon} = 0 \in (\mathrm{Coind}_H^G \mathbb{Z})_{G'}$ for some finitely generated subgroup G' of G. Using the double coset formula (analogue of Proposition III.5.6[1]) and treating $\mathbb{Z}$ and other modules appropriately over specific groups (to ignore restriction), we must have $\bar{\varepsilon}_g = 0 \in (\mathrm{Coind}_{G' \cap gHg^{-1}}^{G'} \mathbb{Z})_{G'}$ for all $g \in G$, where ε_g denotes the component of the augmentation map in the specific summand of coinduction. Now if $|G' : G' \cap gHg^{-1}| < \infty$ then $\mathrm{Coind}_{G' \cap gHg^{-1}}^{G'} \mathbb{Z} \cong \mathrm{Ind}_{G' \cap gHg^{-1}}^{G'} \mathbb{Z} \cong \mathbb{Z}[G'/G' \cap gHg^{-1}]$, giving $\varepsilon_g = \sum_{g' \in K} g' \otimes \varepsilon_g(g') = (\sum g') \otimes 1$ where K is the set of coset representatives for the quotient $G'/G' \cap gHg^{-1}$. Then $g''\varepsilon_g = \varepsilon_g \ \forall\, g'' \in G'$, so $\bar{\varepsilon}_G \neq 0 \in (\mathrm{Coind}_{G' \cap gHg^{-1}}^{G'} \mathbb{Z})_{G'}$. Thus we must have $|G' : G' \cap gHg^{-1}| = \infty$, so (iii) implies (i).

5.5: Let G be a finite group and let k be a field, and consider the free module kG. We have $kG \cong kG \otimes_k k = \mathrm{Ind}_{\{1\}}^G k \cong \mathrm{Coind}_{\{1\}}^G k = \mathrm{Hom}_k(kG, k)$, where the second-to-last equation follows from the analogue over k of Proposition III.5.9[1] since $|G| < \infty$. Then $\mathrm{Hom}_{kG}(-, kG) \cong \mathrm{Hom}_{kG}(-, \mathrm{Hom}_k(kG, k)) \cong \mathrm{Hom}_k(-, k)$ where the last isomorphism follows from the universal property of co-induction. It is a fact that every k-vector space is an injective k-module [if the vector space W with basis $\mathfrak{B}$ is a subspace of a vector space V, then we can extend $\mathfrak{B}$ to a basis of V and then $V = W \oplus U$ where U is the vector space spanned by the additional basis vectors extended from $\mathfrak{B}$]; thus k is injective $\Rightarrow \mathrm{Hom}_k(-, k)$ is exact $\Rightarrow \mathrm{Hom}_{kG}(-, kG)$ is exact $\Rightarrow kG$ is self-injective as a kG-module.

A Noetherian ring is a commutative ring which satisfies the Ascending Chain Condition on ideals (i.e. no infinite increasing chain of ideals), and any field k is Noetherian because the only ideals are $\{0\}$ and k (giving $\{0\} \subseteq k$) by Proposition 7.4.9[2]. Now kG is Noetherian (for G finite) because it is a finite-dimensional k-vector space and so any infinite ascending chain of subspaces would require cofinitely many of those subspaces to have dimension greater than $|G|$, a contradiction; since ideals of kG are necessarily k-subspaces, the result follows. It is a fact that a ring R is Noetherian iff an arbitrary direct sum of injective R-modules is injective. Thus the free module $\mathfrak{F} = \bigoplus_i kG$ is injective and so any projective kG-module is kG-injective (because a projective module is a direct summand of a free module, and a direct summand of an injective module is injective).
Assuming the claim is true that any kG-module is a submodule of a kG-projective module, then by definition of "injective" it follows that any injective kG-module is a direct summand of a kG-projective module, hence kG-projective; it remains to prove this claim. We have a canonical kG-module injective map $M \to \mathrm{Hom}_{kH}(kG, M)$ where the kG-module M can be regarded as a kH-module by restriction of scalars (see pg64 of [1]). But $\mathrm{Hom}_{kH}(kG, M) = \mathrm{Coind}_H^G M \cong \mathrm{Ind}_H^G M = kG \otimes_{kH} M$, where this second-to-last equation follows from the analogue over k of Proposition III.5.9[1] since $|G| < \infty$. Using $H = \{1\}$, this says that M is a submodule of $kG \otimes_k M$. But M is treated as a k-vector space ($M \cong \bigoplus_i k$), so $kG \otimes_k M \cong \bigoplus_i (kG \otimes_k k) \cong \bigoplus_i kG$; i.e. $kG \otimes_k M$ is a free [hence projective] kG-module.

6.1(a): We first note that *an arbitrary direct sum of projective resolutions is projective*, which follows from the fact that an arbitrary direct sum of projective modules is projective and from the exactness of the row for each summand. We then note that *homology commutes with direct sums*, and this follows from the obvious facts $\mathrm{Ker}(\bigoplus_i d_i) = \bigoplus_i \mathrm{Ker}(d_i)$ and $\mathrm{Im}(\bigoplus_i d_i) = \bigoplus_i \mathrm{Im}(d_i)$. From these two facts and noting that the tensor product commutes with direct sums, we see that *the Tor-functor commutes with direct sums*,

$$\mathrm{Tor}_*^R(M, \bigoplus_i N_i) = H_*(M \otimes_R (\bigoplus_i N_i)) \cong H_*(\bigoplus_i (M \otimes_R N_i)) \cong \bigoplus_i H_*(M \otimes_R N_i) = \bigoplus_i \mathrm{Tor}_*^R(M, N_i).$$

For the amalgamation $G = G_1 *_A G_2$ consider the short exact sequence of permutation modules $0 \to \mathbb{Z}[G/A] \to \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2] \to \mathbb{Z} \to 0$. By Proposition III.6.1[1] we have the long exact sequence

$$\begin{tikzcd}
\arrow[r] & H_n(G, \mathbb{Z}[G/A]) \arrow[r] \arrow[d, "\cong"] & H_n(G, \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2]) \arrow[r] \arrow[d, "\cong"] & H_n(G, \mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & {} \\
& H_n(A) & H_n(G_1) \oplus H_n(G_2) & H_n(G) &
\end{tikzcd}$$

where the vertical isomorphisms follow from the fact $H_*(H) = H_*(G, \mathbb{Z}[G/H])$, and the middle isomorphim utilizes commutativity of direct sums which follows from above because $H_*(G, -) = \mathrm{Tor}_*^G(\mathbb{Z}, -)$. This long exact sequence is the Mayer-Vietoris sequence for the amalgam G.

6.1(b): For the amalgamation $G = G_1 *_A G_2$ consider the short exact sequence of permutation modules $0 \to \mathbb{Z}[G/A] \to \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2] \to \mathbb{Z} \to 0$. Applying $- \otimes M$ still yields a short exact sequence because all of the modules are free $\mathbb{Z}$-modules (we consider the sequence of permutation modules as a free resolution of $\mathbb{Z}$, hence a homotopy equivalence by Corollary I.7.6[1]); this homotopy equivalence for $\mathbb{Z}$ gives a homotopy equivalence for $\mathbb{Z} \otimes M = M$ (since functors preserve identities), hence a weak equivalence (which can be stated as a resolution). By Proposition III.5.6[1] we have $\mathbb{Z}[G/A] \otimes M \cong \mathrm{Ind}_A^G \mathrm{Res}_A^G M$, and by Shapiro's Lemma we apply $H_*(G, -)$ to obtain $H_*(G, \mathrm{Ind}_A^G \mathrm{Res}_A^G M) \cong H_*(A, \mathrm{Res}_A^G M)$. Similar results follow for the other modules, and so by Proposition III.6.1[1] the exact sequence (which resulted from the sequence of permutation modules after application of $- \otimes M$) yields a long exact sequence [the Mayer-Vietoris sequence for homology with coefficients]

$$H_n(A, \mathrm{Res}_A^G M) \longrightarrow H_n(G_1, \mathrm{Res}_{G_1}^G M) \oplus H_n(G_2, \mathrm{Res}_{G_2}^G M) \longrightarrow H_n(G, M)$$

Now consider the original sequence of permutation modules, but instead apply $\mathrm{Hom}(-, M)$ which still yields a short exact sequence (same reason as mentioned above). By the analogue over co-induction of Proposition III.5.6[1] (or the result of Exercise III.5.2(a) above) we have $\mathrm{Hom}(\mathbb{Z}[G/A], M) \cong \mathrm{Coind}_A^G \mathrm{Res}_A^G M$, and by Shapiro's Lemma we apply $H^*(G, -)$ to obtain $H^*(G, \mathrm{Coind}_A^G \mathrm{Res}_A^G M) \cong H^*(A, \mathrm{Res}_A^G M)$. Similar results follow for the other modules, and so by Proposition III.6.1[1] the exact sequence (which resulted from the sequence of permutation modules after application of $\mathrm{Hom}(-, M)$ yields a long exact sequence [the Mayer-Vietoris sequence for cohomology with coefficients]

$$H^n(G,M) \longrightarrow H^n(G_1, \mathrm{Res}^G_{G_1} M) \oplus H^n(G_2, \mathrm{Res}^G_{G_2} M) \longrightarrow H^n(A, \mathrm{Res}^G_A M)$$

7.1(a): Consider the exact sequence $\cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} M \to 0$, where each C_i is H_*-acyclic. We can apply the dimension-shifting technique using the short exact sequences $0 \to \mathrm{Ker}\varepsilon \to C_0 \to M \to 0$ and $0 \to \mathrm{Ker}\partial_i \to C_i \to \mathrm{Ker}\partial_{i-1} \to 0$ to obtain the isomorphism $H_n(G,M) \cong H_1(G, \mathrm{Ker}\partial_{n-2}) \cong \mathrm{Ker}\{H_0(G, \mathrm{Ker}\partial_{n-1}) \to H_0(G, C_{n-1})\} = \mathrm{Ker}\{(\mathrm{Ker}\partial_{n-1})_G \to (C_{n-1})_G\}$. Now consider the diagram below concerning C_G

$$\begin{tikzcd}
(C_{n+1})_G \arrow[rr, "\bar{\partial}_{n+1}"] \arrow[dr, two heads, "\gamma_1"] & & (C_n)_G \arrow[rr, "\bar{\partial}_n"] \arrow[dr, two heads, "\beta"] & & (C_{n-1})_G \\
& (Z_n)_G \arrow[ur, "\alpha"] & & (Z_{n-1})_G \arrow[ur, "\gamma_2"] &
\end{tikzcd}$$

with $Z_i = \mathrm{Ker}\partial_i$, noting that the composition $\beta\alpha$ is exact [right-exactness of $(-)_G$ on $0 \to \mathrm{Ker}\partial_n \to C_n \to \mathrm{Ker}\partial_{n-1} \to 0$] and $\mathrm{Ker}\gamma_2 \cong H_n(G,M)$. Thus $\mathrm{Im}\bar{\partial}_{n+1} = \mathrm{Im}\alpha = \mathrm{Ker}\beta$ (since γ_1 is surjective) and so $H_n(C_G) = \mathrm{Ker}\bar{\partial}_n/\mathrm{Ker}\beta$. Noting that $(Z_{n-1})_G = (C_n)_G/\mathrm{Ker}\beta$ by the First Isomorphism Theorem, take the kernel of γ_2 [denoted $K/\mathrm{Ker}\beta$] and take its preimage under β to obtain K in $(C_n)_G$. Since K maps to zero under $\gamma_2\beta = \bar{\partial}_n$ we have $K \subseteq \mathrm{Ker}\bar{\partial}_n$. We also have $\mathrm{Ker}\bar{\partial}_n \subseteq K$ [hence they are equal] because by commutativity of the maps $\mathrm{Ker}\bar{\partial}_n$ maps to $K/\mathrm{Ker}\beta$ and hence lies in K. Thus $\mathrm{Ker}\bar{\partial}_n/\mathrm{Ker}\beta = \mathrm{Ker}\gamma_2$ and so $H_*(G,M) \cong H_*(C_G)$.

7.1(b): Consider the exact sequence $0 \to M \to C^0 \xrightarrow{\delta_0} C^1 \xrightarrow{\delta_1} \cdots$, where each C^i is H^*-acyclic. We can apply the dimension-shifting technique using the short exact sequences $0 \to M \to C_0 \to \mathrm{Ker}\delta_0 \to 0$ and $0 \to \mathrm{Ker}\delta_{i-1} \to C_i \to \mathrm{Ker}\delta_i \to 0$ to obtain the isomorphism $H^n(G,M) \cong H^1(G, \mathrm{Ker}\delta_{n-2}) \cong \mathrm{Coker}\{H^0(G, C_{n-1}) \to H^0(G, \mathrm{Ker}\delta_{n-1})\} = \mathrm{Coker}\{(C_{n-1})^G \to (\mathrm{Ker}\delta_{n-1})^G\}$. Using a similar approach as in part(a) above, we see that $H^*(G,M) \cong H^*(C^G)$.

7.2: This will reprove Proposition III.2.2[1] on isomorphic functors.

Method 1: Let M be $\mathbb{Z}$-torsion-free, so that $M \otimes -$ is an exact functor (M is $\mathbb{Z}$-flat). Then $H_*(G, M \otimes -)$ is a homological functor because given a short exact sequence of modules $\mathcal{C}$, $M \otimes \mathcal{C}$ is a short exact sequence and $F \otimes_G (M \otimes \mathcal{C})$ is a short exact sequence of chain complexes (F is projective, hence flat), so the corresponding long exact homology sequence gives us the desired property (by Lemma 24.1[4] and Theorem 24.2[4]). Similarly, $\mathrm{Tor}^G_*(M,-) = H_*(F' \otimes_G -)$ is a homological functor, where $F' \to M$ is a projective resolution. Both functors are effaceable [erasible] in positive dimensions, since the chain complexes $F \otimes_G (M \otimes P)$ and $F' \otimes_G P$ are exact for P projective. In dimension 0, $H_0(G, M \otimes N) \cong \mathbb{Z} \otimes_G (M \otimes N) \cong (M \otimes N)_G = M \otimes_G N \cong \mathrm{Tor}^G_0(M,N)$. Therefore, by Theorem III.7.3[1] we have an isomorphism of ∂-functors $H_*(G, M \otimes -) \cong \mathrm{Tor}^G_*(M,-)$. [The case for cohomology is similar].

Method 2: The chain complex associated to the group $\mathrm{Tor}^G_*(M,N)$ is given by $\cdots \to F'_0 \otimes_G N \to M \otimes_G N \to 0$, where $F' \to M$ is a projective resolution. This can be rewritten as $\cdots \to (F'_0 \otimes N)_G \to (M \otimes N)_G \to 0$ which yields $\mathrm{Tor}^G_*(M,N) = H_*(C_G)$, where C is the chain complex $(F'_i \otimes N)$. This is indeed an exact sequence because the universal coefficient theorem yields $H_*(F' \otimes N) = H_*(F') \otimes N = N$ (nontrivial only in dimension 0). Now F'_i is projective and hence a summand of a free module $\mathfrak{F} = F'_i \oplus K = \bigoplus_j \mathbb{Z}G$. Then $H_*(G, \mathfrak{F} \otimes N) \cong H_*(G, F'_i \otimes N) \oplus H_*(G, K \otimes N)$ and $H_*(G, \mathfrak{F} \otimes N) \cong \bigoplus_j H_*(G, \mathbb{Z}G \otimes N) = 0$, noting that induced modules are H_*-acyclic by Corollary III.6.6[1]. Thus $H_*(G, F'_i \otimes N) = 0$ and each $F'_i \otimes N$ is H_*-acyclic. We can now apply Exercise III.7.1(a) which implies $\mathrm{Tor}^G_*(M,N) = H_*(C_G) \cong H_*(G, M \otimes N)$. [The case for cohomology is similar].

7.3: For dimension-shifting in homology, we can choose the induced module $\overline{M} = \mathbb{Z}G \otimes_{\mathbb{Z}} M$ which maps onto M by $\varphi(r \otimes m) = rm$; it is an H_*-acyclic module by Corollary III.6.6[1]. This map is $\mathbb{Z}$-split because it composes with the natural map $i : M \to \overline{M}$ to give the identity, $m \mapsto 1 \otimes m \mapsto 1m = m$.
For dimension-shifting in cohomology, we can choose the coinduced module $\overline{\overline{M}} = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G, M)$ which provides the embedding $M \to \overline{\overline{M}}$ given by $m \mapsto (r \mapsto rm)$; it is an H^*-acyclic module by Corollary III.6.6[1]. This map is $\mathbb{Z}$-split because it composes with the natural map $\pi : \overline{\overline{M}} \to M$ to give the identity, $m \mapsto (r \mapsto rm) \mapsto [r \mapsto rm](1) = 1m = m$.

8.1: Let H be a central subgroup of G and let M be an abelian group with trivial G-action. Then the isomorphism $c(g) : (H, M) \to (gHg^{-1}, M)$ becomes the identity on (H, M) given by $(h \mapsto ghg^{-1} = h, m \mapsto gm = m)$. By Corollary III.8.2[1], this conjugation action of G on (H, M) induces an action of G/H on $H_*(H, M)$ given by $gH \cdot z = c(g)_* z$. Letting $\alpha : H \to gHg^{-1} = H$ denote the group map of $c(g)$, and letting F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, we can choose the chain map $\tau : F \to F$ to be the identity, since it satisfies the condition $\tau(hx) = hx = \alpha(h)x = \alpha(h)\tau(x)$. Thus the chain map $F \otimes_H M \to F \otimes_H M$ given by $x \otimes m \mapsto x \otimes gm = x \otimes m$ is the identity, and so the induced map $c(g)_*$ is the identity on $H_*(H, M)$ which gives the trivial G/H-action $gH \cdot z = c(g)_* z = z$. Similarly, by Corollary III.8.4[1], we have an induced action of G/H on $H^*(H, M)$ given by $gH \cdot z = (c(g)^*)^{-1} z$. Using the same chain map τ, we have the cochain map $\mathcal{H}om_H(F, M) \to \mathcal{H}om_H(F, M)$ given by $f \mapsto [x \mapsto gf(x) = f(x)]$ which is the identity. Thus the induced map $c(g)^*$ is the identity on $H^*(H, M)$ which gives the trivial G/H-action $gH \cdot z = (c(g)^*)^{-1} z = z$.
Alternatively, the trivial G/H-action follows immediately from the fact that functors preserve identities, where H_* and H^* are the functors in question and $c(g)$ is the identity map in question.

8.2: Let $\alpha : H \hookrightarrow G$ be an inclusion, let M be an H-module, let $i : M \to \mathrm{Ind}_H^G M$ be the canonical H-map $i(m) = 1 \otimes m$, and let $\pi : \mathrm{Coind}_H^G M \to M$ be the canonical H-map $\pi(f) = f(1)$. We can take the chain map $\tau : F \to F$ to be the identity (F is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$) since $\tau(hx) = hx = \alpha(h)x = \alpha(h)\tau(x)$ and F can be regarded as a free resolution over $\mathbb{Z}H$.
Consider $(\alpha, i)_*$ on the chain level, induced by the map $F \otimes_H M \to F \otimes_G \mathrm{Ind}_H^G M$ given by $x \otimes m \mapsto x \otimes (1 \otimes m)$. This map is a homotopy equivalence because we can use the universal property of tensor products to define its inverse $x \otimes (g \otimes m) = xg \otimes (1 \otimes m) \mapsto xg \otimes m$, the composite map being $x \otimes (g \otimes m) \mapsto xg \otimes m \mapsto xg \otimes (1 \otimes m) = x \otimes g \cdot (1 \otimes m) = x \otimes (g \otimes m)$. In particular we have a weak equivalence which yields the isomorphism $H_*(H, M) \cong H_*(G, \mathrm{Ind}_H^G M)$ of Shapiro's Lemma given by $(\alpha, i)_*$.
Now consider $(\alpha, \pi)^*$ on the cochain level, induced by the map $\mathcal{H}om_H(F, M) \to \mathcal{H}om_G(F, \mathrm{Coind}_H^G M)$ given by $[x \mapsto f(x)] \mapsto [x \mapsto (g \mapsto gf(x))]$ with $g \in \mathbb{Z}G$. This map is a homotopy equivalence because we can use the universal property of co-induction to define its inverse $[x \mapsto (g \mapsto gf(x))] \mapsto [x \mapsto (1 \mapsto f(x))] = [x \mapsto f(x)]$. In particular we have a weak equivalence which yields the isomorphism $H^*(H, M) \cong H^*(G, \mathrm{Coind}_H^G M)$ of Shapiro's Lemma given by $(\alpha, \pi)^*$.

9.1: Considering homology, let $x \otimes_H m$ represent $z \in H_*(H, M)$. Computing $\mathrm{cor}_H^G z$ on the chain level yields $x \otimes_H m \mapsto x \otimes_G m$, while computing $\mathrm{cor}_{gHg^{-1}}^G gz$ on the chain level yields $gx \otimes_{gHg^{-1}} gm \mapsto gx \otimes_G gm = x \otimes_G m$. Since the images are equal, $\mathrm{cor}_{gHg^{-1}}^G gz = \mathrm{cor}_H^G z$.
Considering homology, let $x \otimes_G m$ represent $z \in H_*(G, M)$. Computing $\mathrm{res}_{gHg^{-1}}^G z$ on the chain level yields $x \otimes_G m \mapsto \sum_{g' \in gHg^{-1} \backslash G} g' \cdot (x \otimes_{gHg^{-1}} m)$. Computing $g \cdot \mathrm{res}_H^G z$ on the chain level yields $x \otimes_G m \mapsto \sum_{g' \in H \backslash G} g' \cdot (x \otimes_H m) \mapsto \sum_{g' \in H \backslash G} (gg') \cdot (x \otimes_{gHg^{-1}} m) = \sum_{g'' \in gHg^{-1} \backslash G} g'' \cdot (x \otimes_{gHg^{-1}} m)$, where this last equality arises from $g(Hg') = gHg^{-1}gg' = (gHg^{-1})g''$ with $g'' = gg'$ being the coset representative. Since the images are equal (the sums are the same for all coset representatives), $g \cdot \mathrm{res}_H^G z = \mathrm{res}_{gHg^{-1}}^G z$.
Considering cohomology, let f_G represent $z \in H^*(G, M)$. Computing $g \cdot \mathrm{res}_H^G z$ on the chain level yields $[x \mapsto f_G(x)] = [x \mapsto g^{-1} f_G(gx)] \mapsto [x \mapsto g^{-1} f_H(gx)] \mapsto [x \mapsto gg^{-1} f_{gHg^{-1}}(g^{-1}gx)] = [x \mapsto f_{gHg^{-1}}(x)]$, while computing $\mathrm{res}_{gHg^{-1}}^G z$ on the chain level yields $[x \mapsto f_G(x)] \mapsto [x \mapsto f_{gHg^{-1}}(x)]$. Since the images are equal, $g \cdot \mathrm{res}_H^G z = \mathrm{res}_{gHg^{-1}}^G z$.
Considering cohomology, let f represent $z \in H^*(H, M)$. Computing $\mathrm{cor}_H^G z$ on the chain level yields $f \mapsto \sum_{g' \in G/H} g' \cdot f$, where G acts diagonally on $f \in \mathcal{H}om(F, M)$. Computing $\mathrm{cor}_{gHg^{-1}}^G gz$ on the chain level yields $g \cdot f \mapsto \sum_{g' \in G/gHg^{-1}} (g'g) \cdot f = \sum_{g'' \in G/H} g'' \cdot f$, where this last equality arises from $g'(gHg^{-1})g = g'gH = g''H$ with $g'' = g'g$ being the coset representative. Since the images are equal (the sums are the same for all coset representatives), $\mathrm{cor}_{gHg^{-1}}^G gz = \mathrm{cor}_H^G z$.

9.2: The transfer map $H_1(G) \to H_1(H)$ can be regarded as a map of abelian groups $G_{ab} \to H_{ab}$. If $\bar{g}$ denotes the representative of Hg then $\rho g = g\bar{g}^{-1}$, where $\rho : G \to H$ is the unique map of left H-sets which sends ever coset representative to 1. Now the transfer map is induced by the composite chain map $F(G)_G \xrightarrow{tr} F(G)_H \to F(H)_H$, where the latter map concerns the chain map $\tau : F(G) \to F(H)$ given by $(g_0, g_1) \mapsto (\rho g_0, \rho g_1)$. Using bar notation, this composite chain map is given by $[g] \mapsto \sum_{g' \in E} g'[g] \mapsto$

$\sum_{g'\in E}\tau(g',g'g)=\sum_{g'\in E}(g'\overline{g'}^{-1},\rho(g'g))=\sum_{g'\in E}(1,\rho(g'g))=\sum_{g'\in E}[\rho(g'g)]$, where E is a set of representatives for the right cosets Hg' and we note then that $\overline{g'}=g'$. Note that for homology classes, $[g_1]+[g_2]=[g_1g_2]$ because of the boundary map $\partial_2[g_1|g_2]=[g_2]-[g_1g_2]+[g_1]$; thus (as a homology class) $\sum_{g'\in E}[\rho(g'g)]=[\prod_{g'\in E}\rho(g'g)]$. Using the isomorphism $H_1(G)\to G_{ab}$ given by $[g]\mapsto g \bmod [G,G]$, the transfer map $G_{ab}\to H_{ab}$ is computed as $g \bmod [G,G]\mapsto \prod_{g'\in E} g'g(\overline{g'g})^{-1} \bmod [H,H]$.

Example: The transfer map $\mathbb{Z}\to n\mathbb{Z}$ is multiplication by n, since $x\mapsto [1x(\overline{1x})^{-1}][xx(\overline{xx})^{-1}]\cdots[x^{n-2}x(\overline{x^{n-2}x})^{-1}][x^{n-1}x(\overline{x^{n-1}x})^{-1}]=1\cdot 1\cdots 1\cdot x^n(1)^{-1}=x^n$, where x is the generator of $\mathbb{Z}$.

10.1: The symmetric group $G=S_3$ on three letters is the group of order $3!=6$ whose elements are the permutations of the set $\{1,2,3\}$. The Sylow 3-subgroup is generated by the cycle $(1\,2\,3)$, and a Sylow 2-subgroup is generated by the cycle $(1\,2)$. Noting the semi-direct product representation $S_3=\mathbb{Z}_3\rtimes\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\mathbb{Z}_3$ by conjugation, we have $H^*(S_3)=H^*(S_3)_{(2)}\oplus H^*(S_3)_{(3)}\cong H^*(S_3)_{(2)}\oplus H^*(\mathbb{Z}_3)^{\mathbb{Z}_2}$ by Theorem III.10.3[1]. Now S_3 is the unique nonabelian group of order 6, so $D_6\cong S_3$ and we can use Exercise AE.9 which implies that the $\mathbb{Z}_2$-action on $H_{2i-1}(\mathbb{Z}_3)\cong H^{2i}(\mathbb{Z}_3)$ is multiplication by $(-1)^i$ (we can pass this action to cohomology by naturality of the UCT). Thus $H^n(\mathbb{Z}_3)^{\mathbb{Z}_2}$ is isomorphic to $\mathbb{Z}_3$ for $n=2i$ where i is even, and is trivial for n odd and $n=2i$ where i is odd. Taking any Sylow 2-subgroup $H\cong\mathbb{Z}_2$, Theorem III.10.3[1] states that $H^*(S_3)_{(2)}$ is isomorphic to the set of S_3-invariant elements of $H^*(H)$. In particular we have the monomorphism $H^{2i-1}(S_3)_{(2)}\hookrightarrow H^{2i-1}(H)=0$, so $H^{2i-1}(S_3)_{(2)}=0$. An S_3-invariant element $z\in H^{2i}(H)\cong\mathbb{Z}_2$ must satisfy the equation $\mathrm{res}^H_K z=\mathrm{res}^{gHg^{-1}}_K gz$, where K denotes $H\cap gHg^{-1}$. If $g\in H$ then $gHg^{-1}=H$ and the above condition is trivially satisfied for all z ($hz=z$ by Proposition III.8.1[1]). If $g\notin H$ then $K=\{1\}$ because H is not normal in S_3 and only contains two elements, so the intersection must only contain the trivial element. But then the image of both restriction maps is zero, so the condition is satisfied for all z; thus $H^{2i}(S_3)_{(2)}=\mathbb{Z}_2$. Alternatively, a theorem of Richard Swan states that if G is a finite group such that $\mathrm{Syl}_p(G)$ is abelian and M is a trivial G-module, then $\mathrm{Im}(\mathrm{res}^G_{\mathrm{Syl}_p(G)})=H^*(\mathrm{Syl}_p(G),M)^{N_G(\mathrm{Syl}_p(G))}$. It is a fact that $N_{S_3}(\mathbb{Z}_2)=\mathbb{Z}_2$ (refer to pg51[2]), so taking $G=S_3$ and $H=\mathrm{Syl}_2(S_3)\cong\mathbb{Z}_2$ and $M=\mathbb{Z}$ we have $\mathrm{Im}(\mathrm{res}^{S_3}_H)=(\mathbb{Z}_2)^{\mathbb{Z}_2}=\mathbb{Z}_2$ in the even-dimensional case. Since any invariant is in the image of the above restriction map (by Theorem III.10.3[1]), the result $H^{2i}(S_3)_{(2)}=\mathbb{Z}_2$ follows.

$$\Rightarrow\quad H^n(S_3)\cong\begin{cases}\mathbb{Z} & n=0\\ \mathbb{Z}_6 & n\equiv 0 \bmod 4\,,\ n\neq 0\\ \mathbb{Z}_2 & n\equiv 2 \bmod 4\\ 0 & \text{otherwise}\end{cases}$$

10.2(a): Let H be a subgroup of G of finite index, let C be the double coset HgH, and let $T(C)$ be the endomorphism $H^*(G,f(C))$ of $H^*(H,M)$ where $f(C)$ is the G-endomorphism of $\mathrm{Ind}^G_H M$ given by $1\otimes m\mapsto\sum_{c\in C/H}c^{-1}\otimes cm$. To show that $T(C)z=\mathrm{cor}^H_{H\cap gHg^{-1}}\mathrm{res}^{gHg^{-1}}_{H\cap gHg^{-1}}gz$, it suffices to check this equation in dimension 0 (by Theorem III.7.5[1]). The right side maps $m\in M^H$ to $\sum_{h\in H/H\cap gHg^{-1}}h(gm)=\sum_{hg\in HgH/H}(hg)m=\sum_{g'\in C/H}g'm\in M^H$, where $g'=hg$ as a coset representative. Now $T(C)$ in dimension 0 is given by the composite map

$$H^0(H,M)\xrightarrow{\alpha}H^0(G,\mathrm{Coind}^G_H M)\xrightarrow{\beta}H^0(G,\mathrm{Ind}^G_H M)\xrightarrow{f^*}H^0(G,\mathrm{Ind}^G_H M)\xrightarrow{\beta^{-1}}H^0(G,\mathrm{Coind}^G_H M)\xrightarrow{\alpha^{-1}}H^0(H,M)$$

where α is the Shapiro isomorphism $m\mapsto(s\mapsto m)$, and β is the canonical isomorphism $F\mapsto\sum_{x\in G/H}x\otimes F(x^{-1})$, and f^* is induced by $f(C)$, and α^{-1} is the inverse $F\mapsto F(1)$, and β^{-1} is the inverse $x\otimes m\mapsto\beta^{-1}[x\otimes m](s\cdot x)$ which is sxm if $sx\in H$ and is 0 if $sx\notin H$. This composite is given by

$$T(C):m\mapsto(s\mapsto m)\mapsto\textstyle\sum_{x\in G/H}x\otimes m\mapsto\sum_x\sum_{c\in C/H}x(c^{-1}\otimes cm)=\sum_x\sum_c xc^{-1}\otimes cm\mapsto$$
$$\textstyle\sum_x\sum_c\beta^{-1}[xc^{-1}\otimes cm](s\cdot xc^{-1})\mapsto\sum_x\sum_c\beta^{-1}[xc^{-1}\otimes cm](xc^{-1})\in M^H$$

To simplify this last term, note that the image of $\beta^{-1}[xc^{-1}\otimes cm]$ is nontrivial iff $xc^{-1}\in H\ \Leftrightarrow\ x\in C$, and for each x there is at most one c such that $xc^{-1}\in H$. Thus the double sum reduces to

$\sum_{x\in G/H\,,\,x\in C} xc^{-1}\cdot cm = \sum_{x\in C/H} xm$, and this is precisely the image of the right-side map.
Note: α was determined by noting that any element f of $(\mathrm{Coind}_H^G M)^G$ must satisfy $f(xg) = f(x)$ and hence f is determined by $f(1) = m$. So f is given by $g \mapsto m$, but it must also commute with the H-action which means that $hg \mapsto hm$ and hence $hm = m$, i.e. $m \in M^H$. Thus $\alpha(m) = (s \mapsto m)$, $s \in G$.

10.2(b): If $z \in H^*(H, M)$ is G-invariant where $H \subseteq G$ is of finite index as above, then $T(C)z = \mathrm{cor}^H_{H\cap gHg^{-1}}\mathrm{res}^{gHg^{-1}}_{H\cap gHg^{-1}} gz = \mathrm{cor}^H_{H\cap gHg^{-1}}\mathrm{res}^H_{H\cap gHg^{-1}} z = |H : H\cap gHg^{-1}| = a(C)z$, where the second-to-last equality follows from Proposition III.9.5[1].

10.2(c): Let $X = \{z \in H^*(H, M) \mid T(C)z = a(C)z \quad \forall\ C\}$ where C is any H-H double coset and $a(C) = |C/H| = |H : H\cap gHg^{-1}|$. Since the image of the restriction map res^G_H lies in the set of G-invariant elements of $H^*(H, M)$, and such elements lie in X by part(b) above, we have $\mathrm{Im}(\mathrm{res}^G_H) \subseteq X$. In the situation of Theorem III.10.3[1] and Proposition III.10.4[1], consider the element $w = \mathrm{cor}^G_H z \in H^n(G, M)$ where z is an arbitrary element of X. Then either $H^n(H, M)$ is annihilated by $|H|$ [$H = \mathrm{Syl}_p(G)$] in which case $w \in H^n(G, M)_{(p)}$, or $|G : H|$ is invertible in M [hence in $H^n(G, M)$]. Using Proposition III.9.5[1] we obtain $\mathrm{res}^G_H w = \sum_{g\in H\backslash G/H} \mathrm{cor}^H_{H\cap gHg^{-1}}\mathrm{res}^{gHg^{-1}}_{H\cap gHg^{-1}} gz = \sum_{g\in H\backslash G/H} T(C)z = \sum_{g\in H\backslash G/H} a(C)z = (\sum_{g\in H\backslash G/H} |H : H\cap gHg^{-1}|)z = |G : H|z$. Since either $|G : H|$ is prime to p or is invertible in M, it follows that $z = \mathrm{res}^G_H w'$ where $w' = w/|G : H|$. Thus $X \subseteq \mathrm{Im}(\mathrm{res}^G_H) \;\Rightarrow\; X = \mathrm{Im}(\mathrm{res}^G_H)$.

4 Chapter IV: Low-Dimensional Cohomology and Group Extensions

2.1: If d is a derivation [crossed homomorphism], then $d(1) = d(1 \cdot 1) = d(1) + 1 \cdot d(1) = 2 \cdot d(1)$ and so $d(1) = 0$.

2.2: Let I be the augmentation ideal of $\mathbb{Z}G$ and let $D : G \to I$ be the derivation defined by $g \mapsto g-1$ (this is the principal derivation $G \to \mathbb{Z}G$ corresponding to $1 \in \mathbb{Z}G$). Given any G-module A and any derivation $d : G \to A$, we can extend d to an additive map $\bar{d} : \mathbb{Z}G \to A$ such that $\bar{d}(rs) = \bar{d}r \cdot \varepsilon(s) + r \cdot \bar{d}s$, where ε is the augmentation map and $r, s \in \mathbb{Z}G$. This map is well-defined because $\bar{d}(rs \cdot t) = \bar{d}(rs) \cdot \varepsilon(t) + rs \cdot \bar{d}t = \bar{d}r \cdot \varepsilon(s) \cdot \varepsilon(t) + r \cdot \bar{d}s \cdot \varepsilon(t) + rs \cdot \bar{d}t = \bar{d}r \cdot \varepsilon(st) + r \cdot (\bar{d}s \cdot \varepsilon(t) + s \cdot \bar{d}t) = \bar{d}r \cdot \varepsilon(st) + r \cdot \bar{d}(st) = \bar{d}(r \cdot st)$, and $\bar{d}g = \bar{d}(g1) = \bar{d}g \cdot \varepsilon(1) + g \cdot \bar{d}1 = dg \cdot 1 + g \cdot d1 = dg + g \cdot 0 = dg$. The restriction f of $\bar{d}$ to I is a G-module homomorphism since $f(r \cdot s) = f(r) \cdot \varepsilon(s) + r \cdot f(s) = f(r) \cdot 0 + r \cdot f(s) = r \cdot f(s)$, and $f(g-1) = \bar{d}g - \bar{d}1 = dg - d1 = dg - 0 = dg$, so f is the unique module map $I \to A$ such that $d = fD$. This means D is the universal derivation on G, and $\mathrm{Der}(G, A) \cong \mathrm{Hom}_{\mathbb{Z}G}(I, A)$.

2.3(a): Let $F = F(S)$ be the free group generated by the set S, and consider the F-module A with a family of elements $(a_s)_{s \in S}$. For the set map $S \to A \rtimes F$ defined by $s \mapsto (a_s, s)$, there is a unique extension to a homomorphism $\varphi : F \to A \rtimes F$ by the universal mapping property of F. This is a splitting of $1 \to A \to A \rtimes F \xrightarrow{\pi} F \to 1$ because $\pi\varphi(f) = \pi(df, f) = f$, where $d : F \to A$ is some function which maps $s \in S$ to $a_s \in A$. Since derivations $F \to A$ correspond to splittings of the above group extension, d is the unique derivation such that $ds = a_s \ \ \forall\, s \in S$.

2.3(b): Given any function $\phi : S \to A$ where A is a $\mathbb{Z}F$-module, there is a unique map $d : F \to A$ by part(b) above, hence a unique $\mathbb{Z}F$-module homomorphism $f : I \to A$ such that $fD = d$ by Exercise IV.2.2 above. In particular, I satisfies the universal property of free modules, $\phi = fD|_S$, and so the augmentation ideal I of $\mathbb{Z}F$ is a free $\mathbb{Z}F$-module with basis $(Ds)_{s \in S} = (s-1)_{s \in S}$.
Note that this reproves ***Exercise II.5.3(b)****.*

2.3(c): By part(b) above the universal derivation $D : F \to I$ satisfies $Df = \sum_{s \in S} (\partial f / \partial s) ds$, where I is the augmentation ideal of $\mathbb{Z}F$. By Exercise IV.2.2 any derivation $d : F \to M$ (where M is an F-module) corresponds to a unique F-module map $\varphi : I \to M$ and hence satisfies $df = \varphi(Df) = \sum_{s \in S} (\partial f / \partial s) \varphi(Ds) = \sum_{s \in S} (\partial f / \partial s) ds$, where $\partial f / \partial s$ lies in $\mathbb{Z}F$.
Note that this reproves ***Exercise II.5.3(c)****.*

2.4(a): Let $G = F/R$ where $F = F(S)$ and R is the normal closure of some subset $T \subseteq F$. For any G-module A, derivations $d : G \to A$ correspond to splittings of $1 \to A \to A \rtimes G \to G \to 1$; they are of the form $s(g) \mapsto (dg, g) \in A \rtimes G$. Consider the homomorphism $\varphi : F \to A \rtimes G$ given by $f \mapsto (df, g)$ where g is the image of f under the projection map $p : F \to F/R$, which is the extension of the set map $S \to A \rtimes G$, $s \mapsto (ds, p(s))$, by the universal mapping property of F. Now $r = ftf^{-1} \in R$ is mapped to $\varphi(r) = df + f \cdot (dt + t \cdot df^{-1}) = df + f \cdot dt - ftf^{-1} \cdot df = (1-r) \cdot df + f \cdot dt = dt$, so φ induces a homomorphism $G \to A \rtimes G$ iff $dt = 0 \ \ \forall\, t \in T$ [note: $1 - r = 1 - 1 = 0$ when computing the G-action on $df \in A$]. This homomorphism is a splitting iff d is a derivation, and so derivations $G \to A$ correspond to derivations $d : F \to A$ such that $d(T) = 0$.

2.4(b): From part(a) above and Exercise IV.2.3(c) we see that derivations $G \to A$ correspond to derivations $d : F \to A$ such that $dt = \sum_{s \in S} (\overline{\partial t / \partial s}) ds = 0$ for all $t \in T$, where $\overline{\partial t / \partial s}$ is the image of $\partial t / \partial s$ under $\mathbb{Z}F \to \mathbb{Z}G$ due to the G-action on A (restriction of scalars). By Exercise IV.2.3(a) these correspond to families $(a_s)_{s \in S}$ of elements of A such that $\sum_{s \in S} (\overline{\partial t / \partial s}) a_s = 0$ for all $t \in T$.

