NO HOMOTOPY 4-SPHERE INVARIANTS USING \( ECH = SWF \)

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Abstract. In relation to the 4-dimensional smooth Poincaré conjecture we construct an invariant of smooth homotopy 4-spheres using embedded contact homology (and Seiberg-Witten theory). But they turn out to vanish, for good reason.

1. (Near-)symplectic geometry

Let \( X \) be a smooth homotopy 4-sphere. In particular, it is a closed simply-connected orientable 4-manifold with \( b_2^+ (X) = 0 \) so that the Seiberg-Witten invariants do not apply. As a topological manifold it is homeomorphic to \( S^4 \) by the solved Poincaré conjecture [8]. Whether \( X \) has an exotic smooth structure is the result of

Conjecture 1.1 (4-dimensional smooth Poincaré conjecture). If \( X \) is a smooth homotopy 4-sphere then \( X \) is diffeomorphic to \( S^4 \).

Denote by \( \Gamma_4 \) the abelian group of orientation-preserving diffeomorphisms of the sphere \( S^3 \) modulo those that extend to a diffeomorphism of the ball \( D^4 \). Cerf’s theorem below implies that all orientation-preserving diffeomorphisms of \( S^3 \) are smoothly isotopic to the identity, and in particular, there are no “twisted” exotic 4-spheres (i.e. those which are obtained from two standard 4-balls by gluing their boundary via a diffeomorphism of \( S^3 \)).

Theorem 1.2 (Cerf [4,6,10]). \( \Gamma_4 = 0 \).

Let \( X^* \) be the noncompact manifold obtained by removing a small standard 4-ball from \( X \) and smoothly attaching an end of the form \([0, \infty) \times S^3\). Then \( X^* \) is an exotic \( \mathbb{R}^4 \) if and only if \( X \) is an exotic \( S^4 \), because by Cerf’s theorem there is only one way to remove or replace a standard 4-ball. We also say that \( X^* \) is asymptotically Euclidean, i.e. the complement of some compact set is diffeomorphic to the complement in the standard \( \mathbb{R}^4 \) of a ball.

The reason we pass from \( X \) to \( X^* \) is that \( X \) does not admit symplectic 2-forms nor near-symplectic 2-forms but \( X^* \) admits both. Moreover, we will see momentarily that there are suitably nice near-symplectic forms to equip \( X^* \) with, and that there is a crucial characterization of the symplectic forms. This characterization is known as Gromov’s “recognition of \( \mathbb{R}^4 \),” stated below. It implies that there are no exotic symplectic structures on \( \mathbb{R}^4 \) that are asymptotically standard (up to compactly supported symplectomorphisms), i.e. such that they agree with the standard symplectic form on \( \mathbb{R}^4 \) outside a compact

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\[ ^1\text{That is, } \pi_*(X) \cong \pi_*(S^4), \text{ hence } H_*(X; \mathbb{Z}) \cong H_*(S^4; \mathbb{Z}) \text{ by the Hurewicz theorem.} \]

\[ ^2\text{It is a small exotic } \mathbb{R}^4: \text{ every compact 4-submanifold is surrounded by some smoothly embedded } S^3. \]
set. For the record, by “standard symplectic $\mathbb{R}^4$” we mean the total space of $T^*\mathbb{R}^2$ with canonical coordinates $(x_1, x_2, y_1, y_2)$ and symplectic form $\omega_{\text{std}} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

**Theorem 1.3** (Gromov [13 §0.3.C]). Let $(M, \omega)$ be a connected noncompact symplectic 4-manifold whose reduced integral homology is trivial. Suppose there are compact sets $K_M \subset M$ and $K_{\mathbb{R}^4} \subset \mathbb{R}^4$ with a symplectomorphism $\phi : (M - K_M, \omega) \to (\mathbb{R}^4 - K_{\mathbb{R}^4}, \omega_{\text{std}})$. Then $\phi$ extends to a symplectomorphism $(M, \omega) \to (\mathbb{R}^4, \omega_{\text{std}})$ after removing slightly bigger compact sets.

**Remark 1.4.** Gromov also proved that the group of compactly supported symplectomorphisms of the standard symplectic $(\mathbb{R}^4, \omega_{\text{std}})$ is contractible, so the space of asymptotically standard symplectic forms on $\mathbb{R}^4$ is homotopy equivalent to the group of all compactly supported diffeomorphisms. Eliashberg asked about the topology of this group in [5, §7]. Related results are given in [7, 20].