2.4(c): The identity map $id_I : I \to I$ [where I is the augmentation ideal of $\mathbb{Z}G$] corresponds to a derivation $d : G \to I$ such that $d\bar{s} = id_I D(\bar{s}) = id_I(\bar{s} - 1) = \bar{s} - 1$ by Exercise IV.2.2, and this corresponds to a family $(\bar{s} - 1)_{s \in S}$ of elements of I such that $\sum_{s \in S} (\overline{\partial t / \partial s})(\bar{s} - 1) = 0$ for all $t \in T$ by part(b) above. Thus there is an exact sequence $\mathbb{Z}G^{(T)} \xrightarrow{\partial_2} \mathbb{Z}G^{(S)} \xrightarrow{\partial_1} I$ where $\partial_2 e_t = \sum_{s \in S} (\overline{\partial t / \partial s}) e_s$ and $\partial_1 e_s = \bar{s} - 1$ (e_t and e_s are basis elements of their respective groups). Now ∂_1 is surjective because

the elements $\bar{s}-1$ generate I as a left $\mathbb{Z}G$-ideal by Exercise I.2.1(b), where $\bar{S}$ is a set of generators for $G = F(S)/R$, and so we have the desired exact sequence $\mathbb{Z}G^{(T)} \xrightarrow{\partial_2} \mathbb{Z}G^{(S)} \xrightarrow{\partial_1} I \to 0$.
Note that this reproves the second part of ***Exercise II.5.3(d)****.*

2.4(d): Since the augmentation ideal I of $\mathbb{Z}F$ is free (by Exercise IV.2.3(b)), we have a free resolution $0 \to I \to \mathbb{Z}F \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ of $\mathbb{Z}$ over $\mathbb{Z}F$. Taking R-coinvariants and noting that $(\mathbb{Z}F)_R \cong \mathbb{Z}[F/R] = \mathbb{Z}G$ by Exercise II.2.1, we obtain a complex $I_R \to \mathbb{Z}G \to \mathbb{Z} \to 0$ whose homology is H_*R because $H_1(R,\mathbb{Z}) \cong \mathrm{Ker}(I_R \to \mathbb{Z}G)$ by the dimension-shifting technique; $\mathbb{Z}F$ is an H_*-acyclic module by Proposition III.6.1[1] since it is free as a $\mathbb{Z}R$-module by Exercise I.8.2, and it maps onto $\mathbb{Z}$ with kernel I. Since I is free with basis $(s-1)_{s\in S}$, $I \cong \mathbb{Z}F^{(S)}$ and we have the exact sequence $0 \to H_1R \xrightarrow{\varphi} \mathbb{Z}G^{(S)} \xrightarrow{\partial_1} \mathbb{Z}G \to \mathbb{Z} \to 0$, where we note that taking coinvariants is a right-exact functor.
Now we can map the standard (bar) resolution of $\mathbb{Z}$ over $\mathbb{Z}R$ to the aforementioned free resolution:

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "\partial_2"] \arrow[d, "\phi_2"'] & F_1 \arrow[r, "\partial_1"] \arrow[d, "\phi_1"'] & F_0 \arrow[r, "\partial_0"] \arrow[d, "\phi_0"'] & \mathbb{Z} \arrow[r] \arrow[d, "id"'] & 0 \\
& 0 \arrow[r] & I \arrow[r, "i"] & \mathbb{Z}F \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

where $\phi_{n>1} = 0$, $\phi_1[r] = r-1 = Dr$, and $\phi_0[\ ] = 1$. This is a commutative diagram because $\phi_0\partial_1[r] = \phi_0(r[\ ]-1) = r-1 = i(r-1) = i\phi_1[r]$ and $\phi_1\partial_2[r_1|r_2] = \phi_1(r_1[r_2] - [r_1r_2] + [r_1]) = r_1(r_2-1) - (r_1r_2-1) + r_1 - 1 = r_1r_2 - r_1 - r_1r_2 + 1 + r_1 - 1 = 0$. By applying the coinvariants functor and noting that $\phi_1|_R = D|_R$ and $H_1R = \mathrm{Ker}(\partial_1)_R/\mathrm{Im}(\partial_2)_R \cong R_{ab}$, we have a commutative diagram

$$\begin{tikzcd}
R \arrow[r, "D|_R"] \arrow[d] & I \arrow[d] \\
H_1R \arrow[r, "\varphi"] & \mathbb{Z}G^{(S)}
\end{tikzcd}$$

where the vertical arrows are quotient maps. The composite $F \xrightarrow{D} I \to \mathbb{Z}G^{(S)}$ is a derivation such that $s \mapsto e_s$ because $s_1s_2 \mapsto (s_1-1) + s_1\cdot(s_2-1) \mapsto e_{s_1} + \bar{s}_1e_{s_2}$. Thus the map φ is given by $r \bmod [R,R] \mapsto \sum_{s\in S}(\overline{\partial r/\partial s})e_s$, and we have the desired exact sequence $0 \to R_{ab} \xrightarrow{\theta} \mathbb{Z}G^{(S)} \xrightarrow{\partial} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ where $\partial e_s = \bar{s}-1$ and $\theta(r \bmod [R,R]) = \sum_{s\in S}(\overline{\partial r/\partial s})e_s$.
Note that this reproves the first part of ***Exercise II.5.3(d)****.*

3.1(a): Let $0 \to A \to E \xrightarrow{\pi} G \to 1$ be an extension and let $\alpha : G' \to G$ be a group homomorphism, and consider the pull-back (fiber-product) $E \times_G G' = \{(e,g') \in E \times G' \mid \pi(e) = \alpha(g')\}$. The kernel of the canonical projection $p : E \times_G G' \to G'$ corresponds to $g' = 0 \Rightarrow \alpha(g') = 0 = \pi(e) \Rightarrow \mathrm{Ker}\pi \cong A$, and thus we have an extension $0 \to A \to E \times_G G' \xrightarrow{p} G' \to 1$ which by definition fits into the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & E \arrow[r, "\pi"] & G \arrow[r] & 1 \\
0 \arrow[r] & A \arrow[r] & E \times_G G' \arrow[r, "p"] \arrow[u] & G' \arrow[r] \arrow[u, "\alpha"] & 1
\end{tikzcd}$$

This extension is classified up to equivalence (by fitting into the above commutative diagram) because given another such extension [corresponding to E'] of G' by A, commutativity of the right-hand square implies there is a unique map $\phi : E' \to E \times_G G'$ by the universal property of the pull-back, and this gives commutativity of the right-half of the diagram below:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i_1"] \arrow[d, equal] & E \arrow[r, "\pi"] & G \arrow[r] & 1 \\
0 \arrow[r] & A \arrow[r, "i_2"] \arrow[d, equal] & E \times_G G' \arrow[r, "p"] \arrow[u, "\varphi"] & G' \arrow[r] \arrow[u, "\alpha"] \arrow[d, equal] & 1 \\
0 \arrow[r] & A \arrow[r, "i_3"] & E' \arrow[r] \arrow[u, "\phi"] & G' \arrow[r] \arrow[uu, bend right] & 1
\end{tikzcd}$$

Note that α for the E'-extension yields the identity map $G' \to G'$. It suffices to show that the left-side

of the diagram also commutes, for then we can apply the Five-Lemma which states ϕ is an isomorphism ($E' \cong E \times_G G'$). Now $\phi i_3(a) = i_2(b)$ for some $b \in A$ because $i_3(a)$ maps to $0 \in G'$ by exactness of the bottom row and hence lies in the kernel of $E \times_G G'$ which is contained in $i_2(A)$. Then $\varphi\phi i_3(a) = \varphi i_2(b)$, and $\varphi\phi i_3(a) = i_1(a)$ by commutativity of the outer left-hand square while $\varphi i_2(b) = i_1(b)$ by commutativity of the top left-hand square. Thus $i_1(a) = i_1(b) \Rightarrow a = b$ by injectivity of the inclusion, and this yields $\phi i_3(a) = i_2(a)$ which gives commutativity of the bottom left-hand square and completes the proof. Therefore, α induces a map $\mathcal{E}(G, A) \to \mathcal{E}(G', A)$, and this corresponds to $H^2(\alpha, A) : H^2(G, A) \to H^2(G', A)$ under the bijection of Theorem IV.3.12[1].

3.1(b): Let $0 \to A \xrightarrow{i} E \xrightarrow{p} G \to 1$ be an extension and let $f : A \to A'$ be a G-module homomorphism, and consider the largest quotient E' of $A' \rtimes E$ such that the left-hand square in the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"] & E \arrow[r, "p"] \arrow[d, "\phi"] & G \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & A' \arrow[r, "i'"] & E' \arrow[r, "p'"] & G \arrow[r] & 1
\end{tikzcd}$$

Explicitly, $E' = A' \rtimes E/\sim$ with the equivalence relation $(a_1', e_1) \sim (a_2', e_2)$ iff $a_1' + f(a_1) = a_2' + f(a_2)$ and $e_1 - i(a_1) = e_2 - i(a_2)$ for some $a_1, a_2 \in A$; this relation is obviously reflexive and symmetric. It is transitive because if $a_1' + f(a_1) = a_2' + f(a_2)$ and $a_2' + f(c) = a_3' + f(a_3)$, then $a_1' + f(a_1) = a_3' + f(a_3 + a_2 - c)$ and $e_1 - i(a_1) = [e_3 - i(a_3) + i(c)] - i(a_2) = e_3 - i(a_3 + a_2 - c)$. Define i' by $i'(a') = (a', 0)$ and define p' by $p'(a', e) = p(e)$ and define ϕ by $\phi(e) = (0, e)$. The map p' is well-defined because for $(a_1', e_1) \sim (a_2', e_2)$ we have $p'(a_2', e_2) = p(e_2) = p(e_1) + p[i(a_2 - a_1)] = p(e_1) + 0 = p(e_1) = p'(a_1', e_1)$. Now $p'(a', e) = 0 \Rightarrow p(e) = 0 \Rightarrow \exists\, a \mid i(a) = e \Rightarrow (a', e) = (a', i(a)) \sim (a' + f(a), 0) = i'(a' + f(a)) \Rightarrow \mathrm{Ker}p' \subseteq \mathrm{Im}i'$, and $p'[i'(a')] = p'(a', 0) = p(0) = 0 \Rightarrow \mathrm{Im}i' \subseteq \mathrm{Ker}p'$, so $\mathrm{Ker}p' = \mathrm{Im}i'$ and the bottom row is an E'-extension. This extension is classified up to equivalence (by fitting into the above commutative diagram) because given another such extension [corresponding to E''] of G by A', we get a diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"] \arrow[dd, bend right] & E \arrow[r, "p"] \arrow[d, "\phi"] \arrow[dd, bend right] & G \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & A' \arrow[r, "i'"] \arrow[d, equal] & E' \arrow[r, "p'"] \arrow[d, "\varphi"] & G \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & A' \arrow[r, "i''"] & E'' \arrow[r, "p''"] & G \arrow[r] & 1
\end{tikzcd}$$

where $E \to E''$ is the map Φ and the identity map $A' \to A'$ is induced from the f for the E''-extension. We obtain an induced map $\varphi : E' \to E''$ given by $\varphi(a', e) = i''(a') + \Phi(e)$ which is well-defined because if $(a_1', e_1) \sim (a_2', e_2)$ then $\varphi(a_2', e_2) = i''(a_1') + i''f(a_1) - i''f(a_2) + \Phi(e_1) - \Phi i(a_1) + \Phi i(a_2) = [i''(a_1') + \Phi(e_1)] + i''f(a_1) - i''f(a_2) - i''f(a_1) + i''f(a_2) = \varphi(a_2', e_2) + 0 + 0 = \varphi(a_1', e_1)$, noting that $\Phi i = i''f$ by commutativity of the outer left-hand square. The bottom right-hand square is commutative because $p''\varphi(a', e) = p''i''(a') + p''\Phi(e) = 0 + p(e) = p(e) = p'(a', e)$. The bottom left-hand square [hence the whole diagram] is also commutative because $\varphi i'(a') = \varphi(a', 0) = i''(a') + \Phi(0) = i''(a') + 0 = i''(a')$. We can now apply the Five-Lemma which states φ is an isomorphism ($E'' \cong E'$).
Therefore, f induces a map $\mathcal{E}(G, A) \to \mathcal{E}(G, A')$, and this corresponds to $H^2(G, f) : H^2(G, A) \to H^2(G, A')$ under the bijection of Theorem IV.3.12[1].

3.2(a): Let $0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$ be a short exact sequence of G-modules and let $d : G \to A''$ be a derivation, and consider the set-theoretic pull-back $E = \{(a, g) \in A \rtimes G \mid p(a) = d(g)\}$ where we note that $A \rtimes G = A \times G$ as sets. If $\bar{d} : E \to A$ and $\pi : E \to G$ are the canonical projections, then $d\pi[(a_1, g_1)(a_2, g_2)] = d(g_1 g_2) = d(g_1) + g_1 \cdot d(g_2) = p(a_1) + g_1 \cdot p(a_2)$ and the group law on E can be that of the semi-direct product due to the agreement $p\bar{d}(a_1 + g_1 \cdot a_2, g_1 g_2) = p(a_1 + g_1 \cdot a_2) = p(a_1) + g_1 \cdot p(a_2) = d\pi(a_1 + g_1 \cdot a_2, g_1 g_2)$. Thus E can be regarded as a subgroup of $A \rtimes G$, and $\bar{d}$ is a derivation because $\bar{d}[(a_1, g_1)(a_2, g_2)] = \bar{d}(a_1 + g_1 \cdot a_2, g_1 g_2) = a_1 + g_1 \cdot a_2 = \bar{d}(a_1, g_1) + g_1 \cdot \bar{d}(a_2, g_2)$. Mimicking the proof of Exercise IV.3.1(a) verbatim, for each derivation d there is an extension $0 \to A' \to E \to G \to 1$ characterized by the fact that it fits into a commutative diagram with derivation $\bar{d}$

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "i"] \arrow[d, equal] & A \arrow[r, "p"] & A'' \arrow[r] & 0 \\
0 \arrow[r] & A' \arrow[r, "j"] & E \arrow[r, "\pi"] \arrow[u, "\bar{d}"] & G \arrow[r] \arrow[u, "d"] & 1
\end{tikzcd}$$

This construction gives a map $\mathrm{Der}(G, A'') \to \mathcal{E}(G, A')$.

3.2(b): What follows will be set-theoretic, and we use the same notation/maps as in part(a). A lifting of $d : G \to A''$ to a function $l : G \to A$ is given by $l(g) = p^{-1}d(g)$, where $p^{-1}(x) = 0$ if $x \notin \mathrm{Im}p$. This yields a cross-section $s : G \to E$ of π given by $s(g) = \bar{d}^{-1}l(g)$, because $\pi s = \pi\bar{d}^{-1}l = \pi\bar{d}^{-1}p^{-1}d = \pi(p\bar{d})^{-1}d = \pi(d\pi)^{-1}d = \pi\pi^{-1}d^{-1}d = id_G$. Consider the diagram

$$\begin{tikzcd}
\mathrm{Der}(G, A'') \arrow[r] \arrow[d] & \mathcal{E}(G, A') \arrow[d, "\cong"] \\
H^1(G, A'') \arrow[r, "\delta"] & H^2(G, A')
\end{tikzcd}$$

where the top-horizontal map is the constructed map of part(a) and the left-vertical map is the natural quotient; the isomorphism (right-vertical map) sends an extension to a factor set $f : G \times G \to A'$. The connecting homomorphism $\delta([d]) = [w_2]$ is defined by lifting the element d of $Z^1(G, A'') = \mathrm{Der}(G, A'')$ [equality comes from Exercise III.1.2] to an element w_1 in $C^1(G, A)$ and then taking its image under the coboundary map $\partial^* w_2$ and noting that it must be the image of some $w_2 \in Z^2(G, A')$ under the injection $i^* : C^2(G, A') \to C^2(G, A)$; all of this is well-defined. From above, w_1 in this scenario is l and so $\partial^* l = L : G \times G \to A$ is defined by $L(g, h) = gl(h) - l(gh) + l(g)$ via the equation on pg59[1]. Then we have $i^*(w_2) = L$, and $w_2 : G \times G \to A'$ is a factor set which corresponds to a section S via $S(g)S(h) = j(w_2(g, h))S(gh)$ by the normalization condition on pg91[1]. I claim that S is the section s defined above and that w_2 is the factor set corresponding to s. Assuming this claim holds for the moment, the connecting homomorphism sends the [class of the] derivation d to the factor set w_2 which is precisely the image of the extension under the vertical-isomorphism, and that extension is the image of d under the top-horizontal map, so commutativity of the diagram is satisfied. Therefore, it suffices to prove the claim, i.e. we must check the validity of the equation $s(g)s(h) = j(w_2(g, h))s(gh)$. Note that $j \circ w_2 = \bar{d}^{-1}L$ by commutativity of the left-side of the diagram in part(a), and $s = \bar{d}^{-1}l$, so the equation reduces to $l(g) + [\bar{d}^{-1}l(g)] \cdot l(h) = L(g, h) + [\bar{d}^{-1}L(g, h)] \cdot l(gh)$ after we apply the derivation $\bar{d}$ to both sides. The action of E on A is given by $e \cdot a = \pi(e)a$, so $[\bar{d}^{-1}l(g)] \cdot l(h) = \pi[s(g)]l(h) = gl(h)$. Thus the equation becomes $l(g) + gl(h) = [gl(h) - l(gh) + l(g)] + [\bar{d}^{-1}L(g, h)] \cdot l(gh) \Rightarrow l(gh) = [\bar{d}^{-1}L(g, h)] \cdot l(gh)$. It now suffices to show that the element $\pi[\bar{d}^{-1}L(g, h)]$ fixes $l(gh)$, i.e. $\bar{d}^{-1}L(g, h) \in A' \subset E$. But this is immediate because $\bar{d}^{-1}L(g, h) = j(w_2(g, h)) \in A'$ where A' has lazily been identified with $j(A')$.

3.3: Let G be a finite group which acts trivially on $\mathbb{Z}$. For any homomorphism $G \to \mathbb{Q}/\mathbb{Z}$ we can construct a central extension of G by $\mathbb{Z}$ by pulling back the canonical extension (the top row)

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathbb{Q} \arrow[r] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & E \arrow[r] \arrow[u] & G \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

Thus we have a map $\varphi : \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) \to \mathcal{E}(G, \mathbb{Z})$, which by Exercise IV.3.2 can be identified with the boundary map $\delta : H^1(G, \mathbb{Q}/\mathbb{Z}) \to H^2(G, \mathbb{Z})$ since $H^1(G, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$ by Exercise III.1.2. Now $H_1G = G_{ab}$ is finite, and so is H_2G by Exercise AE.16 [G is finite and $\mathbb{Z}$ is finitely generated]; thus $\mathrm{Hom}(H_2G, \mathbb{Q}) = 0 = \mathrm{Hom}(H_1G, \mathbb{Q})$ because there are no nontrivial homomorphisms from a finite group into the infinite group $\mathbb{Q}$. Also, $\mathbb{Q}$ is $\mathbb{Z}$-injective and hence $\mathrm{Ext}^1_{\mathbb{Z}}(H_0G, \mathbb{Q}) = 0 = \mathrm{Ext}^1_{\mathbb{Z}}(H_1G, \mathbb{Q})$ by Proposition 17.1.9[2]. The universal coefficient sequence of Exercise III.1.3 now yields $H^1(G, \mathbb{Q}) = 0 = H^2(G, \mathbb{Q})$; alternatively, we could have simply noted that $H^n(G, \mathbb{Q}) = 0 \quad \forall\ n > 0$ by Corollary III.10.2[1] since $|G|$ is invertible in $\mathbb{Q}$. By the long exact cohomology sequence (Proposition III.6.1[1]), δ is an isomorphism. Thus φ is a bijection.

3.4(a): Let E be a group which contains a central subgroup $C \subseteq Z(E)$ of finite index n.

Method 1: We have a central extension $1 \to C \to E \to G \to 1$ by taking $G = E/C$. The nth power map $C \to C$ (where C is considered as an abelian group) induces multiplication by n on $H^2(G, C)$

because the cochain $f : G^2 \to C$ is sent to $nf : G^2 \to nC \subseteq C$, and this is the zero-map because $n = |G|$ annihilates $H^2(G, C)$ by Corollary III.10.2[1]. Now Exercise IV.3.1(b) implies the above central extension fits uniquely into a commutative diagram with the split extension [the trivial element in $\mathcal{E}(G, C)$]

$$\begin{tikzcd}
1 \arrow[r] & C \arrow[r] \arrow[d, "n"] & E \arrow[r] \arrow[d, "\phi"'] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & C \arrow[r] & C \times G \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

where we note that $C \rtimes G \cong C \times G$ because C is central. Looking at the left-hand square with E restricted to C, commutativity implies $\phi(c) = (c^n, 1)$, so the C-component of $E \to C \times G$ gives us the desired homomorphism $E \to C$ whose restriction to C is the nth power map.

Method 2: The abelianization map $\rho : E \to E_{ab}$ composed with the transfer map $tr : E_{ab} \to C_{ab} = C$ is given by $\varphi : e \mapsto e \bmod [E, E] \mapsto \prod_{g \in C \backslash E} ge(\overline{ge})^{-1}$. Now $|C \backslash E| = n$ and $c \in C$ commutes with all elements of E, so $\varphi(c) = \prod_{g \in C \backslash E} gc(\overline{gc})^{-1} = c^n \prod g(\overline{cg})^{-1} = c^n \prod gg^{-1} = c^n$. Thus $\varphi = tr \circ \rho : E \to C$ is the desired homomorphism whose restriction to C is the nth power map.

3.4(b): Given a finitely generated group E, suppose the commutator subgroup $[E, E]$ of E is finite. Then there are finitely many nontrivial elements $g^{-1}(ege^{-1})$ where g is a generator of E (and $e \in E$ is arbitrary), so there are only finitely many conjugates ege^{-1} of each generator g of E. Consider the inner automorphism group $\mathrm{Inn}(E) \cong E/C$, where $C = Z(E)$ is the center of E. An arbitrary element is a function $f_e(x) = exe^{-1}$ [with $e \in E$ fixed] which is determined by where it sends the generators of E. Since there are finitely many generators and finitely many conjugates of each generator, there are only finitely many non-identity maps $f_e \in \mathrm{Inn}(G)$. Thus $|\mathrm{Inn}(G)|$ is finite, and so the center C of E has finite index.

Conversely, suppose the center C of the finitely generated group E has finite index. Then part(a) above gives us a homomorphism $\varphi : E \to C$ such that $\varphi|_C$ is the nth power map $C \xrightarrow{n} C$. It suffices to show that $\mathrm{Ker}\varphi$ is finite, for then $E/\mathrm{Ker}\varphi$ is abelian (being isomorphic to a subgroup of the abelian group C by the First Isomorphism Theorem) and hence $[E, E] \subseteq \mathrm{Ker}\varphi$ by Proposition 5.4.7[2], so the commutator subgroup $[E, E]$ of E is finite. Now φ is the C-component of the map $\phi : E \to C \times G$ given in part(a), and the kernel of ϕ is contained in the kernel of the projection $\pi : E \to G$ by commutativity of that diagram in part(a), so $\mathrm{Ker}\varphi \subseteq \mathrm{Ker}\phi \subseteq \mathrm{Ker}\pi = C$ and hence $\mathrm{Ker}\varphi \subseteq \mathrm{Ker}(C \xrightarrow{n} C)$. It suffices to show that $C \subseteq E$ is a finitely generated abelian group, for then $\mathrm{Ker}(C \xrightarrow{n} C)$ is a subgroup of finite exponent in a finitely generated abelian group (hence finite by the Fundamental Theorem of Finitely Generated Abelian Groups). The *Nielsen-Schreier Theorem* (Theorem 85.1[6]) states that every subgroup of a free group is free. The *Schreier Index Formula* (Theorem 85.3[6]) states that for a free group F of finite rank with a subgroup H of finite index, $rk_{\mathbb{Z}}H = |F : H|(rk_{\mathbb{Z}}F - 1) + 1$. The finitely generated group E has the presentation F/R with F free of finite rank, and $C \subseteq F$ corresponds bijectively to some $H \subseteq F$ with $H/R = C$ by the Lattice [4th] Isomorphism Theorem; H is free and finitely generated by the Nielsen-Schreier Theorem and the Schreier index formula. Since quotients of finitely generated groups are finitely generated (the generators of the quotient are the images of the generators under the projection), C is finitely generated (and abelian since it lies in the center of E).

3.5(a): Let E be a group which contains an infinite cyclic central subgroup of finite index.

Method 1: The group E of an extension in $\mathcal{E}(G, \mathbb{Z})$ gives us a homomorphism $\varphi : G \to \mathbb{Q}/\mathbb{Z}$ by Exercise IV.3.3 (where $G = E/\mathbb{Z}$), along with a map $\phi : E \to \mathbb{Q}$ which is injective on $\mathbb{Z} \subseteq E$. Now the nontrivial finitely generated subgroups of $\mathbb{Q}$ are of the form $\frac{m}{n}\mathbb{Z}$ with $m, n \in \mathbb{N}$, so $\mathrm{Im}\phi = \frac{m}{n}\mathbb{Z} \cong \mathbb{Z}$ [note: E is finitely generated because both $\mathbb{Z}$ and $E/\mathbb{Z}$ are]. Thus we have a surjective map $\tilde{\phi} : E \to \mathrm{Im}\phi \cong \mathbb{Z}$ which is simply ϕ rephrased. Its kernel $F := \mathrm{Ker}\tilde{\phi} = \mathrm{Ker}\phi$ is finite because $|F \cap \mathbb{Z}| = 1$ and $|F\mathbb{Z}/\mathbb{Z}| = |F : F \cap \mathbb{Z}| = |F|$ by the 2nd Isomorphism Theorem, where we note that $F\mathbb{Z}/\mathbb{Z} \subseteq G$ is finite [$F\mathbb{Z}$ is a subgroup of E because $F \lhd E$]. We then have an extension $0 \to \mathrm{Ker}\tilde{\phi} \to E \xrightarrow{\tilde{\phi}} \mathbb{Z} \to 0$ which must split (by Exercise AE.27) because $\mathbb{Z}$ is free. Thus $E \cong F \rtimes \mathbb{Z}$ with F finite.

Method 2: By Exercise IV.3.4(a) we have a map $\phi : E \to \mathbb{Z}$ which has a finite kernel as proved in Exercise IV.3.4(b). Letting $\tilde{\phi} : E \to \phi(E)$ be the surjective map induced from ϕ, we have $\phi(E) = m\mathbb{Z} \cong \mathbb{Z}$ for some $m \in \mathbb{N}$ and so $\tilde{\phi} : E \to \mathbb{Z}$ is a surjective map with finite kernel. We then have an extension $0 \to \mathrm{Ker}\tilde{\phi} \to E \to \mathbb{Z} \to 0$ which must split (by Exercise AE.27) because $\mathbb{Z}$ is free. Thus $E \cong F \rtimes \mathbb{Z}$ with

F finite (taking $F = \mathrm{Ker}\tilde{\phi} = \mathrm{Ker}\phi$).

3.5(b): Let E be a torsion-free group which has an infinite cyclic subgroup $\mathbb{Z}$ of finite index. Let E act by left multiplication on the finite set T of left cosets of $\mathbb{Z}$ in E and let $\pi_{\mathbb{Z}} : E \to S_{|T|}$ be the associated permutation representation afforded by this action. By Theorem 4.2.3[2], the kernel of the action is the *core* subgroup $\mathrm{Ker}\pi_{\mathbb{Z}} = \bigcap_{e\in E} e\mathbb{Z}e^{-1} \equiv A$ and so $A \subseteq \mathbb{Z}$ is a normal subgroup of E of finite index (A is necessarily nontrivial and hence infinite cyclic). The group action of E on $A \cong \mathbb{Z}$ by conjugation is a homomorphism $E \to \mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$, and the kernel is a subgroup $E' \subseteq E$ of index 1 or 2 such that $A \subseteq Z(E')$. By part(a) we have $E' \cong F \rtimes A$ with F finite, but since E is torsion-free, $F = \{1\}$ and hence $E' \cong \mathbb{Z}$. We therefore have an extension $0 \to \mathbb{Z} \to E \to G \to 1$ with $|G| \leq 2$ (take $G = E/E'$). If $G = \{1\}$ we are done. If $G = \mathbb{Z}_2$ acts non-trivially on $\mathbb{Z}$ then $H^2(G = \mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z}^{\mathbb{Z}_2}/N\mathbb{Z} = 0/N\mathbb{Z} = 0$, where $N \in \mathbb{Z}G$ is the norm element. In view of Theorem IV.3.12[1] we see that the extension splits (so $E \cong \mathbb{Z} \rtimes G$), contradicting the assumption that E is torsion-free. Hence G acts trivially, so $\mathbb{Z}$ is central in E and $E \cong \mathbb{Z}$ by part(a).

3.6: Let E be a finitely generated torsion-free group which contains an abelian subgroup of finite index. This subgroup is isomorphic to $\mathbb{Z}^n$ for some n by the Fundamental Theorem of Finitely Generated Abelian Groups. Note that $\mathbb{Z}^n$ is a *polycyclic group* because it is solvable (it has the series $1 \lhd \mathbb{Z} \lhd \mathbb{Z}^n$ with $\mathbb{Z}^n/\mathbb{Z} = \mathbb{Z}^{n-1}$ abelian) and every subgroup is finitely generated; an equivalent definition of a polycyclic group is that it has a subnormal series with each quotient cyclic (so for $\mathbb{Z}^n$ we have the series $1 \lhd \mathbb{Z} \lhd \mathbb{Z}^2 \lhd \cdots \lhd \mathbb{Z}^{n-1} \lhd \mathbb{Z}^n$ with each quotient $\mathbb{Z}^i/\mathbb{Z}^{i-1} = \mathbb{Z}$ cyclic). Thus E is a *virtually polycyclic group* because it has a polycyclic subgroup of finite index [note: E is also *virtually abelian*]. Let E act by left multiplication on the finite set T of left cosets of $\mathbb{Z}^n$ in E and let $\pi_{\mathbb{Z}^n} : E \to S_{|T|}$ be the associated permutation representation afforded by this action. By Theorem 4.2.3[2], the kernel of the action is $\mathrm{Ker}\pi_{\mathbb{Z}^n} = \bigcap_{e\in E} e\mathbb{Z}^n e^{-1} =: A$ and so $A \subseteq \mathbb{Z}^n$ is a normal subgroup of E of finite index; A is necessarily nontrivial and hence isomorphic to $\mathbb{Z}^n$ because it is a subgroup of $\mathbb{Z}^n$ of finite index ($|E : \mathbb{Z}^n| = |(E/A)/(\mathbb{Z}^n/A)|$ is finite and $|E : A|$ is finite). We therefore have an extension $0 \to \mathbb{Z}^n \to E \xrightarrow{p} G \to 1$ with G finite. The group action of G on $\mathbb{Z}^n$ is a homomorphism $\rho : G \to \mathrm{Aut}(\mathbb{Z}^n) \cong GL_n(\mathbb{Z})$, and $GL_n(\mathbb{Z})$ contains only integral matrices with determinant ± 1 as deduced from the surjective map $\det : GL_n(\mathbb{Z}) \to \mathbb{Z}^\times = \{\pm 1\}$ or from the fact that an integral matrix is invertible iff its determinant is a unit in $\mathbb{Z}$. Consider the finite kernel $K := \mathrm{Ker}\rho$ and its preimage $E' := p^{-1}(K) \subseteq E$ under p, and note that we have $E' \lhd E$ by the Lattice Isomorphism Theorem because $E'/\mathbb{Z}^n = K \lhd G = E/\mathbb{Z}^n$. Now this torsion-free group E' is finitely generated because its subgroup $\mathbb{Z}^n$ and its quotient K are both finitely generated [if $e \in E' - \mathbb{Z}^n$ then it can be written as a finite sum with generators of K, and if $e \in \mathbb{Z}^n$ then it can be written as a finite sum with generators of $\mathbb{Z}^n$]. Also, the corresponding K-action on $\mathbb{Z}^n$ is trivial (as K is the kernel of the G-action) and hence $\mathbb{Z}^n$ lies in the center of E', so by Exercise IV.3.4(b) the commutator subgroup $[E', E']$ is finite. But the only finite subgroup of a torsion-free group is the trivial group, so $[E', E'] = 0$ and $E'_{ab} = E'/[E', E'] = E'$ (i.e. E' is abelian, hence isomorphic to $\mathbb{Z}^n$ by the Fundamental Theorem of Finitely Generated Abelian Groups, noting that $|E' : \mathbb{Z}^n| = |K|$ is finite). Letting $F = E/E' \cong (E/\mathbb{Z}^n)/(E'/\mathbb{Z}^n) = G/K$ where the isomorphism follows from the 3rd Isomorphism Theorem, we have a group extension $0 \to \mathbb{Z}^n \to E \to F \to 1$ coupled with the faithful group action $F \hookrightarrow GL_n(\mathbb{Z})$; this action is faithful because we modded the map ρ by its kernel and injected $A \cong \mathbb{Z}^n$ into $E' \cong \mathbb{Z}^n$.
Since $|F| = r$ annihilates $H^2(F, \mathbb{Z}^n)$ by Corollary III.10.2[1], the rth power map $\mathbb{Z}^n \to \mathbb{Z}^n$ induces the zero-map on $H^2(F, \mathbb{Z}^n)$. Exercise IV.3.1(b) then implies the above extension fits uniquely into a commutative diagram with the split extension [the trivial element in $\mathcal{E}(F, \mathbb{Z}^n)$]

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^n \arrow[r, "i_1"] \arrow[d, "r"] & E \arrow[r, "p_1"] \arrow[d, "\phi"'] & F \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & \mathbb{Z}^n \arrow[r, "i_2"] & \mathbb{Z}^n \rtimes F \arrow[r, "p_2"] & F \arrow[r] & 1
\end{tikzcd}$$

The map ϕ is injective because if $\phi(e) = 0$ then $p_1 e = 0 \;\Rightarrow\; \exists\, z \mid i_1 z = e \;\Rightarrow\; i_2(rz) = \phi(i_1 z) = 0 \;\Rightarrow\; rz = 0 \;\Rightarrow\; z = 0 \;\Rightarrow\; e = i_1 z = i_1 0 = 0$. Alternatively, we could simply note that since the r-map is injective, $\mathrm{Ker}\phi \cap \mathbb{Z}^n = \{1\}$ and hence $\mathrm{Ker}\phi$ injects into F, which means $\mathrm{Ker}\phi$ is trivial by commutativity of the right-side diagram.
A *crystallographic group* is a discrete cocompact subgroup of the group $\mathbb{R}^n \rtimes O_n$ of isometries of some

Euclidean space; in general, $V \subseteq W$ is *cocompact* if W/V is compact. Note that $\mathbb{R}^n$ has the usual topology (its basis consists of the open n-balls), and the relative topology for the subspace $\mathbb{Z}^n$ is the discrete topology (any point $\mathbf{x} \in \mathbb{Z}^n$ is equal to the intersection $\mathbb{Z}^n \cap B_x$ where B_x is the ball of radius $\frac{1}{3}$ centered at $\mathbf{x}$). The orthogonal group $O_n = \{M \in GL_n(\mathbb{R}) \mid M^T M = 1\}$ has the relative topology induced from the matrix group $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$.
It is a fact that a faithful action $F \hookrightarrow GL_n(\mathbb{R})$ can also be considered as a faithful action $F \hookrightarrow O_n$, so in our case we have an injection $F \hookrightarrow O_n$ since $GL_n(\mathbb{Z}) \subset GL_n(\mathbb{R})$. The product topology on $\mathbb{Z}^n \rtimes F \subset \mathbb{R}^n \rtimes O_n$ is the discrete topology [thus $\mathbb{Z}^n \rtimes F$ is discrete] because $\mathbb{Z}^n$ has the discrete topology as mentioned above and the relative topology on F is discrete since F is finite. Since a subgroup of a discrete group is discrete (the relative topology induced from the discrete topology is discrete), E can be embedded (by ϕ) as a discrete subgroup of $\mathbb{R}^n \rtimes O_n$.
The $\mathbb{Z}^n$-action on $\mathbb{R}^n$ given by translation $\mathbf{x} \mapsto \mathbf{x} + \mathbf{z}$ is properly discontinuous, so the quotient map $p : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n \cong \mathbb{T}^n$ is a regular covering map (see Exercise I.4.2) where $\mathbb{T}^n \equiv S^1 \times \cdots \times S^1$ is the compact n-torus. Now O_n is a compact space by the Heine-Borel Theorem because it is a closed bounded subspace of $\mathbb{R}^{n^2}$ (see pg292[3]), so $\mathbb{Z}^n$ is a cocompact subgroup of $\mathrm{Isom}(\mathbb{R}^n) := \mathbb{R}^n \rtimes O_n$ because the quotient $\mathbb{T}^n \rtimes O_n$ is compact. But $\mathbb{Z}^n$ is a subgroup of E of finite index, so E is also cocompact. This completes the proof that E is a crystallographic group.

3.7(a): Let G be a perfect group (so $H_1 G = 0$) and let A be an abelian group with trivial G-action. The universal coefficient sequence of Exercise III.1.3 then implies $H^2(G, A) \cong \mathrm{Hom}(H_2 G, A)$.

3.7(b): Let G be a perfect group and let A be any abelian group with trivial G-action. Yoneda's lemma from Exercise I.7.3(a) states that a natural transformation $\varphi : \mathrm{Hom}(H_2 G, -) \to H^2(G, -)$ is determined by where it sends $id_{H_2 G} \in \mathrm{Hom}(H_2 G, H_2 G)$, and so for the isomorphism φ of part(a) there is an element $u \in H^2(G, H_2 G)$ such that $\varphi(id_{H_2 G}) = u$. Now for any $v \in H^2(G, A)$ there is a unique map $f : H_2 G \to A$ such that $\varphi(f) = v$ because φ is an isomorphism, and Yoneda's lemma gives $\varphi(f) = H^2(G, f)u$. Thus u is the "universal" cohomology class of $H^2(G, H_2 G)$, in the sense that for any $v \in H^2(G, A)$ there is a unique map $f : H_2 G \to A$ such that $v = H^2(G, f)u$.

3.7(c): In view of Theorem IV.3.12[1], part(b) can be reinterpreted as saying that the [perfect] group G admits a "universal central extension" $0 \to H_2 G \to E \xrightarrow{\pi} G \to 1$ characterized by a certain property. This property states that given any abelian group A and any central extension $0 \to A \to E' \xrightarrow{\pi'} G \to 1$, there is a unique map $f : H_2 G \to A$ such that the extension is the image of the universal extension under $H^2(G, f) = \mathcal{E}(G, f)$. By Exercise IV.3.1(b) this latter part means there is a map $E \to E'$ making the following diagram commute

$$\begin{tikzcd} E \arrow[r, "\pi"] \arrow[d] & G \\ E' \arrow[ur, "\pi'"'] & \end{tikzcd}$$

and we assert that this map is unique. Indeed, two such maps $h_1, h_2 : E \rightrightarrows E'$ which induce the same map f must differ by a homomorphism $\phi : E \to A$ because $\pi' h_1(e) = \pi(e) = \pi' h_2(e) \;\Rightarrow\; h_1 e = h_2 e \cdot a$ with $a \in \mathrm{Ker}\pi' = A$, giving $\phi(e) = a$ [it is obviously a homomorphism since h_1 and h_2 are]; ϕ must also factor through G (i.e. ϕ is equal to $E \to E/H_2 G \cong G \to A$) because h_1 and h_2 must agree on where it sends $H_2 G$ by commutativity of the diagram in Exercise IV.3.1(b). Now any homomorphism $\psi : G \to A$ satisfies $[G, G] \subseteq \mathrm{Ker}\psi$ by Proposition 5.4.7[2]; but G is perfect, so $G = \mathrm{Ker}\psi$ and there are no non-trivial maps $G \to A$ (hence $\phi = 0 \;\Rightarrow\; h_1 = h_2$).

Note: It happens to be true that E in the universal central extension is necessarily perfect, so there is another way to show uniqueness of the map $E \to E'$ [this is presented in John Milnor's *Introduction to Algebraic K-Theory*]. For any $y, z \in E$ we have $h_1 y = h_2 y \cdot c$ and $h_1 z = h_2 z \cdot c'$ with $c, c' \in \mathrm{Ker}\pi' \subseteq Z(E')$. Thus $h_1(yzy^{-1}z^{-1}) = h_2(yzy^{-1}z^{-1})$ by basic rearrangements of the two previous equations, noting that we can move c and c' around as they lie in the center of E'. Since $E = [E, E]$, it is generated by commutators and hence $h_1 = h_2$.

3.8(a): Let $0 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1$ be a central extension with G abelian. The *commutator pairing* associated to the extension is the map $c : G \times G \to A$ defined by $c(g, h) = i^{-1}([\tilde{g}, \tilde{h}]) = i^{-1}(\tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1})$, where $\tilde{g}$

and $\tilde{h}$ are lifts of g and h to E. Since $A \triangleleft E$ and $G = E/A$ is abelian, $[E, E] \subseteq A$ by Proposition 5.4.7[2] and so $i^{-1}([\tilde{g}, \tilde{h}])$ is defined. Any other lift of g to E is of the form $\tilde{g}a = a\tilde{g}$, and $[a\tilde{g}, a'\tilde{h}] = [\tilde{g}, \tilde{h}]$ because A lies in the center of E. Thus c is well-defined, and it is alternating because $c(g, g) = i^{-1}([\tilde{g}, \tilde{g}]) = i^{-1}(1) = 0$. Now $\tilde{g}\tilde{h}$ is a lifting of $g + h$ because $\pi(\tilde{g}\tilde{h}) = \pi(\tilde{g}) + \pi(\tilde{h}) = g + h$, and $[\tilde{g}\tilde{h}, \tilde{k}] = \tilde{g}\tilde{h}\tilde{k}\tilde{h}^{-1}\tilde{g}^{-1}\tilde{k}^{-1} = \tilde{g}[\tilde{h}, \tilde{k}]\tilde{g}^{-1}[\tilde{g}, \tilde{k}] = [\tilde{g}, \tilde{k}][\tilde{h}, \tilde{k}]$ where the last equality follows from $[E, E] \subseteq A \subseteq Z(E)$. We then have $c(g + h, k) = c(g, k) + c(h, k)$ because i is injective, giving $i^{-1}([\tilde{g}, \tilde{k}][\tilde{h}, \tilde{k}]) = i^{-1}([\tilde{g}, \tilde{k}]) + i^{-1}([\tilde{h}, \tilde{k}])$. An analogous computation gives $c(k, g + h) = c(k, g) + c(k, h)$, so c is $\mathbb{Z}$-bilinear. Since c is alternating and bilinear, $c(g, h) = -c(h, g)$ and hence c can be viewed as a map $\bigwedge^2 G \to A$, where $\bigwedge^2 G = G \otimes G/\langle\{g \otimes g\}\rangle$ is the second exterior power of G.

3.8(b): Let f be a factor set to the central extension in part(a). To show that $c(g, h) = f(g, h) - f(h, g)$ it suffices to show that $[\tilde{g}, \tilde{h}] = i[f(g, h)]i[f(h, g)]^{-1}$ because i is injective. Given the section $s : G \to E$, f satisfies $s(g)s(h) = i[f(g, h)]s(gh)$ and $s(h)s(g) = i[f(h, g)]s(gh)$. Thus we have $[s(g), s(h)] = s(g)s(h)s(g)^{-1}s(h)^{-1} = i[f(g, h)]i[f(h, g)]^{-1}$. Since $s(g)$ is a lifting of g for all $g \in G$, we have the desired $[s(g), s(h)] = [\tilde{g}, \tilde{h}]$.

3.8(c): Let $\theta : H^2(G, A) \to \mathrm{Hom}(\bigwedge^2 G, A)$ be the map which sends the class of a cocycle f to the alternating map $f(g, h) - f(h, g)$. This map is well-defined because $[\delta c](g, h) - [\delta c](h, g) = c(g) + gc(h) - c(gh) - c(h) - hc(g) + c(hg) = c(g) + c(h) - c(gh) - c(h) - c(g) + c(gh) = 0$, where we note that G is abelian and the G-action on the cochain c is trivial since A is central. In view of Theorem IV.3.12[1], part(b) implies this image is the commutator pairing $c(g, h)$, and $f \in \mathrm{Ker}\theta$ iff $\theta f = c(g, h) = 0$. Now $c(g, h) = i^{-1}([\tilde{g}, \tilde{h}]) = 0$ iff $[\tilde{g}, \tilde{h}] = 0$ which is equivalent to E being abelian, i.e. $[E, E] = 0$. Thus there is a bijection $\mathrm{Ker}\theta \cong \mathcal{E}_{ab}(G, A)$, where $\mathcal{E}_{ab}(G, A)$ is the set of equivalence classes of abelian extensions of G by A.

4.1: Let Q_{2^n} be a generalized quaternion group. It is a fact that Q_{2^n} has a unique element of order 2, hence a unique subgroup of order 2. Any subgroup of Q_{2^n} is also a 2-group (by Lagrange's Theorem) and so it has an element of order 2 by Cauchy's Theorem; this element is then unique in each subgroup (giving a unique $\mathbb{Z}_2$ subgroup). Therefore, by Theorem IV.4.3[1] every subgroup of Q_{2^n} is either a cyclic group or a generalized quaternion group.
Alternatively, $Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^4 = 1,\ yxy^{-1} = x^{-1}\rangle$ has the property that any subgroup G is a 2-group with a unique $\mathbb{Z}_2$ subgroup (as mentioned above). If G is abelian, then by the Fundamental Theorem of Finitely Generated Abelian Groups, $G \cong \mathbb{Z}_{2^{\alpha_1}} \times \cdots \times \mathbb{Z}_{2^{\alpha_i}}$ which has more than one $\mathbb{Z}_2$ subgroup if $i > 1$, and so $G \cong \mathbb{Z}_{2^\alpha}$ is a cyclic subgroup. Suppose G is nonabelian, which means G contains elements of the form x^k (they necessarily form a cyclic subgroup $H \subset \langle x\rangle$) and elements of the form $x^k y$ [note: each element of Q_{2^n} can be written uniquely in the form $x^i y^j$ for $0 \le i \le 2^{n-1} - 1$ and $0 \le j \le 1$, and any element in $Q_{2^n} - \langle x\rangle$ has order 4 because $(x^k y)^4 = (x^k y x^k y)^2 = (x^k x^{-k} y y)^2 = (y^2)^2 = y^4 = 1$]. Let $X = \{x^{k_1} y, x^{k_2} y\} \subset G$ be the elements of the form $x^k y$ and let x^i be the generator of H. Now $x^{k_2} y \cdot x^{k_1} y = x^{k_2 - k_1} y^2 = x^{\Delta k + 2^{n-2}}$ and $(x^{k_1} y)^2 = y^2$, so $x^{\Delta k} y^2 \cdot y^2 = x^{\Delta k}$ and $x^{\Delta k} \cdot x^{k_1} y = x^{k_2} y$. Thus $x^{k_1} y$ is a generator of G (of order 4) and the other generator is $x^r = x^{gcd(i, \Delta k)}$ which is cyclic [note: if X includes other elements (up to $x^{k_m} y$) then the above still applies with $r = gcd(i, k_m - k_1, \ldots, k_m - k_{m-1})$, and if either $i = 0$ or $k_1 = \cdots = k_m = 0$ then omit those integers/differences in the gcd-term]. Since $(x^{k_1} y) x^r (x^{k_1} y)^{-1} = x^{k_1}(y x^r y^{-1}) x^{-k_1} = x^{k_1} x^{-r} x^{-k_1} = x^{-r}$, the presentation is complete and G is a generalized quaternion group.

4.2: The dihedral group $D_{2^n} = \mathbb{Z}_{2^{n-1}} \rtimes \mathbb{Z}_2 = \langle x, y \mid x^{2^{n-1}} = y^2 = 1,\ xy = yx^{-1}\rangle$ has the property that each element can be written uniquely in the form $y^i x^j$ for $0 \le i \le 1$ and $0 \le j \le 2^{n-1} - 1$. Any element not in $\mathbb{Z}_{2^{n-1}} = \langle x\rangle$ has order 2 because $(yx^k)^2 = yx^k yx^k = (yx^k y)x^k = x^{-k}x^k = 1$. If a subgroup contains only elements of $\langle x\rangle$ then it is cyclic, so we consider the only other subgroups G, and these contain elements of the form yx^k and elements in $\langle x\rangle$ (which necessarily form a subgroup $H \subseteq \langle x\rangle$). Let $X = \{yx^{k_1}, \ldots, yx^{k_m}\} \subset G$ be the elements of the form yx^k and let x^i be the generator of H. The given subset X implies G contains the elements $x^{k_m - k_1}, \ldots, x^{k_m - k_{m-1}}$. Then $x^r = x^{gcd(i, k_m - k_1, \ldots, k_m - k_{m-1})}$ generates the elements in G of the form x^k (because k would be a multiple of r), and $yx^{k_1} \cdot x^{(k_m - k_1) - (k_m - k_j)} = yx^{k_j}$ for any $j \le m$ [note: if $i = 0$ (meaning the only elements of G are of the form yx^k) then $r = gcd(k_m - k_1, \ldots, k_m - k_{m-1})$, and if $k_1 = \cdots = k_m = 0$ (meaning y is

the only element in G of the form yx^k) then $r = i$]. Thus $G = \langle x^r, yx^{k_1}\rangle$ where yx^{k_1} is of order 2 and $(yx^{k_1})x^r(yx^{k_1}) = (yx^{k_1+r}y)x^{k_1} = x^{-k_1-r}x^{k_1} = x^{-r}$, i.e. G is a dihedral group.
The non-cyclic abelian subgroups of D_{2^n} are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 = D_4$. The group D_8 contains two non-cyclic abelian normal subgroups, $\langle x^2, y\rangle$ and $\langle x^2, yx\rangle$. We assert that such subgroups are not normal if $n > 3$ (i.e. if $|D_{2^n}| > 8$). From the subgroup presentations above we see that the non-cyclic abelian subgroups are $H_i = \langle x^{2^{n-2}}, yx^i\rangle$ for all $i \in \{0, \ldots, 2^{n-1} - 1\}$. It suffices to show that there is an integer j such that the element $x^j(yx^i)x^{-j} = yx^{i-2j}$ does not lie in H_i (i.e. $i - 2j$ is neither i nor $i + 2^{n-2}$ modulo 2^{n-1}). The first condition implies $2j \not\equiv 0 \bmod 2^{n-1} \Rightarrow j \neq 2^{n-2}m$ for any $m \in \mathbb{Z}$, and the latter condition implies $2j + 2^{n-2} \not\equiv 0 \bmod 2^{n-1} \Rightarrow j \neq 2^{n-2}m - 2^{n-3}$. Thus we can take $j = 2^{n-2} - 1$, and $x^{2^{n-2}-1} \in D_{2^n}$ will yield the non-normality $(x^{2^{n-2}-1})H_i(x^{1-2^{n-2}}) \not\subseteq H_i$ [note: if $n = 3$ then j cannot be an even integer $2m$ nor can it be an odd integer $2m - 1$, which means j does not exist].

4.3: Let $G = \mathbb{Z}_q \rtimes \mathbb{Z}_2$ with $q = 2^n$ ($n \geq 3$) and let $A \subset \mathbb{Z}_q$ be the subgroup of order 2. Note that $A \rtimes \mathbb{Z}_2 = A \times \mathbb{Z}_2$ is a non-cyclic abelian subgroup of G since $\mathbb{Z}_2$ acts trivially on A [multiplication by $-1 + 2^{n-1}$ is the action, and $(-1 + 2^{n-1})2^{n-1} = -2^{n-1} + 2^{n-2}2^n = 2^{n-1} + 0 = 2^{n-1}$ for the generator $2^{n-1} \in A$]. If $H \subset G$ is a non-cyclic proper subgroup, then $H \not\subseteq \mathbb{Z}_q$ and $H\mathbb{Z}_q$ is a subgroup of G by Corollary 3.2.15[2], so $H\mathbb{Z}_q = G$. Thus $2q = |G| = |H\mathbb{Z}_q| = |H|q/|H \cap \mathbb{Z}_q| \Rightarrow |H : H \cap \mathbb{Z}_q| = 2$, and since $H \cap \mathbb{Z}_q \lhd H$ we have $H/(H \cap \mathbb{Z}_q) \cong \mathbb{Z}_2$ and this yields the extension $0 \to H \cap \mathbb{Z}_q \to H \to \mathbb{Z}_2 \to 0$. The generator of $\mathbb{Z}_2$ acts as multiplication by -1 on $H \cap \mathbb{Z}_q$ because $-1 + 2^{n-1} \equiv -1 \bmod 2^m$ with $m < n$ [note: $2^m = |H \cap \mathbb{Z}_q|$ with $m < n$ because $H \cap \mathbb{Z}_q$ is a subgroup of $\mathbb{Z}_q$ and hence has order 2^m, and if $m = n$ then $\mathbb{Z}_q \subseteq H$ which means $H = G$, contradicting the assumption that H is proper]. If H were abelian then $H \cap \mathbb{Z}_q$ would be central in H, and if $H \cap \mathbb{Z}_q$ were central in H then H would be abelian since $H/(H \cap \mathbb{Z}_q) \cong \mathbb{Z}_2$ is cyclic; this latter fact holds because $H/(H \cap \mathbb{Z}_q) = \{1(H \cap \mathbb{Z}_q), x(H \cap \mathbb{Z}_q)\}$ and so any $h \in H$ has a representation $h = x^a z$ for $z \in H \cap \mathbb{Z}_q$, giving $h_1h_2 = x^{a_1}z_1x^{a_2}z_2 = x^{a_1}x^{a_2}z_1z_2 = x^{a_2}x^{a_1}z_2z_1 = x^{a_2}z_2x^{a_1}z_1 = h_2h_1$. Now $H \cap \mathbb{Z}_q$ is central in H iff the $\mathbb{Z}_2$-action is trivial ($-z = z$) iff $H \cap \mathbb{Z}_q = A$, and so H is abelian iff $H \cap \mathbb{Z}_q = A$.
We have $H^1(\mathbb{Z}_2, \mathbb{Z}_q) = \mathrm{Ker}\overline{N}$ where $\overline{N} : (\mathbb{Z}_q)_{\mathbb{Z}_2} \to (\mathbb{Z}_q)^{\mathbb{Z}_2}$ is the norm map. But $(\mathbb{Z}_q)_{\mathbb{Z}_2}$ is a quotient of a cyclic group (it is then necessarily cyclic of order $m = 2^r$ where $m|q$), and in particular we must have $(-1 + 2^{n-1})1 = 1 \bmod m \Rightarrow 2^{n-2} - 1 = ms = 2(2^{r-1}s)$ which is impossible because an even number cannot equal an odd number, so $(\mathbb{Z}_q)_{\mathbb{Z}_2} = 0$ and the kernel of the norm map is trivial. Thus $H^1(\mathbb{Z}_2, \mathbb{Z}_q) = 0$, and by Proposition IV.2.3[1] this means that the extension $0 \to \mathbb{Z}_q \to G \to \mathbb{Z}_2 \to 0$ has a unique splitting (up to conjugacy) and hence that G contains only two conjugacy classes of subgroups of order 2 (specifically, $0 \times \mathbb{Z}_2$ lies in one class, and $A \times 0$ is the other class because the number of conjugates of A is $|G : N_G(A)| = |G : G| = 1$ by Proposition 4.3.6[2]). As an abelian non-cyclic subgroup, H contains at least two subgroups of order 2 (one necessarily being A) and hence H can be $A \times \mathbb{Z}_2$ and its conjugates.
We assert that $H = A \times \mathbb{Z}_2$ is not normal in G. Indeed, the element $(1, x) \in G$ where x is the generator of $\mathbb{Z}_2$ can be used with $(2^{n-1}, 1) \in H$ to obtain $(1, x)(2^{n-1}, 1)(1, x)^{-1} = (1 + x \cdot 2^{n-1}, x)(-1, x) = (1 + 2^{n-1} + x \cdot (-1), x^2) = (2 + 2^{n-1}, 1)$ which is not in H since $2 + 2^{n-1}$ is neither 0 nor 2^{n-1} modulo 2^n for $n \geq 3$.

4.4: Let G be a p-group such that every abelian normal subgroup is cyclic and choose a maximal abelian normal subgroup $\mathbb{Z}_q \subset G$ with $q = p^n$, so we have the corresponding extension $0 \to \mathbb{Z}_q \to G \to H \to 1$. If $|H| = 1$ then $G \cong \mathbb{Z}_q$ is cyclic, hence of type (A). If $|H| = p$ then Theorem IV.4.1[1] states G is of type (A),(D),(E), or (F) because groups of type (C) have a non-cyclic abelian subgroup of index p by Proposition IV.4.4[1] (such subgroups are necessarily normal by Corollary 4.2.5[2]) and groups of type (B) are non-cyclic abelian normal subgroups of themselves [note: if G is of type (D) then we must have $|G| \geq 16 = 2^4$ because $G = D_8$ contains two non-cyclic abelian normal subgroups and $G = D_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is a non-cyclic abelian normal subgroup of itself]. Suppose, then, that $|H| \geq p^2$, and consider the normal subgroups $H' \subseteq H$ of order p. If such an H' acted trivially on $\mathbb{Z}_q$, then the inverse image $G' \subset G$ of $H' = G'/\mathbb{Z}_q$ would be a central extension of $H' \cong \mathbb{Z}_p$ by $\mathbb{Z}_q$, hence an abelian subgroup of G [as explained in Exercise IV.4.3] bigger than $\mathbb{Z}_q$. But $G' \lhd G$ by the Lattice Isomorphism Theorem (since $H' \lhd H$), so it contradicts the maximality of $\mathbb{Z}_q$ and hence H' cannot act trivially on $\mathbb{Z}_q$. Now G' is a p-group with a cyclic subgroup $\mathbb{Z}_q$ of index p, so by Theorem IV.4.1[1] it is of type (C),(D),(E), or (F) because types (A) and (B) were eliminated by the previous statement. Also, if H' acted as in (C)

then we would have a unique non-cyclic abelian subgroup G'' of G' of index p by Proposition IV.4.4[1]; this applies to all p and $n \geq 2$ *except* for the case $p = 2 = n$. Now G acts on G' by conjugation (since $G' \triangleleft G$) and hence maps G'' to another non-cyclic p-index subgroup. But G'' is the only such subgroup, so conjugation sends G'' to itself (i.e. $G'' \triangleleft G$) and G is a non-cyclic abelian normal subgroup of G, contradicting the hypothesis. Thus the only possibility is that $p = 2$ since groups of type (D),(E), and (F) are 2-groups, and the non-trivial element of H' acts as -1 [corresponding to (D),(E), and (C)] or $-1 + 2^{n-1}$ with $n \geq 3$ [corresponding to (F)]. This means that H' embeds in $\mathbb{Z}_{2^n}^* \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}$ as either $\mathbb{Z}_2 \times \{0\}$ or $\{(0,0),(1,2^{n-3})\}$. But the composite $H \to \mathbb{Z}_{2^n}^* \to \{\pm 1\}$ [where the first map is the action-representation and the latter map is the projection] has a non-trivial kernel, and we can simply take H' to be contained in the kernel; this implies H' is embedded as $\{0\} \times 2^{n-3}\mathbb{Z}/2^{n-2}\mathbb{Z} \subset \mathbb{Z}_{2^n}^*$ which is not any of the aforementioned subgroups. Thus we do not have $|H| \geq p^2$ and the proof is complete.

6.1: Suppose N is a group with trivial center.

Method 1: The center of N is $C = \{1\}$, and so $H^2(G,C) = 0 = H^3(G,C)$ for any group G. Then any homomorphism $\psi : G \to \mathrm{Out}(N)$ gives rise to an obstruction in $H^3(G,C) = 0$ which necessarily vanishes, thus $\mathcal{E}(G,N,\psi) \neq \varnothing$ by Theorem IV.6.7[1]. Therefore, by Theorem IV.6.6[1], $\mathcal{E}(G,N,\psi) \cong H^2(G,C) = 0$ and hence there is exactly one extension of G by N (up to equivalence) corresponding to any homomorphism $G \to \mathrm{Out}(N)$.

Method 2: Note that N is an $\mathrm{Aut}(N)$-crossed module via the canonical map $\alpha : N \to \mathrm{Aut}(N)$ and the canonical action of $\mathrm{Aut}(N)$ on N, and $\mathrm{Ker}\alpha = Z(N) = \{1\}$. Thus any extension of G by N fits into a commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r, "i"] \arrow[d, equal] & E \arrow[r, "\pi"] \arrow[d] & G \arrow[r] \arrow[d, "\psi"] & 1 \\
1 \arrow[r] & N \arrow[r, "\alpha"] & \mathrm{Aut}(N) \arrow[r] & \mathrm{Out}(N) \arrow[r] & 1
\end{tikzcd}$$

where ψ is determined by the E-extension. By Exercise IV.3.1(a), this is a pull-back diagram and hence all such extensions for a given ψ are equivalent (under the above diagram). Thus there is exactly one extension of G by N (up to equivalence) corresponding to any homomorphism $G \to \mathrm{Out}(N)$.

6.2: Let $1 \to N \to E \to G \to 1$ be an extension of finite groups such that $gcd(|N|,|G|) = 1$. Corollary IV.3.13[1] states that such an extension must split if N is abelian, and it is indeed true that this result could be generalized to the non-abelian case. Now one might hope to deduce this generalization directly from Theorem IV.6.6[1] in view of the vanishing of $H^2(G,C)$ [note that $H^2(G,C) = 0$ by the cohomology analogue of Exercise AE.16 because $gcd(|C|,|G|) = 1$, where C is the center of N]. However, this does not work because $\mathcal{E}(G,N,\psi)$ doesn't contain the semidirect product (hence the split extension) $N \rtimes G$ unless $\psi : G \to \mathrm{Out}(N)$ [which is determined by the above extension] lifts to a homomorphism $\varphi : G \to \mathrm{Aut}(N)$.

5 Chapter V: Products

1.1: Let $G = \langle t \rangle$ be a finite cyclic group of order m and let F be the periodic resolution $\cdots \to \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$, where $N = \sum_{i=0}^{m-1} t^i$ is the norm element (note that $F_n = \mathbb{Z}G$ for all $n \geq 0$). Let $\Delta : F \to F \otimes F$ be the map whose (p,q)-component $\Delta_{pq} : F_{p+q} \to F_p \otimes F_q$ is given by

$$\Delta_{pq}(1) = \begin{cases} 1 \otimes 1 & p \text{ even} \\ 1 \otimes t & p \text{ odd, } q \text{ even} \\ \sum_{0 \leq i < j \leq m-1} t^i \otimes t^j & p \text{ odd, } q \text{ odd} \end{cases}$$

Now Δ is a G-module map of degree 0, where $\Delta(g \cdot x) = (g,g) \cdot \Delta(x)$ with the action given by restriction of scalars with respect to the diagonal embedding $G \to G \times G$, so in order to prove that it is a diagonal approximation it suffices to show that it commutes with the boundary maps (making it a chain map) and is augmentation-preserving. Note that $(F \otimes F)_n = \bigoplus_{p+q=n} F_p \otimes F_q$ with differential $D(f \otimes f') = df \otimes f' + (-1)^{\deg f} f \otimes df'$, where d is the boundary operator of F. In dimension zero we have $(\varepsilon \otimes \varepsilon)[\Delta_{00}(1)] = \varepsilon(1) \otimes \varepsilon(1) = 1 \otimes 1 \equiv 1 = \varepsilon(1)$, so Δ is augmentation-preserving [the equivalence equation is from $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z}$]. Moving up a dimension, we have $\Delta_{00}[(t-1)1] = (t,t)[1 \otimes 1] - [1 \otimes 1] = t \otimes t - 1 \otimes 1$ and $D_1[\Delta_{10}(1) + \Delta_{01}(1)] = D_1(1 \otimes t) + D_1(1 \otimes 1) = (t-1) \otimes t + (-1)^1 1 \otimes 0 + 0 \otimes 1 + (-1)^0 1 \otimes (t-1) = t \otimes t - 1 \otimes t - 0 + 0 + 1 \otimes t - 1 \otimes 1 = t \otimes t - 1 \otimes 1$, so commutativity is satisfied. Moving up another dimension, we have $(\Delta_{10} + \Delta_{01})[N1] = \sum_k (t^k, t^k)[1 \otimes t] + \sum_k (t^k, t^k)[1 \otimes 1] = \sum_k t^k \otimes t^{k+1} + \sum_k t^k \otimes t^k$ and $D_2[\Delta_{02}(1) + \Delta_{11}(1) + \Delta_{20}(1)] = D_2(1 \otimes 1) + \sum_{i<j} D_2(t^i \otimes t^j) + D_2(1 \otimes 1) = 0 \otimes 1 + (-1)^0 1 \otimes N1 + \sum_{i<j} [(t-1)t^i \otimes t^j + (-1)^1 t^i \otimes (t-1)t^j] + N1 \otimes 1 + (-1)^2 1 \otimes 0 = 0 + \sum_k 1 \otimes t^k + \sum_{i<j} t^{i+1} \otimes t^j - \sum_{i<j} t^i \otimes t^j - \sum_{i<j} t^i \otimes t^{j+1} + \sum_{i<j} t^i \otimes t^j + \sum_k t^k \otimes 1 + 0 = \sum_k 1 \otimes t^k + (\sum_k t^k \otimes t^k - 1 \otimes 1) - (\sum_k 1 \otimes t^k - 1 \otimes t) + (\sum_k t^k \otimes t^{k+1} - 1 \otimes t - t^{m-1} \otimes 1) - (\sum_k t^k \otimes 1 - 1 \otimes 1 - t^{m-1} \otimes 1) = \sum_k t^k \otimes t^{k+1} + \sum_k t^k \otimes t^k$, so commutativity is satisfied. Thus we can proceed by induction.
For even integers $p+q = 2c$ ($c \in \mathbb{N}$) there are $c+1$ $(1 \otimes 1)$-elements, 0 $(1 \otimes t)$-elements, and c $(\sum_{i<j} t^i \otimes t^j)$-elements. For odd integers $p+q = 2c+1$ ($c \in \mathbb{N}$) there are $c+1$ $(1 \otimes 1)$-elements, 0 $(\sum_{i<j} t^i \otimes t^j)$-elements, and $c+1$ $(1 \otimes t)$-elements. This then gives $D_{2c}[\Delta_{2c}(1)] = c[N \otimes 1 + 1 \otimes N] + c \sum_{i<j} [t^{i+1} \otimes t^j - t^i \otimes t^{j+1}]$ and $\Delta_{2c-1}(N1) = c \sum_k [t^k \otimes t^{k+1} + t^k \otimes t^k]$, and these are equal because they are just a multiple of the low-dimensional case; the $c[N \otimes 1 + 1 \otimes N]$ came from $(c+1)[N \otimes 1 + 1 \otimes N]$ minus $N \otimes 1 + 1 \otimes N$ which accounts for the two zero-dimensional tensor components which give trivial boundary, i.e. $D(1 \otimes 1 + 1 \otimes 1) = (0 \otimes 1 + N \otimes 1) + (N \otimes 1 + 1 \otimes 0)$. Commutativity in the next dimension is also satisfied [similar calculation], and the proof is complete.

2.1: Let M (resp. M') be an arbitrary G-module (resp. G'-module), let F (resp. F') be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ (resp. $\mathbb{Z}G'$), and consider the map $(F \otimes_G M) \otimes (F' \otimes_{G'} M') \to (F \otimes F') \otimes_{G \times G'} (M \otimes M')$ given by $(x \otimes m) \otimes (x' \otimes m') \mapsto (x \otimes x') \otimes (m \otimes m')$. Note that $(F \otimes_G M) \otimes (F' \otimes_{G'} M') = (F \otimes M)_G \otimes (F' \otimes M')_{G'}$, which is the quotient of $F \otimes M \otimes F' \otimes M'$ by the subgroup generated by elements of the form $gx \otimes gm \otimes g'x' \otimes g'm'$. The isomorphism $F \otimes M \xrightarrow{\cong} M \otimes F$ of chain complexes (M in dimension 0) is given by $x \otimes m \mapsto (-1)^{\deg m \cdot \deg x} m \otimes x = m \otimes x$, and so the aforementioned quotient is isomorphic to $F \otimes F' \otimes M \otimes M'$ modulo the subgroup generated by elements of the form $(gx \otimes g'x' \otimes gm \otimes g'm') = (g,g') \cdot (x \otimes x' \otimes m \otimes m')$ where this latter action is the diagonal $(G \times G')$-action. Now this is precisely $(F \otimes F' \otimes M \otimes M')_{G \times G'} = (F \otimes F') \otimes_{G \times G'} (M \otimes M')$ and hence the considered map is an isomorphism.
Assuming now that either M or M' is $\mathbb{Z}$-free, we have a corresponding Künneth formula
$\bigoplus_{p=0}^{n} H_p(G,M) \otimes H_{n-p}(G',M') \hookrightarrow H_n(G \times G', M \otimes M') \twoheadrightarrow \bigoplus_{p=0}^{n-1} \operatorname{Tor}_1^{\mathbb{Z}}(H_p(G,M), H_{n-p-1}(G',M'))$ by Proposition I.0.8[2]. Note that in order to apply the proposition we needed one of the chain complexes (say, $F \otimes_G M$) to be dimension-wise $\mathbb{Z}$-free (and so with a free resolution F this means we needed M to be $\mathbb{Z}$-free). *Actually, the general Künneth theorem has a more relaxed condition and it suffices to choose M (or M') as a $\mathbb{Z}$-torsion-free module.*

*Must require some conditions on groups/actions: We have assumed G acts trivially on M' while G' acts trivially on M. Since these actions don't mix, we cannot deduce a Universal Coefficients theorem (UCT) with nontrivial actions.

2.2: [[no proofs, just notes]] (*for more info, see topological analog in* §60[4])
Let M (resp. M') be an arbitrary G-module (resp. G'-module), let F (resp. F') be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ (resp. $\mathbb{Z}G'$), and consider the cochain cross-product $\mathcal{H}om_G(F,M)\otimes\mathcal{H}om_{G'}(F',M') \to \mathcal{H}om_{G\times G'}(F\otimes F', M\otimes M')$ which maps the cochains u and u' to $u\times u'$ given by $\langle u\times u', x\otimes x'\rangle = (-1)^{\deg u'\cdot\deg x}\langle u,x\rangle\otimes\langle u',x'\rangle$. This map is an isomorphism under the hypothesis that either $H_*(G,M)$ or $H_*(G',M')$ is of *finite type*, that is, the ith-homology group is finitely generated for all i (alternatively, we could simply require the projective resolution F or F' to be finitely generated). For example, if $M=\mathbb{Z}$ then $\mathcal{H}om_G(-,\mathbb{Z})$ commutes with finite direct sums, so we need only consider the case $F=\mathbb{Z}G$. An inverse to the above map is given by $t\mapsto\varepsilon\otimes\phi$, where ε is the augmentation map and $\phi: F'\to M'$ is given by $\phi(f')=t(1\otimes f')$. Note that this does not hold for infinitely generated $P=\bigoplus^\infty\mathbb{Z}G$ because $\mathcal{H}om_G(P,\mathbb{Z})\cong\prod^\infty\mathbb{Z}$ is not $\mathbb{Z}$-projective (i.e. free abelian).
Assuming now that either M or M' is $\mathbb{Z}$-free, we have a corresponding Künneth formula
$\bigoplus_{p=0}^n H^p(G,M)\otimes H^{n-p}(G',M') \hookrightarrow H^n(G\times G', M\otimes M') \twoheadrightarrow \bigoplus_{p=0}^{n+1}\mathrm{Tor}_1^{\mathbb{Z}}(H^p(G,M), H^{n-p+1}(G',M'))$ by Proposition I.0.8[2].

*Must require some conditions on groups/actions: We have assumed G acts trivially on M' while G' acts trivially on M. Since these actions don't mix, we cannot deduce a Universal Coefficients theorem (UCT) with nontrivial actions.

3.1: Let $m\in H^0(G,M)=M^G$ and $u\in H^q(G,N)$, and let $f_m: H^*(G,N)\to H^*(G,M\otimes N)$ denote the map induced by the coefficient homomorphism $n\mapsto m\otimes n$. This homomorphism is also given by $n\mapsto 1\otimes n\mapsto m\otimes n$ where the former map in the composition is the canonical isomorphism $N\xrightarrow{\cong}\mathbb{Z}\otimes N$, the latter map in the composition is $F_m\otimes id_N:\mathbb{Z}\otimes N\to M\otimes N$, and $F_m:\mathbb{Z}\to M$ is given by $F(1)=m$. Then using two properties of the cup product (existence of identity element and naturality with respect to coefficient homomorphisms) we obtain $f_m(u)=(F_m\otimes id_N)^*(1\smile u)=F_m^*(1)\smile id_N^*(u)=m\smile u$.
Now let $m\in H^0(G,M)=M^G$ and $z\in H_q(G,N)$, and let $f_m: H_*(G,N)\to H_*(G,M\otimes N)$ denote the map induced by the same coefficient homomorphism $n\mapsto m\otimes n$. Then using two properties of the cap product (existence of identity element and naturality with respect to coefficient homomorphisms) we obtain $f_m(z)=(F_m\otimes id_N)^*(1\frown z)=F_m^*(1)\frown id_N^*(z)=m\frown z$.

3.2: Consider the diagonal transformation Δ presented in Exercise V.1.1, along with the cohomology groups $H^{2r}(G,M)\cong M^G/NM$ and $H^{2r+1}(G,M')\cong\mathrm{Ker}(N:M'\to M')/IM'$ of the finite cyclic group $G=\langle\sigma\rangle$ of order n, where $I=\langle\sigma-1\rangle$ is the augmentation ideal of G. The cup product in $H^*(G,M\otimes M')$ is given by $u\smile v=(u\times v)\circ\Delta$ with $\langle u\times v, x\otimes x'\rangle=(-1)^{\deg v\cdot\deg x}\langle u,x\rangle\otimes\langle v,x'\rangle$. Choose representatives $\langle u,x\rangle=m\in M$ of $H^i(G,M)$ with $m\in M^G$ for i even and $m\in\mathrm{Ker}(N:M\to M)$ for i odd, and choose representatives $\langle v,x'\rangle=m'\in M'$ of $H^j(G,M')$ with $m'\in M'^G$ for j even and $m'\in\mathrm{Ker}(N:M'\to M')$ for j odd. If i is even then $(-1)^{\deg v\cdot\deg x}=(-1)^{\deg v\cdot i}=1$, and if j is even then $(-1)^{\deg v\cdot\deg x}=(-1)^{-j\cdot\deg x}=1$, and if both i and j are odd then $(-1)^{\deg v\cdot\deg x}=(-1)^{-j\cdot i}=-1$. Thus the cup product element of $H^{i+j}(G,M\otimes M')$ is represented by $m\otimes m'$ for i or j even and is represented by $-\sum_{0\le p<q\le n-1}t^pm\otimes t^qm'$ when i and j are both odd.

3.3(a): Let G be a finite group which acts freely on S^{2k-1}, and consider the exact sequence of G-modules from pg20[1], $0\to\mathbb{Z}\to C_{2k-1}\to\cdots\to C_1\to C_0\to\mathbb{Z}\to 0$ where each $C_i=C_i(S^{2k-1})$ is free. Tensoring the sequence with an arbitrary G-module M gives an exact sequence (as explained on pg61[1]) $0\to M\to C_{2k-1}\otimes M\xrightarrow{\partial_{2k-1}}\cdots\to C_1\otimes M\xrightarrow{\partial_1}C_0\otimes M\xrightarrow{\partial_0}M\to 0$. We can break this up into short exact sequences $0\to\mathrm{Ker}\partial_0\to C_0\otimes M\to M\to 0$ and $0\to\mathrm{Ker}\partial_i\to C_i\otimes M\to\mathrm{Ker}\partial_{i-1}\to 0$ for all i, and we can then apply the $H^*(G,-)$ functor to obtain corresponding long exact cohomology sequences. First note that $\mathcal{H}om_G(F,C_*\otimes M)\to\mathcal{H}om_G(F,M)$ is a weak equivalence by Theorem I.8.5[1], so $H^n(G,C_i\otimes M)\cong H^n(\mathcal{H}om_G(F,M))=0$ for all i with $n>0$ (M is considered a chain complex concentrated in dimension 0). Thus we can apply the dimension-shifting argument to the above short exact sequences and obtain $H^i(G,M)\cong H^{i+1}(G,\mathrm{Ker}\partial_0)\cong H^{i+2}(G,\mathrm{Ker}\partial_1)\cong\cdots\cong H^{i+2k}(G,M)$ for $i>0$ where we note that $\mathrm{Ker}\partial_{2k-1}=M$. For $i=0$ we use the first short exact sequence mentioned above to obtain the exact sequence $H^0(G,M)\to H^1(G,\mathrm{Ker}\partial_0)\to H^1(G,C_0\otimes M)=0$, with $H^1(G,\mathrm{Ker}\partial_0)\cong H^{2k}(G,M)$. Thus there is an iterated coboundary map $d: H^i(G,M)\to H^{i+2k}(G,M)$ which is an isomorphism for $i>0$ and an epimorphism for $i=0$.

3.3(b): Consider the "periodicity map" d from part(a) above with the finite group G which acts freely on the sphere S^{2k-1}. Since d is simply an iteration of coboundary maps δ, we can use Property V.3.3[1] which states $d(w \smile v) = d(w) \smile v$ for any $w \in H^*(G,\mathbb{Z})$ and $v \in H^*(G,M)$. Choosing $w = 1 \in H^0(G,\mathbb{Z})$ and noting that $1 \smile v = v$ by Property V.3.4[1], $d(v) = d(1 \smile v) = d(1) \smile v$. But d is an isomorphism from $H^0(G,\mathbb{Z})$ to $H^{2k}(G,\mathbb{Z})$ by part(a) above, so there exists an element $u \in H^{2k}(G,\mathbb{Z})$ satisfying $d(1) = u$. Thus there is an element $u \in H^{2k}(G,\mathbb{Z})$ such that the "periodicity map" d of $H^*(G,M)$ is given by $d(v) = u \smile v$ for all $v \in H^*(G,M)$.

3.3(c): Let G be a finite cyclic group of order $|G| = n$, and note that it acts freely on the circle S^1 by rotations. By part(b) above, the periodicity isomorphism d maps $1 \in H^0(G,\mathbb{Z}) \cong \mathbb{Z}$ to a generator $\alpha \in H^2(G,\mathbb{Z}) \cong \mathbb{Z}_n$. For the ring structure on $H^*(G,\mathbb{Z})$ with multiplication being the cup product, α^2 is a generator of $H^4(G,\mathbb{Z}) \cong \mathbb{Z}_n$ since d is an isomorphism and $d(\alpha) = \alpha \smile \alpha = \alpha^2$. Generalizing, α^m is a generator of $H^{2m}(G,\mathbb{Z}) \cong \mathbb{Z}_n$ and hence the cohomology ring is the polynomial ring $H^*(G,\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(n\alpha)$ with $|\alpha| = 2$.
Alternatively, we could note that the infinite-dimensional lens space L^∞ is a $K(G,1)$-complex, so $H^*(G,\mathbb{Z}) \cong H^*(L^\infty)$ by the cohomological analog of Proposition II.4.1[1]. The ring structure was calculated in Example 3.41 on pg251[3], giving $H^*(L^\infty) \cong \mathbb{Z}[\alpha]/(n\alpha)$ with $|\alpha| = 2$.

3.4: Let G be cyclic of order n and consider the endomorpism $\alpha(m)$ of G given by $\alpha(m)g = g^m$, for any $m \in \mathbb{Z}_n$. Since $\alpha(m)^*(u \smile v) = \alpha(m)^*u \smile \alpha(m)^*v$ and $H^*(G,\mathbb{Z})$ consists of elements of the form $\sum_i z_i\beta^i$ with $z_i \in \mathbb{Z}$ and $|\beta| = 2$ by Exercise V.3.3(c), it suffices to calculate $\alpha(m)^*$ on $H^2(G,\mathbb{Z})$ in order to calculate $\alpha(m)^* : H^*(G,\mathbb{Z}) \to H^*(G,\mathbb{Z})$. By Exercise III.1.3 we have the universal coefficient isomorphism $H^2(G,\mathbb{Z}) \cong \mathrm{Ext}(H_1G,\mathbb{Z})$ since $\mathrm{Hom}(H_1G,\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}_n,\mathbb{Z}) = 0$. The map $\alpha(m)$ is multiplication by m on $G_{ab} = G$ and hence is multiplication by m on H_1G by Exercise II.6.3(a). Now $f(mg) = mf(g)$ for any group homomorphism f, so given a free resolution F of the abelian group $G_{ab} = H_1G = G$ (namely, its presentation) we have the commutative diagram

$$\begin{tikzcd}
0 & \mathrm{Hom}(F_1,\mathbb{Z}) \arrow[l] \arrow[d, "m"] & \mathrm{Hom}(F_0,\mathbb{Z}) \arrow[l, "\phi"'] \arrow[d, "m"] & \mathrm{Hom}(G,\mathbb{Z}) \arrow[l] \arrow[d, "m"] \\
0 & \mathrm{Hom}(F_1,\mathbb{Z}) \arrow[l] & \mathrm{Hom}(F_0,\mathbb{Z}) \arrow[l, "\phi"'] & \mathrm{Hom}(G,\mathbb{Z}) \arrow[l]
\end{tikzcd}$$

Thus $\alpha(m)$ induces multiplication by m on $\mathrm{Coker}\phi \equiv \mathrm{Ext}(H_1G,\mathbb{Z})$ and hence $\alpha(m)^*$ is multiplication by m on $H^2(G,\mathbb{Z})$ by naturality of the universal coefficient formula. Alternatively, we could have used the interpretation of H^2 in terms of group extensions (Exercise IV.3.1(a)) which states that $\alpha(m)$ induces a map $\mathcal{E}(G,\mathbb{Z}) \to \mathcal{E}(G,\mathbb{Z})$, sending the extension E to the fiber-product $E \times_G G$ which correponds to mE (i.e. the elements $e^m \in E$) and hence gives the m-multiplication map in cohomology by Theorem IV.3.12[1]. Referring back to the cohomology ring, $\alpha(m)^*$ is multiplication by m^i for the $(2i)^{\text{th}}$-dimension since $\sum_i z_i\beta^i \mapsto \sum_i z_i\alpha(m)^*(\beta \smile \cdots \smile \beta) = \sum_i z_i[\alpha(m)^*\beta] \smile \cdots \smile [\alpha(m)^*\beta] = \sum_i z_i(m\beta) \smile \cdots \smile (m\beta) = \sum_i z_i m^i \beta^i$.

3.5: The symmetric group S_3 on three letters has a semi-direct product representation $S_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\mathbb{Z}_3$ by conjugation. Thus $H^*(S_3) = H^*(S_3)_{(2)} \oplus H^*(S_3)_{(3)} \cong H^*(S_3)_{(2)} \oplus H^*(\mathbb{Z}_3)^{\mathbb{Z}_2}$ by Theorem III.10.3[1], with $H^*(S_3)_{(2)}$ isomorphic to the set of S_3-invariant elements of $H^*(\mathbb{Z}_2)$. Exercise III.10.1 showed that $H^*(S_3)_{(2)} \cong \mathbb{Z}_2$, so it suffices to compute $H^*(\mathbb{Z}_3)^{\mathbb{Z}_2}$ where we know that $\mathbb{Z}_2$ acts by conjugation on $\mathbb{Z}_3$ ($1 \mapsto 1$, $x \mapsto x^2$, $x^2 \mapsto x$). But this action can be considered as the endomorphism $\alpha(2)$ from Exercise V.3.4 above since $(1)^2 = 1$ and $(x)^2 = x^2$ and $(x^2)^2 = x^4 = x$, and that exercise implies that the induced map on the $(2i)^{\text{th}}$-cohomology is multiplication by 2^i [we know that the cohomology is trivial in odd dimensions]. Now $2^1 \equiv 2 \bmod 3$ and $2^2 \equiv 1 \bmod 3$, so by multiplying both of those statements by 2 repeatedly we see that $2^i \equiv 1 \bmod 3$ for i even and $2^i \equiv 2 \bmod 3$ for i odd. Thus the largest $\mathbb{Z}_2$-submodule of $H^{2i}(\mathbb{Z}_3) \cong \mathbb{Z}_3$ on which $\mathbb{Z}_2$ acts trivially is $\mathbb{Z}_3$ (itself) for i even, and is 0 for i odd. It now follows that the integral cohomology $H^*(S_3)$ is the same as that which was deduced in Exercise III.10.1, namely, it is $\mathbb{Z}_2$ in the $2 \bmod 4$ dimensions and is $\mathbb{Z}_6$ in the $0 \bmod 4$ dimensions and is 0 otherwise (besides the 0th-dimension in which it is $\mathbb{Z}$).