We now remind the reader of the definition of a near-symplectic form. Using $M$ to temporarily denote either $X^*$ or any closed 4-manifold, a closed 2-form $\omega : M \to \bigwedge^2 T^*M$ is near-symplectic if for all points $x \in M$ either $\omega^2(x) > 0$, or $\omega(x) = 0$ and the rank of the gradient $\nabla \omega_x : T_x M \to \bigwedge^2 T^*_x M$ is three. In other words, $\omega$ is symplectic on $M - \omega^{-1}(0)$ and vanishes transversely on its zero set $\omega^{-1}(0)$ which consists of a finite disjoint union of smooth embedded circles.

**Theorem 1.5** (Luttinger, Taubes [26]). There exist (exact) near-symplectic forms $\omega$ on $X^*$ which are asymptotically standard.

**Remark 1.6.** We cannot apply [9 Corollary 8] to obtain an asymptotically standard near-symplectic form on $X^*$ because that corollary requires the existence of a homologically-nontrivial surface in $X^*$ with positive self-intersection (whereas $X^*$ has trivial homology).

The zero-circles of a near-symplectic form are not all the same. They come in two “types” depending on the behavior of $\omega$ near them (see [11]), called untwisted and twisted. By work of Luttinger (see [21]), $\omega$ can be modified so that $\omega^{-1}(0)$ has any positive number of components, but as noted by Gompf (see [21, Theorem 1.8]) in the case of closed 4-manifolds, the parity of the number of untwisted zero-circles is a priori determined by the cohomology of the closed 4-manifold. It follows that an asymptotically standard near-symplectic form on $X^*$ must have an even number of untwisted zero-circles: The reason is that we can glue this 2-form into any near-symplectic closed 4-manifold (by the Darboux theorem) to build another near-symplectic form, so it must preserve the parity of the number of untwisted zero-circles of the original near-symplectic form.

**Question 1.** If $X^*$ is symplectic then an asymptotically standard near-symplectic form $\omega$ can be modified to remove its zero-set, but can this be done locally? When $\omega^{-1}(0)$ is a single circle (necessarily of twisted type), is there a unique pseudoholomorphic disk bounding it? When $\omega^{-1}(0)$ is two circles (necessarily of the same type, twisted or untwisted), is there a unique pseudoholomorphic cylinder bounding it?

**Example 1.7.** The standard symplectic form on $\mathbb{C}^2$ with complex coordinates $(z, w)$ is $\omega_{\text{std}} = \frac{i}{2} (dz \wedge d\bar{z} + dw \wedge d\bar{w})$. A version of Luttinger’s birth model [19] is a family of
near-symplectic forms on $\mathbb{C}^2$

$$\omega_\varepsilon = \frac{i}{2} \left[ (-\varepsilon + |z|^2 - |w|^2)(dz \wedge d\bar{z} + dw \wedge d\bar{w}) + (Rw - z\bar{w})dz \wedge dw - (R\bar{w} - zw)d\bar{z} \wedge d\bar{w} \right]$$

with parameter $\varepsilon \in \mathbb{R}$ and fixed $R \gg 1$. For $0 < \varepsilon \ll 1$ there are two zero-circles, one of which dies as $\varepsilon$ crosses 0, namely $Z_\varepsilon = \{(z, 0) \in \mathbb{C}^2 \mid |z|^2 = \varepsilon\}$. For fixed such $\varepsilon$ we can modify a constant multiple of $\omega_\varepsilon$ on the complement of the radius $\sqrt{2\varepsilon}$ ball about the origin containing $Z_\varepsilon$, in such a way that the other zero-circle is destroyed and that it agrees with $\omega_{\text{std}}$ on the complement of a radius $32\sqrt{2\varepsilon}$ ball (see [26, §2] for details). The result is an asymptotically standard near-symplectic form with a single twisted zero-circle, and $\{(z, 0) \in \mathbb{C}^2 \mid |z|^2 \leq \varepsilon\}$ is an $i$-holomorphic disk bounding $Z_\varepsilon$. With respect to Question [1], Taubes suggested that the existence of certain pseudoholomorphic cylinders between zero-circles may be used to cancel them, analogous to the “Morse cancellation lemma” for certain gradient flowlines between critical points of a Morse function [23–25]. If successful this may lead to a proof of Conjecture [1.1]. The methodology taken in this paper is the opposite, we suggest that the existence of pseudoholomorphic curves may prevent the cancellation of zero-circles. That is, we would like to build invariants of $X$ by counting pseudoholomorphic curves in a completion of $X^* - \omega^{-1}(0)$ for any asymptotically standard near-symplectic form $\omega$. The significance is that they would give obstructions to removing the zero-circles of $\omega$, and may detect counterexamples to Conjecture [1.1].