4.1:

4.2:

5.1(a): Let G be an abelian group so that $\mathbb{Z}G$ is a commutative ring, and let $M \otimes_{\mathbb{Z}G} N$ be the tensor product with $\mathbb{Z}G$-module structure defined by $r \cdot (m \otimes n) = rm \otimes n = m \otimes rn$, where $r \in \mathbb{Z}G$. There exists a Pontryagin product given by $H_*(G, M) \otimes H_*(G, N) \xrightarrow{\times} H_*(G \times G, M \otimes N) \xrightarrow{\mu_*} H_*(G, M \otimes_{\mathbb{Z}G} N)$. The former map is the homology cross product (see pg109[1]), where $G \times G$ acts on $M \otimes N$ via $(g, g') \cdot (m \otimes n) = gm \otimes g'n$. The latter map in the composition is induced by $\mu = (\alpha, f) : (G \times G, M \otimes N) \to (G, M \otimes_{\mathbb{Z}G} N)$, where $\alpha(g, g') = gg'$ and $f(m \otimes n) = m \otimes n$. As explained on pg79[1], μ_* is a well-defined map as long as $f(rm) = \alpha(r)f(m)$ for $r \in \mathbb{Z}G$. This is indeed the case because $\alpha(g, g')f(m \otimes n) = gg' \cdot (m \otimes n) = g \cdot (m \otimes g'n) = gm \otimes g'n = f(gm \otimes g'n) = f[(g, g') \cdot (m \otimes n)]$.

5.1(b): Take $G = \mathbb{Z}_3 = \langle g \rangle$ and take the modules $M = N = \mathbb{Z}_7$ with nontrivial $\mathbb{Z}_3$-action, i.e. $g \cdot x^i = x^{2i}$ and $g^2 \cdot x^i = x^{4i}$ with $g^3 = 1$ [consistency check: $g^3 \cdot x^i = x^{8i} = (x^8)^i = x^i$]. Note that there is only one nontrivial action in this scenario because an action is a homomorphism $\mathbb{Z}_3 \to \mathrm{Aut}(\mathbb{Z}_7) \cong \mathbb{Z}_6$ and there are two such maps, the trivial map and the inclusion. The tensor product $\mathbb{Z}_7 \otimes_{\mathbb{Z}_3} \mathbb{Z}_7$ can be interpreted as the group $(\mathbb{Z}_7 \otimes \mathbb{Z}_7)_{\mathbb{Z}_3}$ where $\mathbb{Z}_3$ acts diagonally, $g \cdot (m \otimes n) = gm \otimes gn$. This group contains only idempotent elements since $(x^i \otimes x^j)^2 = x^{2i} \otimes x^{2j} = gx^i \otimes gx^j = g \cdot (x^i \otimes x^j) \sim x^i \otimes x^j$. Now the tensor product $\mathbb{Z}_7 \otimes_{\mathbb{Z}[\mathbb{Z}_3]} \mathbb{Z}_7$ can be interpreted as the group $(\mathbb{Z}_7 \otimes \mathbb{Z}_7)_{\mathbb{Z}_3}$ where $\mathbb{Z}_3$ acts anti-diagonally, $g \cdot (m \otimes n) = g^{-1}m \otimes gn = g^2 m \otimes gn$. This group, however, does contain elements which are not idempotent. For example, $(x \otimes x)^2 = x^2 \otimes x^2 = g^2 x^4 \otimes gx \sim x^4 \otimes x = (x \otimes x)(x^3 \otimes 1)$ and $(x^3 \otimes 1) \nsim 1 \otimes 1$. Thus $\mathbb{Z}_7 \otimes_{\mathbb{Z}[\mathbb{Z}_3]} \mathbb{Z}_7 \ncong \mathbb{Z}_7 \otimes_{\mathbb{Z}_3} \mathbb{Z}_7$.

5.2: *If G and G' are abelian and k is a commutative ring, then the cohomology cross-product $H^*(G, k) \otimes_k H^*(G', k) \xrightarrow{\times} H^*(G \times G', k \otimes k)$ is a k-algebra homomorphism. Moreover, $z \times z' = p_* z \smile p'_* z'$ for any $z \in H^*(G, k)$ and $z' \in H^*(G', k)$, where $p : G \times G' \to G$ and $p' : G \times G' \to G'$ are the projections.*
To prove these two statements, note first that the cross-product is given by $\langle z \times z', x \otimes x' \rangle = (-1)^{|z'||x|} z(x) \otimes z'(x')$ for $x \in F$ and $x' \in F'$, with F and F' being the resolutions for G and G', respectively. Then

$$\langle (z_1 \otimes z_1')(z_2 \otimes z_2'), x \otimes x' \rangle = \langle (-1)^{|z_1'||z_2|} z_1 z_2 \times z_1' z_2', x \otimes x' \rangle = (-1)^{|z_1'||z_2|}(-1)^{|z_1' z_2'||x|} z_1(x) z_2(x) \otimes z_1'(x') z_2'(x') = (-1)^{|z_1'||x|}(-1)^{|z_1'||x|}[z_1(x) \otimes z_1'(x')][z_2(x) \otimes z_2'(x')] = \langle z_1 \times z_1', x \otimes x' \rangle \cdot \langle z_2 \times z_2', x \otimes x' \rangle$$

and so the cross-product is a k-algebra homomorphism (it obviously commutes with addition). Since $z \otimes z' = (z \otimes 1) \cdot (1 \otimes z')$ with $1 \in H^0(G, \mathbb{Z}) = \mathbb{Z}$, it follows that $z \times z' = (z \times 1) \smile (1 \times z')$, where we make the identifications $\mathbb{Z} \otimes k = k = k \otimes \mathbb{Z}$ to switch from tensor-multiplication to cup product. I claim that $z \times 1 = p_* z$; indeed, by naturality of the cross-product it suffices to check this for $G' = 1$. But this is obvious because $p_* : H^*(G, k) \to H^*(G \times 1, k)$ maps z to $z \times 1$, noting that $F' = 1$. Similarly, $1 \times z' = p'_* z'$, whence the proposition.

5.3(a): A *directed set* D is a partially-ordered set having the property that for each pair $\alpha, \beta \in D$ there exists $\gamma \in D$ such that $\alpha, \beta \leq \gamma$. A *directed system* of groups is a family of groups $\{G_\alpha\}$ indexed by a directed set D along with a family of homomorphisms $\{f_{\alpha\beta} : G_\alpha \to G_\beta\}$ such that $f_{\alpha\alpha} = id_{G_\alpha}$ and $f_{\alpha\gamma} = f_{\beta\gamma} \circ f_{\alpha\beta}$ for $\alpha \leq \beta \leq \gamma$. The *direct limit* $G = \varinjlim G_\alpha$ of this directed system is defined to be $\coprod_\alpha G_\alpha / \sim$ where $g_\alpha \sim g_\beta$ if there exists some $\gamma \in D$ such that $f_{\alpha\gamma}(g_\alpha) = f_{\beta\gamma}(g_\beta)$. Now for any G-module M, we have a compatible family of maps $H_*(G_\alpha, M) \to H_*(G, M)$, hence a map $\varphi : \varinjlim H_*(G_\alpha, M) \to H_*(G, M)$. It is true that homology commutes with direct limits of chain complexes (see Albrecht Dold's *Lectures on Algebraic Topology*, Proposition VIII.5.20), so to prove that φ is an isomorphism it suffices to show that $\varinjlim(F_\alpha \otimes_{G_\alpha} M) = F \otimes_G M$ where F_α is the standard resolution for G_α and F is the standard resolution for G. The obvious maps $F_\alpha \to F_\beta$ are given by $(g_1, \ldots, g_n) \mapsto (f_{\alpha\beta}(g_1), \ldots, f_{\alpha\beta}(g_n))$, and F is obviously the direct limit of F_α. Thus $F \otimes_G M = (\coprod_\alpha F_\alpha / \sim) \otimes_G M$, and since we can switch actions from G to G_α via restriction of scalars, $(\coprod_\alpha F_\alpha / \sim) \otimes_G M = \coprod_\alpha (F_\alpha \otimes_{G_\alpha} M) / \sim$ where $\sim$ is simply altered by tensoring each tuple with an element of M. But $\coprod_\alpha (F_\alpha \otimes_{G_\alpha} M) / \sim$ is by definition the direct limit of $F_\alpha \otimes_{G_\alpha} M$, so $\varinjlim(F_\alpha \otimes_{G_\alpha} M) = F \otimes_G M$ and hence group homology commutes with direct limits of groups.

5.3(b): It is a fact that any group is the direct limit of its finitely generated subgroups. So for any abelian

group $G = \varinjlim G_\alpha$ and commutative ring k, the homology ring $H_*(G,k)$ is isomorphic to $\varinjlim H_*(G_\alpha, k)$ by part(a) above. If each of those rings $H_*(G_\alpha, k)$ is strictly anti-commutative, then $H_*(G,k)$ will obviously be strictly anti-commutative since it is a quotient of the direct sum of those rings. Thus we can reduce to the case where G is a finitely generated abelian group, hence isomorphic to a finite product of cyclic groups. Let's argue by induction on the number of cyclic factors. The infinite cyclic group has resolution $F = \bigwedge(x)$ which is strictly anti-commutative as explained in subsection 5.2 on pg118[1], and the finite cyclic group has resolution $F = \bigwedge(e_1) \otimes_{\mathbb{Z}G} \Gamma(e_2)$ which is strictly anti-commutative by property (iii) of subsection 5.3 on pg119[1]. Since the admissible product on F induces a k-bilinear product on $F \otimes_G k$ via $(f \otimes k)(f' \otimes k') = ff' \otimes kk'$, the complex $F \otimes_G k$ is strictly anti-commutative; thus $H_*(G,k)$ is strictly anti-commutative for G cyclic. Applying the inductive hypothesis, we can attach another cyclic factor by the method of subsection 5.4 on pg119[1]: F and F' are resolutions with admissible product for G (cyclic) and G' (inductive group), so $F \otimes F'$ is a resolution with admissible product for $G \times G'$. This resolution is strictly anti-commutative because $(x \otimes y)(x \otimes y) = (-1)^{|x| \cdot |y|} x^2 \otimes y^2 = 0$ if either x^2 or y^2 is 0, so for x (resp. y) of odd degree and y (resp. x) of even degree we have $x \otimes y$ of odd degree which satisfies $(x \otimes y)^2 = 0$. Then $(F \otimes F') \otimes_{G \times G'} k$ is strictly anti-commutative and so is $H_*(G \times G', k)$, completing the inductive process. Thus the ring $H_*(G,k)$ is strictly anti-commutative for any abelian group G and commutative ring k.

5.4: Let $n = p + q + r$, and let $\sigma \in S_n$ be a permutation with *signature* $\operatorname{sgn}(\sigma)$ being the number of inversions of σ; an *inversion* of σ is a pair of elements (i,j) such that $i < j$ and $\sigma(i) > \sigma(j)$ [it also indicates the number of swaps needed to give the original sequence ordering]. A permutation σ is called a (p,q,r)-*shuffle* if $\sigma(i) < \sigma(j)$ for $1 \le i < j \le p$ and for $p+1 \le i < j \le p+q$ and for $p+q+1 \le i < j \le p+q+r$. But this permutation is clearly the composition of a (p,q)-shuffle τ_1 and a $(p+q,r)$-shuffle τ_2 since the former shuffle will give $\tau_1(i) < \tau_1(j)$ for $1 \le i < j \le p$ and for $p+1 \le i < j \le p+q$, and the latter shuffle will give $\tau_2(i) < \tau_2(j)$ for $p+q+1 \le i < j \le p+q+r$ and will preserve the ordering of the former shuffle via $1 \le i < j \le p+q$. Then $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\tau_1) + \operatorname{sgn}(\tau_2)$ because the inversions of τ_2 give the original ordering of the sequence up to changes in $\{1, \dots, p+q\}$ and the inversions of τ_1 give the original ordering of that set. Therefore (in the bar resolution),

$$[g_1|\cdots|g_p] \cdot [g_{p+1}|\cdots|g_{p+q}] \cdot [g_{p+q+1}|\cdots|g_{p+q+r}] = \textstyle\sum_\sigma \sigma[g_1|\cdots|g_{p+q+r}]$$

where σ ranges over the (p,q,r)-shuffles.
The notation is $[g_1|\cdots|g_n] \cdot [g_{n+1}|\cdots|g_{n+m}] = \sum_\tau \tau[g_1|\cdots|g_{n+m}]$ where τ ranges over all (p,q)-shuffles, and $\sigma[g_1|\cdots|g_n] = (-1)^{\operatorname{sgn}\sigma}[g_{\sigma^{-1}(1)}|\cdots|g_{\sigma^{-1}(n)}]$.
Generalizing, we have

$$[g_1|\cdots|g_{a_1}] \cdot [g_{a_1+1}|\cdots|g_{a_1+a_2}] \cdots [g_{a_1+\cdots+a_{n-1}+1}|\cdots|g_{a_1+\cdots+a_n}] = \textstyle\sum_\sigma \sigma[g_1|\cdots|g_{a_1+\cdots+a_n}]$$

where σ ranges over the $(a_1, \dots, a_n)$-shuffles.

6.1: Let G be an abelian group (written additively) with $n \in \mathbb{Z}$, and consider the endomorphism $g \mapsto ng$ of G. To see what it induces on the rational homology ring $H_*(G, \mathbb{Q})$ it suffices to figure out what the endomorphism induces on the exterior algebra $\bigwedge^*(G \otimes \mathbb{Q})$ by Theorem V.6.4[1] (they're isomorphic), noting that the isomorphism in said theorem is natural. Now the induced map on $\bigwedge^*(G \otimes \mathbb{Q})$ is uniquely determined by the induced map $\varphi : G \otimes \mathbb{Q} \to \bigwedge^1(G \otimes \mathbb{Q}) = G \otimes \mathbb{Q}$ by the universal mapping property of exterior algebras (pg122[1]), and this map φ is given by $g \otimes q \mapsto ng \otimes q = n(g \otimes q)$, i.e. multiplication by n. Then on the i-fold tensor product of $G \otimes Q$ with itself (hence the exterior algebra) we have the induced map as $f_1 \otimes \cdots \otimes f_i \mapsto nf_1 \otimes \cdots \otimes nf_i = n^i(f_1 \otimes \cdots \otimes f_i)$ where $f_1, \dots, f_i \in G \otimes \mathbb{Q}$. Thus the original endomorphism on G induces multiplication by n^i on $H_*(G, \mathbb{Q})$ for the ith-dimension.

6.2: Let A and B be strictly anti-commutative graded k-algebras, where k is a commutative ring. We assert that $A \otimes_k B$ is the *sum* (i.e. *coproduct*) of A and B in the category of strictly anti-commutative graded k-algebras, via the maps $f_A : A \to A \otimes_k B$ and $f_B : B \to A \otimes_k B$ with $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$. The coproduct refers to the pair $(A \otimes_k B, \{f_A, f_B\})$ satisfying the universal property that given a family of algebra homomorphisms $\{g_A : A \to C,\ g_B : B \to C\}$, there exists a unique algebra homomorphism $h : A \otimes_k B \to C$ such that $hf_A = g_A$ and $hf_B = g_B$. We define a k-bilinear map $A \times B \to C$ given by $a \times b \mapsto g_A(a)g_B(b)$. Then Corollary 10.4.16[2] gives us a unique homomorphism $h : A \otimes_k B \to C$ given by $a \otimes b \mapsto g_A(a)g_B(b)$,

and this is clearly an algebra homomorphism since $h[(a_1 \otimes b_1)(a_2 \otimes b_2)] = h[(-1)^{|a_2||b_1|}a_1a_2 \otimes b_1b_2] = (-1)^{|a_2||b_1|}g_A(a_1a_2)g_B(b_1b_2) = (-1)^{|a_2||b_1|}g_A(a_1)[(-1)^{|a_2||b_1|}g_B(b_1)g_A(a_2)]g_B(b_2) = h(a_1 \otimes b_1)h(a_2 \otimes b_2)$. Now $hf_A(a) = h(a \otimes 1) = g_A(a)g_B(1) = g_A(a)$, with a similar calculation for B, and so the universal property of tensor products (Theorem 10.4.10[2]) guarantees that $A \otimes_k B$ is the categorical sum $A + B$.

6.3(a): Let A be a strictly anti-commutative graded ring with a differential ∂ and a system of divided powers, so that there is a family of functions $\varphi_i : A_{2n} \to A_{2ni}$ denoted $x \mapsto x^{(i)}$ satisfying the properties on pg124[1]. Note that $x^{(i)}$ is a cycle if x is a cycle because $\partial x^{(i)} = x^{(i-1)}\partial x = x^{(i-1)} \cdot 0 = 0$. Thus φ_i restricted to the kernel $Z_{2n} \subseteq A_{2n}$ gives a function $Z_{2n} \to Z_{2ni}$. These functions inherit the same properties associated with A because every term in all of the properties are cycles x^i and $x^{(j)}$, assuming x is a cycle. Now suppose $x^{(i)}$ is a boundary whenever x is a boundary. Then φ_i induces a function $H_{2n}A \to H_{2ni}A$ because $x = x + \partial y \mapsto x^{(i)} + (\partial y)^{(i)} = x^{(i)} + \partial w = x^{(i)}$, and the properties remain untouched. Therefore, we have an induced system of divided powers on H_*A.

6.3(b): Consider the divided polynomial algebra $\Gamma(y)$ with $\deg y = 2$, and assume that y is a boundary, $y = \partial x$ for some x. I claim that $y^{(i)}$ is not a boundary. Indeed, suppose $y^{(i)} = \partial f$ for some element $f = \sum_j z_j y^{(j)} \in \Gamma(y)$, where $z_j \in \mathbb{Z}$. Then $y^i/i! = \sum_j z_j \partial(y^j)/j! = \sum_j z_j[jy^{j-1}\partial y]/j! = \partial y \sum_j z_j y^{j-1}/(j-1)! = \partial^2 x \cdot w = 0 \cdot w = 0$. But this implies $y^i = 0$, a contradiction.

6.4(a): By Theorem V.6.4[1] we have an injection $\psi : \bigwedge^*(G \otimes k) \to H_*(G, k)$ for G abelian and k a PID. This k-algebra map was the unique extension of the isomorphism $G \otimes k \to H_1(G, k)$ in dimension 1. Now $\psi[(g \otimes 1) \wedge (h \otimes 1)] = \psi(g \otimes 1) \cdot \psi(h \otimes 1)$ by definition of an algebra map, where $\cdot$ is the Pontryagin product. On the bar resolution this product is given by the shuffle product, and the isomorphism $\psi : G \otimes k \to H_1(G, k)$ sends $g \otimes 1$ to $[g]$. Thus $(g \otimes 1) \wedge (h \otimes 1) \mapsto [g] \cdot [h] = \sum_\sigma [g|h] = [g|h] - [h|g]$, where σ ran over the two possible $(1, 1)$-shuffles. *Remark:* This map is well-defined because for $(g \otimes 1) \wedge (g \otimes 1) = 0 \in \bigwedge^2(G \otimes k)$, the image is $[g|g] - [g|g] = 0$.

6.4(b): Let k be a PID in which 2 is invertible, let G be an abelian group, and consider the map $C_2(G, k) \to \bigwedge^*(G \otimes k)$ given by $[g|h] \mapsto (g \otimes 1) \wedge (h \otimes 1)/2$. This induces a map $\varphi : H_2(G, k) \to \bigwedge^*(G \otimes k)$ because any 3-coboundary $\partial[r|s|t] = [s|t] - [rs|t] + [r|st] - [r|s]$ is mapped to the trivial element $(s \otimes 1) \wedge (t \otimes 1)/2 - (r + s \otimes 1) \wedge (t \otimes 1)/2 + (r \otimes 1) \wedge (s + t \otimes 1)/2 - (r \otimes 1) \wedge (s \otimes 1)/2 = [(s \otimes 1) \wedge (t \otimes 1)/2] - [(r \otimes 1) \wedge (t \otimes 1)/2] - [(s \otimes 1) \wedge (t \otimes 1)/2] + [(r \otimes 1) \wedge (s \otimes 1)/2] + [(r \otimes 1) \wedge (t \otimes 1)/2] - [(r \otimes 1) \wedge (s \otimes 1)/2] = 0$. Using ψ from part(a) above, φ is its left-inverse because $(g \otimes 1) \wedge (h \otimes 1) \mapsto [g|h] - [h|g] \mapsto (g \otimes 1) \wedge (h \otimes 1)/2 - (h \otimes 1) \wedge (g \otimes 1)/2 = (g \otimes 1) \wedge (h \otimes 1)/2 + (g \otimes 1) \wedge (h \otimes 1)/2 = (g \otimes 1) \wedge (h \otimes 1)$, where we note in the last equality that the exterior algebra is strictly anti-commutative.

6.5: Let G be abelian and let A be a G-module with trivial G-action. In view of the isomorphism $H_2G \cong \bigwedge^2 G$, the universal coefficient theorem gives us a split exact sequence $0 \to \mathrm{Ext}(G, A) \to H^2(G, A) \xrightarrow{\theta} \mathrm{Hom}(\bigwedge^2 G, A) \to 0$. The isomorphism $\psi : \bigwedge^2 G \to H_2G$ is given by $g \wedge h \mapsto [g|h] - [h|g]$ by Exercise V.6.4(a). Now the map $\beta : H^2(G, A) \to \mathrm{Hom}(H_2G, A)$ in the universal coefficient sequence sends the class $\|f\|$ of the cocycle f to $(\beta\|f\|)([g_1|g_2]) = f(g_1, g_2)$. Then θ is given by $\|f\| \mapsto \beta\|f\| \mapsto (\beta\|f\| \circ \psi)(g \wedge h) = (\beta\|f\|)([g|h] - [h|g]) = f(g, h) - f(h, g)$, and this element is an alternating map. Thus θ coincides with the map θ in Exercise IV.3.8(c), and so we see that every alternating map comes from a 2-cocycle. It also follows that $\mathrm{Ext}(G, A) \cong \mathcal{E}_{ab}(G, A)$, whence the name "Ext."

6 Chapter VI: Cohomology Theory of Finite Groups

2.1: Suppose $|G:H| < \infty$ and that M is a $\mathbb{Z}G$-module with a relative injective resolution Q, and $\eta : M \to Q^0$ is the canonical admissible injection (i.e. $Q^0 = \mathrm{Coind}_H^G \mathrm{Res}_H^G M$). If M is free as a $\mathbb{Z}H$-module then Q^0 is $\mathbb{Z}G$-free by Corollary VI.2.2[1]. Since η is H-split, the exact sequence $0 \to M \xrightarrow{\eta} Q^0 \to \mathrm{Coker}\eta \to 0$ is split-exact and hence $Q^0 \cong M \oplus \mathrm{Coker}\eta$. Since M and Q^0 are both $\mathbb{Z}H$-free (a free $\mathbb{Z}G$-module is $\mathbb{Z}H$-free by Exercise I.3.1), $\mathrm{Coker}\eta$ is by definition *stably free*. Let us find this cokernel explicitly. The canonical injection is given by $M \hookrightarrow \mathrm{Hom}_H(\mathbb{Z}G, \mathrm{Res}_H^G M) \cong \mathrm{Ind}_H^G \mathrm{Res}_H^G M = \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathrm{Res}_H^G M$, $m \mapsto [s \mapsto sm] \mapsto \sum_{g \in G/H} g \otimes g^{-1}m$. But $\mathrm{Ind}_H^G \mathrm{Res}_H^G M \cong \mathbb{Z}[G/H] \otimes M$ with the mapping $g \otimes m \mapsto g \otimes gm$, by Proposition III.5.6[1]. Thus the canonical injection is given by $m \mapsto \sum_{g \in G/H} g \otimes m$. We can now consider M as a free $\mathbb{Z}H$-module, and without loss of generalization we can assume $M = \mathbb{Z}H$ (since direct sums commute with tensor products). We can also regard $\mathbb{Z}[G/H] = \bigoplus_{g \in G/H} \mathbb{Z}[g]$ as a direct sum of integers via the isomorphism $\mathbb{Z}[g] \cong \mathbb{Z}$, $g \mapsto 1$. Thus the H-split injection is $\eta : \mathbb{Z}H \hookrightarrow \bigoplus^{|G:H|} \mathbb{Z} \otimes \mathbb{Z}H \cong \bigoplus^{|G:H|} \mathbb{Z}H$ with the mapping $m \mapsto (g_1, \ldots, g_{|G:H|}) \otimes m \mapsto (1, \ldots, 1) \otimes m \mapsto (m, \ldots, m)$. The cokernel of this map is $\bigoplus^{|G:H|} \mathbb{Z}H/\mathbb{Z}H \cong \bigoplus^{|G:H|-1} \mathbb{Z}H$, a free $\mathbb{Z}H$-module. Indeed, For any finite direct sum $\bigoplus^n X$, the quotient of this group by its diagonal subgroup $X = \{(x, \ldots, x)\}$ is the direct sum $\bigoplus^{n-1} X$ because any element $(x_1, \ldots, x_{n-1}, x_n)X$ is equivalent in the quotient group to $(x_1 x_n^{-1}, \ldots, x_{n-1}x_n^{-1}, 1)X$ via the element $(x_n^{-1}, \ldots, x_n^{-1}, x_n^{-1})$. Since $\mathrm{Coker}\eta$ is free as a $\mathbb{Z}H$-module, we can apply Corollary VI.2.2[1] to get an admissible injection $\mathrm{Coker}\eta \hookrightarrow Q^1$ with Q^1 free as a $\mathbb{Z}G$-module. Continuing in this way, we obtain the resolution Q as a complex of free $\mathbb{Z}G$-modules.

3.1: Let G act freely on S^{2k-1}, as on pg20[1]. From this we have a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ which is periodic of period $2k$:

$$\cdots \to C_1 \to C_0 \to C_{2k-1} \to \cdots \to C_1 \to C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $C_* = C_*(X)$ and $X \approx S^{2k-1}$ is a free G-complex that makes each C_i finitely generated. Then by Proposition VI.3.5[1] we can take the above resolution $\varepsilon : C \to \mathbb{Z}$, form the dual (backwards resolution) of its suspension $\varepsilon^* : \mathbb{Z}^* = \mathbb{Z} \to \overline{\Sigma C}$, and then splice together C and $\overline{\Sigma C}$ to form a complete resolution F for G. This resolution is obviously periodic of period $2k$ because ε is periodic of period $2k$ and the suspension just shifts the resolution (leaving the period unaltered) and the dual functor forms a periodic resolution of the same period.
If $G = \langle t \rangle$ is finite cyclic of order n, and $k = 1$, then G acts by rotations on the circle (n vertices/edges) and we have a periodic resolution of period 2:

$$\cdots \to \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $N = 1 + t + \cdots + t^{n-1}$ is the norm element. Now $\varepsilon(1) = 1$, so by Proposition VI.3.4[1] the dual $\varepsilon^* : \mathbb{Z} \to \mathbb{Z}G$ is given by $\varepsilon^*(1) = \sum_{g \in G} g = N$. The maps $\mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G$ and $\mathbb{Z}G \xrightarrow{N} \mathbb{Z}G$ are invariant under the dual functor because $\mathrm{Hom}_G(\mathbb{Z}G, \mathbb{Z}G) \cong \mathbb{Z}G$. Therefore, the explicit complete resolution is:

$$\cdots \to \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \to \cdots .$$

5.1: We have a natural map $H^* \to \widehat{H}^*$ which is an isomorphism in positive dimensions and an epimorphism in dimension 0. The cup product for both functors agrees in dimension 0 because $\smile$: $H^0(G, M) \otimes H^0(G, N) \to H^0(G, M \otimes N)$ is the map $M^G \otimes N^G \to (M \otimes N)^G$ induced by the inclusions $M^G \hookrightarrow M$ and $N^G \hookrightarrow N$, and $\smile$: $\widehat{H}^0(G, M) \otimes \widehat{H}^0(G, N) \to \widehat{H}^0(G, M \otimes N)$ is induced by $M^G \otimes N^G \to (M \otimes N)^G$ via the surjection $H^0 \twoheadrightarrow \widehat{H}^0$. From this compatibilism in dimension 0 we can deduce that the diagram

$$\begin{tikzcd}
H^p(G, M) \otimes H^q(G, M) \arrow[r, "\smile"] \arrow[d] & H^{p+q}(G, M \otimes N) \arrow[d] \\
\widehat{H}^p(G, M) \otimes \widehat{H}^q(G, M) \arrow[r, "\smile"] & \widehat{H}^{p+q}(G, M \otimes N)
\end{tikzcd}$$

commutes for all $p, q \in \mathbb{Z}$. Indeed, embed M in the (co)induced module $\bar{M} = \mathrm{Hom}(\mathbb{Z}G, M) \cong \mathbb{Z} \otimes M$ (noting that G is finite for our purposes) and let $0 \to M \to \bar{M} \to C \to 0$ be the canonical $\mathbb{Z}$-split exact sequence (see Exercise III.7.3). For any G-module N the sequence $0 \to M \otimes N \to \bar{M} \otimes N \to C \otimes N \to 0$ is

exact, and the module $\bar{M} \otimes N$ is induced (see Exercise III.5.2(b)). We therefore have dimension-shifting isomorphisms $\delta : H^p(G, C) \to H^{p+1}(G, M)$ and $\delta : H^p(G, C \otimes N) \to H^{p+1}(G, M \otimes N)$ which commutes with the cup product (see pg110[1]). The compatibilism for $p = 0, q = 0$ allows us to prove by ascending induction on p that the above diagram is commutative for $p \geq 0$ and $q = 0$. Embedding N in an induced module, we then see that it commutes for $q \geq 0$. The scenario $p, q < 0$ is trivial because $H^p = 0$ for $p < 0$. Thus the cup product on $\widehat{H}^*$ is compatible with that defined originally on H^*, and the natural map $H^* \to \widehat{H}^*$ preserves products.

5.2(a): Let $\widehat{H}^*(G)_{(p)}$ be the p-primary component of $\widehat{H}^*(G) = \widehat{H}^*(G, \mathbb{Z})$, so that we have $\widehat{H}^*(G) = \bigoplus_{p \mid |G|} \widehat{H}^*(G)_{(p)}$. Since $\widehat{H}^*(G)_{(p)}$ is a subgroup of $\widehat{H}^*(G)$, in order to show that it is an ideal in $\widehat{H}^*(G)$ it suffices to show that $\alpha \smile \beta \in \widehat{H}^*(G)_{(p)}$ for $\alpha \in \widehat{H}^*(G)_{(p)}$ and $\beta \in \widehat{H}^*(G)$. But this is trivial because if $p^s\alpha = 0$ then $p^s(\alpha \smile \beta) = (p^s\alpha) \smile \beta = 0 \smile \beta = 0$. Now $\bigoplus_{q \neq p} \widehat{H}^*(G)_{(q)}$ is also an ideal because a sum of ideals is an ideal. Consequently, $\widehat{H}^*(G)_{(p)}$ is a quotient ring of $\widehat{H}^*(G)$ via the projection $\widehat{H}^*(G) \twoheadrightarrow \widehat{H}^*(G)_{(p)}$.

Note: The p-primary ring $\widehat{H}^*(G)_{(p)}$ is *not* a subring of $\widehat{H}^*(G)$ because the inclusion $\widehat{H}^*(G)_{(p)} \hookrightarrow \widehat{H}^*(G)$ does not preserve identity elements. The identity of $\widehat{H}^*(G)_{(p)}$ is $1 \in \mathbb{Z}_{p^r} = \widehat{H}^0(G)_{(p)}$ while the identity of $\widehat{H}^*(G)$ is $1 = (1, \ldots, 1) \in \mathbb{Z}_{|G|} = \widehat{H}^0(G)$, where we have used the factorization of $\mathbb{Z}_{|G|}$ into the direct sum of its p-primary components and $|G| = p^r m$ with $p \nmid m$. The inclusion will send the identity 1 to $(1, 0, 0, \ldots, 0) \neq 1$, where we take the first summand of $\mathbb{Z}_{|G|}$ to be $\mathbb{Z}_{p^r}$.

5.2(b): Consider the group isomorphism $\varphi : \widehat{H}^*(G) \xrightarrow{\cong} \prod_{p \mid |G|} \widehat{H}^*(G)_{(p)}$, where each factor on the right is a ring via part(a), and multiplication in the product is done componentwise [*note*: we switch from direct sum to direct product notation in order to emphasize the fact that we are dealing categorically with rings]. The map is given by $\varphi(\alpha) = (\ldots, \alpha_p, \ldots)$, where $\alpha = \sum_p \alpha_p$ in the decomposition $\widehat{H}^*(G) = \bigoplus_{p \mid |G|} \widehat{H}^*(G)_{(p)}$. In order to show that this map is a *ring* isomorphism it suffices to show that $(\alpha\beta)_p = \alpha_p\beta_p$, where $\alpha\beta = \alpha \smile \beta$, for then $\varphi(\alpha \smile \beta) = (\ldots, (\alpha\beta)_p, \ldots) = (\ldots, \alpha_p\beta_p, \ldots) = (\ldots, \alpha_p, \ldots)(\ldots, \beta_p, \ldots) = \varphi(\alpha) \smile \varphi(\beta)$. Writing $\alpha = \sum_p \alpha_p$ and $\beta = \sum_{p'} \beta_{p'}$ we have $\alpha\beta = \sum_{p,p'} \alpha_p\beta_{p'} = \sum_p \alpha_p\beta_p$, because $\alpha_p\beta_{p' \neq p}$ is annihilated by both p and $p' \neq p$, and hence annihilated by $1 = gcd(p, p')$ [note that $gcd(p, p') = mp + np'$ for some $m, n \in \mathbb{Z}$ by the Euclidean Algorithm]. It is now obvious that $(\alpha\beta)_p = (\sum_q \alpha_q\beta_q)_p = \alpha_p\beta_p$.

6.1:

6.2: Let R be a ring, let C be a chain complex of finitely generated projective R-modules, and let $\bar{C}$ be the dual complex $\mathcal{H}om_R(C, R)$ of finitely generated projective right R-modules. For any $z \in (\bar{C}\widehat{\otimes}_R C)_n$ any any chain complex C', there is a graded map $\psi_z : \mathcal{H}om_R(C, C') \to \bar{C}\widehat{\otimes}_R C'$ of degree n, given by $\psi_z(u) = (id_{\bar{C}}\widehat{\otimes}u)(z)$. Let $z \in (\bar{C}\widehat{\otimes}_R C)_0$ correspond to id_C under the isomorphism $\varphi : \bar{C}\widehat{\otimes}_R C \to \mathcal{H}om_R(C, C)$ from Exercise VI.6.1. This element is indeed a cycle, because $\varphi_{-1}(\partial z) = D_0\varphi_0(z) = D_0(id_C) = d \circ id_C - (-1)^0 id_C \circ d = d - d = 0$ implies that $\partial z = 0$ since φ is an isomorphism. Then $z = (z_p)_{p \in \mathbb{Z}}$, where $z_p \in \bar{C}_{-p} \otimes_R C_p = (C_p)^* \otimes_R C_p$ corresponds to $(-1)^p id_{C_p}$ under the canonical isomorphism $\phi : (C_p)^* \otimes_R C_p \cong \mathrm{Hom}_R(C_p, C_p)$ of Proposition I.8.3[1] given by $\bar{c} \otimes c \mapsto \langle \bar{c}, x \rangle c$. Indeed, denoting $z_p = \bar{c} \otimes c$, the canonical isomorphism gives $\langle \bar{c}, x \rangle c = (-1)^p id_{C_p}$ and the isomorphism $\varphi_0 = (\varphi_{-pp})_{p \in \mathbb{Z}}$ then becomes $(-1)^{-p \cdot p}\langle \bar{c}, x \rangle c = (-1)^{p^2}(-1)^p id_{C_p} = (-1)^{p(p+1)} id_{C_p} = id_{C_p}$, agreeing with our choice for z. Now ψ_z is induced by maps $\psi_{pq} : \mathrm{Hom}_R(C_{-p}, C'_q) \to \bar{C}_p \otimes_R C'_q$. Since $u \in \mathrm{Hom}_R(C_{-p}, C'_q)$ is of degree $p + q$ and $id_{\bar{C}_p}$ is in dimension p, these maps are clearly given by $\psi_{pq}(u) = (-1)^{(q+p)p}(id_{\bar{C}_p} \otimes u)(z_{-p})$; see the definition of a map between completed tensor products on pg137[1]. Then Exercise I.8.7 states that $\varphi_{pq}^{-1} = \psi_{\varphi^{-1}(id_{C_p})} = \psi_r$, where $\psi_r : \mathrm{Hom}_R(C_{-p}, C'_q) \to \bar{C}_p \otimes_R C'_q$ is defined by $\psi_r(u) = (id_{\bar{C}_p} \otimes_R u)(r)$. Since $r = \varphi^{-1}(id_{C_p}) = (-1)^{pq}\phi^{-1}(id_{C_p}) = (-1)^{pq}(-1)^p z_{-p}$, we have $\varphi_{pq}^{-1}(u) = \psi_r(u) = (id_{\bar{C}_p} \otimes_R u)((-1)^{pq}(-1)^p z_{-p}) = (-1)^{(pq+p^2)}(id_{\bar{C}_p} \otimes u)(z_{-p}) = \psi_{pq}(u)$, noting that $(-1)^p = (-1)^{p^2}$. Therefore, ψ_z is the inverse of the isomorphism $\varphi : \bar{C}\widehat{\otimes}_R C' \to \mathcal{H}om_R(C, C')$ of Exercise VI.6.1.

6.3: Let G be a finite group, let F be an acyclic chain complex of projective $\mathbb{Z}G$-modules, and let $\varepsilon' : F' \to \mathbb{Z}$ be a complete resolution. Part(b) of Proposition VI.6.1[1] states that if F is of finite type then $\varepsilon' \otimes M$ induces a weak equivalence $F\widehat{\otimes}_G(F' \otimes M) \to F\widehat{\otimes}_G M = F \otimes_G M$. The proof of the proposition considers the dual $\bar{F} = \mathcal{H}om_G(F, \mathbb{Z}G)$ and utilizes the fact that it is projective. However, if we did not impose the finiteness hypothesis on F then its dual would not necessarily by projective. Indeed, the dual of an infinite direct sum is an infinite direct product, and $\mathcal{H}om_G(\bigoplus^\infty \mathbb{Z}G, \mathbb{Z}G) \cong \prod^\infty \mathbb{Z}G$ is not $\mathbb{Z}G$-projective. If it were $\mathbb{Z}G$-projective, then by Exercise I.8.2 it would also be $\mathbb{Z}$-projective (i.e. free abelian). But any subgroup of a free abelian group is free abelian by Theorem I.7.3[5], and $\prod^\infty \mathbb{Z}$ is a subgroup of $\prod^\infty \mathbb{Z}G$ which is not free abelian, giving the desired contradiction. Thus the finiteness hypothesis is necessary *for the given proof* – this does not guarantee that the finiteness hypothesis is necessary for the *statement* of the proposition.

6.4:

7.1(a): Let $\varphi : \widehat{H}^i(G, M) \xrightarrow{\cong} \widehat{H}_{-1-i}(G, M)$ be the isomorphism established in the proof of Proposition VI.7.2[1], which on the chain level has the inverse $\varphi^{-1} : \mathcal{H}om_G(F, \mathbb{Z}G) \otimes_G M \to \mathcal{H}om_G(F, M)$ given by $u \otimes m \mapsto [x \mapsto u(x) \cdot m]$, and let $z = \varphi(1) \in \widehat{H}_{-1}(G, \mathbb{Z})$. For an arbitrary G-module coefficient homomorphism $h : M \to N$ we have a commutative diagram

$$\begin{tikzcd}
\mathcal{H}om_G(F, \mathbb{Z}G) \otimes_G M \arrow[r, "\varphi^{-1}"] \arrow[d, "\alpha"] & \mathcal{H}om_G(F, M) \arrow[d, "\beta"] \\
\mathcal{H}om_G(F, \mathbb{Z}G) \otimes_G N \arrow[r, "\varphi^{-1}"] & \mathcal{H}om_G(F, N)
\end{tikzcd}$$

where $\alpha(u \otimes m) = u \otimes h(m)$ and $\beta(f) = h \circ f$, because $\beta\varphi^{-1}(u \otimes m) = \beta[u(x) \cdot m] = h(u(x) \cdot m) = u(x) \cdot h(m) = \varphi^{-1}(u \otimes h(m)) = \varphi^{-1}\alpha(u \otimes m)$. Thus φ^{-1} is natural and hence so is φ. Since φ is natural the following diagram with short exact rows is commutative (suppressing the end 0's)

$$\begin{tikzcd}
\mathcal{H}om_G(F, M') \arrow[r] \arrow[d, "\varphi"] & \mathcal{H}om_G(F, M) \arrow[r] \arrow[d, "\varphi"] & \mathcal{H}om_G(F, M'') \arrow[d, "\varphi"] \\
\mathcal{H}om_G(F, \mathbb{Z}G) \otimes_G M' \arrow[r] & \mathcal{H}om_G(F, \mathbb{Z}G) \otimes_G M \arrow[r] & \mathcal{H}om_G(F, \mathbb{Z}G) \otimes_G M''
\end{tikzcd}$$

and so φ is compatible with connecting homomorphisms in long exact sequences by Proposition I.0.4[1].

7.1(b): By definition of z, φ and $\frown z$ agree on $1 \in \widehat{H}^0(G, \mathbb{Z})$ since $1 \frown z = z$. If now M and $u \in \widehat{H}^0(G, M)$ are arbitrary, there is a coefficient homomorphism $\mathbb{Z} \to M$ such that $1 \mapsto u$ under the induced map $\alpha : \widehat{H}^0(G, \mathbb{Z}) \to \widehat{H}^0(G, M)$, noting that this cohomology map is induced from $H^0(G, \mathbb{Z}) = \mathbb{Z}^G \to M^G = H^0(G, M)$. Since the cap product is natural with respect to coefficient homomorphisms we have a commutative diagram

$$\begin{tikzcd}
\widehat{H}^0(G, \mathbb{Z}) \arrow[r, "\frown z"] \arrow[d, "\alpha"] & \widehat{H}_{-1}(G, \mathbb{Z}) \arrow[d, "\beta"] \\
\widehat{H}^0(G, M) \arrow[r, "\frown z"] & \widehat{H}_{-1}(G, M)
\end{tikzcd}$$

which defines $\beta(z) = \beta(1 \frown z) = \alpha(1) \frown z = u \frown z$. Thus by naturality of φ from part(a) we have an analogous commutative diagram as above (replacing $\frown z$ with φ), and this yields $\varphi(u) = \varphi\alpha(1) = \beta\varphi(1) = \beta(z) = u \frown z$.

7.1(c): The maps φ and $\frown z$ agree in dimension 0 (referring to the domain) by part(b), and φ is δ-compatible by part(a). Thus we can use dimension-shifting [the δ boundary isomorphisms] to deduce that φ and $\frown z$ agree in all dimensions, up to sign. Indeed, we have the commutative diagram

$$\begin{tikzcd}
\widehat{H}^0(G, M) \arrow[r, "\varphi"] \arrow[d, "\cong"', "\delta"] & \widehat{H}_{-1}(G, M) \arrow[d, "\cong"', "\delta"] \\
\widehat{H}^n(G, K) \arrow[r, "\varphi"] & \widehat{H}_{-1-n}(G, K)
\end{tikzcd}$$

where the vertical maps are due to iterations of the dimension-shifting technique 5.4 on pg136[1]. These isomorphisms provide ambiguity in the sign, so $\varphi(u) = \pm u \frown z$ in any dimension and hence Proposition VI.7.2[1] has been reproved (the isomorphism is given by the cap product with the *fundamental class* z).

7.2: Let A be an abelian torsion group, and consider the injective resolution $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ of $\mathbb{Z}$. Applying $\mathrm{Hom}(A, -)$ gives the cochain complex $0 \to \mathrm{Hom}(A, \mathbb{Z}) \to \mathrm{Hom}(A, \mathbb{Q}) \xrightarrow{\delta^0} \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\delta^1} 0$, and we have $\mathrm{Ext}(A, \mathbb{Z}) \equiv \mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) = \mathrm{Ker}\delta^1/\mathrm{Im}\delta^0 = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})/\mathrm{Im}\delta^0 = A'/\mathrm{Im}\delta^0$. But $\mathbb{Q}$ is torsion-free, so $\mathrm{Hom}(A, \mathbb{Q}) = 0$ and $\mathrm{Im}\delta^0 = 0$. Thus $\mathrm{Ext}(A, \mathbb{Z}) = A'$.
Equivalently, Theorem 17.1.10[2] provides us with a long exact sequence $0 \to \mathrm{Hom}(A, \mathbb{Z}) \to \mathrm{Hom}(A, \mathbb{Q}) \to \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Ext}(A, \mathbb{Z}) \to \mathrm{Ext}(A, \mathbb{Q})$. But $\mathrm{Hom}(A, \mathbb{Q}) = 0$ as mentioned above, and $\mathrm{Ext}(A, \mathbb{Q}) = 0$ by Proposition 17.1.9[2] since $\mathbb{Q}$ is $\mathbb{Z}$-injective. Thus we have a desired isomorphism $A' = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \mathrm{Ext}(A, \mathbb{Z})$.

7.3: Let G be a finite group, let M be a G-module which is free as an abelian group, and let F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Note that $M^* = \mathrm{Hom}(M, \mathbb{Z})$ by Proposition VI.3.4[1]. Consider the split exact coefficient sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. Since M is $\mathbb{Z}$-free, applying the functor $\mathrm{Hom}(M, -)$ yields the exact sequence $0 \to M^* \to \mathrm{Hom}(M, \mathbb{Q}) \to M' \to 0$ where $M' = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$; this sequence is $\mathbb{Z}$-split exact because the original coefficient sequence is split exact (Hom commutes with direct sums). I claim that $\widehat{H}^*(G, \mathrm{Hom}(M, \mathbb{Q})) = 0$ and $\widehat{H}^*(G, \mathrm{Hom}(M, \mathbb{Q}) \otimes M) = 0$. Assuming this for the moment, we then have dimension-shifting isomorphisms $\delta : \widehat{H}^j(G, M') \to \widehat{H}^{j+1}(G, M^*)$ for all j. It is a fact that the tensor product of a G-module with a $\mathbb{Z}$-split exact sequence is exact, so $0 \to M^* \otimes M \to \mathrm{Hom}(M, \mathbb{Q}) \otimes M \to M' \otimes M \to 0$ is an exact sequence. Thus we also have dimension-shifting isomorphisms $\delta : \widehat{H}^j(G, M' \otimes M) \to \widehat{H}^{j+1}(G, M^* \otimes M)$. Moreover, we have a commutative diagram

$$\begin{tikzcd}
\widehat{H}^{i-1}(G, M') \otimes \widehat{H}^{-i}(G, M) \arrow[r, "\smile"] \arrow[d, "\cong"', "\delta \otimes id"] & \widehat{H}^{-1}(G, M' \otimes M) \arrow[r, "\alpha"] \arrow[d, "\cong"', "\delta"] & \widehat{H}^{-1}(G, \mathbb{Q}/\mathbb{Z}) \arrow[d, "\cong"', "\delta"] \\
\widehat{H}^{i}(G, M^*) \otimes \widehat{H}^{-i}(G, M) \arrow[r, "\smile"] & \widehat{H}^{0}(G, M^* \otimes M) \arrow[r, "\beta"] & \widehat{H}^{0}(G, \mathbb{Z})
\end{tikzcd}$$

where the left-side square follows from compatibility with δ (see pg110[1]) and the right-side square follows from naturality of the long exact cohomology sequence (see pg72[1]); α and β are induced by the evaluation maps. Since the top row is a duality pairing by Corollary VI.7.3[1], so is the bottom row.
It suffices to prove the claim. The analog of Proposition III.10.1[1] for Tate cohomology states that if $\widehat{H}^n(H, M) = 0$ for some n with $H \subseteq G$, then $H^n(G, M)$ is annihilated by $|G : H|$. Taking $H = \{1\}$ and M a rational vector space, the norm map $\overline{N} : M_H \to M^H$ is an isomorphism ($\overline{N} = id_M$). Then $\widehat{H}^{-1}(\{1\}, M) = \mathrm{Ker}\overline{N} = 0 = \mathrm{Coker}\overline{N} = \widehat{H}^0(\{1\}, M)$, so $\widehat{H}^{-1}(G, M)$ and $\widehat{H}^0(G, M)$ are annihilated by $|G|$ and are thus trivial groups since $|G|$ is invertible in M. The claim is now justified since $\mathrm{Hom}(M, \mathbb{Q})$ and $\mathrm{Hom}(M, \mathbb{Q}) \otimes M$ are both rational vector spaces.

7.4: Let k be an arbitrary commutative ring and Q an injective k-module. Let $A' = \mathrm{Hom}_k(A, Q)$ for any k-module A. If M is a kG-module, then the pairing $\widehat{H}^i(G, M') \otimes \widehat{H}^{-1-i}(G, M) \xrightarrow{\smile} \widehat{H}^{-1}(G, M' \otimes M) \to \widehat{H}^{-1}(G, Q) \hookrightarrow Q$ induces an isomorphism $\widehat{H}^i(G, M') \cong \widehat{H}^{-1-i}(G, M)'$. Indeed, this is simply the analog of Corollary VI.7.3[1], and the proof of that corollary goes through untouched if we replace $\mathbb{Q}/\mathbb{Z}$ by Q (as both are injective) and $\mathbb{Z}$ by k (as both are commutative ring coefficients).

8.1: Let G be a group and M a $\mathbb{Z}G$-module such that $\widehat{H}^*(G, M) = 0$ but M is not cohomologically trivial. If G is cyclic, then by Theorem VI.8.7[1] it cannot be a p-group, so $G = \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$ for distinct primes p and q. The complete resolution from Exercise VI.3.1 then implies that $\widehat{H}^n(G, M) = \mathrm{Coker}\overline{N}$ for n even and $\widehat{H}^n(G, M) = \mathrm{Ker}\overline{N}$ for n odd (see pg58[1]), so $\overline{N} : M_G \to M^G$ is an isomorphism. Also, Proposition VI.8.8[1] implies that either $\widehat{H}^i(\mathbb{Z}_p, M) \neq 0$ for some i or $\widehat{H}^j(\mathbb{Z}_q, M) \neq 0$ for some j (or both). As mentioned on pg150[1] as a consequence of Theorem VI.8.5[1], $\widehat{H}^i(\mathbb{Z}_p, \mathbb{Z}_p) \neq 0$ for all $i > 0$. Therefore, let us consider $M = \mathbb{Z}_p$. It suffices to find a $\mathbb{Z}_{pq}$-action on $\mathbb{Z}_p$ such that $\overline{N} : (\mathbb{Z}_p)_{\mathbb{Z}_{pq}} \to (\mathbb{Z}_p)^{\mathbb{Z}_{pq}}$ is an isomorphism. But $\mathbb{Z}_p$ is simple, so either $\mathbb{Z}_{pq}$ acts trivially on $\mathbb{Z}_p$ or $(\mathbb{Z}_p)_{\mathbb{Z}_{pq}} = (\mathbb{Z}_p)^{\mathbb{Z}_{pq}} = 0$. But if

$\mathbb{Z}_{pq}$ acts trivially on $\mathbb{Z}_p$ then $\overline{N} : \mathbb{Z}_p \to \mathbb{Z}_p$ is the zero map (hence has nontrivial kernel) since multiplication by $|\mathbb{Z}_{pq}| = pq$ annihilates $\mathbb{Z}_p$. Thus we must have $(\mathbb{Z}_p)_{\mathbb{Z}_{pq}} = (\mathbb{Z}_p)^{\mathbb{Z}_{pq}} = 0$. Taking p to be an odd prime, this condition is satisfied by the $\mathbb{Z}_{pq}$-action $x \cdot m^i = m^{2i}$, where $\mathbb{Z}_p = \langle m \rangle$ and $\mathbb{Z}_{pq} = \langle x \rangle$, as long as $2^{pq} \equiv 1 \bmod p$ (because we must have $m = x^{pq} \cdot m = m^{2^{pq}}$). For example, $\{p = 3, q = 2\}$ works, as does $\{p = 7, q = 3\}$.
As a result, a desired example is the group $G = \mathbb{Z}_6 = \langle x \rangle$ and the $\mathbb{Z}_6$-module $M = \mathbb{Z}_3 = \langle m \rangle$ coupled with the action $x \cdot m^i = m^{2i}$.

8.2: Suppose M is a G-module which is $\mathbb{Z}$-free and cohomologically trivial (G is of course finite). Then M is $\mathbb{Z}G$-projective by Theorem VI.8.10[1], so $\mathfrak{F} = M \oplus N$ for some projective module N and free module $\mathfrak{F}$. For any G-module K, the module $\mathrm{Hom}(\mathfrak{F}, K)$ is an induced module by Exercise III.5.2(b) (since $\mathfrak{F} \cong \mathbb{Z}G \otimes \mathfrak{F}' = \mathrm{Ind}_{\{1\}}^G \mathfrak{F}'$ where $\mathfrak{F}'$ is a free $\mathbb{Z}$-module of the same rank) and hence cohomologically trivial. Since the Hom-functor commutes with direct sums, $\mathrm{Hom}(M, K)$ is also cohomologically trivial for any G-module K.
Alternatively, for any G-module K choose an exact sequence $0 \to L \to F \to K \to 0$ with F free (such sequences exist because every module is a quotient of a free module). Since M is $\mathbb{Z}$-free, we can apply $\mathrm{Hom}(M, -)$ to get the exact sequence $0 \to \mathrm{Hom}(M, L) \to \mathrm{Hom}(M, F) \to \mathrm{Hom}(M, K) \to 0$. Since F and L are also $\mathbb{Z}$-free, $\mathrm{Hom}(M, L)$ and $\mathrm{Hom}(M, F)$ are cohomologically trivial by Lemma VI.8.11[1]; L is free because it embeds in the free $\mathbb{Z}$-module F and any submodule of a $\mathbb{Z}$-free module is free. Thus $\mathrm{Hom}(M, K)$ is cohomologically trivial by the long exact Tate cohomology sequence.

8.3(a): Let M and P be $\mathbb{Z}G$-modules such that M is $\mathbb{Z}$-free and P is $\mathbb{Z}P$-projective, and consider any exact sequence $0 \to P \xrightarrow{i} E \xrightarrow{\varphi} M \to 0$. The obstruction to splitting the sequence lies in $H^1(G, \mathrm{Hom}(M, P)) \cong \mathrm{Ext}^1_{\mathbb{Z}G}(M, P)$, where the isomorphism follows from Proposition III.2.2[1]. More precisely, we have a short exact sequence of G-modules $0 \to \mathrm{Hom}(M, P) \to \mathrm{Hom}(M, E) \to \mathrm{Hom}(M, M) \to 0$ since M is $\mathbb{Z}$-free, and this yields the sequence $\mathrm{Hom}_G(M, E) \to \mathrm{Hom}_G(M, M) \xrightarrow{\delta} H^1(G, \mathrm{Hom}(M, P))$ via the long exact cohomology sequence, where we recall that $\mathrm{Hom}_G(\cdot, \cdot) = \mathrm{Hom}(\cdot, \cdot)^G = H^0(G, \mathrm{Hom}(\cdot, \cdot))$. Hence the extension splits iff $\delta(id_M) = 0$, because if the extension splits then there is a section $s : M \to E$ which maps onto id_M (i.e. $s \mapsto \varphi \circ s = id_M$) and so $id_M \in \mathrm{Ker}\delta$ by exactness, and if $\delta(id_M) = 0$ then by exactness there exists a map $M \to E$ which maps onto id_M and that map is then the desired section. It suffices to show that $\mathrm{Hom}(M, P)$ is cohomologically trivial, for then $H^1(G, \mathrm{Hom}(M, P)) = 0$ and $\delta = 0$. By additivity [Hom commutes with direct sums], it suffices to show that $\mathrm{Hom}(M, \mathfrak{F})$ is cohomologically trivial for any free $\mathbb{Z}G$-module $\mathfrak{F}$, since the projective P is a direct summand of some $\mathfrak{F}$. But $\mathfrak{F} \cong \mathrm{Coind}_{\{1\}}^G \mathfrak{F}'$ where $\mathfrak{F}'$ is a free $\mathbb{Z}$-module of the same rank (by Corollary VI.2.3[1]), so $\mathrm{Hom}(M, \mathfrak{F})$ is induced (by Exercise III.5.2(b)) and hence cohomologically trivial.
Alternatively, since M is $\mathbb{Z}$-free the original exact sequence in consideration is $\mathbb{Z}$-split (see Exercise AE.27), so the injection $i : P \to E$ of G-modules is a $\mathbb{Z}$-split injection, hence admissible. Since P is G-projective, it is relatively injective by Corollary VI.2.3[1] and so the mapping problem

$$\begin{tikzcd} P \arrow[r, "i"] \arrow[d, "id_P"'] & E \arrow[dl, dotted, "f"] \\ P & \end{tikzcd}$$

can be solved (i.e. there exists a map $f : E \to P$ such that $f \circ i = id_P$). But this just means that $f : E \to P$ is a $\mathbb{Z}G$-splitting homomorphism for the sequence $0 \to P \xrightarrow{i} E \xrightarrow{\varphi} M \to 0$, and so this sequence splits.

8.3(b): Let M be a $\mathbb{Z}G$-module such that $\mathrm{proj\,dim}\, M < \infty$, and consider the projective resolution $0 \to P_n \to \cdots \to P_0 \to M \to 0$. We can break this up into short exact sequences $Z_i \to P_i \to Z_{i-1} \to 0$, where Z_i is the kernel of $P_i \to P_{i-1}$. Now Z_i ($i \geq 0$) is $\mathbb{Z}$-free because it is a submodule of a $\mathbb{Z}G$-projective module which is a submodule of a $\mathbb{Z}G$-free module $\mathfrak{F}$, and $\mathfrak{F}$ is also necessarily $\mathbb{Z}$-free and any subgroup (in particular, Z_i) of a $\mathbb{Z}$-free group is $\mathbb{Z}$-free. Therefore, for the sequence $0 \to P_n = Z_{n-1} \to P_{n-1} \to Z_{n-2} \to 0$ with P_n G-projective and Z_{n-2} $\mathbb{Z}$-free, part(a) above implies that this sequence splits and hence $P_{n-1} \cong P_n \oplus Z_{n-2}$. Since P_{n-1} is G-projective, so is Z_{n-2}. One now sees by descending induction on i that Z_i is G-projective for $i \geq 0$, so that $0 \to Z_0 \to P_0 \to Z_{-1} = M \to 0$

is a projective resolution of length 1. Thus proj dim $M \leq 1$.

8.4: Let G be a group such that there exists a free, finite G-CW-complex X with $H_*(X) \cong H_*(S^{2k-1})$. Since $H_i(S^{2k-1}) = 0$ for $i \neq 0$ and $i \neq 2k-1$, the augmented cellular chain complex $C_* = C_*(X)$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ up to dimension $2k-1$. From the chain sequence $C_{2k-1} \overset{\partial_{2k-1}}{\to} C_{2k-2} \overset{\partial_{2k-2}}{\to} C_{2k-3}$ we have $\mathrm{Ker}\partial_{2k-2} = \mathrm{Im}\partial_{2k-1} \cong C_{2k-1}/\mathrm{Ker}\partial_{2k-1}$ [the isomorphism is due to the 1st Isomorphism Theorem] and hence we have an exact sequence $0 \to C_{2k-1}/\mathrm{Ker}\partial_{2k-1} \hookrightarrow C_{2k-2} \overset{\partial_{2k-2}}{\to} C_{2k-3} \to \cdots$. Now $\mathrm{Im}\partial_{2k} = B$ is the module of $(2k-1)$-boundaries of C_*, and we have a surjection $C_{2k-1}/B \twoheadrightarrow C_{2k-1}/\mathrm{Ker}\partial_{2k-1}$ with kernel $\mathrm{Ker}\partial_{2k-1}/B \cong \mathbb{Z}$ [note: this isomorphism comes from the fact that $\mathbb{Z} \cong H_{2k-1}(S^{2k-1})$]. Thus we have an exact sequence S of G-modules $0 \to \mathbb{Z} \to C_{2k-1}/B \to C_{2k-2} \overset{\partial_{2k-2}}{\to} C_{2k-3} \to \cdots \to C_0 \overset{\varepsilon}{\to} \mathbb{Z} \to 0$ where each C_i is G-free. I claim that C_{2k-1}/B is $\mathbb{Z}$-free and has finite projective dimension. Assuming this claim holds, C_{2k-1}/B is cohomologically trivial by Theorem VI.8.12[1] and hence is $\mathbb{Z}G$-projective by Theorem VI.8.10[1]. Thus we can splice together an infinite number of copies of S (which forms an acyclic complex of projective G-modules) and we can then apply Proposition VI.3.5[1] to obtain a complete resolution which is periodic of period $2k$. It suffices to prove the claim. Since C_{2k-1} is $\mathbb{Z}G$-free, it is necessarily $\mathbb{Z}$-free and hence any subgroup (in particular, B) must also be $\mathbb{Z}$-free; thus C_{2k-1}/B is $\mathbb{Z}$-free. As X is a finite complex, $C_*(X)$ stops after $C_n(X)$ for some $2k-1 < n < \infty$. Thus we have a finite projective resolution $0 \to C_n \overset{\partial_n}{\to} \cdots \to C_{2k} \overset{\partial_{2k}}{\to} C_{2k-1} \to C_{2k-1}/B \to 0$ and proj dim $C_{2k-1}/B < \infty$.

9.1: Let G be a nontrivial finite group which has periodic cohomology of period d. Then there is an element $u \in \widehat{H}^d(G)$ which is invertible in the ring $\widehat{H}^*(G)$, so cup product with u gives a periodicity isomorphism $v \mapsto u \smile v$. Taking $v = u$ and using anti-commutativity of the cup product, $u \smile u = (-1)^{d^2}(u \smile u)$. If d is not even, then $2u^2 = 0$. If $|G| = 2$ then G is cyclic (of order 2) and hence has period $d = 2$ (which is even), so we must have $|G| \geq 3$. But then $2u^2 = 0$ implies that $u \smile u = u^2 = 0$ and hence $u = 0$ by the periodicity isomorphism. This contradicts the fact that u is nontrivial (it is invertible), so d must be even.

9.2: If G is cyclic then $\widehat{H}^*(G)$ is periodic of period 2 because G admits a 2-dimensional fixed-point-free representation as a group of rotations (see pg154[1]); we could also just note that $\widehat{H}^i(G)$ is $\mathbb{Z}_{|G|}$ for i even and is 0 for i odd. Conversely, if a group G has periodic cohomology of period 2, then $G_{ab} = H_1G = \widehat{H}^{-2}(G) \cong \widehat{H}^0(G) = \mathbb{Z}_{|G|}$. Now $|G| = |G_{ab}| = |G|/|[G,G]| \Rightarrow |[G,G]| = 1$ and hence $G = G_{ab} \cong \mathbb{Z}_{|G|}$, which is cyclic.

9.3: Suppose $\widehat{H}^*(G)$ is periodic of period 4. We have $\widehat{H}^{-1}(G) = 0$ and $\widehat{H}^0(G) = \mathbb{Z}_{|G|}$, as explained on pg135[1]. We also have $\widehat{H}^{-2}(G) = H_1G = G_{ab}$, and $\widehat{H}^1(G) = H^1G = \mathrm{Hom}(G,\mathbb{Z}) = 0$ by Exercise III.1.2 (noting that G is finite by hypothesis). Thus, since $\widehat{H}^n(G) \cong \widehat{H}^{n+4}(G)$ for all n,

$$H_n(G) \cong \begin{cases} \mathbb{Z} & n = 0 \\ G_{ab} & n \equiv 1 \bmod 4 \\ \mathbb{Z}_{|G|} & n \equiv 3 \bmod 4 \\ 0 & \text{otherwise} \end{cases} \qquad H^n(G) \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}_{|G|} & n \equiv 0 \bmod 4\,,\ n \neq 0 \\ G_{ab} & n \equiv 2 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

Note that this reproves ***Exercise II.5.7(a)****, because the finite subgroup $G \subset S^3 \subset \mathbb{H}^*$ has $\widehat{H}^*(G)$ periodic of period 4 (see pg155[1]) and hence $H_2G = 0$.*

9.4: Suppose the finite group G has periodic cohomology. Now $\widehat{H}^*(G) \cong \prod_{p\,|\,|G|} \widehat{H}^*(G)_{(p)}$ by Exercise VI.5.2(b), and each factor $\widehat{H}^*(G)_{(p)}$ embeds in $\widehat{H}^*(\mathrm{Syl}_p(G))$ by the Tate cohomology version of Theorem III.10.3[1], so $\widehat{H}^i(G) = 0$ for i odd if $\widehat{H}^i(\mathrm{Syl}_p(G)) = 0$ for each prime p. Thus to show that $\widehat{H}^i(G) = 0$ for i odd it suffices to do this when G is a p-group. For the p-group G (of order p^r) with periodic cohomology, G is either a cyclic group $\mathbb{Z}_{p^r}$ or a generalized quaternion group Q_{2^r} by Proposition VI.9.3[1]. If G is cyclic then it has period 2 by Exercise VI.9.2, so since $\widehat{H}^{-1}(G) = 0$, $\widehat{H}^i(G) = 0$ for i odd. If G is generalized quaternion then it has period 4 as explained on pg155[1] (it is a finite subgroup of $\mathbb{H}^*$), so by Exercise VI.9.3, $\widehat{H}^i(G) = 0$ for i odd.

9.5: Suppose G is a p-group which has a unique subgroup C of order p; note that C is necessarily cyclic. Choose a fixed-point-free representation of C on a 2-dimensional vector space W (as a group of rotations), and form the induced module $V = \mathbb{Z}G \otimes_{\mathbb{Z}C} W$. Since C is unique, it is normal in G and hence $\mathrm{Res}^G_C V \cong \bigoplus_{g \in G/C} gW$ by Proposition III.5.6[1]; each gW is clearly a fixed-point-free representation of C ($c \cdot gw = (cg)w \neq gw$). Consequently, V is a fixed-point-free represenation of C. But then V is also a fixed-point-free representation of G, for a nontrivial isotropy group $G_v (v \in V - \{0\})$ would contain an element x of order p (by Cauchy's Theorem, Theorem 3.2.11[2]) and hence would contain C (uniqueness implies $C = \langle x \rangle$), contradicting the fact that C acts freely on $V - \{0\}$. Thus G admits a periodic complete resolution of period $2|G : C|$ as explained on pg154[1], so G has periodic cohomology.

9.6: Let $G = \mathbb{Z}_m \rtimes \mathbb{Z}_n$, where m and n are relatively prime and $\mathbb{Z}_n$ acts on $\mathbb{Z}_m$ via a homomorphism $\mathbb{Z}_n \to \mathbb{Z}_m^*$ whose image has order k. If a prime q divides n, then a Sylow q-subgroup H lies in $\mathbb{Z}_n$ and is necessarily central (in $\mathbb{Z}_n$) because cyclic groups are abelian. By Theorem III.10.3[1] we have $\widehat{H}^*(\mathbb{Z}_n)_{(q)} \cong \widehat{H}^*(H)^{\mathbb{Z}_n/H}$ and by Exercise III.8.1 we know that $\mathbb{Z}_n/H$ acts trivially on $\widehat{H}^*(H)$, so $\widehat{H}^*(H) \cong \widehat{H}^*(\mathbb{Z}_n)_{(q)}$. By Theorem III.10.3[1] we also have $\widehat{H}^*(G)_{(q)} \subseteq \widehat{H}^*(H) \cong \widehat{H}^*(\mathbb{Z}_n)_{(q)}$. Since $\widehat{H}^*(\mathbb{Z}_n)_{(q)} \subseteq \widehat{H}^*(G)_{(q)}$ by Exercise AE.55, we must have $\widehat{H}^*(G)_{(q)} \cong \widehat{H}^*(\mathbb{Z}_n)_{(q)}$. Now if a prime p divides m, then the same argument yields $\widehat{H}^*(H) \cong \widehat{H}^*(\mathbb{Z}_m)_{(p)}$ where H is now a Sylow p-subgroup of $\mathbb{Z}_m \subset G$. We have $H \triangleleft G$ because H is the unique subgroup of $\mathbb{Z}_m \triangleleft G$ (hence $gHg^{-1} \cong H$ for all $g \in G$), so Theorem III.10.3[1] implies $\widehat{H}^*(G)_{(p)} \cong \widehat{H}^*(H)^{G/H} \cong \widehat{H}^*(\mathbb{Z}_m)^{G/H}_{(p)} = \widehat{H}^*(\mathbb{Z}_m)^{(\mathbb{Z}_m/H)\rtimes\mathbb{Z}_n}_{(p)}$. We have a $\mathbb{Z}_n$-action on $\mathbb{Z}_m$ given by $\varphi : \mathbb{Z}_n \to \mathrm{Aut}(\mathbb{Z}_m) = \mathbb{Z}_m^*$, and we have a trivial $\mathbb{Z}_m/H$-action $\psi : \mathbb{Z}_m/H \to \mathrm{Aut}(\mathbb{Z}_m)$ given by $\psi(g) = id_{\mathbb{Z}_m} \equiv 1$. We then have a $(\mathbb{Z}_m/H) \rtimes \mathbb{Z}_n$-action $\psi \rtimes \varphi$ which is precisely the action $\varphi : \mathbb{Z}_n \to \mathbb{Z}_m^*$ because $(\psi \rtimes \varphi)(g, z) = \psi(g) \cdot \varphi(z) = 1 \cdot \varphi(z) = \varphi(z)$. Thus we can consider the G/H-action on $\mathbb{Z}_m$ (hence on its Tate cohomology) as the $\mathbb{Z}_n$-action, so $\widehat{H}^*(G)_{(p)} \cong \widehat{H}^*(\mathbb{Z}_m)^{\mathbb{Z}_n}_{(p)}$. Since $\widehat{H}^*(G)$ is the direct sum of its primary components and p (resp. q) ranges over prime divisors of m (resp. n), we have the isomorphism $\widehat{H}^*(G) \cong \widehat{H}^*(\mathbb{Z}_n) \oplus \widehat{H}^*(\mathbb{Z}_m)^{\mathbb{Z}_n}$.
Let us examine the $\mathbb{Z}_n$-action a little more carefully. The image of $\mathbb{Z}_n$ under the action-homomorphism consists solely of automorphisms $f : \mathbb{Z}_m \to \mathbb{Z}_m$ such that $f^k = id_{\mathbb{Z}_m}$. Such a map induces $f_* = \widehat{H}^2(f)$ on $\widehat{H}^2(\mathbb{Z}_m) \cong \mathbb{Z}_m$. If α generates the 2nd-dimension cohomology, then in the cohomology ring, $f_*(\alpha) = \lambda\alpha$ for some $\lambda \in \mathbb{Z}_m$. But $\alpha = id_*(\alpha) = f_*^k(\alpha) = \lambda^k\alpha$, so $\lambda^k \equiv 1 \bmod m$. Noting that $\widehat{H}^i(\mathbb{Z}_m) = 0$ for i odd, but is nontrivial for i even. Now $f_*(\alpha^i) = f_*(\alpha \smile \cdots \smile \alpha) = f_*(\alpha) \smile \cdots \smile f_*(\alpha) = (\lambda\alpha) \smile \cdots \smile (\lambda\alpha) = \lambda^i\alpha^i$ which is the identity only when i is a multiple of $2k$. Thus $\widehat{H}^i(\mathbb{Z}_m)^{\mathbb{Z}_n}$ is nontrivial only when $i \in 2k\mathbb{Z}$ (in which case it is $\mathbb{Z}_m$ since the action is trivial). Therefore,

$$\widehat{H}^i(\mathbb{Z}_m \rtimes \mathbb{Z}_n) \cong \widehat{H}^i(\mathbb{Z}_n) \oplus \widehat{H}^i(\mathbb{Z}_m)^{\mathbb{Z}_n} \cong \begin{cases} \mathbb{Z}_{mn} & i \equiv 0 \bmod 2k \\ 0 & i \text{ odd} \\ \mathbb{Z}_n & \text{otherwise} \end{cases}$$

for the case $gcd(m, n) = 1$. This means that the period of $H^*(G)$ is $2k$.

9.7: Let $\mathbb{F}_q$ be a field with q elements, where q is a prime power. The special linear group $G = SL_n(\mathbb{F}_q)$ is the kernel of the surjective determinant homomorphism $\det : GL_n(\mathbb{F}_q) \to \mathbb{F}_q^*$, i.e. it is the group of matrices with determinant 1. Let us first assume that $n \geq 3$. Then the cyclic groups $A = \langle \mathrm{diag}(a, a^{-1}, 1, \ldots, 1) \rangle$ and $B = \langle \mathrm{diag}(1, \ldots, 1, b, b^{-1}) \rangle$ form a non-cyclic abelian subgroup $A \times B \subseteq G$, noting that $\mathbb{F}_q$ is commutative. If we now let $n = 2$ then we will assume that q is not prime. Then the cyclic groups $A = \langle \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \rangle$ and $B = \langle \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \rangle$ form a non-cyclic abelian subgroup $A \times B \subseteq G$, since $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & a+b \\ 0 & 1 \end{smallmatrix}\right)$ with b not equal to any multiple of a (and vice versa). By Theorem VI.9.5[1], $G = SL_n(\mathbb{F}_q)$ does not have periodic cohomology if $n \geq 3$ or if q is not prime.
Note that if $n = 2$ and q is prime then $SL_2(\mathbb{F}_q)$ does have periodic cohomology, as explained on pg157[1].

9.8:

9.9: Suppose that G has p-periodic cohomology. Let $P \subseteq G$ be a subgroup of order p, let $N(P)$ (resp. $C(P)$) be the normalizer (resp. centralizer) of P in G, and let $W = N(P)/C(P)$; note that

if $C(P) = P$ then W is called the *Weyl group*. Choose a Sylow p-subgroup H containing P. Since G has p-periodic cohomology, H is either cyclic or generalized quaternion by Theorem VI.9.7[1]. I assert that $H \subseteq C(P) \subseteq N(P)$. Indeed, if H is cyclic then it is necessarily abelian so the result follows, and if H is generalized quaternion (must have $p = 2$) then it has a unique element of order 2 (as stated on pg98[1]) and this is then the generator for $P \cong \mathbb{Z}_2$ which must be central in H, so the result follows. Denote by X_G the G-invariant elements of $\widehat{H}^*(H, M)$ and similarly for $X_{N(P)}$, where an element $z \in \widehat{H}^*(H, M)$ is G-invariant if $\mathrm{res}^H_{H\cap gHg^{-1}} z = \mathrm{res}^{gHg^{-1}}_{H\cap gHg^{-1}} gz$ for all $g \in G$. Theorem III.10.3[1] states that $\widehat{H}^*(G, M)_{(p)} \cong X_G$ and $\widehat{H}^*(N(P), M)_{(p)} \cong X_{N(P)}$. Now trivially, $X_G \subseteq X_{N(P)}$, so it suffices to show that $X_{N(P)} \subseteq X_G$, for then $\widehat{H}^*(G, M)_{(p)} \cong X_G = X_{N(P)} \cong \widehat{H}^*(N(P), M)_{(p)}$. Note that P is the unique subgroup in H of order p because if H is generalized quaternion then the reasoning is as stated above and if H is cyclic then every subgroup has unique order (by Theorem 2.3.7[2]). If $H \cap gHg^{-1}$ is trivial then every element in $H^*(H, M)$ is clearly invariant for such $g \in G$. If $H \cap gHg^{-1}$ is not trivial then its order is at least p and the intersection contains P. This implies $P \subseteq gHg^{-1} \;\Rightarrow\; g^{-1}Pg \subseteq H \;\Rightarrow\; g^{-1}Pg = P$ and hence $g \in N(P)$, so the question of G-invariance reduces to the question of $N(P)$-invariance, i.e. $X_{N(P)} \subseteq X_G$.
Also, $X_{N(P)} \subseteq X_{C(P)}$ trivially, so by Theorem III.10.3[1] we have the inclusion (up to isomorphism) $\widehat{H}^*(N(P), M)_{(p)} \subseteq \widehat{H}^*(C(P), M)_{(p)}$. Since H is a Sylow p-subgroup contained in $C(P)$, $|W|$ and p are relatively prime and hence $\mathrm{cor}^{N(P)}_{C(P)}\mathrm{res}^{N(P)}_{C(P)} = |N(P) : C(P)| = |W|$ is an isomorphism on $\widehat{H}^*(N(P), M)_{(p)}$, where the equality is due to Proposition III.9.5[1]. Thus the restriction map induces a monomorphism $\widehat{H}^*(N(P), M)_{(p)} \hookrightarrow \widehat{H}^*(C(P), M)$. But as explained on pg84[1], if $z = \mathrm{res}^{N(P)}_{C(P)} u$ then z is $N(P)$-invariant; let the $N(P)$-invariants be denoted by $Y \subseteq \widehat{H}^*(C(P), M)$. Thus $\mathrm{res}^{N(P)}_{C(P)}$ maps $\widehat{H}^*(N(P), M)_{(p)}$ monomorphically into Y. Since $C(P) \triangleleft N(P)$, $Y = \widehat{H}^*(C(P), M)^W$ as noted on pg84[1]. Thus $\widehat{H}^*(N(P), M)_{(p)} \subseteq \widehat{H}^*(C(P), M)^W$, and so combining the two inclusions we see that $\widehat{H}^*(N(P), M)_{(p)} \subseteq \widehat{H}^*(C(P), M)_{(p)} \cap \widehat{H}^*(C(P), M)^W = \widehat{H}^*(C(P), M)^W_{(p)}$. For the other direction, if $z \in Y_{(p)} = \widehat{H}^*(C(P), M)^W_{(p)}$ then consider the element $w = \mathrm{cor}^{N(P)}_{C(P)} z$. Since $\widehat{H}^*(C(P), M)_{(p)}$ is annihilated by a power of p, $w \in \widehat{H}^*(N(P), M)_{(p)}$. Regurgitating the proof of Theorem III.10.3[1] using the double-coset formula, we deduce that $z = \mathrm{res}^{N(P)}_{C(P)} w'$ where $w' = w/|W| \in \widehat{H}^*(N(P), M)_{(p)}$. This means that $H(C(P), M)^W_{(p)} \subseteq \widehat{H}^*(N(P), M)_{(p)}$ because $\mathrm{res}^{N(P)}_{C(P)}$ maps $\widehat{H}^*(N(P), M)_{(p)}$ monomorphically into the $N(P)$-invariants. Thus $\widehat{H}^*(G, M)_{(p)} \cong \widehat{H}^*(N(P), M)_{(p)} \cong \widehat{H}^*(C(P), M)^W_{(p)}$.

9.10: For any finite group G, the augmentation ideal $I \subset \mathbb{Z}G$ is a cyclic G-module if G is cyclic group by Exercise I.2.1(b); we could also just note that if $G = \langle s \rangle$ then $I = \mathbb{Z}G \cdot (s - 1)$ because I consists of elements of the form $s^k - 1$ [we then form the elements $s^i - s^j \in I$ via summation], and $s^k - 1 = N \cdot (s - 1)$ where $N = s^{k-1} + \cdots + s + 1 \in \mathbb{Z}G$. Conversely, if I is cyclic as a G-module, so that $I = \mathbb{Z}G \cdot x$, then I claim that G admits a periodic resolution of period 2. Assuming this for the moment, G then has periodic cohomology (of period 2) by Theorem VI.9.1[1] and hence G is cyclic by Exercise VI.9.2. It suffices to prove the claim. The multiplication map $\mathbb{Z}G \to \mathbb{Z}G$ given by $r \mapsto rx$ has image I and kernel K, so we can form the exact sequence $0 \to K \to \mathbb{Z}G \xrightarrow{x} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$. Under the category of abelian groups, the Rank-Nullity Theorem gives $|G| = \dim_{\mathbb{Z}}\mathbb{Z}G = \dim_{\mathbb{Z}} I + \dim_{\mathbb{Z}}\mathbb{Z} = \dim_{\mathbb{Z}} I + 1$ for the augmentation map ε, and gives $|G| = \dim_{\mathbb{Z}}\mathbb{Z}G = \dim_{\mathbb{Z}} K + \dim_{\mathbb{Z}} I$ for the x-multiplication map. The first equation implies $\dim_{\mathbb{Z}} I = |G| - 1$ and the second equation then implies $\dim_{\mathbb{Z}} K = 1$, i.e. $K \cong \mathbb{Z}$ as an abelian group. So with $K = \langle k \rangle$, G acts on K via $gk_0 = zk_0$ (for $z \in \mathbb{Z}$). But then $k_0 = g^{|G|}k_0 = z^{|G|}k_0$ and hence $z = 1$, i.e. the G-action is trivial. Our exact sequence is now $0 \to \mathbb{Z} \to \mathbb{Z}G \xrightarrow{x} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$. Splicing together this sequence infinitely many times, we obtain the desired periodic resolution.

7 Chapter VII: Equivariant Homology and Spectral Sequences

2.1: Let $0 \to C' \to C \to C'' \to 0$ be a short exact sequence of chain complexes, and let $\{F_pC\}$ be the filtration such that $F_0C = 0$, $F_1C = C'$, and $F_2C = C$. Then $E^1_{1q} = H_{1+q}(C'/0) = H_{1+q}(C')$ and $E^1_{2q} = H_{2+q}(C/C') = H_{2+q}(C'')$ and $E^1_{pq} = 0$ for $p \leq 0$ and $p > 2$. The differential $d^1 : E^1_{pq} \to E^1_{p-1,q}$ gives maps $\partial : H_n(C'') \to H_{n-1}(C')$ with $n = 2 + q$ and $\partial = d^1_{2q}$.

Now E^1 is given by $\cdots \to E^1_{3q} = 0 \xrightarrow{d^1_3} E^1_{2q} \xrightarrow{\partial} E^1_{1q} \xrightarrow{d^1_1} E^1_{0q} = 0 \to \cdots$, and from $E^2 = H(E^1)$ we see that $E^2_{1q} = \mathrm{Ker} d^1_1/\mathrm{Im}\partial = H(C')/\mathrm{Im}\partial = \mathrm{Coker}\partial$ and $E^2_{2q} = \mathrm{Ker}\partial/\mathrm{Im} d^1_3 = \mathrm{Ker}\partial$ and $E^2_{pq} = 0$ for $p \leq 0$ and $p > 2$. In E^2 we have $\mathrm{Ker}\partial \xrightarrow{d^2_{2q}} 0$ and $\mathrm{Coker}\partial \xrightarrow{d^2_{1q}} 0$ and trivial maps for $p \neq 1$ and $p \neq 2$, because the differential $d^2 : E^2_{pq} \to E^2_{p-2,q+1}$ is of bidegree $(-2, 1)$. Thus $H(E^2) = E^2$, and so the spectral sequence "collapses" at E^2 (i.e. $E^2 \cong E^\infty$) because $E^{r+1} \cong H(E^r)$ in general.

Since $E^\infty_p = Gr_p\, H(C) := F_pH(C)/F_{p-1}H(C)$ where $F_pH(C) = \mathrm{Im}\{H(F_pC) \xrightarrow{\varphi} H(C)\}$, we see that $E^\infty_p = 0$ for $p \leq 0$ and $p > 2$ and $\mathrm{Coker}\partial = E^2 \cong E^\infty_1 = F_1H(C)$ and $\mathrm{Ker}\partial = E^2 \cong E^\infty_2 = F_2H(C)/F_1H(C)$. Since $F_2H(C) = H(C)$ we have a surjection $H_n(C) \twoheadrightarrow F_2H(C)/F_1H(C) \cong \mathrm{Ker}\partial$, and since $F_1H(C) = \mathrm{Im}\{H(C') \xrightarrow{\varphi} H(C)\}$ we have a commutative diagram

$$\begin{tikzcd}
H_n(C') \arrow[r] \arrow[dr, two heads, "\varphi"'] & H_n(C) \\
& F_1H_n(C) \arrow[u, hook]
\end{tikzcd}$$

hence an injection $\mathrm{Coker}\partial = F_1H_n(C) \hookrightarrow H_n(C)$. We can now rewrite the E^∞ sequence $0 \to F_1H_n(C) \to F_2H_n(C)/F_1H_n(C) \to 0$ as a short exact sequence $0 \to \mathrm{Coker}\partial \hookrightarrow H_n(C) \twoheadrightarrow \mathrm{Ker}\partial \to 0$, and we then have a commutative diagram

$$\begin{tikzcd}
H_n(C') \arrow[rr, "i"] \arrow[dr, two heads, "\varphi"'] & & H_n(C) \arrow[rr, "j"] \arrow[dr, two heads] & & H_n(C'') \\
& \mathrm{Coker}\partial \arrow[ur, hook] & & \mathrm{Ker}\partial \arrow[ur, hook] &
\end{tikzcd}$$

The top row is exact at $H_n(C)$ because $\mathrm{Ker} j = \mathrm{Ker}\{H_n(C) \twoheadrightarrow \mathrm{Ker}\partial\} = \mathrm{Coker}\partial = \mathrm{Im}\varphi = \mathrm{Im} i$. Since $\mathrm{Im} j = \mathrm{Ker}\partial$ and $\mathrm{Ker} i = \mathrm{Ker}\varphi = \mathrm{Im}\partial$, the top row of this commutative diagram extends to a long exact homology sequence.