In this direction, if we are given an explicit handlebody decomposition of $X$ involving a single 4-handle and no 3-handles, then [22] builds explicit near-symplectic 2-forms on $X^*$ based on 2-handle data. Candidates for exotic 4-spheres have been built from surgeries along 2-spheres (by Cappell-Shaneson, Ghuck, and others) and 2-tori (by Fintushel-Stern, Iwase, and Nash) and projective 2-planes (by Price), but many were shown standard by Akbulut, Gompf, and Kirby.

We build such an invariant by mimicking the construction of the Gromov invariants of closed near-symplectic 4-manifolds [11,12], in turn using embedded contact homology and its known isomorphism with monopole Floer homology. Unfortunately, it is not sensitive enough. We clarify and summarize this as follows:

**Main Result.** For a suitable neighborhood $\mathcal{N}$ of $\omega^{-1}(0)$ such that $X^* - \mathcal{N}$ is a symplectic manifold with contact boundary (as described in Lemma [4.1]), there exists a well-defined element $\text{Gr}_{X,\omega}$ in the embedded contact homology $ECH_*(\partial\mathcal{N})$ as described in Theorem [4.3], obtained by a suitable count of punctured pseudoholomorphic curves in a completion of $X^* - \mathcal{N}$ which are asymptotic to specific Reeb orbits in $\partial\mathcal{N}$. There is also a relative Seiberg-Witten invariant associated with $X^*$ as described in Theorem [5.2], and its value does not depend on $X^*$. Then $\text{Gr}_{X,\omega}$ is identified with this relative Seiberg-Witten invariant (Theorem [6.3]), so it is not able to detect potentially exotic 4-spheres.

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3One stumbling block here is the Andrews-Curtis conjecture in group theory [1], which if true would imply that any homotopy sphere given as a handlebody without 3-handles is standard.

4Fox, Gordon, Litherland, Mazur, Melvin, Montesinos, Pao, Plotnick, and Zeeman.
Here is an outlook. This near-symplectic ECH-type invariant $Gr_{X,\omega}$ is seen to be inherently related to Seiberg-Witten theory, and subsequently not helpful. Now there is also a greater machinery to consider, SFT (symplectic field theory), which subsumes ECH in some sense. So we may try to define a near-symplectic SFT-type invariant – but this also turns out not to be helpful (see Section 6.2). These facts may suggest that the existence of pseudoholomorphic curves won’t obstruct the removal of zero-circles and will instead be useful in the removal of them... Otherwise we have to consider more intricate moduli spaces of pseudoholomorphic curves.

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2. Brief review of pseudoholomorphic curve theory

We introduce most of the terminology and notations that appear in this paper. More details are found in [15].

2.1. Orbits. Let $(Y,\lambda)$ be a closed contact 3-manifold, oriented by $\lambda \wedge d\lambda > 0$. Let $\xi = \text{Ker} \lambda$ be the contact structure, oriented by $d\lambda$. Equivalently, $\xi$ is co-oriented by the Reeb vector field $R$ determined by $d\lambda(R,\cdot) = 0$ and $\lambda(R) = 1$. A Reeb orbit is a map $\gamma : \mathbb{R}/\mathbb{Z} \to Y$ for some $T > 0$ with $\gamma'(t) = R(\gamma(t))$, modulo reparametrization, which is necessarily an $m$-fold cover of an embedded Reeb orbit for some $m \geq 1$. A given Reeb orbit is nondegenerate if the linearization of the Reeb flow around it does not have 1 as an eigenvalue, in which case the eigenvalues are either on the unit circle (such $\gamma$ are elliptic) or on the real axis (such $\gamma$ are hyperbolic). Assume from now on that $\lambda$ is nondegenerate, i.e. all Reeb orbits are nondegenerate, which is a generic property.