We have thus deduced the familiar long exact homology sequence from this spectral sequence, and *the spectral sequence of a filtered complex can be regarded as a generalization of the long exact sequence associated to a chain complex and a subcomplex.*

3.1(a): Let C be a first-quadrant double complex such that the associated spectral sequence to $F_p(TC)_n = \bigoplus_{i \leq p} C_{i,n-i}$ has $E^1_{pq} = 0$ for $q \neq 0$, and let D be the chain complex $E^1_{*,0}$ with differential d^1. We have the isomorphisms $H_n(TC) = Ker[\bigoplus_{p \leq n} C_{p,n-p} \to \bigoplus_{p \leq n-1} C_{p,n-1-p}]/Im[\bigoplus_{p \leq n+1} C_{p,n+1-p} \to \bigoplus_{p \leq n} C_{p,n-p}] = [\bigoplus_{p \leq n-1} H_{n-p}(C_{p,*})] \oplus X = [\bigoplus_{p \leq n-1} E^1_{p,n-p}] \oplus X = 0 \oplus X$, where $X = C_{n,0}/(Im[C_{n+1,0} \to C_{n,0}] \oplus Im[C_{n,1} \to C_{n,0}]) = H_n(C_{*,0}/Im[C_{*,1} \to C_{*,0}]) = H_n(H_0(C_{*,*})) = H_n(E^1_{*,0}) = H_n(D)$. Thus $H_n(TC) \cong H_n(D)$.

3.1(b): Take $\tau : TC \to D$ to be the canonical surjection, and note that this can be viewed as a map of double complexes $C \to D$ (where D is regarded as a double complex concentrated on the line $q = 0$); this is obviously a filtration-preserving chain map. Now $E^1_{pq}(D) = H_q(E^1_{p,*} \equiv E^1_{p,0})$ which is 0 if $q \neq 0$, and is $\mathrm{Ker}\partial_0 \equiv E^1_{p,0}$ if $q = 0$ (since $E^1_{p,-1} = 0$). This is precisely the spectral sequence associated to C, so the induced map on spectral sequences from τ is an isomorphism at the E^1-level. Thus by Proposition VIII.2.6, τ induces an isomorphism $H_*(TC) \to H_*(TD) \cong H_*(D)$ and hence is a weak equivalence.

4.1: Suppose X is the union of subcomplexes X_α such that every non-empty intersection $X_{\alpha_0} \cap \cdots \cap X_{\alpha_p}$ ($p \geq 0$) is acyclic, and let K be the nerve of the covering $\{X_\alpha\}$. Let C be the double complex $C_{pq} = \bigoplus_{\sigma \in K^{(p)}} C_q(X_\sigma)$, and let T be the [total] chain complex TC. As shown on pg167[1], we have a spectral sequence with E^1_{pq} equal to 0 if $q \neq 0$ and equal to $C_p(X)$ if $q = 0$. Then Exercise 3.1(b)

implies that we have a weak equivalence $T \to C(X)$. Moreover, we have another spectral sequence with $E^1_{pq} = C_p(K, \mathcal{H}_q)$ where $\mathcal{H}_q \equiv \{H_q(X_\sigma)\}$ is a coefficient system on K. Since each X_σ is acyclic, E^1_{pq} is 0 for $q \neq 0$ and is $C_p(X, \mathbb{Z})$ for $q = 0$. Then Exercise 3.1(b) implies that we have a weak equivalence $T \to C(K)$. Thus we have an isomorphism $H_*(X) \cong H_*(K)$.

7.2: Let X be a G-complex such that for each cell σ of X, the isotropy group G_σ fixes σ pointwise; in this case the orbit space X/G inherits a CW-structure. Assume further that each G_σ is finite. We have a spectral sequence $E^1_{pq} = \bigoplus_{\sigma \in \Sigma_p} H_q(G_\sigma, M_\sigma) \Rightarrow H^G_{p+q}(X, M)$, where Σ_p is a set of representatives for X_p/G (X_p is the set of p-cells of X) and $M_\sigma = \mathbb{Z}_\sigma \otimes M$ ($\mathbb{Z}_\sigma$ is the G_σ-module additively isomorphic to $\mathbb{Z}$, on which G_σ acts by the *orientation character* $\chi_\sigma : G_\sigma \to \{\pm 1\}$). Since G_σ fixes σ pointwise, $\chi_\sigma(G_\sigma) = \{1\}$ and hence $\mathbb{Z}_\sigma = \mathbb{Z}$ and hence $\mathbb{Q}_\sigma = \mathbb{Z} \otimes \mathbb{Q} = \mathbb{Q}$ (where we now take rational coefficients $M = \mathbb{Q}$). Now for all $q > 0$, $H_q(G_\sigma, \mathbb{Q}) = H_q(G_\sigma) \otimes \mathbb{Q} = 0$ where the first equality is proved in Exercise AE.6 and the latter equality follows from the fact that $H_q(G_\sigma)$ is finite (proved in Exercise AE.16). The E^1 term is then concentrated on the line $q = 0$, and the spectral sequence therefore collapses at $E^2 = H_*(E^1)$ to yield $H^G_*(X, \mathbb{Q}) \cong H_*(\bigoplus_{\sigma \in \Sigma_p} H_0(G_\sigma, \mathbb{Q}_\sigma)) \overset{*}{=} H_*(H_0(G, \bigoplus_{\sigma \in \Sigma_p} \mathrm{Ind}^G_{G_\sigma} \mathbb{Q}_\sigma)) = H_*(H_0(G, C_p(X; \mathbb{Q}))) = H_*(C_p(X; \mathbb{Q})_G) = H_*(X/G; \mathbb{Q})$, where the starred equality is the result of Shapiro's lemma and the last equality is given by Proposition II.2.4[1].
If X is also contractible, then X is necessarily acyclic. Thus the above result and Proposition VII.7.3[1] imply $H_*(G, \mathbb{Q}) \cong H^G_*(X, \mathbb{Q}) \cong H_*(X/G; \mathbb{Q})$.

Note: The hypothesis that G_σ fixes σ pointwise is not very restrictive in practice. In the case of a simplicial action, it can always be achieved by passage to the barycentric subdivision $\tilde{X}$. Indeed, for $\sigma' \subset \tilde{X}$ which lies in $\sigma \subset X$, $G_{\sigma'} \subseteq G_\sigma$. If $G_{\sigma'}$ did not fix σ' pointwise then this would break continuity of the G-action on σ (consider two such simplices of $\tilde{X}$ which lie in σ and have a common face).

7.3: Let X be a G-complex and E a free contractible G-complex. There is a G-map $X \times E \to X$ which is a homotopy equivalence, so Proposition VII.7.3 implies that $H^G_*(X \times E) \to H^G_*(X)$ is an isomorphism. As G acts freely on $X \times E$, $H^G_*(X \times E) \cong H_*((X \times E)/G)$. Thus we have the equivalence $H^G_*(X) \cong H_*(X \times^G E)$.

7.5: Let X be a G-complex and let N be a normal subgroup of G which acts freely on X. Let $\mathbb{Z}_\sigma$ be the orientation module and let X_p denote the set of p-cells of X. Let Σ_p be a set of representatives for X_p/G and let Σ'_p be a set of representatives for $(X/N)_p/(G/N)$; it is easy to see that both sets are in bijective correspondence. To prove that $H^G_*(X, M) \cong H^{G/N}_*(X/N, M)$ with any G/N-module coefficient M, it suffices to show that $\bigoplus_{\sigma \in \Sigma_p} H_q(G_\sigma, M_\sigma) \cong \bigoplus_{\sigma' \in \Sigma'_p} H_q((G/N)_{\sigma'}, M_{\sigma'})$; this is because we have a spectral sequence $E^1_{pq} = \bigoplus_{\sigma \in \Sigma_p} H_q(G_\sigma, M_\sigma) \Rightarrow H^G_{p+q}(X, M)$ and so the said isomorphism will give isomorphic associated graded modules (since $E^r = H(E^{r-1})$ and E^∞ is the associated graded module), and this will give $H^G_*(X, M) \cong H^{G/N}_*(X/N, M)$ by Lemma VII.2.1[1]. In view of the bijection $\Sigma_p \to \Sigma'_p$, $\sigma \mapsto \sigma'$, it suffices to show that $H_q(G_\sigma, M_\sigma) \cong H_q((G/N)_{\sigma'}, M_{\sigma'})$. First note that $M_\sigma \equiv \mathbb{Z}_\sigma \otimes M = \mathbb{Z}_{\sigma'} \otimes M \equiv M_{\sigma'}$ because the G_σ-action and the $(G/N)_{\sigma'}$-action on $\mathbb{Z}$ coincide (if $g \in G_\sigma$ preserves the orientation of $\sigma \in X$, $g\sigma = +\sigma$, then gN preserves the orientation of $\sigma' = \sigma = n\sigma \in X/N$ because $+\sigma' = +\sigma = g\sigma = g \cdot n\sigma = (gn)\sigma = (gn)\sigma' \in X/N$ for all $n \in N$) and because the two actions coincide on M by definition (since M is a G/N-module). Thus it suffices to show (due to the Künneth formula) that $H_q(G_\sigma) \cong H_q((G/N)_{\sigma'})$ where the homology is now using integer coefficients, and in turn it suffices to show that $G_\sigma \cong (G/N)_{\sigma'}$ where σ' is the image of σ under the quotient $X \to X/N$. Consider the obvious monomorphism $\varphi : G_\sigma \to (G/N)_{\sigma'}$ given by $g \mapsto gN$; it suffices to show that φ is surjective. But this is immediate, because if $gN \in (G/N)_{\sigma'}$ then $gN\sigma' = \sigma'$ which implies $gn_1\sigma = n_2\sigma$ for some $n_1, n_2 \in N$ which implies $n_2^{-1}gn_1 \in G_\sigma$, and then $\varphi(n_2^{-1}gn_1) = n_2^{-1}gn_1N = n_2^{-1}gN = n_2^{-1}g(g^{-1}n_2g)N = gN$ where we note that $N \lhd G$.

10.1: The proof of Theorem VII.10.5[1] used the assumption that $|G| = p$ (p prime) to state that $\dim_{\mathbb{Z}_p} \widehat{H}^n(G, \mathbb{Z})_p = 1$ for all $n \in \mathbb{Z}$, because $G \cong \mathbb{Z}_p$ has Tate cohomology group $\mathbb{Z}_p$ in every dimension. The proof also used that assumption in order to apply Proposition VII.10.1[1] to the G-invariant subcomplex X^G of X; the isotropy group G_σ for every cell $\sigma \in X - Y$ cannot equal G (because X^G is the largest subcomplex on which G acts trivially) and hence must be the trivial group (the only proper

subgroup of $G \cong \mathbb{Z}_p$).

10.2: The extended theorem holds for $|G| = p^1$ by the original Theorem VII.10.5[1], so we proceed by induction, assuming the extended theorem holds for $|G| = p^r$. Let $|G| = p^{r+1}$ and remember the hypothesis that X^H is a subcomplex for all $H \subseteq G$. Since G is a p-group we can choose a maximal normal subgroup N of index p, so that $X^G = (X^N)^{G/N}$. Via induction (since $|N| = p^r$) we can apply the extended theorem to X with X^N [which is subcomplex by hypothesis], so every condition of the theorem on X is also satisfied on X^N. Since $|G/N| = p$ we can apply the original theorem to X^N with $(X^N)^{G/N}$ [which is a subcomplex since it is equal to X^G], so every condition of the theorem on X^N is also satisfied on $(X^N)^{G/N}$. Thus every condition of the extended theorem on X is also satisfied on $(X^N)^{G/N} = X^G$, and the proof is complete.

10.3: Let X be a finite-dimensional free G-complex (G finite) with $H_*(X) \cong H_*(S^{2k})$. Proposition VII.10.1[1] (with $Y = \varnothing$) implies that $\widehat{H}_*^G(X, M) = 0$, noting that G_σ is trivial for all σ since X is G-free. On the other hand, we have a spectral sequence $E^2_{pq} = \widehat{H}_p(G, H_q(X, M)) \Rightarrow \widehat{H}^G_{p+q}(X, M)$. Since the spectral sequence is concentrated on the horizontal lines $q = 0$ and $q = 2k$ (the only nonzero homology terms of the $2k$-sphere), it follows that the differential $d^{2k+1} : \widehat{H}_p(G, H_{2k}(X, M)) \to \widehat{H}_{p-(2k+1)}(G, H_{2k+(2k+1)-1}(X, M)) = \widehat{H}_{p-2k-1}(G, H_{4k}(X, M)) = 0$ is an isomorphism. To prove that every nontrivial element of G acts nontrivially on $H_{2k}(X)$ it suffices to show that every cyclic subgroup of G (generated by the elements of G) acts nontrivially on $H_{2k}(X)$, so we are immediately reduced to the case where G is cyclic and nontrivial. Using the isomorphism d^{2k+1} with $M = \mathbb{Z}$ and p odd, we conclude that $\widehat{H}_{odd}(G, H_{2k}(X)) = 0$. This means G acts nontrivially on $H_{2k}(X)$, otherwise $\widehat{H}_{odd}(G, H_{2k}(X)) = \widehat{H}_{odd}(G, \mathbb{Z}) = \mathbb{Z}_{|G|}$ which is not equal to 0. The proof is now complete. Note that a nontrivial G-action on $H_{2k}(X) \cong \mathbb{Z}$ means that $G = \mathbb{Z}_2$, so having every nontrivial element of G act nontrivially on $H_{2k}(X)$ means that $|G| \leq 2$ (the case $|G| = 1$ is satisfied vacuously since there are no nontrivial elements).

8 Chapter VIII: Finiteness Conditions

2.1: By definition, $\operatorname{cd}\Gamma = 0$ iff $\mathbb{Z}$ admits a projective resolution $0 \to P \to \mathbb{Z} \to 0$ of length 0, i.e. $\mathbb{Z} \cong P$ and $\mathbb{Z}$ is $\mathbb{Z}\Gamma$-projective. But Exercise I.8.1 states that only the trivial group $\Gamma = \{1\}$ makes $\mathbb{Z}$ a projective module. Thus the trivial group is the only group of cohomological dimension zero.

2.2: Take $\Gamma = \mathbb{Z}_2$. Then $\operatorname{cd}\Gamma = \infty$ by Corollary 2.5. Now a free $\mathbb{Z}\Gamma$-module F is a direct sum $\bigoplus \mathbb{Z}\Gamma$ where the Γ-action would be a parity-permutation on all or some of the summands, so $H^n(\Gamma, F) \cong [\bigoplus H^n(\Gamma, (\mathbb{Z}\Gamma)^2)] \oplus [\bigoplus H^n(\Gamma, \mathbb{Z}\Gamma)]$ where the first collection of summands has the nontrivial Γ-action. But then that module $(\mathbb{Z}\Gamma)^2$ is an induced module $\operatorname{Ind}_{\{1\}}^{\Gamma}\mathbb{Z}\Gamma$, and so are the other modules $\mathbb{Z}\Gamma$ (trivially). So these modules are H^*-acyclic and hence $H^n(\Gamma, F) = 0$ for all $n > 0$, i.e. $\sup\{n \mid H^n(\Gamma, F) \neq 0 \text{ for some } F\} = 0$.

2.7(a): Induced Γ-modules $\mathbb{Z}\Gamma \otimes A$ are cohomologically trivial (as noted on pg148[1]) and hence have projective dimension ≤ 1 by Theorem VI.8.12[1].

2.7(b): If $\operatorname{proj\,dim}_R M \leq n$, then $\operatorname{Ext}_R^{n+1}(M, -) = 0$ by Lemma VII.2.1[1]. For any direct summand M' of M, we must then have $\operatorname{Ext}_R^{n+1}(M', -) = 0$ since $\operatorname{Ext}_R^{n+1}(-, -)$ commutes with direct sums. Thus $\operatorname{proj\,dim}_R M' \leq n$ by Lemma VII.2.1[1].

2.7(c): Suppose Γ is finite and M is a Γ-module in which $|\Gamma|$ is invertible. It suffices to show that M is a direct summand of an induced module $\mathbb{Z}\Gamma \otimes A$, for then $\operatorname{proj\,dim}_{\mathbb{Z}\Gamma}(\mathbb{Z}\Gamma \otimes A) \leq 1$ by part(a) and hence $\operatorname{proj\,dim}_{\mathbb{Z}\Gamma} M \leq 1$ by part(b). Take $A = M_0$, where M_0 is the underlying abelian group of M. By Corollary III.5.7[1] there is a Γ-module isomorphism $\varphi : \mathbb{Z}\Gamma \otimes M \to \mathbb{Z}\Gamma \otimes M_0$ given by $g \otimes m \mapsto g \otimes g^{-1}m$, where $\mathbb{Z}\Gamma \otimes M$ has the diagonal Γ-action. The inclusion $i : M \to \mathbb{Z}\Gamma \otimes M$ given by $m \mapsto \sum_{g\in\Gamma} g \otimes m$ is a Γ-module homomorphism because $\gamma \cdot i(m) = \gamma \cdot (\sum_g g \otimes m) = \sum_g \gamma g \otimes \gamma m = \sum_g g \otimes \gamma m = i(\gamma \cdot m)$. The Γ-module map $\pi : \mathbb{Z}\Gamma \otimes M_0 \to M$ defined by $g \otimes m \mapsto \frac{1}{|\Gamma|} gm$ is a Γ-splitting to φi [note: the action on $\mathbb{Z}\Gamma \otimes M_0$ is $\gamma \cdot (g \otimes m) = \gamma g \otimes m$]. Indeed, $\pi[\varphi i](m) = \pi[\varphi(\sum_g g \otimes m)] = \pi[\sum_g g \otimes g^{-1}m] = \frac{1}{|\Gamma|}\sum_g gg^{-1}m = \frac{1}{|\Gamma|}|\Gamma|m = m = id_M(m)$. Thus the injection $\varphi i : M \to \mathbb{Z}\Gamma \otimes M_0$ splits, and M is then (by definition of a splitting homomorphism) a direct summand of $\mathbb{Z}\Gamma \otimes M_0$ as a Γ-module.

4.1: If $\mathbb{Z}$ is finitely presented as a $\mathbb{Z}\Gamma$-module then $\mathbb{Z}$ is finitely generated and every surjection $P \twoheadrightarrow \mathbb{Z}$ (with P finitely generated and projective) has a finitely generated kernel (Proposition VIII.4.1[1]). In particular, the augmentation map $\varepsilon : \mathbb{Z}\Gamma \to \mathbb{Z}$ has kernel I which then must be finitely generated (obviously noting that $\mathbb{Z}\Gamma$ is free of rank 1). Exercise I.2.1(d) then implies Γ is a finitely generated group. Conversely, suppose Γ is a finitely generated abelian group, so that $\Gamma = F(S)/R$ is a group presentation for Γ with $|S| < \infty$. Then there is an exact sequence $(\mathbb{Z}\Gamma)^{|S|} \to \mathbb{Z}\Gamma \to \mathbb{Z} \to 0$ by Exercise IV.2.4(d) and hence $\mathbb{Z}$ is finitely presented as a $\mathbb{Z}\Gamma$-module by Proposition VIII.4.1[1].

6.1: Let Γ be of type FL and $\operatorname{cd}\Gamma = n$. Then Γ is of type FP and hence of type FP_∞ by Proposition VIII.6.1, and so there is a partial resolution $F_m \to \cdots \to F_0 \to \mathbb{Z} \to 0$ with each F_i free of finite rank by Proposition VIII.4.3 (for all $m \geq 0$). Due to its cohomological dimension, we can make a finite projective resolution $0 \to P \to F_{n-1} \to \cdots \to F_0 \to \mathbb{Z} \to 0$ with each F_i free and P projective. Then Proposition VIII.6.5 implies that P is stably free, and so there is some free module F of finite rank such that $P \oplus F$ is free. Take the free resolution $0 \to F \to F \to 0 \to \cdots \to 0$ and consider its direct sum with the finite projective resolution. This gives us a finite free resolution for $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ of length n.

6.3: Let Γ be of type FP and $\operatorname{cd}\Gamma = n$. Then we have a finite projective resolution $P_n \to \cdots \to P_0 \to \mathbb{Z} \to 0$, and taking the Hom-dual we obtain the resolution for cohomology which behaves as $\cdots \to Hom_{\mathbb{Z}\Gamma}(P_n, \mathbb{Z}\Gamma) \to 0$. Since P_n is a finitely generated projective module, so is its Hom-dual; thus $H^n(\Gamma, \mathbb{Z}\Gamma)$ is a finitely generated Γ-module.

9 Chapter IX: Euler Characteristics

1.1: For a ring A, suppose there is a $\mathbb{Z}$-valued function r on finitely generated projective A-modules, satisfying $r(P \oplus Q) = r(P) + r(Q)$ and $r(A) = 1$ and $r(P) > 0$ if $P \neq 0$. I claim that A is *indecomposable*, i.e. that A cannot be decomposed as the direct sum of two non-zero left ideals. Indeed, a decomposition $A = I \oplus J$ yields the equation $1 = r(I) + r(J)$ because I and J are projective A-modules (direct summands of the free module A). Since both ideals are non-zero, $r(I) \geq 1$ and $r(J) \geq 1$, and this yields the desired contradiction $1 \geq 1 + 1 = 2$.

2.1: Let P be a finitely generated projective (left) A-module and let $P^* = \mathrm{Hom}_A(P, A)$ be its dual. Then P^* is a right A-module and we have $P^* \otimes_A P \cong \mathrm{Hom}_A(P, P)$ by Proposition I.8.3. Consider the diagram

$$\begin{tikzcd} P^* \otimes_A P \arrow[rr, "\cong"] \arrow[dr, "ev"'] & & \mathrm{Hom}_A(P,P) \arrow[dl, "tr"] \\ & T(A) & \end{tikzcd}$$

where $ev(u \otimes x) = \overline{u(x)}$ is the evaluation map. The isomorphism is given by $u \otimes x \mapsto [p \mapsto u(p) \cdot x]$, and on basis elements $e_i \in P$ this image homomorphism is $e_i \mapsto \sum_j u(e_i) r_j \cdot e_j$, where $x = \sum_j r_j e_j$. Thus the composition is $u \otimes x \mapsto tr[p \mapsto u(p) \cdot x] = \overline{\sum_i u(e_i) r_i} = \overline{\sum_i r_i \cdot u(e_i)} = \overline{\sum_i u(r_i e_i)} = \overline{u(x)} = ev(u \otimes x)$, and the diagram is commutative.

2.4: Let F be a finitely generated free module and $e : F \to F$ a projection operator of F onto a direct summand isomorphic to P (this is *idempotent* since $e^2 = e$). Then $e = i \circ id_P \circ \pi$, where i and π are the inclusion and projection maps between F and P, so $tr(e) = tr(i \circ id_P \circ \pi) = tr(id_P) = R(P)$. *Thus $R(P)$ is equal to the trace of an idempotent matrix defining P.*

2.5: Let Γ be a group and $\varphi : \mathbb{Z}\Gamma \to \mathbb{Z}$ the augmentation map, and let P be a finitely generated projective Γ-module. Then Proposition 2.3 implies that $tr_{\mathbb{Z}}(\mathbb{Z} \otimes_\Gamma id_P) = T(\varphi)(tr_{\mathbb{Z}\Gamma}(id_P)) = T(\varphi)(R_{\mathbb{Z}\Gamma}(P))$. As this is an element of $T(\mathbb{Z}) = \mathbb{Z}$, it is immediate that $tr_{\mathbb{Z}}(\mathbb{Z} \otimes_\Gamma id_P)$ is precisely the $\mathbb{Z}$-rank of $P_\Gamma = \mathbb{Z} \otimes_\Gamma P$. Thus $\varepsilon_\Gamma(P) = T(\varphi)(R_{\mathbb{Z}\Gamma}(P))$.

2.6(a): Let Γ be a finite group. From the definition, $tr_{\mathbb{Z}\Gamma/\mathbb{Z}} : T(\mathbb{Z}\Gamma) \to T(\mathbb{Z})$ lifts to the map $\mathbb{Z}\Gamma \to T(\mathbb{Z}) = \mathbb{Z}$ given by $\gamma \mapsto tr_{\mathbb{Z}}(\mu_\gamma)$, where $\mu_\gamma : \mathbb{Z}\Gamma \to \mathbb{Z}\Gamma$ is right-multiplication by γ. Taking μ_γ as a matrix over $\mathbb{Z}$, it is the identity matrix for $\gamma = 1$ and is a matrix with zeros on the diagonal for $1 \neq \gamma \in \Gamma$ (γ permutes the basis elements). Thus $tr_{\mathbb{Z}\Gamma/\mathbb{Z}}(\bar{1}) = |\Gamma|$ and $tr_{\mathbb{Z}\Gamma/\mathbb{Z}}(\bar{\gamma}) = 0$ for $1 \neq \gamma \in \Gamma$. Consequently, there is a well-defined homomorphism $\tau : T(\mathbb{Z}\Gamma) \to \mathbb{Z}$ such that $\tau(\bar{1}) = 1$ and $\tau(\bar{\gamma}) = 0$ for $1 \neq \gamma \in \Gamma$, and one has $tr_{\mathbb{Z}\Gamma/\mathbb{Z}} = |\Gamma| \cdot \tau$.

2.6(b): Applying Proposition 2.4 to $\alpha = id_P$ and $\varphi : \mathbb{Z} \hookrightarrow \mathbb{Z}\Gamma$, and following the same method as in the proof of Exercise 2.5, and using the result of part(a), we have that $rk_{\mathbb{Z}}(P) = |\Gamma| \cdot \tau(R_{\mathbb{Z}\Gamma}(P))$ for any finitely generated projective Γ-module P. Thus $\rho_\Gamma(P) = \tau(R_{\mathbb{Z}\Gamma}(P))$.

2.7: The previous exercise shows that $\rho_\Gamma(P) \in \mathbb{Z}$, for Γ a finite group. It is obvious from the definition of ρ that $\rho_\Gamma(P) > 0$ if $P \neq 0$ (it must be greater than or equal to $1/|\Gamma|$). Now for two finitely generated projective Γ-modules P and Q, they are necessarily free $\mathbb{Z}$-modules and hence $rk_{\mathbb{Z}}(P \oplus Q) = rk_{\mathbb{Z}}(P) + rk_{\mathbb{Z}}(Q)$. Furthermore, $\rho_\Gamma(\mathbb{Z}\Gamma) = rk_{\mathbb{Z}}(\mathbb{Z}\Gamma)/|\Gamma| = |\Gamma|/|\Gamma| = 1$. Thus $\mathbb{Z}\Gamma$ is indecomposable by Exercise 1.1.

4.1: The proof of Theorem IX.4.4[1] used the assumption that Γ was finite in order to apply Proposition IX.4.1[1] by replacing Γ by the cyclic subgroup $\Gamma' = \langle \gamma \rangle$. The proposition requires Γ' to be of finite index (for all $\gamma \in \Gamma$), and this will hold in general if $|\Gamma| < \infty$.

4.3(b): It is a fact from representation theory that a finitely generated $k\Gamma$-module is determined up to isomorphism by its character. By part(a), the character is in bijective correspondence with the Hattori-Stallings rank. Thus two finitely generated $k\Gamma$-modules are isomorphic iff they have the same

Hattori-Stallings rank.

4.3(c): Take $k = \mathbb{Q}$. If Γ is finite and P is a finitely generated projective $\mathbb{Z}\Gamma$-module then $\mathbb{Q} \otimes_{\mathbb{Z}} P$ is a finitely generated projective $\mathbb{Q}\Gamma$-module, hence a finitely generated projective $\mathbb{Z}\Gamma$-module by Exercise I.8.2. Then by Theorem 4.4 there is an integer r such that $R_\Gamma(\mathbb{Q} \otimes_{\mathbb{Z}} P) = r \cdot [1]$. But this Hattori-Stallings rank is precisely that of $(\mathbb{Q}\Gamma)^r$, so $\mathbb{Q} \otimes_{\mathbb{Z}} P$ is a free $\mathbb{Q}\Gamma$-module by part(b).

4.4: Taking $\rho_\Gamma(P) = R_\Gamma(P)(1)$ as a definition, the result is precisely Proposition 4.1 applied to $\gamma = 1$.

10 Additional Exercises

1: Find a counterexample to the statement $M_G \cong M^G$ for a G-module M.

For an arbitrary group G we have $(\mathbb{Z}G)_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}G \cong \mathbb{Z}$. Alternatively, $(\mathbb{Z}G)_G \cong \mathbb{Z}G/[I \cdot \mathbb{Z}G] = \mathbb{Z}G/I \cong \mathbb{Z}$ (where the latter isomorphism follows from application of the 1st Isomorphism Theorem on the augmentation map). If G is finite, then the norm element N exists, and the integer multiples zN ($z \in \mathbb{Z}$) are the only elements of $\mathbb{Z}G$ fixed by all $g \in G$ (assuming left-multiplication action), so $(\mathbb{Z}G)^G = \mathbb{Z} \cdot N$ [$\cong \mathbb{Z}$] is the ideal generated by N. But if G is infinite then there is no norm element, and $(\mathbb{Z}G)^G = 0$. To prove this last statement, take any nonzero element $x = \sum_i r_i g_i$ of $\mathbb{Z}G$ (since it's a finite sum we can assume $0 \leq i \leq n$ and all $r_i \in \mathbb{Z}$ are nonzero) and consider the set $S = \{g_0, \ldots, g_n\}$. To show that $x \notin (\mathbb{Z}G)^G$ it suffices to show that there is at least one nontrivial g with $gS \neq S$. We can assume $1 \notin S$, otherwise for any $g \notin S$ we have $g \cdot 1 = g \notin S$. Suppose that $gS = S \ \ \forall\, g \in G$. Then $g_0 \cdot g_0 = g_0^2 \in S$, and so through trivial induction we see that $g_0^i \in S$ ($1 \leq i \leq n+1$) and $g_0 \cdot g_0^{n+1} = g_0^{n+2} \notin S$ (otherwise $|S| > n+1$), noting that $g_0^i \neq 1$ since $1 \notin S$. Thus we have arrived at a contradiction, and so the choices $G = \mathbb{Z}$ and $M = \mathbb{Z}G$ suffice.

Another solution uses the choices $G = \mathbb{Z}_2 = \langle x \rangle$ and $M = \mathbb{Z}$ where G acts by $x \cdot n = -n$. Then $x \cdot n = n \ \Rightarrow\ -n = n \ \Rightarrow\ 2n = 0$, so the largest quotient on which G acts trivially is $\mathbb{Z}_{\mathbb{Z}_2} = \mathbb{Z}_2$, and $2n = 0$ only holds for $n = 0 \in \mathbb{Z}$, hence $\mathbb{Z}^{\mathbb{Z}_2} = 0$.

2: Let $\mathbb{Z}_2 = \langle x \rangle$ act on the additive complex numbers $\mathbb{C}$ by $x \cdot z = z^*$, where $z^* = x - iy$ is the complex conjugate of $z = x + iy$. [*Note*: $\mathbb{C}^{\mathbb{Z}_2} \cong \mathbb{C}_{\mathbb{Z}_2} \cong \mathbb{R}$]. Find $H^1(\mathbb{Z}_2, \mathbb{C})$.

For a derivation $f : \mathbb{Z}_2 \to \mathbb{C}$ we have $f(x^0) = f(x^2) = f(1) = 0$ and $f(x)$ determines f. Now $f(x^2) = f(x) + x \cdot f(x) = f(x) + f(x)^*$ and so we must have $f(x) = -f(x)^*$ (i.e. a pure-imaginary number). Since $iy = -(iy)^*$ and $0 + iy = (iy)/2 + (iy)/2 = (iy)/2 - (-iy)/2 = (iy)/2 - (iy)^*/2$, we have $f(x) = \frac{1}{2}f(x) - \frac{1}{2}f(x)^* = -\frac{1}{2}f(x)^* - x \cdot [\frac{1}{2}f(x)] = x \cdot [\frac{1}{2}f(x)^*] - [\frac{1}{2}f(x)^*]$. Therefore, f is a principal derivation and hence $H^1(\mathbb{Z}_2, \mathbb{C}) = 0$, using the result of Exercise III.1.2 above.

3: Let the multiplicative cyclic group $C_{2k} = \langle x \rangle$ act on $\mathbb{Z}$ by $x \cdot n = (-1)^k n$. Calculate $H^1(C_{2k}, \mathbb{Z})$ under this action.

For $k = 2m$ even, C_{2k} acts trivially on $\mathbb{Z}$, so $H^1(C_{2k}, \mathbb{Z}) \cong \mathrm{Hom}(C_{2k}, \mathbb{Z}) = 0$ where this equation follows from Exercise III.1.2 above. For $k = 2m+1$ odd, we start by viewing the 1-cocycles as $Z^1 = \mathrm{Der}(C_{2k}, \mathbb{Z})$ and the 1-coboundaries as $B^1 = \mathrm{PDer}(C_{2k}, \mathbb{Z})$. For $f \in Z^1$, $f(x^{2i}) = f(x) + x \cdot f(x^{2i-1}) = f(x) - f(x^{2i-1})$ and $f(x^{2i}) = f(x^{2i-1}) + x^{2i-1} \cdot f(x) = f(x^{2i-1}) + (-1)^{(2i-1)(2m+1)} f(x) = f(x^{2i-1}) - f(x)$, so $2f(x) = 2f(x^{2i-1}) \ \Rightarrow\ f(x) = f(x^{2i-1})$ and $f(x^{2i}) = 0$. So Z^1 consists of the derivations which are determined by $f(x)$ and satisfy the derived properties, hence $Z^1 \cong \mathbb{Z}$. For $f \in B^1$, $f(x^i) = x^i \cdot n - n = (-1)^i n - n$ which satisfies $f(x^i) = 0 \ \ \forall\ i = 2j$ and $f(x^i) = -2n \ \ \forall\ i = 2j+1$, and so $B^1 \cong 2\mathbb{Z}$. Thus $H^1(C_{2k}, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$ when k is odd.
Alternatively (for k odd), we know that $H^1(C_{2k}, \mathbb{Z}) = \mathrm{Ker}\overline{N}$ where the map $\overline{N} : \mathbb{Z}_{C_{2k}} \to \mathbb{Z}^{C_{2k}}$ is the norm map induced by multiplication on $\mathbb{Z}$ by the norm element N. Now $Nm = 1 \cdot m + x \cdot m + x^2 \cdot m + \cdots + x^{2k-1}m = m - m + m - m + \cdots + m - m = 0$ and so $N\mathbb{Z} = 0 \ \Rightarrow\ \mathrm{Ker}\overline{N} = \mathbb{Z}_{C_{2k}}$. Since $x^i \cdot n = (-1)^i n$, the action is trivial for all $g \in C_{2k}$ if $-n = n$, hence the largest quotient on which C_{2k} acts trivially is $\mathbb{Z}_{C_{2k}} = \mathbb{Z}_2$.

4: Noting that the only nontrivial map $\varphi : \mathbb{Z}_i = \langle t \rangle \to \mathbb{Z}_{ij} = \langle s \rangle$ is the canonical inclusion defined by $t \mapsto s^j$, show that the induced map under H_{2n-1} is the same inclusion. This is the corestriction map $\mathrm{cor}_{\mathbb{Z}_i}^{\mathbb{Z}_{ij}} : H_*(\mathbb{Z}_i) \to H_*(\mathbb{Z}_{ij})$.

Considering the two periodic free resolutions of $\mathbb{Z}$, there exists an augmentation-preserving chain map f between them by Theorem I.7.5[1] and we look at the commutative diagram in low dimensions:

$$\begin{tikzcd}
\mathbb{Z}[\mathbb{Z}_i] \arrow[r, "t-1"] \arrow[d, "f_1"] & \mathbb{Z}[\mathbb{Z}_i] \arrow[r, "\varepsilon"] \arrow[d, "f_0"] & \mathbb{Z} \arrow[r] \arrow[d, "id_{\mathbb{Z}}"'] & 0 \\
\mathbb{Z}[\mathbb{Z}_{ij}] \arrow[r, "s-1"] & \mathbb{Z}[\mathbb{Z}_{ij}] \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

The right square yields $f_0(1) = 1$, and the left square yields $(s-1)f_1(1) = f_0(t-1) = t \cdot f_0(1) - f_0(1) = \varphi(t)1 - 1 = s^j - 1 = (s-1)(1+s+\cdots+s^{j-1}) \;\Rightarrow\; f_1(1) = 1+s+\cdots+s^{j-1}$. We assert that f is given by $f_n(1) = 1$ for n even and $f_n(1) = 1+s+\cdots+s^{j-1}$ for n odd. Indeed, assuming inductively that the chain map is valid up to n, it suffices to check commutativity $(1+s+\cdots+s^{ij-1})f_{n+1}(1) = f_n(1+t+\cdots+t^{i-1})$ for n odd and $(s-1)f_{n+1}(1) = f_n(t-1)$ for n even. For the latter case [n even] we have $(s-1)(1+s+\cdots+s^{j-1}) = (s^j-1)$ and $f_n(t-1) = (s^j-1)f_n(1) = s^j-1$, so commutativity is satisfied. For the former case [n odd] we have $f_n(1+t+\cdots+t^{i-1}) = (1+s^j+\cdots+s^{ij-j})f_n(1) = (1+s^j+\cdots+s^{ij-j})(1+s+\cdots+s^{j-1}) = 1+s+\cdots+s^{ij-1}$ and $(1+s+\cdots+s^{ij-1})f_{n+1}(1) = (1+s+\cdots+s^{ij-1})$, so commutativity is satisfied.
Using this chain map, and after moving to quotients, the cycle elements (for odd-dimensional homology) are mapped via $\varphi_*(1) = 1+1+\cdots+1 = j$ while the boundary elements are mapped via $\varphi_*(1) = 1$; thus the result follows.

5: Let $d : G \to A$ be a derivation. Prove the relation $d(x^n) = \frac{x^n-1}{x-1}dx$ for $x \in G$.

For $n = 0$, $d(1) = d(x^0) = \frac{1-1}{x-1}dx = 0$, and we argue by induction on n. Assuming the relation at $n = k$ holds, $d(x^{k+1}) = d(x \cdot x^k) = d(x) + x \cdot d(x^k) = [1 + x\frac{x^k-1}{x-1}]dx = [\frac{x-1+x^{k+1}-x}{x-1}]dx = \frac{x^{k+1}-1}{x-1}dx$ and we are finished.
Alternatively, we note that $\frac{x^n-1}{x-1} = 1+x+\cdots+x^{n-2}+x^{n-1}$, so the relation immediately follows by successive calculations $d(x^n) = d(x) + x \cdot d(x^{n-1}) = d(x) + x[d(x) + x \cdot d(x^{n-2})] = d(x) + x \cdot d(x) + x^2[d(x) + x \cdot d(x^{n-3})] = [1+x+x^2]d(x) + x^3 \cdot d(x^{n-3})$, etc..

6: What information do we obtain about the homology of a group G by computing its homology with rational coefficients?

Assuming $\mathbb{Q}$ is an abelian group with trivial G-action, we can apply the result of Exercise III.1.2 to obtain the short exact sequence $0 \to H_n(G) \otimes \mathbb{Q} \to H_n(G, \mathbb{Q}) \to \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(G), \mathbb{Q}) \to 0$. Since $\mathbb{Q}$ is torsion-free we have the equality $\mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(G), \mathbb{Q}) = 0$ and hence the isomorphism $H_n(G, \mathbb{Q}) \cong H_n(G) \otimes \mathbb{Q}$. Now $A \otimes \mathbb{Q} = 0$ for any torsion abelian group A because $q \otimes a = \frac{|a|q}{|a|} \otimes a = \frac{q}{|a|} \otimes |a|a = \frac{q}{|a|} \otimes 0 = 0$. Therefore, $\dim_{\mathbb{Q}}(H_n(G, \mathbb{Q})) = \mathrm{rk}_{\mathbb{Z}}(H_nG)$. Moreover, if $H_n(G, \mathbb{Q})$ is nontrivial then H_nG is torsion-free.

7: Let the multiplicative cyclic group $C_n = \langle x \rangle$ act on $M = \bigoplus_{j=1}^n \mathbb{Z}_2$ by $x \cdot a_i = a_{i+1}$ where a_j generates the jth $\mathbb{Z}_2$-summand. Compute $H^1(C_n, M)$.

For $f \in Z^1 = \mathrm{Der}(C_n, M)$ we have $0 = f(x^n) = f(x) + x^{n-1} \cdot f(x) \;\Rightarrow\; x^{n-1} \cdot f(x) = -f(x) = f(x)$ [we can drop the negative sign because the maximum order for elements is 2]. So f is determined by $f(x) = (z_1, \ldots, z_n)$ and we must have $(z_1, \ldots, z_n) = x^{n-1} \cdot (z_1, \ldots, z_n) = (z_2, \ldots, z_n, z_1) \;\Rightarrow\; z_1 = z_2 = \cdots = z_n$. Thus $f(x)$ is 0 or $(1, 1, \ldots, 1)$, and $Z^1 = \mathbb{Z}_2$. But $f(x^i) = f(x) + x^{i-1} \cdot f(x) = f(x) + f(x) = 2f(x) = 0$, and $f(x) = (1, 1, \ldots, 1) = x \cdot n - n$ where $n \in M$ is the element consisting of alternating 1's and 0's [the case $f(x) = 0$ is trivial]. Thus $f \in B^1 = \mathrm{PDer}(C_n, M)$, and so $H^1(C_n, M) = 0$.