An orbit set is a finite set of pairs $\Theta = \{(\Theta_i, m_i)\}$ where the $\Theta_i$ are distinct embedded Reeb orbits and the $m_i$ are positive integers (which may be empty). An orbit set is admissible if $m_i = 1$ whenever $\Theta_i$ is hyperbolic. Its homology class is defined by

$$[\Theta] := \sum_i m_i [\Theta_i] \in H_1(Y;\mathbb{Z})$$

For a given $\Gamma \in H_1(Y;\mathbb{Z})$, the ECH chain complex $ECC_*(Y,\lambda,J,\Gamma)$ is freely generated over $\mathbb{Z}/2\mathbb{Z}$ by admissible orbit sets representing $\Gamma$. The differential $\partial_{ECH}$ will be defined momentarily.

2.2. Curves. Given two contact manifolds $(Y_+,\lambda_+)$, possibly disconnected or empty, a strong symplectic cobordism from $(Y_+,\lambda_+)$ to $(Y_-,\lambda_-)$ is a compact symplectic manifold $(X,\omega)$ with oriented boundary

$$\partial X = Y_+ \sqcup -Y_-$$

such that $\omega|_{Y_\pm} = d\lambda_\pm$. We can always find neighborhoods $N_\pm$ of $Y_\pm$ in $X$ diffeomorphic to $(-\varepsilon,0] \times Y_+$ and $[0,\varepsilon) \times Y_-$, such that $\omega|_{N_\pm} = d(e^{\pm s}\lambda_\pm)$ where $s$ denotes the coordinate on $(-\varepsilon,0]$. We then glue symplectization ends to $X$ to obtain the completion

$$\overline{X} := ((-\infty,0] \times Y_-) \cup_{Y_-} X \cup_{Y_+} ([0,\infty) \times Y_+)$$
of $X$, a noncompact symplectic 4-manifold whose symplectic form is also denoted by $\omega$. We will also use the notation $\overline{X}$ to denote the symplectization $\mathbb{R} \times Y$ of $(Y, \lambda)$, with $\omega = d(\epsilon^* \lambda)$.

An almost complex structure $J$ on a symplectization $(\mathbb{R} \times Y, d(\epsilon^* \lambda))$ is symplectization-admissible if it is $\mathbb{R}$-invariant; $J(\partial_\mathbb{R}) = \partial_\mathbb{R}$; and $J(\xi) \subseteq \xi$ such that $d\lambda(v, Jv) \geq 0$ for $v \in \xi$. An almost complex structure $J$ on the completion $\overline{X}$ is cobordism-admissible if it is $\omega$-compatible on $X$ and agrees with symplectization-admissible almost complex structures on the ends $[0, \infty) \times Y_+$ and $(-\infty, 0] \times Y_-$. 

Given a cobordism-admissible $J$ on $\overline{X}$ and orbit sets $\Theta^+ = \{(\Theta_i^+, m_i^+)\}$ in $Y_+$ and $\Theta^- = \{(\Theta_i^-, m_i^-)\}$ in $Y_-$, a $J$-holomorphic curve $C$ in $\overline{X}$ from $\Theta^+$ to $\Theta^-$ is defined as follows. It is a $J$-holomorphic map $C \to \overline{X}$ whose domain is a possibly disconnected punctured compact Riemann surface, defined up to composition with biholomorphisms of the choice of $C \in M$ element $C$ of elements in $M$. We will also use the notation $\Theta$ to denote the completion $\overline{X}$ is a small ball $B^3 \subset Y$ and disagrees with $\xi$ on $B^3$ by a map $(B^3, \partial B^3) \to (SO(3), \{1\})$ of degree $2n$.

2.3. Homology. The $ECH$ index $I(C)$ of a current $C \in M(\Theta^+, \Theta^-)$ is an integer depending only on its relative class in $H_2(X, \Theta^+, \Theta^-)$, and is the local expected dimension of this moduli space of $J$-holomorphic currents (see [15, §3]). Denote by $M_I(\Theta^+, \Theta^-)$ the subset of elements in $M(\Theta^+, \Theta^-)$ that have $ECH$ index $I$.

Given admissible orbit sets $\Theta^\pm$ of $(Y, \lambda)$, the coefficient $<\partial_{ECH} \Theta^+, \Theta^->$ in $\mathbb{Z}/2\mathbb{Z}$ is the count (modulo 2) of elements in $M_I(\Theta^+, \Theta^-)$ on the symplectization $\overline{X} = \mathbb{R} \times Y$. If $J$ is generic then $\partial_{ECH}$ is well-defined and $\partial_{ECH}^2 = 0$. The resulting homology is independent of the choice of $J$, depends only on $\xi$ and $\Gamma$, and is denoted by $ECH_*(Y, \xi, \Gamma)$.

The total sum

$ECH_*(Y, \xi) := \bigoplus_{\Gamma \in H_1(Y, \mathbb{Z})} ECH_*(Y, \xi, \Gamma)$

has an absolute grading by homotopy classes of oriented 2-plane fields on $Y$ (see [14, §3]), the set of which is denoted by $J(Y)$. There is a transitive $\mathbb{Z}$-action on $J(Y)$, namely, if $[\xi] \in J(Y)$ then $[\xi] + n$ is the homotopy class of a 2-plane field which agrees with $\xi$ outside a small ball $B^3 \subset Y$ and disagrees with $\xi$ on $B^3$ by a map $(B^3, \partial B^3) \to (SO(3), \{1\})$ of degree $2n$. 

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2.4. **L-flat approximations.** The symplectic action of an orbit set Θ = {((θ_i, m_i))} is defined by

\[ \mathcal{A}(\Theta) := \sum_i m_i \int_{\theta_i} \lambda \]

The symplectic action induces a filtration on the ECH chain complex. For a positive real number \( L \), the \( L \)-filtered \( ECH \) is the homology of the subcomplex \( ECC^{\leq}_L(Y, \lambda, J, \Gamma) \) spanned by admissible orbit sets of action less than \( L \). The ordinary \( ECH \) is recovered by taking the direct limit over \( L \), via maps induced by inclusions of the filtered chain complexes.

For a fixed \( L > 0 \) it is convenient (and possible) to modify \( \lambda \) and \( J \) on small tubular neighborhoods of all Reeb orbits of action less than \( L \), in order to relate \( J \)-holomorphic curves to Seiberg-Witten theory most easily. The desired modifications of \( (\lambda, J) \) are called \( L \)-flat approximations, and were introduced by Taubes in [28, Appendix]. They induce isomorphisms on the \( L \)-filtered \( ECH \) chain complex, and the key fact here is that \( L \)-flat orbit sets are in bijection with Seiberg-Witten solutions of “energy” less than \( 2\pi L \) (see Section 3.3).

3. **Brief review of gauge theory**

We introduce most of the terminology and notations that appear in this paper. More details are found in [16,18].

3.1. **Contact 3-manifolds.** Let \( (Y, \lambda) \) be a closed oriented connected contact 3-manifold, and choose an almost complex structure \( J \) on \( \xi \) that induces a symplectization-admissible almost complex structure on \( \mathbb{R} \times Y \). There is a compatible metric \( g \) on \( Y \) such that \( |\lambda| = 1 \) and \( *\lambda = \frac{1}{2}d\lambda \), with \( g(v, w) = \frac{1}{2}d\lambda(v, Jw) \) for \( v, w \in \xi \).

View a spin-c structure \( s \in \text{Spin}^c(Y) \) on \( Y \) as an isomorphism class of a pair \((S, \text{cl})\) consisting of a rank 2 Hermitian vector bundle \( S \to Y \) (the spinor bundle) and Clifford multiplication \( \text{cl} : TY \to \text{End}(S) \). The contact structure \( \xi \) (and more generally, any oriented 2-plane field on \( Y \)) picks out a canonical spin-c structure \( S_\xi = (S_\xi, \text{cl}) \) with \( S_\xi = \mathbb{C} \oplus \xi \), where \( \mathbb{C} \to Y \) denotes the trivial line bundle, and \( \text{cl} \) is defined as follows. Given an oriented orthonormal frame \( \{e_1, e_2, e_3\} \) for \( T_yY \) such that \( \{e_2, e_3\} \) is an oriented orthonormal frame for \( \xi_y \), then in terms of the basis \( (1, e_2) \) for \( S_\xi \),

\[ \text{cl}(e_1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \text{cl}(e_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{cl}(e_3) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \]

There is then a canonical isomorphism

\[ H^2(Y; \mathbb{Z}) \to \text{Spin}^c(Y) \]

where the 0 class corresponds to \( s_\xi \). Specifically, there is a canonical decomposition \( S = E \oplus \xi E \) into \( \pm i \) eigenbundles of \( \text{cl}(\lambda) \), where \( E \to Y \) is the complex line bundle corresponding to a given class in \( H^2(Y; \mathbb{Z}) \).