8: Let $GL_n(\mathbb{Z})$ act on $\mathbb{Z}^n$ by left matrix multiplication, where we consider $\mathbb{Z}^n$ as an $n \times 1$ column vector with integral entries. Compute the induced map $\psi : H_*(GL_n(\mathbb{Z}), \mathbb{Z}^n) \to H_*(GL_n(\mathbb{Z}), \mathbb{Z}^n)$ under the action of $-[\delta_{ij}]$ on $z \in \mathbb{Z}^n$.

The anti-identity matrix $m = -[\delta_{ij}]$ is in the center $Z(GL_n(\mathbb{Z}))$ and so the conjugation action on $GL_n(\mathbb{Z})$ by m is the identity. Thus we can rewrite the map of the action $m \cdot z$ as $(g \mapsto mgm^{-1} = g\,,\, z \mapsto m \cdot z = -z) \in (GL_n(\mathbb{Z}), \mathbb{Z}^n)$. By Proposition III.8.1[1] this map induces the identity on the respective homology with coefficients, hence $\psi = id_*$. [*But clearly* $\psi = -id_*$ *because it's induced from* $z \mapsto -z \in \mathbb{Z}^n$. *Thus* $2 \cdot id_* = 0$ *and* $H_*(GL_n(\mathbb{Z}), \mathbb{Z}^n)$ *is all* 2*-torsion*].

9: The cyclic group C_m is a normal subgroup of the dihedral group $D_m = C_m \rtimes C_2$ (of symmetries of the regular m-gon). There is a C_2-action on $C_m = \langle\sigma\rangle$ given by $\sigma \mapsto \sigma^{-1}$. Determine the action of C_2 on the homology $H_{2i-1}(C_m, \mathbb{Z})$, noting that there is an element $g \in D_m$ such that $g\sigma g^{-1} = \sigma^{-1}$.

Letting $c(g) : C_m \to C_m$ be conjugation by g, we can apply Corollary III.8.2[1] to obtain the induced action of $D_m/C_m \cong C_2$ on $H_*(C_m, \mathbb{Z})$ given by $z \mapsto c(g)_* z$. It suffices to compute $c(g)_*$ on the chain level, using the periodic free resolution P of C_m, and using the trivial action on $\mathbb{Z}$. Using the condition $\tau(hx) = [c(g)](h)\tau(x) = h^{-1}\tau(x)$ on the chain map $\tau : P \to P$ (for $h \in C_m$), we claim that $\tau_{2i-1}(x) = \tau_{2i}(x) = (-1)^i\sigma^{m-i}x$ for $i \in \mathbb{N}$ and $\tau_0(x) = x$. Assuming this claim holds, the chain map $P \otimes_{C_m} \mathbb{Z} \to P \otimes_{C_m} \mathbb{Z}$ [in odd dimensions] is given by $x \otimes y \mapsto (-1)^i\sigma^{m-i}x \otimes gy = (-1)^i\sigma^{m-i}x \otimes y = (-1)^i x \otimes \sigma^{i-m}y = (-1)^i x \otimes y$, and so $c(g)_*$ [hence the C_2-action] is multiplication by $(-1)^i$ on $H_{2i-1}(C_m, \mathbb{Z})$. It suffices to prove the claim. Seeing that $N\tau_{2i}(1) = \tau_{2i-1}(N) = N\tau_{2i-1}(1)$ where N is the norm element, we can restrict our attention to τ_{2i-1} and use induction on i since $(\sigma - 1)\tau_1(1) = (\sigma - 1)(-\sigma^{m-1}) = \sigma^{m-1} - 1 = \sigma^{-1} - 1 = \tau_0(\sigma - 1)$. This chain map must satisfy commutativity $(\sigma - 1)\tau_{2i-1}(1) = \tau_{2(i-1)}(\sigma - 1)$, and this is indeed the case because $(\sigma - 1)\tau_{2i-1}(1) = (-1)^i(\sigma^{m-i+1} - \sigma^{m-i})$ and $\tau_{2(i-1)}(\sigma - 1) = (\sigma^{-1} - 1)(-1)^{i-1}\sigma^{m-i+1} = (-1)^i(\sigma^{m-i+1} - \sigma^{m-i})$.

10: Assuming $|G : H| < \infty$, we define $tr : M_G \to M_H$ by $tr(\overline{m}) = \sum_{g \in H\backslash G} \overline{\overline{gm}}$, and we then define the transfer map $\mathrm{res}^G_H : H_*(G, -) \to H_*(H, -)$ to be the unique extension of tr to a map of homological functors (using Theorem III.7.3[1]). Show that this agrees with the map defined by applying $H_*(G, -)$ to the canonical injection $M \to \mathrm{Coind}^G_H M \cong \mathrm{Ind}^G_H M$ and using Shapiro's lemma. Assume that we know this latter map is compatible with ∂.

By Theorem III.7.3[1] it suffices to verify that the latter map equals tr in dimension zero.
We first want an explicit isomorphism for $\mathrm{Coind}^G_H M \cong \mathrm{Ind}^G_H M$, asserting that it is given by $\psi(f) = \sum_{g \in G/H} g \otimes f(g^{-1})$. Following the proof of Proposition III.5.9[1], there is an H-map $\varphi_0 : M \to \mathrm{Coind}^G_H M$ given by $[\varphi_0(m)](g) = \{gm \quad \forall\ g \in H \quad , 0 \text{ otherwise}\}$, and by the universal property of induction this extends to a G-map $\varphi : \mathrm{Ind}^G_H M \to \mathrm{Coind}^G_H M$ given by $\varphi(g \otimes m) = g\varphi_0(m)$. Now we have $\psi\varphi : g' \otimes m \mapsto g'[\varphi_0(m)] \mapsto \sum_{g \in G/H} g \otimes [\varphi_0(m)](g^{-1}g') = g'' \otimes g''^{-1}g'm = g''g''^{-1}g' \otimes m = g' \otimes m$, noting that there is one and only one $g'' \in G/H$ such that $g''^{-1}g' \in H$ (otherwise $g_1^{-1}g' = h_1$ and $g_2^{-1}g' = h_2$ gives $g_1h_1 = g_2h_2$ which contradicts their coset representations). In the other direction we have $\varphi\psi : f \mapsto \sum_{g \in G/H} g \otimes f(g^{-1}) \mapsto \sum_{g \in G/H} g[\varphi_0(f(g^{-1}))](g') = [\varphi_0(f(g''^{-1}))](g'g'') = g'g''f(g''^{-1}) = f(g'g''g''^{-1}) = f(g') = f$, also noting that there is only one $g'' \in G/H$ such that $g'g'' \in H$. Thus $\varphi = \psi^{-1}$ and ψ is the desired isomorphism.
Applying the zeroth homology functor $H_0(G, -)$ to the aforementioned map $M \to \mathrm{Coind}^G_H M \to \mathrm{Ind}^G_H M$ given by $m \mapsto (g \mapsto gm) \mapsto \sum_{g \in G/H} g \otimes g^{-1}m$, and using the isomorphism from Shapiro's lemma, we obtain the chain map $x \otimes_G m \mapsto \sum_{g \in G/H} x \otimes_G (g \otimes_H g^{-1}m) = \sum_{g \in G/H} xg \otimes_G (1 \otimes_H g^{-1}m) \cong \sum_{g \in G/H} xg \otimes_H g^{-1}m = \sum_{g \in G/H} g^{-1}x \otimes_H g^{-1}m = \sum_{g \in H\backslash G} gx \otimes_H gm = \sum_{g \in H\backslash G} g \cdot (x \otimes_H m)$. Using the natural isomorphism $H_0(G, -) \cong (-)_G$ of Proposition III.6.1[1], this chain map yields the trace map tr described above.

11: Prove that $H_n(G, M) \cong \mathrm{Tor}^G_{n-1}(I, M)$ where I is the augmentation ideal of $\mathbb{Z}G$.

We have the short exact sequence of G-modules, $0 \to I \hookrightarrow \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$. By an analogue of Theorem 17.1.15[2] we have a long exact sequence of abelian groups $\cdots \to \mathrm{Tor}^G_n(\mathbb{Z}G, M) \to \mathrm{Tor}^G_n(\mathbb{Z}, M) \to \mathrm{Tor}^G_{n-1}(I, M) \to \mathrm{Tor}^G_{n-1}(\mathbb{Z}G, M) \to \cdots$. It is a fact that if P is R-projective then $\mathrm{Tor}^R_n(P, B) = 0$ for any R-module B ($n \geq 1$). Therefore, $\mathrm{Tor}^G_n(\mathbb{Z}G, M) = 0$ since $\mathbb{Z}G$ is a free [hence projective] $\mathbb{Z}G$-module, and so we obtain the isomorphisms $\mathrm{Tor}^G_n(\mathbb{Z}, M) \to \mathrm{Tor}^G_{n-1}(I, M)$. Since $H_n(G, -) = \mathrm{Tor}^G_n(\mathbb{Z}, -)$, the result follows.

12: Given $|G : H| < \infty$ and $z \in H(G, M)$ where $H(-, -)$ is either H_* or H^*, show that $\mathrm{cor}^G_H \mathrm{res}^G_H z = |G : H|z$.

Referring to Exercise AE.10 above, we have the map $M \to \mathrm{Coind}_H^G M \to \mathrm{Ind}_H^G M$ which yields the chain map $x \otimes_G m \mapsto \sum_{g \in H\backslash G} gx \otimes_H gm$, and this induces the transfer map res_H^G in homology. Composing this with the corestriction map on the chain level $x \otimes_H m \mapsto x \otimes_G m$, we obtain the chain map $x \otimes_G m \mapsto \sum_{g \in H\backslash G} gx \otimes_G gm = \sum_{g \in H\backslash G} x \otimes_G m = |G:H|(x \otimes_G m)$ which induces $\mathrm{cor}_H^G \mathrm{res}_H^G$ as multiplication by $|G:H|$.
On the other hand, we have the chain map $\mathcal{H}om(F,M)^H \to \mathcal{H}om(F,M)^G$ given by $f \mapsto \sum_{g \in G/H}[x \mapsto gf(g^{-1}x)]$ which induces the transfer map cor_H^G in cohomology; G acts diagonally. Composing this with the restriction map on the chain level, we obtain the chain map $f \mapsto f \mapsto \sum_{g \in G/H}[x \mapsto gf(g^{-1}x)] = \sum_{g \in G/H}[x \mapsto f(x)] = |G:H|[x \mapsto f(x)]$, noting that the domain element f is a G-invariant homomorphism. This induces the aforementioned map $\mathrm{cor}_H^G \mathrm{res}_H^G$.

13: Give another proof that $H(G, \bigoplus_i^m M_i) \cong \bigoplus_i^m H(G, M_i)$ where $H(-,-)$ is either H_* or H^*.

Applying Proposition III.6.1[1] to the short exact sequence of G-modules $0 \to M_1 \xrightarrow{\alpha} M_1 \oplus M_2 \xrightarrow{\beta} M_2 \to 0$, we obtain the standard long exact sequence in (co)homology. Since α is an injection onto a direct summand, the induced map α_* under the covariant functor $H_*(G,-)$ is an injection (refer to Exercise II.7.3); similarly, the induced map $\beta^* = H^*(G,\beta)$ is also an injection. Thus the long exact sequence breaks up into short exact sequences $0 \to H_n(G,M_1) \to H_n(G, M_1 \oplus M_2) \to H_n(G,M_2) \to 0$ (similar for cohomology). These are split exact sequences because $\gamma_* \alpha_* = id_*$ and $\Gamma^* \beta^* = id^*$, where γ is the projection $M_1 \oplus M_2 \to M_1$ and Γ is the inclusion $M_2 \to M_1 \oplus M_2$. The result now follows by trivial induction on m.

14: Prove that if the G-module M has exponent p [prime] then $H^n(G,M)$ and $H_n(G,M)$ are $\mathbb{Z}_p$-vector spaces.

Showing that these groups are the specified modules is equivalent to showing that they are annihilated by p (i.e. have exponent dividing p). Considering homology, a chain $z \in C_n(G,M)$ is a finite sum of terms of the form $m \otimes [g_1|\cdots|g_n]$, so $pz = \sum p(m \otimes [g_1|\cdots|g_n]) = \sum (pm) \otimes [g_1|\cdots|g_n] = \sum 0 = 0$. Thus $pH_n(G,M) = 0 \ \ \forall\, n \geq 0$. Considering cohomology, a cochain $f \in C^0(G,M)$ is an element of M and hence $pf = 0$ trivially. If $n \geq 1$ then $f \in C^n(G,M)$ is a function $G^n \to M$, so pf is a function $G^n \to M \to pM = 0$ and hence $pf = 0$. Thus $pH^n(G,M) = 0 \ \ \forall\, n \geq 0$.

15: If G is a finite group, show that $H_n(G,M)$ is a $\mathbb{Z}_{|G|}$-module for $n > 0$. Thus we have the primary decomposition $H_n(G,M) = \bigoplus_p H_n(G,M)_{(p)}$ where p ranges over the primes dividing $|G|$.

By Proposition III.9.5[1], $\mathrm{cor}_{\{1\}}^G \mathrm{res}_{\{1\}}^G z = |G:\{1\}|z = |G|z$ where $z \in H_*(G,M)$. But this composition factors through $H_*(\{1\},M)$ which is trivial in positive dimensions. Thus $\mathrm{cor}_{\{1\}}^G \mathrm{res}_{\{1\}}^G z = 0 \ \Rightarrow\ |G|z = 0$, and so $H_n(G,M)$ is annihilated by $|G|$ for $n > 0$.

16: Show that $H_n(G,M) = 0 \ \ \forall\ n > 0$ if M is a finite abelian group and G is finite with relatively prime order, $gcd(|G|,|M|) = 1$. Furthermore, show that $H_n(G,M)$ is a finite group (for $n > 0$) if G is finite and M is a finitely generated abelian group.

By Exercises AE.14+15 above, $H_n(G,M)$ is annihilated by both $|G|$ and $|M|$, and hence must be annihilated by a common factor. But $gcd(|G|,|M|) = 1$, so the common factor is 1 and $H_n(G,M) = 0$ for $n > 0$.
By Exercise AE.15 above (assuming $n > 0$), $|G|H_n(G,M) = 0$ and so $H_n(G,M)$ is all torsion. Since $|G| < \infty$, F_n is a free G-module of finite rank $r = |G|^n$ (F is the bar resolution of $\mathbb{Z}$ over $\mathbb{Z}G$), where we note that a G-basis for F_n is the $(n+1)$-tuples whose first element is 1. Thus $F_n \otimes_G M \cong (\bigoplus_i^r \mathbb{Z}G) \otimes_G M \cong \bigoplus_i^r (\mathbb{Z}G \otimes_G M) \cong \bigoplus_i^r M$ is finitely generated since M is finitely generated, which implies that $H_n(G,M)$ is also finitely generated. A finitely generated torsion group is finite, so the result follows.

17: Why does res_H^G induce a monomorphism $H^n(G,M)_{(p)} \hookrightarrow H^n(H,M)$, where H is the Sylow p-subgroup of G?

Consider an element $z \in H^n(G, M)_{(p)}$ which lies in the kernel of res^G_H. Composing this with the corestriction map gives you multiplication by $|G : H|$ by Proposition III.9.5[1], and so $|G : H|z = 0$. But the order of z is $|z| = p^\alpha$ and p does not divide $|G : H|$. Since p^α and $|G : H|$ both annihilate z, there is a common factor between them which also annihilates z, so $z = 0$ (the only common factor is 1). Thus res^G_H is a monomorphism when restricted to the p-primary component.

18: Compute the homology group $H_1(G)$ for any nonabelian simple group G.

The commutator group $[G, G]$ is either 0 or G because the commutator group is a normal subgroup of G and a normal subgroup of a simple group must be either the trivial group or G itself. Since G is nonabelian, we cannot have $[G, G] = 0$, and so $[G, G] = G$ (i.e. G is perfect). Thus $H_1(G) \cong G/[G, G] = G/G = 0$.

In particular, $H_1(A_n) = 0 \ \ \forall\ n \geq 5$ where A_n is the alternating group on n letters (subgroup of S_n with index 2), by Theorem 4.6.24[2].

19: Show that $I/I^2 \cong G_{ab}$ where I is the augmentation ideal of $\mathbb{Z}G$.

Apply Proposition III.6.1[1] to the short exact sequence $0 \to I \to \mathbb{Z}G \to \mathbb{Z} \to 0$ to obtain the exact sequence $H_1(G, \mathbb{Z}G) \to H_1(G) \to I_G \to (\mathbb{Z}G)_G \to \mathbb{Z} \to 0$ in low dimensions. The latter map is an isomorphism (by the First Isomorphism Theorem) because the map is surjective and $(\mathbb{Z}G)_G \cong \mathbb{Z}$. By Proposition III.6.1[1], $H_1(G, \mathbb{Z}G) = 0$ because $\mathbb{Z}G$ is free [hence projective]. Thus $H_1(G) \cong I_G = I/I^2$ by exactness, and we know that $H_1(G) \cong G_{ab}$, so the result follows.

An explicit isomorphism is given by $g[G, G] \mapsto (g - 1) + I^2$.

20: Find a module M with trivial action such that $H^1(D_{2^n}, M)$ is nonzero, where D_{2^n} is the dihedral group with respect to a regular 2^{n-1}-gon.

A presentation for this group is $D_{2^n} = \langle \alpha, \beta \mid \alpha^2 = \beta^2 = (\alpha\beta)^{2^{n-1}} = 1 \rangle$. By the result of Exercise III.1.2, $H^1(D_{2^n}, M) = \text{Hom}(D^{ab}_{2^n}, M)$. Now D_{2^n} quotiented by $[D_{2^n}, D_{2^n}]$ forces the trivial relation $1 = (\alpha\beta)^{2^{n-1}} = \alpha^{2^{n-1}}\beta^{2^{n-1}} = 1 \cdot 1 = 1$, so $D^{ab}_{2^n} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Thus $H^1(D_{2^n}, M) = \text{Hom}(\mathbb{Z}_2, M) \oplus \text{Hom}(\mathbb{Z}_2, M)$, which is nonzero if $M = \mathbb{Z}_2$, giving $H^1(D_{2^n}, \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

21: Prove that a finitely generated abelian group G is cyclic if and only if $H_2(G, \mathbb{Z}) = 0$.

If G is a cyclic group $\mathbb{Z}_n$, then $H_2(\mathbb{Z}_n, \mathbb{Z}) = 0$ as explained on pg35[1].
If $H_2(G, \mathbb{Z}) = 0$ then Hopf's formula (Theorem II.5.3[1]) gives $R \cap [F, F] \subseteq [F, R]$, where F/R is a presentation for G. Since G is abelian we have $0 = [G, G] = [F/R, F/R] = [F, F]R/R$ which implies $[F, F] \subseteq R$. Alternatively, since $G = F/R$ is abelian, $[F, F] \subseteq R$ by Proposition 5.4.7[2]. Thus $[F, F] = [F, R]$, and $|F| < \infty$ since G is finitely generated. If $|F| = 1$ then $R = \{x^n\}$ for some $n \geq 0$, in which case G is obviously cyclic.
It seems we're stuck in terms of extracting any more information, but alternatively we can refer to Theorem V.6.3[1] which says $H_2(G)$ is isomorphic to the second exterior power $\bigwedge^2 G$ for any abelian group. Thus it suffices to show that G is cyclic if $G \otimes G \cong \langle \{g \otimes g\} \rangle \cong G$. Now if $rk_{\mathbb{Z}}(G) = n$ then $rk_{\mathbb{Z}}(G \otimes G) = n^2$, so $n^2 = n$ gives $n = 1$ or $n = 0$. Also, $\mathbb{Z}_i \otimes \mathbb{Z}_j \cong \mathbb{Z}_{gcd(i,j)}$, so G must contain at most one $\mathbb{Z}_m$-summand (for each m) because the tensor product contains at least the squared-amount of those summands. Assume that G is not cyclic; then all m's must be relatively prime (otherwise the tensor product will contain additional summands not in G due to greatest common divisors). Thus $G \cong \mathbb{Z} \oplus \mathbb{Z}_n$ (for some $n \geq 2$) since $\mathbb{Z}_i \oplus \mathbb{Z}_j \cong \mathbb{Z}_{ij}$ iff $gcd(i, j) = 1$ [note that G must have the $\mathbb{Z}$-summand in order to not be cyclic]. But $(\mathbb{Z} \oplus \mathbb{Z}_n) \otimes (\mathbb{Z} \oplus \mathbb{Z}_n) \cong \mathbb{Z} \oplus \mathbb{Z}_n \oplus \mathbb{Z}_n \oplus \mathbb{Z}_n$ which is not isomorphic to G, a contradiction (hence G is cyclic).
Aside: A finitely generated abelian group $G = F/R$ is cyclic iff $[F, F] = [F, R]$.

22: Let $\mathbb{Z}_m = \langle t \rangle$, $\mathbb{Z}_{m^2} = \langle s \rangle$, and define a $\mathbb{Z}_m$-action on $\mathbb{Z}_{m^2}$ by $t \cdot s = s^{m+1}$. Compute the resulting $\mathbb{Z}_m$-module structure on $H_j(\mathbb{Z}_{m^2})$, and then compute $H_i(\mathbb{Z}_m, H_j(\mathbb{Z}_{m^2}))$ where m is an odd prime.

First we must compute the change-of-rings map in integral homology for the map $\mathbb{Z}_a = \langle x \rangle \to \mathbb{Z}_b = \langle y \rangle$, $x \mapsto y^r$, where $b|ra$. Mimicking the solution to Exercise AE.4, the induced map in odd-dimensional homology is easily seen to be multiplication by r. Applying this result to the case where $r = m+1$ and $a = b = m^2$, the $\mathbb{Z}_m$-action on $H_j(\mathbb{Z}_{m^2})$ is multiplication by $p(m+1)$ for t^p. If j is even, then the homologies in question are trivial. By a result on pg58[1], $H_{odd}(\mathbb{Z}_m, \mathbb{Z}_{m^2}) = \mathrm{Coker}(\overline{N} : (\mathbb{Z}_{m^2})_{\mathbb{Z}_m} \to (\mathbb{Z}_{m^2})^{\mathbb{Z}_m})$ and $H_{even}(\mathbb{Z}_m, \mathbb{Z}_{m^2}) = \mathrm{Ker}(\overline{N} : (\mathbb{Z}_{m^2})_{\mathbb{Z}_m} \to (\mathbb{Z}_{m^2})^{\mathbb{Z}_m})$. Since m is an odd prime, the only nontrivial proper subgroup/quotient of $\mathbb{Z}_{m^2}$ is $\mathbb{Z}_m$ which is not fixed by the $\mathbb{Z}_m$-action, so the norm map is $0 \to 0$ and thus $H_i(\mathbb{Z}_m, H_j(\mathbb{Z}_{m^2})) = 0 \quad \forall$ nonzero i, j.

23: Prove that if Q is injective then $\mathcal{E}(G, Q)$ is trivial.

Any short exact sequence $0 \to Q \to E \to G \to 0$ splits for Q injective, so there is only one equivalence class of group extensions (namely, the split extension) and $\mathcal{E}(G, Q) = 0$. Alternatively, Proposition III.6.1[1] and Theorem IV.3.12[1] imply $\mathcal{E}(G, Q) \cong H^2(G, Q) = 0$ because Q is injective.

24: Prove that $H^2(A_5, \mathbb{Z}_2) \neq 0$.

We have the central group extension $1 \to \mathbb{Z}_2 \to SL_2(\mathbb{F}_5) \to A_5 \to 1$ as in Exercise I.5.7(b). It suffices to show that $E = \mathbb{Z}_2 \rtimes A_5$ is not isomorphic to $SL_2(\mathbb{F}_5)$, for then the above extension is nonsplit and hence $H^2(A_5, \mathbb{Z}_2) \cong \mathcal{E}(A_5, \mathbb{Z}_2) \neq 0$ by Theorem IV.3.12[1]. First note that $\mathbb{Z}_2$ must have the trivial A_5-action, so $E = \mathbb{Z}_2 \times A_5$. Now it is a fact that $SL_2(\mathbb{F}_5)$ is perfect, while $[E, E] \subseteq A_5$ by Proposition 5.4.7[2] because $E/A_5 \cong \mathbb{Z}_2$ is abelian (hence E is not perfect). Thus $H_1(E) \neq 0 = H_1(SL_2(\mathbb{F}_5))$ and so $\mathbb{Z}_2 \rtimes A_5 \ncong SL_2(\mathbb{F}_5)$.

25: Let $A = \mathbb{Z}_2 \times \mathbb{Z}_2$ and let $\mathrm{Aut}(A) \cong S_3$ act on A in the natural fashion. Prove that $H^1(S_3, \mathbb{Z}_2 \times \mathbb{Z}_2) = 0$.

In the semi-direct product $E = A \rtimes S_3 = \mathrm{Hol}(A)$ [called the *holomorph* of A] we have a Sylow 3-subgroup $P \cong \mathbb{Z}_3$ by Sylow's Theorem, where $|E| = 24 = 2^3 \cdot 3$. By Sylow's Theorem, $n_3 \mid 8$ and $n_3 \equiv 1 \bmod 3$, where n_3 is the number of Sylow 3-subgroups of E. Thus n_3 is either 1 or 4, and this implies $|N_E(P)|$ is either 24 or 6 by the fact $n_3 = |E : N_E(P)|$. But we can exhibit at least two such subgroups [noting that $S_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2$], namely $P_1 = (0 \times 0) \rtimes (\mathbb{Z}_3 \rtimes 0)$ and $P_2 = \{(0,0,0,0), (1,0,1,0), (1,0,2,0)\}$. Thus $|N_E(P)| = 6$ which corresponds to $n_3 = 4$, and S_3 is then the normalizer of the Sylow 3-subgroup $P = 0 \rtimes \mathbb{Z}_3$ of E because $P \lhd S_3$ and $|S_3| = 6$. Given a complement G to A in E [a group G is a *complement* to A in E if $E = A \rtimes G$], one Sylow 3-subgroup is $Q \lhd G$ and hence $P = eQe^{-1}$ for some $e \in E$ by Sylow's Theorem. Now $S_3 = N_E(P) = N_E(eQe^{-1}) = eN_E(Q)e^{-1} = eGe^{-1}$, and so all complements are conjugate. Noting that the conjugacy classes of complements to A in E are the A-conjugacy classes of splittings of E, we have $H^1(S_3, \mathbb{Z}_2 \times \mathbb{Z}_2) = 0$ by Proposition IV.2.3[1] because there is only one conjugacy class.

26: In the proof of Theorem IV.3.12[1], all extensions were assumed to have normalized sections. Explain why this simplification does not affect the result of the theorem.

Given a factor set (2-cocycle) $f : G \times G \to A$, we assert that this lies in the same cohomology class as a normalized factor set (2-cocycle). Let $\delta_1 c$ be the coboundary of the constant function $c : G \to A$ defined by $c(g) = f(1,1)$. It suffices to show that $F = f - \delta_1 c$ is a normalized factor set, for then it belongs to the same cohomology class as the arbitrary f [it differs by a coboundary]. Now $F(1,1) = f(1,1) - [\delta_1 c](1,1) = f(1,1) - [c(1) + 1 \cdot c(1) - c(1 \cdot 1)] = f(1,1) - f(1,1) = 0$ and so $F : G \times G \to A$ is normalized. It is indeed a 2-cocycle (factor set) because $gF(h,k) - F(gh,k) + F(g,hk) - F(g,h) = [gf(h,k) - f(gh,k) + f(g,hk) - f(g,h)] - \{g[c(h) + hc(k) - c(hk)] - [c(gh) + ghc(k) - c(ghk)] + [c(g) + gc(hk) - c(ghk)] - [c(g) + gc(h) - c(gh)]\} = 0 - gc(h) - ghc(k) + gc(hk) + c(gh) + ghc(k) - c(ghk) - c(g) - gc(hk) + c(ghk) + c(g) + gc(h) - c(gh) = [-gc(h) + gc(h)] + [-ghc(k) + ghc(k)] + [gc(hk) - gc(hk)] + [c(gh) - c(gh)] + [-c(ghk) + c(ghk)] + [-c(g) + c(g)] = 0 + 0 + 0 + 0 + 0 + 0 = 0$.
Since the bijection in the theorem is dependent on the cohomology *class*, and every class has a normalized factor set (hence normalized section), we are able to restrict our attention to those sections satisfying the normalization condition.

27: Show that $H^2(F, A) = 0$ for F free by appealing to group extensions.

For any group extension $0 \to A \to E \xrightarrow{\pi} F \to 1$ define the set map $S \to E$ by $s \mapsto \tilde{s}$, where $\tilde{s} \in \pi^{-1}(s)$ is a lifting of $s \in F = F(S)$. Then by the universal mapping property of free groups, this set map extends uniquely to a homomorphism $\varphi : F \to E$ which satisfies $\pi\varphi = id_F$ by construction. Thus the extension splits, and by Theorem IV.3.12[1] we have $H^2(F, A) \cong \mathcal{E}(F, A) = 0$.
Note that this also follows from direct computation using the free resolution $0 \to I \to \mathbb{Z}F \to \mathbb{Z} \to 0$, where the augmentation ideal I of $\mathbb{Z}F$ is free by Exercise IV.2.3(b). Indeed, the coboundary map is $\delta_n : \mathcal{H}om_F(0, A) \to \mathcal{H}om_F(0, A)$ for all $n > 1$ and hence $H^n(F, A) = 0 \quad \forall\, n > 1$.

28: Compute $H^2(Q_8, \mathbb{Z}_2)$, and determine the number of group extensions of $\mathbb{Z}_2$ by Q_8. [These two problems are *independent* of each other].

The quaternion group Q_8 is a non-abelian group of order 8 with presentation $\langle x, y \mid x^4 = 1, x^2 = y^2 = (xy)^2\rangle$. Its center is $C := Z(Q_8) \cong \mathbb{Z}_2$ which is also its commutator subgroup, and $\mathrm{Inn}(Q_8) = Q_8/Z(Q_8) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Also, $\mathrm{Aut}(Q_8) \cong S_4$ and hence $\mathrm{Out}(Q_8) = \mathrm{Aut}(Q_8)/\mathrm{Inn}(Q_8) \cong S_3$. As explained on pg102+104[1], Q_8 is an S_4-crossed module via the canonical map $Q_8 \to \mathrm{Aut}(Q_8)$, and such an extension gives rise to a homomorphism $\psi : \mathbb{Z}_2 \to \mathrm{Out}(Q_8) \cong S_3$. By Theorem IV.6.6[1] the set $\mathcal{E}(\mathbb{Z}_2, Q_8, \psi)$ of equivalence classes of extensions giving rise to ψ is either empty or in bijective correspondence with $H^2(\mathbb{Z}_2, C)$, where C is a $\mathbb{Z}_2$-module via ψ [note: it is a fact that the center of a group is a characteristic subgroup, so $\mathrm{Aut}(Q_8)$ acts naturally on C and hence $\mathrm{Out}(Q_8)$ acts on C because any inner automorphism leaves C fixed]. Now the only possible action on $C \cong \mathbb{Z}_2$ is the trivial one (giving $C_{\mathbb{Z}_2} = C = C^{\mathbb{Z}_2}$), so $H^2(\mathbb{Z}_2, C) = \mathrm{Coker}\overline{N} = C/NC = C/0 \cong \mathbb{Z}_2$. There are two possible choices of ψ, namely, the trivial map and the injection $\mathbb{Z}_2 \hookrightarrow S_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$. If ψ is the trivial map then it automatically lifts to the trivial homomorphism $\mathbb{Z}_2 \to S_4$ and we have a direct product extension $E \cong Q_8 \times \mathbb{Z}_2$. For the injective ψ we have the semi-direct product $Q_8 \rtimes \mathbb{Z}_2$ where $\mathbb{Z}_2$ switches the generators of Q_8. Thus $\mathcal{E}(\mathbb{Z}_2, Q_8, \psi)$ is nonempty, and Theorem IV.6.6[1] implies there are a total of 4 group extensions $1 \to Q_8 \to E \to \mathbb{Z}_2 \to 1$.
To compute the second cohomology group of the group Q_8 with coefficients in $\mathbb{Z}_2$, embed $\mathbb{Z}_2$ in the H^*-acyclic module $\mathrm{Hom}(\mathbb{Z}Q_8, \mathbb{Z}_2)$ by $z \mapsto (q \mapsto qz)$, and note that $\mathrm{Coker}(\mathbb{Z}_2 \to \mathrm{Hom}(\mathbb{Z}Q_8, \mathbb{Z}_2)) = \mathbb{Z}_2$ because the evaluation map $\mathrm{Hom}(\mathbb{Z}Q_8, \mathbb{Z}_2) \to \mathbb{Z}_2$ given by $f \mapsto f(2)$ composes with the embedding to give the trivial map $z \mapsto (q \mapsto qz) \mapsto 2z = 0$. The *dimension-shifting argument* then implies $H^2(Q_8, \mathbb{Z}_2) \cong H^1(Q_8, \mathbb{Z}_2)$, and by Exercise III.1.2 we have $H^1(Q_8, \mathbb{Z}_2) \cong \mathrm{Hom}(H_1(Q_8), \mathbb{Z}_2)$. Now $H_1(Q_8) = Q_8/[Q_8, Q_8] = Q_8/C \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and so $\mathrm{Hom}(H_1(Q_8), \mathbb{Z}_2) \cong \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) \oplus \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, $H^2(Q_8, \mathbb{Z}_2) \cong H^1(Q_8, \mathbb{Z}_2) \cong H_1(Q_8) \cong \mathbb{Z}_2^2 \equiv \mathbb{Z}_2 \times \mathbb{Z}_2$.

29: Let p be a prime, let A be an abelian normal p-subgroup of a finite group G, and let P be a Sylow p-subgroup of G. Prove that G is a split extension of G/A by A iff P is a split extension of P/A by A [Note: it is a fact that a normal p-subgroup is contained in every Sylow p-subgroup, so $A \subseteq P$].
This result is known as Gaschütz's Theorem.

Suppose G splits over A, so that $G \cong A \rtimes G/A$. Then the subgroup $A \rtimes P/A$ is a Sylow p-subgroup of G and hence $A \rtimes P/A \cong P$ by Sylow's Theorem, so P also splits over A. Conversely, suppose P splits over A (i.e. $P \cong A \rtimes P/A$). Note that $P/A = Syl_p(G/A)$ and multiplication by $|G/A : P/A| = |G : P|$ is an automorphism of A [hence of $H^2(G/A, A)$] since $|A|$ is relatively prime to $|G : P|$. The composition $H^2(G/A, A) \xrightarrow{\text{res}} H^2(P/A, A) \xrightarrow{\text{cor}} H^2(G/A, A)$ is then an isomorphism by Proposition III.9.5[1] because it is multiplication by $|G : P|$. In particular, the restriction homomorphism $\mathrm{res} : H^2(G/A, A) \to H^2(P/A, A)$ is injective, so the only element of $H^2(G/A, A)$ which corresponds to the trivial element [split extension] of $H^2(P/A, A)$ is the trivial element [split extension]. Thus G splits over A, and the proof is complete.

30: The *Schur multiplier* of a finite group G is defined as $H_2(G, \mathbb{Z}) \cong H^2(G, \mathbb{C}^*)$ where the multiplicative group $\mathbb{C}^* = \mathbb{C} - \{0\}$ is a trivial G-module. Prove that the Schur multiplier (of a finite group) is finite.

Since $|G| < \infty$ and $\mathbb{Z}$ is finitely generated (as an abelian group), $H_2(G, \mathbb{Z})$ is a finite group by Exercise AE.16.
Alternatively, we shall show that every cohomology class contains a cocycle whose values lie in the nth

roots of unity $\langle\zeta\rangle \cong \mathbb{Z}_n$ (where $n = |G|$ and $\zeta = e^{2\pi i/n}$), for then there are only finitely many functions/cocycles $f : G^2 \to \langle\zeta\rangle$ and $|H^2(G, \mathbb{C}^*)| \leq n^{n^2} < \infty$. From the exact sequence $0 \to \mathbb{Z}_n \to \mathbb{C}^* \xrightarrow{n} \mathbb{C}^* \to 0$ we obtain a long exact sequence $\to H^1(G, \mathbb{C}^*) \to H^1(G, \mathbb{C}^*) \to H^2(G, \mathbb{Z}_n) \to H^2(G, \mathbb{C}^*) \to H^2(G, \mathbb{C}^*) \to$ by Proposition III.6.1[1]. But the nth-power map on $\mathbb{C}^*$ induces the n-multiplication map on $H^2(G, \mathbb{C}^*)$ which is the zero map since n annihilates $H^2(G, \mathbb{C}^*)$ by Corollary III.10.2[1], so the above long exact sequence gives us a surjection $H^2(G, \mathbb{Z}_n) \to H^2(G, \mathbb{C}^*) \to 0$. Now $H^2(G, \mathbb{Z}_n)$ is finite by Exercise AE.16, so $H^2(G, \mathbb{C}^*)$ is necessarily finite (by the 1st Isomorphism Theorem) and the proof is complete.

31: Show that $\mathbb{Z}G \otimes \mathbb{Z}G' \cong \mathbb{Z}[G \times G']$ as $(G \times G')$-modules.

The map $\mathbb{Z}G \times \mathbb{Z}G' \to \mathbb{Z}[G \times G']$ defined by $(zg, z'g') \mapsto zz'(g, g')$ is obviously a $\mathbb{Z}$-balanced map and hence gives a rise to a unique group homomorphism $\varphi : \mathbb{Z}G \otimes \mathbb{Z}G' \to \mathbb{Z}[G \times G']$ by Theorem 10.4.10[2]. The obvious group homomorphism $\mathbb{Z}[G \times G'] \to \mathbb{Z}G \otimes \mathbb{Z}G'$ defined by $z(g, g') \mapsto z(g \otimes g') = (zg \otimes g') = (g \otimes zg')$ is the inverse of φ because $z(g, g') \mapsto (zg \otimes g') \mapsto z1(g, g') = z(g, g')$ and $(zg \otimes z'g') \mapsto zz'(g, g') \mapsto zz'(g \otimes g') = (zg \otimes z'g')$. Thus φ is a group isomorphism, and it is a $(G \times G')$-module isomorphism because $\varphi[(h, h') \cdot (zg \otimes z'g')] = \varphi(zhg \otimes z'h'g') = zz'(hg, h'g') = zz'(h, h')(g, g') = (h, h') \cdot \varphi(zg \otimes z'g')$ where $(h, h') \in G \times G'$.

32: Suppose $u_1 \in H^pG$, $u_2 \in H^qG$, $v_1 \in H^rG$, and $v_2 \in H^sG$. Prove that $(u_1 \times v_1) \smile (u_2 \times v_2) = (-1)^{qr}(u_1 \smile u_2) \times (v_1 \smile v_2)$ in $H^{p+q+r+s}(G, \mathbb{Z})$.

Note that $u \smile v := d^*(u \times v)$ where $d : G \to G \times G$ is the diagonal map. Let $D : G \times G \to G^4$ be the analogous diagonal map, and let $P : G^4 \to G^4$ be the permutation $(g_1, g_2, g_3, g_4) \mapsto (g_1, g_3, g_2, g_4)$. Then $D = P \circ (d \times d)$, and we obtain $(u_1 \times v_1) \smile (u_2 \times v_2) = D^*(u_1 \times v_1 \times u_2 \times v_2) = (d \times d)^*[P^*(u_1 \times v_1 \times u_2 \times v_2)] = (-1)^{qr}(d \times d)^*[u_1 \times u_2 \times v_1 \times v_2] = (-1)^{qr} d^*(u_1 \times u_2) \times d^*(v_1 \times v_2) = (-1)^{qr}(u_1 \smile u_2) \times (v_1 \smile v_2)$. Another way is to perform the same calculation using $u \times v := p_1^*(u) \smile p_2^*(v)$ and the fact that $p^*(u \smile v) = p^*(u) \smile p^*(v)$, where p_1 is the projection $G \times G \to G \times \{1\} = G$ and p_2 is the projection $G \times G \to \{1\} \times G = G$.

33: For the cap product, state the property of naturality with respect to group homomorphisms $\alpha : G \to H$. Also, provide the existence of an identity element for the cap product.

Checking definitions, we have $u \frown \alpha_* z = \alpha_*(\alpha^* u \frown z)$ which is associated to the "commutative" diagram

$$\begin{tikzcd}
H^p(G, M) \otimes H_q(G, N) \arrow[r, "\frown"] \arrow[d, bend right, "\alpha_*"] & H_{q-p}(G, M \otimes N) \arrow[d, "\alpha_*"] \\
H^p(H, M) \otimes H_q(H, N) \arrow[r, "\frown"] \arrow[u, bend right, "\alpha^*"] & H_{q-p}(H, M \otimes N)
\end{tikzcd}$$

There is a left-identity element $1 \in H^0(G, \mathbb{Z}) = \mathbb{Z}$ which is represented by the augmentation map ε, regarded as a 0-cocycle in $\mathcal{H}om_G(F, \mathbb{Z})$. Let F be the standard resolution, let $z \in H_*(G, M)$ with representation $z = (g_0, \ldots, g_i) \otimes m$, and take the diagonal approximation Δ to be the Alexander-Whitney map. Then $\varepsilon \otimes z$ maps under $\frown$ to $\sum_{p=0}^{i} (-1)^{\deg \varepsilon \cdot p}(g_0, \ldots, g_p) \otimes \varepsilon(g_p, \ldots, g_i) \otimes n = \sum 0 + (-1)^{0 \cdot i}(g_0, \ldots, g_i) \otimes 1 \otimes n = z$, as explained on pg113[1]. Thus the element satisfies $1 \frown z = z$ for all $z \in H_*(G, M)$, where we make the obvious identification $\mathbb{Z} \otimes M = M$ of coefficient modules.

34: Prove that a finitely generated projective $\mathbb{Z}$-module M is a finitely generated free $\mathbb{Z}$-module; thus the two are actually equivalent (since free modules are projective).

By the Fundamental Theorem (Theorem 12.1.5[2]) M has the decomposition $M \cong \mathbb{Z}^r \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_i}$, where we note that $\mathbb{Z}$ is a Principal Ideal Domain. Since M is projective, all of its direct summands must be projective. Now $\mathbb{Z}^r$ is free, hence projective. But $\mathbb{Z}_n$ is not $\mathbb{Z}$-projective because if it were then it would be a direct summand of a free $\mathbb{Z}$-module F, and F would then have elements of finite order (a contradiction); alternatively we could note that applying the functor $\mathrm{Hom}(\mathbb{Z}_n, -)$ to the exact sequence $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}_n \to 0$ yields the sequence $0 \to 0 \to 0 \to \mathbb{Z}_n \to 0$ which is not exact. Thus $M \cong \mathbb{Z}^r$ for

some $r \in \mathbb{N} \cup \{0\}$ and hence M is $\mathbb{Z}$-free of finite rank.

35: Let $G = PSL_2(\mathbb{Z})$ and let A be a G-module. Show that for every $q \geq 2$ and for every $x \in H^q(G, A)$, we have $6x = 0$.

The *modular group* is the group of Möbius transformations $T(z) = \frac{az+b}{cz+d}$ in the complex plane such that $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$. It is isomorphic to G via the map $T \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and G is called the projective special linear group [the quotient of $SL_2(\mathbb{Z})$ by $Z(SL_2(\mathbb{Z})) \cong \mathbb{Z}_2$]. Now it is a fact that G contains a free subgroup H of index $|G : H| = 6$ [this fact can be found in the paper *The Number of Subgroups of Given Index in the Modular Group* by W. Stothers]. By Exercise AE.27, $H^q(H, A) = 0$ for all $q \geq 2$. Thus we can apply Proposition III.10.1[1] which states $H^q(G, A)$ is annihilated by $|G : H|$, i.e. $6x = 0$ for all $x \in H^q(G, A)$ with $q \geq 2$.

36: Let G be a finite cyclic group and let M be a G-module. The *Herbrand quotient* is defined to be $h(M) = |H^2(G, M)|/|H^1(G, M)|$, assuming both cohomology groups are finite. Show that $h(\mathbb{Z}) = |G|$ where $\mathbb{Z}$ has the trivial G-action, and show that $h(M) = 1$ for M finite.

We know that $H^2(G, M) \cong M^G/NM$ and $H^1(G, M) = {}_NM/IM$, where N is the norm element and $I \equiv \langle \sigma - 1 \rangle$ is the augmentation ideal of $G = \langle \sigma \rangle$ and ${}_NM := \mathrm{Ker}(N : M \to M)$. For $M = \mathbb{Z}$ with trivial action we have $H^2(G, M) \cong \mathbb{Z}_{|G|}$ and $H^1(G, M) = 0$, so $h(\mathbb{Z}) = |\mathbb{Z}_G|/|\{0\}| = |G|/1 = |G|$. If M is finite then $h(M) = (|M^G| \cdot |IM|)/(|NM| \cdot |{}_NM|) = |M^G| \cdot |(\sigma - 1)M|/|M|$, where we note that $M/{}_NM \cong NM$. But the kernel K of the surjective map $M \overset{\sigma-1}{\to} (\sigma - 1)M$ is equal to M^G because $m \in M^G$ maps to $\sigma m - m = m - m = 0$ and if $m \in K$ then $\sigma^{|G|-1}m = \sigma^{|G|-2}m = \cdots = \sigma m = m$, i.e. $m \in M^G$. Thus $|M|/|M^G| = |(\sigma - 1)M|$ and so $h(M) = |M|/|M| = 1$.

37: Given the short exact sequence of G-modules $0 \to A \to B \to C \to 0$ with G finite cyclic, show that $h(B) = h(A)h(C)$ where we assume the cohomology groups for A and C [hence B] are finite. Here h is the Herbrand quotient (defined in the previous exercise).

Consider the long exact cohomology sequence $H^0(G, C) \overset{\delta_0}{\to} H^1(G, A) \overset{\varphi}{\to} H^1(G, B) \to \cdots \to H^2(G, B) \overset{\phi}{\to} H^2(G, C) \overset{\delta_2}{\to} H^3(G, A)$. It is a fact (Exercise V.3.3(b)) that cupping this sequence with the generator of $H^2(G, \mathbb{Z})$ gives an isomorphism of that sequence onto itself (raising dimensions by 2), and hence gives the equality $\delta_0 = \delta_2$ which implies $\mathrm{Ker}\varphi \cong \mathrm{Coker}\phi$. We can break the above sequence and obtain an exact sequence $0 \to \mathrm{Ker}\varphi \to H^1(G, A) \overset{\varphi}{\to} \cdots \overset{\phi}{\to} H^2(G, C) \to \mathrm{Coker}\phi \to 0$. I claim that if $0 \to H_1 \to \cdots \to H_m \to 0$ is an exact sequence of finite abelian groups, then $\prod_{i=1}^m |H_i|^{(-1)^i} = 1$. Assuming this claim holds, and noting that the cohomology groups are finite abelian, we have $1 = \frac{|\mathrm{Coker}\phi|}{|\mathrm{Ker}\varphi|} \frac{|H^1(G,A)|}{|H^1(G,B)|} \frac{|H^1(G,C)|}{|H^2(G,A)|} \frac{|H^2(G,B)|}{|H^2(G,C)|} = 1 \cdot \frac{1}{h(A)} \frac{1}{h(C)} h(B)$ which implies $h(B) = h(A)h(C)$ as desired. It suffices to prove the claim. The case $m = 2$ is trivial since the sequence yields the isomorphism $H_1 \cong H_2$ and hence $|H_1|/|H_2| = 1$, so we proceed by induction on $m > 1$. From the exact sequence $0 \to H_1 \to \cdots \to H_{m-1} \overset{\phi}{\to} H_m \overset{\varphi}{\to} H_{m+1} \to 0$ we obtain the exact sequence $0 \to H_1 \to \cdots H_{m-2} \to H_{m-1} \overset{\phi}{\to} \mathrm{Im}\phi \to 0$. Applying the inductive hypothesis we have $\prod_{i=1}^{m-1} |H_i|^{(-1)^i} \cdot |\mathrm{Im}\phi|^{(-1)^m} = 1$, and by exactness of the original sequence we see that $|\mathrm{Im}\phi| = |\mathrm{Ker}\varphi|$. But $|H_m|/|\mathrm{Ker}\varphi| = |\mathrm{Im}\varphi| = |H_{m+1}|$, so $|\mathrm{Ker}\varphi| = |H_m|/|H_{m+1}|$ and hence $1 = \prod_{i=1}^{m-1} |H_i|^{(-1)^i} \cdot (|H_m|/|H_{m+1}|)^{(-1)^m} = \prod_{i=1}^{m+1} |H_i|^{(-1)^i}$ as desired.

38: If G and H are abelian groups with isomorphic group rings $\mathbb{Z}G \cong \mathbb{Z}H$, show that $G \cong H$.

The homology groups of G and H are independent of the choice of resolution up to canonical isomorphism, and the groups are defined by $H_iG = H_i(F_G)$ where F is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ (similary for H). Since $\mathbb{Z}G \cong \mathbb{Z}F$, the G-modules F_i can be regarded as H-modules via restriction of scalars and hence the projective resolution F for the homology of G can also be used for the homology of H. We have $F_G \cong F_H$ since $\mathbb{Z} \otimes_{\mathbb{Z}G} F \cong \mathbb{Z} \otimes_{\mathbb{Z}H} F$ by the obvious map $1 \otimes f \mapsto 1 \otimes f$ [using the isomorphism $\varphi : \mathbb{Z}G \to \mathbb{Z}H$ we have $1 \otimes f = 1 \otimes gf \mapsto 1 \otimes \varphi(g)f = 1 \otimes f$], and so $H_i(G) \cong H_i(H)$ for all i. In particular, $G/[G, G] \cong H_1(G) \cong H_1(H) \cong H/[H, H]$. Since G and H are abelian groups,

$[G,G] = 0 = [H,H]$ and hence $G \cong H$.

39: Let F be a field and G a finite group of order $|G| > 1$. Show that the group algebra FG has zero divisors (hence is not a field) and show that the augmentation ideal I is a maximal ideal of FG.

Noting that $FG/I \cong F$ from the exact sequence $0 \to I \to FG \xrightarrow{\varepsilon} F \to 0$, I is a maximal ideal by Proposition 7.4.12[2] since F is a field [in a commutative ring R, an ideal M is maximal iff R/M is a field]. Now take $g \in G$ such that $g^m = 1$ with $m > 1$. Then $(1-g)(1+g+\cdots+g^{m-1}) = 1-g^m = 1-1 = 0$ and hence $1-g$ is a zero divisor.

40: Regarding $\mathbb{Z}_2$ as a module over the ring $\mathbb{Z}_4$, construct a resolution of $\mathbb{Z}_2$ by free modules over $\mathbb{Z}_4$ and use this to show that $\mathrm{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2)$ is nonzero for all n.

The 2-multiplication map $\mathbb{Z}_4 \xrightarrow{2} \mathbb{Z}_4$ has kernel $\{0,2\}$ and image $\{0,2\}$, and the projection $\mathbb{Z}_4 \twoheadrightarrow \mathbb{Z}_2$ has kernel $\{0,2\}$ and image $\{0,1\}$. Thus we have a free resolution $\cdots \to \mathbb{Z}_4 \xrightarrow{2} \mathbb{Z}_4 \xrightarrow{2} \mathbb{Z}_4 \twoheadrightarrow \mathbb{Z}_2 \to 0$. Since $\mathrm{Hom}(\mathbb{Z}_4, \mathbb{Z}_2) = \mathbb{Z}_2$, the 2-multiplication maps become trivial maps, and we obtain the sequence $\cdots \leftarrow \mathbb{Z}_2 \xleftarrow{0} \mathbb{Z}_2 \xleftarrow{0} \mathbb{Z}_2$ after applying $\mathrm{Hom}(-, \mathbb{Z}_2)$ to our free resolution. Thus $\mathrm{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2 \neq 0$ for all n.

41: Let H be a subgroup of G of finite index, and let $0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ be an S.E.S. of G-modules. Show that the connecting homomorphisms are consistent with corestriction, i.e. the following diagram commutes.

$$\begin{tikzcd}
H^n(G, A'') \arrow[r, "\delta"] & H^{n+1}(G, A') \\
H^n(H, A'') \arrow[r, "\delta"] \arrow[u, "\mathrm{cor}"] & H^{n+1}(H, A') \arrow[u, "\mathrm{cor}"']
\end{tikzcd}$$

Let F be the resolution associated to G and let F' be the resolution associated to H, and let ∂^* denote the [cochain complex] coboundary map. Consider the exact sequence of cochain complexes $0 \to \mathcal{H}om_H(F'_n, A') \xrightarrow{i_H} \mathcal{H}om_H(F'_n, A) \xrightarrow{j_H} \mathcal{H}om_H(F'_n, A'') \to 0$, with similar notation for the sequence associated to G. The connecting map δ in cohomology is defined as follows. For the cohomology-class representative $\widehat{f} \in \mathcal{H}om_H(F'_n, A'')$, there exists an $f \in \mathcal{H}om_H(F'_n, A)$ which maps onto $\widehat{f}$ via j_H (by surjectivity of j_H). Then it is realized that $\partial^* f = i_H(\overline{f})$ for some $\overline{f} \in \mathcal{H}om_H(F'_n, A')$ which is unique by injectivity of i_H, and $\overline{f}$ is the class representative of $\delta[\widehat{f}]$ where $[\widehat{f}]$ is the cohomology class of $\widehat{f}$. Thus in order for δ to commute with corestriction, it suffices to show that corestriction commutes with $\{\partial^*, i^*, j^*\}$ where i^* and j^* refer to both maps associated to H, G. Let's show commutativity of the diagram

$$\begin{tikzcd}
\mathcal{H}om_G(F_n, A) \arrow[r, "j_G"] & \mathcal{H}om_G(F_n, A'') \\
\mathcal{H}om_H(F'_n, A) \arrow[r, "j_H"] \arrow[u, "\mathrm{cor}"] & \mathcal{H}om_H(F'_n, A'') \arrow[u, "\mathrm{cor}"']
\end{tikzcd}$$

Now $\mathrm{cor}[j_H(f)] = \mathrm{cor}[\widehat{f} = j \circ f] = \sum_{g \in G/H} g\widehat{f}(g^{-1}\quad) = \sum_{g \in G/H} gj[f(g^{-1}\quad)]$, and in the other direction we have $j_G[\mathrm{cor}(f)] = j_G[\sum_{g \in G/H} gf(g^{-1}\quad)] = \sum_{g \in G/H} g\widehat{f(g^{-1}\quad)} = \sum_{g \in G/H} j[gf(g^{-1}\quad)] = \sum_{g \in G/H} gj[f(g^{-1}\quad)]$ where we note that j is a G-module homomorphism and hence commutes with the G-action. Thus the diagram is commutative, and similar calculations give commutativity with i^* and ∂^* since both are G-module homomorphisms.

42: Let p be a prime and let S_p be the symmetric group of degree p. Then each Sylow p-subgroup P is cyclic of order p, one such being the subgroup generated by the cycle $(1\,2\,\cdots\,p)$. Thus, $H^*(P, \mathbb{Z}) \cong \mathbb{Z}[\nu]/(p\nu)$ where $\deg\nu = 2$. Show that $N_{S_p}(P)/P \cong \mathbb{Z}_p^*$ and that it acts on $P \cong \mathbb{Z}_p$ in the obvious way. Conclude that $H^*(S_p, \mathbb{Z})_{(p)} \cong \mathbb{Z}[\nu^{p-1}]/(p\nu)$.

Since P is abelian, $P \subseteq C_{S_p}(P)$. Now $|C_{S_p}(P)| = p(p-p)! = p$ as explained on pg127[2] where we note that P and its generator have the same centralizer, so we must have $C_{S_p}(P) \cong P$. Corollary 4.4.15[2] states that $N_{S_p}(P)/C_{S_p}(P)$ is isomorphic to a subgroup of $\mathrm{Aut}(P)$, and by Proposition 4.4.16[2] we know that $\mathrm{Aut}(P) \cong \mathbb{Z}_p^*$ is cyclic of order $\varphi(p) = p-1$ [φ is the Euler function]. Thus $N_{S_p}(P)/P \cong H \subseteq \mathbb{Z}_p^*$. The number of p-cycles in S_p is $(p-1)!$ as explained on pg127[2], and every conjugate of P contains exactly $p-1$ p-cycles, so there are $(p-1)!/(p-1) = (p-2)!$ conjugates of P which is equal to the index $|S_p : N_{S_p}(P)|$ by Proposition 4.3.6[2]; thus $|N_{S_p}(P)| = p!/(p-2)! = p(p-1)$. This means $|H| = |N_{S_p}(P)/P| = p-1$ and hence $H \cong \mathbb{Z}_p^* \;\Rightarrow\; N_{S_p}(P)/P \cong \mathbb{Z}_p^*$. This group acts by conjugation on P because it is a quotient of the normalizer, and this action is well-defined because conjugation by elements of P is trivial, noting that P is abelian.
Theorem III.10.3[1] along with a theorem of Swan (see Exercise III.10.1) states that $H^*(S_p,\mathbb{Z})_{(p)} \cong H^*(P,\mathbb{Z})^{N_{S_p}(P)}$. By Proposition II.6.2[1] the conjugation action by P induces the identity on $H^*(P,\mathbb{Z})$, so the resulting action (Corollary II.6.3[1]) is the $N_{S_p}(P)/P$-action induced on $H^*(P,\mathbb{Z})$. Thus $H^*(S_p,\mathbb{Z})_{(p)} \cong (\mathbb{Z}[\nu]/(p\nu))^{\mathbb{Z}_p^*}$. For a particular dimension j, the elements of the jth-cohomology group belong to $\mathbb{Z}_p$ and hence the action on an element ν would be $\nu \mapsto z_* \cdot \nu = z\nu$ with $0 < z < p$ and $z_* \in \mathbb{Z}_p^*$. Then in the cohomology ring, the action is given by $\nu^i \mapsto z_* \cdot \nu^i = (z_* \cdot \nu) \smile \cdots \smile (z_* \cdot \nu) = (z\nu)^i = z^i\nu^i$. So for an element to belong to the group of $\mathbb{Z}_p^*$-invariants, we must have $\nu^i = z^i\nu^i \;\Rightarrow\; z^i \equiv 1 \bmod p$ for all $z < p$. This is only satisfied when $i = p-1$ by Fermat's Little Theorem (z is relatively prime to p), so the invariant elements are of the form $\sum_i z_i \nu^{i(p-1)}$. Therefore, $H^*(S_p,\mathbb{Z})_{(p)} \cong \mathbb{Z}[\nu^{p-1}]/(p\nu)$.

43: Prove that the Pontryagin product on $H_*(G,\mathbb{Z})$ for G finite cyclic is the trivial map in positive dimensions ($\mathbb{Z}$ has trivial G-action).

The map is given by $H_iG \otimes H_jG \to H_{i+j}G$ for each i,j. Since $H_nG \cong G$ for n odd and $H_nG = 0$ for n even, the domain of the map [hence the map] is trivial for i or j even. But if both $i := 2c+1$ and $j := 2c'+1$ are odd, then $i+j = 2(c+c'+1)$ is even, so the image of the map [hence the map] is trivial.

44: Compute $H^{2r}(\mathbb{Z}_p \oplus \mathbb{Z}_q)$ where p and q are not necessarily relatively prime.

The cohomology Künneth formula of Exercise V.2.2 gives $H^{2r}(\mathbb{Z}_p \oplus \mathbb{Z}_q) \cong \big[\bigoplus_{i=0}^{2r} H^i(\mathbb{Z}_p) \otimes H^{2r-i}(\mathbb{Z}_q)\big] \oplus \big[\bigoplus_{i=0}^{2r+1} \mathrm{Tor}_1^{\mathbb{Z}}\big(H^i(\mathbb{Z}_p), H^{2r-i+1}(\mathbb{Z}_q)\big)\big] \cong \big[(\mathbb{Z}\otimes\mathbb{Z}_q) \oplus (\mathbb{Z}_p \otimes \mathbb{Z}_q)^{r-1} \oplus (\mathbb{Z}_p \otimes \mathbb{Z})\big] \oplus [0] \cong \mathbb{Z}_p \oplus \mathbb{Z}_q \oplus \mathbb{Z}_{gcd(p,q)}^{r-1}$ for $r \geq 1$.
The Tor-parts were trivial because each summand became $\mathrm{Tor}(H^{even}, H^{odd}) = 0$ and $\mathrm{Tor}(H^{odd}, H^{even}) = 0$. Note that if p and q are relatively prime, then the result becomes $\mathbb{Z}_p \oplus \mathbb{Z}_q$ which agrees with the fact that $\mathbb{Z}_p \oplus \mathbb{Z}_q \cong \mathbb{Z}_{pq}$ is cyclic.

45: Prove that $H^i(\mathbb{Z}^n,\mathbb{Z}) \cong \mathbb{Z}^{\binom{n}{i}}$ for $i,n \in \mathbb{N}$, where we interpret $\binom{n<i}{i} = 0$. Deduce that $H^i(\mathbb{Z}^n,k) \cong k^{\binom{n}{i}}$ for any commutative ring k.

For $n = 1$ we have $H^0(\mathbb{Z},\mathbb{Z}) = H^1(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$ and $H^i(\mathbb{Z},\mathbb{Z}) = 0$ for all $i > 1$ as proven on pg58[1]. This agrees with the proposed formula, so we argue by induction on n (assume the result holds up to and including n). The Künneth formula of Exercise V.2.2 gives $H^i(\mathbb{Z}^{n+1}) = H^i(\mathbb{Z}^n \oplus \mathbb{Z}) \cong [\bigoplus_{p=0}^{i} H^p(\mathbb{Z}^n) \otimes H^{i-p}(\mathbb{Z})] \oplus [\bigoplus_{p=0}^{i+1} \mathrm{Tor}_1^{\mathbb{Z}}\big(H^p(\mathbb{Z}^n), H^{i-p+1}(\mathbb{Z})\big)] \cong \bigoplus_{p=0}^{i} H^p(\mathbb{Z}^n) \otimes H^{i-p}(\mathbb{Z})$, where the latter isomorphism follows from the fact that $\mathrm{Tor}_1^{\mathbb{Z}}(-,\mathbb{Z}) = 0$ and $H^{i-p+1}(\mathbb{Z})$ is either $\mathbb{Z}$ or 0 as stated above. Now $\bigoplus_{p=0}^{i} H^p(\mathbb{Z}^n) \otimes H^{i-p}(\mathbb{Z}) \cong (H^i(\mathbb{Z}^n) \otimes \mathbb{Z}) \oplus (H^{i-1}(\mathbb{Z}^n) \otimes \mathbb{Z}) \cong H^i(\mathbb{Z}^n) \oplus H^{i-1}(\mathbb{Z}^n) \cong \mathbb{Z}^{\binom{n}{i}} \oplus \mathbb{Z}^{\binom{n}{i-1}}$, so $H^i(\mathbb{Z}^{n+1},\mathbb{Z}) \cong \mathbb{Z}^{\binom{n+1}{i}}$ since $\binom{n}{i} + \binom{n}{i-1} = \frac{n!}{i!(n-i)!} + \frac{n!}{(i-1)!(n-i+1)!} = \frac{n!(n-i+1)}{i!(n-i+1)!} + \frac{n!i}{i!(n-i+1)!} = \frac{(n+1)!}{i!(n+1-k)!} = \binom{n+1}{i}$, and the inductive process is complete.
The cohomological analog of the universal coefficient sequence of Exercise III.1.3 gives $H^i(\mathbb{Z}^n,k) \cong (H^i(\mathbb{Z}^n) \otimes k) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H^{i+1}(\mathbb{Z}^n),k) \cong (\mathbb{Z}^{\binom{n}{i}} \otimes_{\mathbb{Z}} k) \oplus 0 \cong k^{\binom{n}{i}}$, where we note that $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^r,-) = 0$.
Note: These integral cohomology groups are dual to the associated integral homology groups; see pg38[1]. This observation of Poincaré duality is reflected in the relation $\binom{n}{i} = \binom{n}{n-i}$.

46: Let p be a prime, G a finite p-group, k a field of characteristic p, and $n \in \mathbb{N}$. If $H^n(G,k) = 0$, prove that $H^{n+1}(G,\mathbb{Z}) = 0$.

The cohomological analog of the universal coefficient sequence of Exercise III.1.3 gives $H^n(G,k) \cong (H^n(G) \otimes k) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H^{n+1}(G),k) = 0$, which in particular implies $\mathrm{Tor}_1^{\mathbb{Z}}(H^{n+1}(G),k) = 0$. Then since k contains the subring $\mathbb{Z}_p$, we must have $\mathrm{Tor}_1^{\mathbb{Z}}(H^{n+1}(G),\mathbb{Z}_p) = 0$ and hence no element in $H^{n+1}(G)$ has order p. But $|G| = p^\alpha$ and so $p^\alpha H^{n+1}(G) = 0$ by Corollary III.10.2[1], which means any element must have order a power of p. The only way this is satisfied with no element having order p is for every element to have order 1 (if an element h had order p^a with $a \geq 2$, then $p^{a-1}h$ would be an element of order p). Only the identity has order 1, so all elements are the same; thus $H^{n+1}(G,\mathbb{Z}) = 0$.

47: Prove that the shuffle product is strictly anti-commutative.

The shuffle product on the bar resolution is given by
$[g_1|\cdots|g_n]\cdot[g_1|\cdots|g_n] = \sum_\sigma (-1)^{\mathrm{sgn}\sigma}[g_{\sigma^{-1}(1)}|\cdots|g_{\sigma^{-1}(n)}|g_{\sigma^{-1}(n+1)}|\cdots|g_{\sigma^{-1}(2n)}]$,
where σ ranges over the (n,n)-shuffles. But given any shuffled tuple $(-1)^s[g_{i_1}|\cdots|g|\cdots|g|\cdots|g_{i_{2n}}]$ we have a corresponding shuffle which simply swaps the two g's, leaving the $2n$-tuple fixed and altering the sign to $(-1)^{s+1}$; the sign change arises from moving the left g an l amount of times and then moving the right g an $l+1$ amount of times (the extra $+1$ is due to moving the right g around the left g) and this gives a total of $2l+1$ amount of moves with $(-1)^{s+2l+1} = (-1)^{s+1}$. So if there are an odd number of shuffled tuples (due to n being even, i.e. the tuple is of even degree) then the sum will consist of paired tuples with different signs plus one extra tuple, giving a nontrivial sum. But if the degree is odd (i.e. an even number of shuffled tuples) then the sum will consist of only paired tuples with different signs, giving a sum of 0. Thus $x^2 = 0$ if $\deg x$ is odd, where $x = [g_1|\cdots|g_n]$, and this is precisely the definition of *strict anti-commutativity.*

48: Let p be a prime and let C_p be the cyclic group of order p with trivial $\mathbb{F}_p$-action. Explain how the fact $H^2(C_p,\mathbb{F}_p) \cong \mathbb{F}_p$ and the classification of extensions of C_p by $\mathbb{F}_p$ matches up with the classification theorem for groups of order p^2.

If P is a group with $|P| = p^2$ then it has nontrivial center, $|Z(P)| \neq 1$, by Theorem 4.3.8[2]. If $|Z(P)| = p^2$ then $P = Z(P)$ and P is abelian. The only other scenario is $|Z(P)| = p$, in which case $|P/Z(P)| = p$; this implies $P/Z(P)$ is cyclic and hence it is a fact that P is abelian. By the Fundamental Theorem of Finitely Generated Abelian Groups, P must then be either the cyclic group C_{p^2} or the elementary abelian group $C_p \times C_p$; another proof of this is given in Corollary 4.3.9[2]. This means there are only two extension groups of C_p by $\mathbb{F}_p$, but this does *not* necessarily mean that there are only two classes of extensions (possible short exact sequences). Theorem IV.3.12[1] states that $\mathcal{E}(C_p,\mathbb{F}_p) \cong \mathbb{F}_p$ and hence there are a total of p classes of group extensions. There is the canonical split extension $0 \to \mathbb{F}_p \to C_p \times C_p \to C_p \to 1$, and so the other extensions must fit into $p-1$ classes and arise from projections $C_{p^2} \twoheadrightarrow C_p$ (this is because the only injection $C_p \hookrightarrow C_{p^2}$ is the canonical inclusion, i.e. the pth-power map). Switching to multiplicative notation, these extensions are $1 \to C_p \equiv \langle a\rangle \to C_{p^2} \equiv \langle b\rangle \xrightarrow{\beta_i} C_p \to 1$ with $a \mapsto b^p$ and $\beta_i(b) = a^i$ for $1 \leq i \leq p-1$.

49: Let G be a finite group, let $H \subseteq G$, and let K be any group. Let $\mathbb{F}$ be a field which acts trivially on G and K, and consider the Künneth isomorphism $\kappa : H^*(G,\mathbb{F}) \otimes_{\mathbb{F}} H^*(K,\mathbb{F}) \xrightarrow{\cong} H^*(G\times K,\mathbb{F})$. Show that $\mathrm{res}^{G\times K}_{H\times K} \circ \kappa = \kappa \circ (\mathrm{res}^G_H \otimes id)$ and $\mathrm{tr}^{G\times K}_{H\times K} \circ \kappa = \kappa \circ (\mathrm{tr}^G_H \otimes id)$.

Both equations are straightforward, so we will only prove the latter (concerning the transfer map). For $u_H \in H^*(H,\mathbb{F})$ and $v \in H^*(K,\mathbb{F})$, κ is defined on $H^*(H,\mathbb{F})\otimes_{\mathbb{F}} H^*(K,\mathbb{F})$ by $\kappa(u_H \otimes v) = \langle u_H \times v, x \otimes y\rangle = \langle u_H, x\rangle \cdot \langle v, y\rangle$, where we hide the factor $(-1)^{\deg v \cdot \deg x}$ for convenience. Then $(\mathrm{tr}\circ k)(u_H \otimes v) = \mathrm{tr}\langle u_H \times v, x \otimes y\rangle = \sum_{(g,k)\in(G\times K)/(H\times K)}\langle (g,k)\cdot u_H \times v, (g,k)^{-1}x \otimes y\rangle = \sum_{(g,1)\in(G/H)\times\{1\}}\langle gu_H \times 1v, g^{-1}x \otimes 1y\rangle = \sum_{g\in G/H}\langle gu_H, g^{-1}x\rangle \cdot \langle v,y\rangle$. But $[\kappa\circ(\mathrm{tr}\otimes id)](u_H\otimes v) = \kappa(\sum_{g\in G/H} gu_H(g^{-1}x)\otimes v(y)) = \sum_{g\in G/H}\langle gu_H, g^{-1}x\rangle \cdot \langle v,y\rangle$, so the two compositions are in fact equal, as desired.

50: For an abelian group G whose order is divisible by the prime p, Theorem V.6.6[1] states that

the isomorphism $\rho : \bigwedge_{\mathbb{Z}_p}(G_p) \otimes_{\mathbb{Z}_p} \Gamma_{\mathbb{Z}_p}({}_pG) \xrightarrow{\cong} H_*(G, \mathbb{Z}_p)$ is natural if $p \neq 2$. Why?

Referring to pg126[1], where ${}_pG = \mathrm{Tor}(G, \mathbb{Z}_p)$ and $G_p = G \otimes \mathbb{Z}_p = G/pG$, we have a split-exact universal coefficient sequence $0 \to \bigwedge^2(G_p) \to H_2(G, \mathbb{Z}_p) \to {}_pG \to 0$. Now $\rho(x \otimes y) = \psi(x)\varphi(y)$, where $\psi : \bigwedge(G_p) \to H_*(G, \mathbb{Z}_p)$ is the natural map of Theorem V.6.4[1] and $\varphi : \Gamma({}_pG) \to H_*(G, \mathbb{Z}_p)$ is the $\mathbb{Z}_p$-algebra homomorphism extended from a splitting $\phi : {}_pG \to H_2(G, \mathbb{Z}_p)$ of the above sequence. Since ψ is natural and φ is an extension of the splitting ϕ, the question of naturality of ρ reduces to the question of naturality of ϕ (in dimension 2). The splitting is made by choice, and if p is odd, we may use the canonical splitting $H_2(G, \mathbb{Z}_p) \to \bigwedge^2(G_p)$ given in Exercise V.6.4(b) since 2 is invertible in $\mathbb{Z}_p$ for $p \neq 2$ (i.e. prime p odd); remember that a splitting can be made on either side of the sequence. Thus the isomorphism ρ is natural if $p \neq 2$. But if $p = 2$ then we do not have a known canonical splitting, and we cannot prove naturality in this case since we do not have an explicit map.

51: Referring to the proof of Proposition VI.2.6[1], if $\eta : M \to Q^0$ is an admissible injection (i.e. a split injection of H-modules) and M is projective as a $\mathbb{Z}H$-module, then why is $\mathrm{Coker}\eta$ projective as a $\mathbb{Z}H$-module?

Since η is H-split, we have a split-exact sequence $0 \to M \xrightarrow{\eta} Q^0 \to \mathrm{Coker}\eta \to 0$, and so $Q^0 \cong M \oplus \mathrm{Coker}\eta$. But Q^0 is $\mathbb{Z}G$-projective by Corollary VI.2.2.[1], hence $\mathbb{Z}H$-projective by Exercise I.8.2. Now a direct sum is projective iff each direct summand is projective, so $\mathrm{Coker}\eta$ must be $\mathbb{Z}H$-projective.

52: Let G be a goup with order r. Show that for each q there exists G-modules C with $\widehat{H}^q(G, C)$ cyclic of order r.

For $q = 0$ we can take $C = \mathbb{Z}$ since $\widehat{H}^0(G, \mathbb{Z}) = \mathbb{Z}/|G|\mathbb{Z} = \mathbb{Z}_r$. Then using the dimension-shifting technique, we can find G-modules C_1 and C_2 such that $\widehat{H}^0(G, \mathbb{Z}) \cong \widehat{H}^1(G, C_1)$ and $\widehat{H}^0(G, \mathbb{Z}) \cong \widehat{H}^{-1}(G, C_2)$; see property 5.4 on pg136[1]. Therefore, through repeatable applications of dimension-shifting, we can range over all q to obtain $\widehat{H}^q(G, C) \cong \mathbb{Z}_r$.

53: Fill in the details to the proof of Proposition VI.7.1[1] which states that the evaluation pairing $\rho : H^i(G, M') \otimes H_i(G, M) \to \mathbb{Q}/\mathbb{Z}$ is a duality pairing, where $M' = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$.

Let F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. The evaluation pairing is obtained by composing the pairing $\langle \cdot, \cdot \rangle : \mathcal{H}om_G(F, M') \otimes (F \otimes_G M) \to M' \otimes_G M$ with the evaluation map $M' \otimes_G M \to \mathbb{Q}/\mathbb{Z}$. The pairing $\langle u, z = x \otimes m \rangle$ is given by $u \otimes (x \otimes m) = u(x) \otimes m$, and composing this with the evaluation map gives $\rho(u \otimes z) = [u(x)](m)$. Note that ρ gives rise to the map $\bar{\rho} : H^i(G, M') \to H_i(G, M)'$ defined by $[\bar{\rho}(u)](z) = \rho(u \otimes z) = [u(x)](m)$. We have $\mathcal{H}om_G(F, M') = \mathcal{H}om_G(F, \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})) \cong \mathcal{H}om_G(F \otimes M, \mathbb{Q}/\mathbb{Z}) \overset{(*)}{=} \mathcal{H}om(F \otimes_G M, \mathbb{Q}/\mathbb{Z}) = (F \otimes_G M)'$. The equality $(*)$ arises from Exercise III.1.3, as $\mathbb{Q}/\mathbb{Z}$ has trivial G-action. Since $(\ )'$ is exact, we can pass to homology to obtain $H^*(G, M') \cong H_*(G, M)'$. This isomorphism resulted from $\mathcal{H}om_G(F, \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})) \cong \mathcal{H}om_G(F \otimes M, \mathbb{Q}/\mathbb{Z})$ which is given by Theorem 10.5.43[2], and this is precisely the map $\bar{\rho}$. Therefore, since $\bar{\rho}$ is an isomorphism, ρ is a duality pairing.

54: Let $G = \mathbb{Z}_2 = \langle g \rangle$ and let $A = \mathbb{Z}_8$, written additively. Make G act on A by $x \mapsto 3x$, and let G act trivially on $B = \mathbb{Z}_2$. Show that A is cohomologically trivial, but $A \otimes B$ is not.

First note that $\widehat{H}^*(\{1\}, M) = 0$ for any M; it is clearly trivial in all dimensions not equal to -1 and 0, and in those two dimensions it is the kernel and cokernel of the norm map $M \to M$ which is the identity (so the kernel and cokernel are trivial). Thus for A to be cohomologically trivial it suffices to show that $\widehat{H}^*(G, A) = 0$. The complete resolution from Exercise VI.3.1 implies that $\widehat{H}^n(G, A) = \mathrm{Coker}\overline{N}$ for n even and $\widehat{H}^n(G, A) = \mathrm{Ker}\overline{N}$ for n odd (see pg58[1]). The action on A gives $A_G = \mathbb{Z}_8/\mathbb{Z}_4 = \{0+\mathbb{Z}_4, 1+\mathbb{Z}_4\} \cong \mathbb{Z}_2$ since $1 \mapsto 3 \equiv 1 \bmod 2$, and $A^G = \mathbb{Z}_2 = \{0, 4\}$ since $4 \mapsto 12 \equiv 4 \bmod 8$. Thus the norm map $\overline{N} : \mathbb{Z}_2 \to \mathbb{Z}_2$ is given by $1 \mapsto N1 = 1 \cdot 1 + g \cdot 1 = 1 + 3 = 4$, which is the identity. Thus $\mathrm{Ker}\overline{N} = \mathrm{Coker}\overline{N} = 0$ and $\widehat{H}^*(G, A) = 0$. However, repeating the above with coefficient module $A \otimes B$ we see that the norm map is the trivial map, $1 \otimes 1 \mapsto 1 \otimes 1 + g \cdot (1 \otimes 1) = 1 \otimes 1 + 3 \otimes 1 = 4 \otimes 1 = 2 \otimes 2 = 0$. Thus

$\widehat{H}^{2i+1}(G, A) = \mathrm{Ker}\overline{N} = \mathbb{Z}_2 \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2$ and $A \otimes B$ is not cohomologically trivial.

55: Let $H \lhd G$ and let M be a G-module. Then M^H is naturally a G/H-module, and the pair $(\rho : G \to G/H, \alpha : M^H \to M)$ is compatible in the sense that $\alpha[\rho(g) \cdot m] = g \cdot \alpha(m)$. Thus we have a homomorphism $\mathrm{inf} : H^n(G/H, M^H) \to H^n(G, M)$ called the *inflation map*.
Using this, show that for the semi-direct product $G = H \rtimes K$ and module M with trivial G-action, the group $H^n(K, M)$ is isomorphic to a subgroup $H^n(G, M)$.

From the inclusion $i : K \hookrightarrow G$ and the surjection $\rho : G \to G/H = K$ we can pass to cohomology to obtain the restriction $\mathrm{res} : H^n(G, M) \to H^n(K, M)$ and the inflation $\mathrm{inf} : H^n(K, M) \to H^n(G, M)$. Since $\rho \circ i = id_K$, the composite $\mathrm{inf} \circ \mathrm{res}$ is also the identity. Thus inf is injective, so $H^n(K, M) \subseteq H^n(G, M)$ up to isomorphism.

55: In Exercise AE.34 it was shown that a module is finitely generated $\mathbb{Z}$-projective iff it is finitely generated $\mathbb{Z}$-free. Weakening the hypothesis, show that $\mathbb{Z}$-projective $=$ $\mathbb{Z}$-free.

Free modules are projective, so it suffices to show that $\mathbb{Z}$-projective implies $\mathbb{Z}$-free. If P is $\mathbb{Z}$-projective then it is a submodule of a $\mathbb{Z}$-free module. But any submodule of a $\mathbb{Z}$-free module is free by Theorem I.7.3[5], so P is $\mathbb{Z}$-free.

57: Prove that the symmetric group S_3 has periodic cohomology, and find its period.

Since $S_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ and $|S_3| = 3! = 6$, its Sylow subgroups are $\mathbb{Z}_2$ and $\mathbb{Z}_3$ which are both cyclic. Then by Theorem VI.9.5[1], S_3 has periodic cohomology.
Alternatively, any proper subgroup must have order 1 or 2 or 3 and hence must be cyclic, so Theorem VI.9.5[1] implies that S_3 has periodic cohomology.
Alternatively, $H^4(S_3, \mathbb{Z}) \cong \mathbb{Z}_6$ by Exercise III.10.1 and so S_3 has periodic cohomology by Theorem VI.9.1[1].
We can see this periodicity via Exercise III.10.1, because $H^n(S_3) \cong H^{n+4}(S_3)$ for all $n > 0$. Thus the period is 4.

58: Let $N : \mathbb{Z} \to \mathbb{Z}G$ denote the G-module homomorphism $z \mapsto Nz$, where $N \in \mathbb{Z}G$ is the norm element, and let $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ be the augmentation map. Prove that if $\alpha : \mathbb{Z} \to \mathbb{Z}G$ is a G-module homomorphism then $\alpha = aN$ for some $a \in \mathbb{Z}$, and if $\beta : \mathbb{Z}G \to \mathbb{Z}$ is a G-module homomorphism then $\beta = b\varepsilon$ for some $b \in \mathbb{Z}$.

First note that α is determined by where it sends the identity, $\alpha(1) = x$. Then for α to be compatible with the G-action we must have $g \cdot x = g \cdot \alpha(1) = \alpha(g \cdot 1) = \alpha(1) = x$, so $x \in (\mathbb{Z}G)^G = \mathbb{Z} \cdot N$ where the equality is shown in Exercise AE.1; thus $\alpha = aN$ for some $a \in \mathbb{Z}$. Now β is also determined by $\beta(1) = z$, so we must have $z = g \cdot z = g \cdot \beta(1) = \beta(g) = z'$ and hence β maps G onto a single integer b; thus $\beta = b\varepsilon$ for some $b \in \mathbb{Z}$ (where we note that $\varepsilon(G) = 1$).

59: If G_1 and G_2 are perfect groups with universal central extensions E_1 and E_2, respectively, prove that $E_1 \times E_2$ is a universal central extension of $G_1 \times G_2$.

We have universal central extensions $0 \to H_2(G_i) \to E_i \to G_i \to 1$ by hypothesis (see Exercise IV.3.7). Since the direct sum of two extensions is an extension, we have a central extension $0 \to H_2(G_1) \oplus H_2(G_2) \to E_1 \times E_2 \to G_1 \times G_2 \to 1$. I claim that this extension is a universal central extension. Indeed, $H_2(G_1) \oplus H_2(G_2)$ is isomorphic to $H_2(G_1 \times G_2)$ by the Künneth formula, since $H_1(G_i) = 0$ by perfectness of G_i. Now, the universal central extension of $G_1 \times G_2$ is $0 \to H_2(G_1 \times G_2) \to E \to G_1 \times G_2 \to 1$ by definition, so $E_1 \times E_2 \cong E$ by the Five-Lemma (applied to the two sequences of $G_1 \times G_2$) and hence $E_1 \times E_2$ is the universal central extension of $G_1 \times G_2$.

60: In the proof of Proposition VIII.2.4[1], with $\Gamma' \subset \Gamma$, where did we use the hypothesis that $|\Gamma : \Gamma'| < \infty$?

We considered a free Γ-module F, and noted that if F' is a free Γ'-module of the same rank then $F \cong \mathrm{Ind}_{\Gamma'}^{\Gamma} F'$. We then applied Shapiro's lemma to yield $H^n(\Gamma', F') \cong H^n(\Gamma, F)$. But Shapiro's Lemma is actually given by $H^n(\Gamma', F') \cong H^n(\Gamma, \mathrm{Coind}_{\Gamma'}^{\Gamma} F')$. We therefore used the isomorphism $\mathrm{Coind}_{\Gamma'}^{\Gamma} F' \cong \mathrm{Ind}_{\Gamma'}^{\Gamma} F'$ of Proposition III.5.9[1], which holds if $|\Gamma : \Gamma'| < \infty$.

11 References

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