A \textit{spin-c connection} is a connection \( A \) on \( S \) which is compatible with Clifford multiplication in the sense that

\[ \nabla_A(\text{cl}(v)\psi) = \text{cl}(\nabla v)\psi + \text{cl}(v)\nabla_A \psi \]
where $\nabla v$ denotes the covariant derivative of $v \in TY$ with respect to the Levi-Civita connection. Such a connection is equivalent to a Hermitian connection (also denoted by $A$) on $\det S$, and determines a Dirac operator
\[ D_A : \Gamma(S) \xrightarrow{\nabla_A} \Gamma(T^*Y \otimes S) \xrightarrow{cl} \Gamma(S) \]
With respect to the decomposition $S = E \oplus \xi E$, the determinant line bundle is $\det S = \xi E^2$ and any section (a spinor) of $S$ can be written as
\[ \psi = (\alpha, \beta) \]
There is a unique connection $A_{\xi}$ on $\xi$ such that its Dirac operator kills the spinor $(1, 0) \in \Gamma(S_{\xi})$, and there is a canonical decomposition
\[ A = A_{\xi} + 2A \]
with Hermitian connection $A$ on $E$. The gauge group $C^\infty(Y, S^1)$ acts on a given pair $(A, \psi)$ by
\[ u \cdot (A, \psi) = (A - u^{-1}du, u\psi) \]
In this paper, a configuration $c$ refers to a gauge-equivalence class of such a pair.

Fix a suitably generic exact 2-form $\mu \in \Omega^2(Y)$ as described in [16, §2.2], and a positive real number $r \in \mathbb{R}$. A configuration $c$ solves Taubes’ perturbed Seiberg-Witten equations when
\[ (3.1) \quad D_A \psi = 0, \quad * F_A = r(\tau(\psi) - i\lambda) - \frac{1}{2} * F_{A_{\xi}} + i * \mu \]
where $F_{A_{\xi}}$ is the curvature of $A_{\xi}$ and $\tau : S \to iT^*Y$ is the quadratic bundle map
\[ \tau(\psi)(\cdot) = \langle cl(\cdot)\psi, \psi \rangle \]
An appropriate change of variables recovers the usual Seiberg-Witten equations (with perturbations) that appear in [18].

Remark 3.1. We have suppressed additional “abstract tame perturbations” to these equations required to obtain transversality of the moduli spaces of its solutions (see [18, §10]), because they do not interfere with the analysis presented in this paper. This is further clarified in [16, §2.1] and [28, §3.h Part 5], where the same suppression occurs.

Denote by $\mathfrak{M}(Y, s)$ the set of solutions to (3.1), called (SW) monopoles. A solution is reducible if its spinor component vanishes, and is otherwise irreducible. The monopoles freely generate the monopole Floer chain complex $\hat{CM}^*(Y, \lambda, s, J, r)$. The chain complex differential will not be reviewed here. Denote by $\hat{CM}_L^*(Y, \lambda, s, J, r)$ the submodule generated by irreducible monopoles $c$ with energy
\[ E(c) := i \int_Y \lambda \wedge F_A < 2\pi L \]
When $r$ is sufficiently large, $\hat{CM}_L^*(Y, \lambda, s, J, r)$ is a subcomplex of $\hat{CM}^*(Y, \lambda, s, J, r)$ and its homology $\hat{HM}_L^*(Y, \lambda, s, J, r)$ is well-defined and independent of $r$ and $\mu$. Taking the direct
limit over \( L > 0 \), we recover the ordinary \( \overline{HM}^*(Y, s) \) in \([18]\) which is independent of \( \lambda \) and \( J \). The total sum
\[
\overline{HM}^*(Y) := \bigoplus_{s \in \text{Spin}^c(Y)} \overline{HM}^*(Y, s)
\]
has an absolute grading by homotopy classes of oriented 2-plane fields on \( Y \) (see \([18, \S 28]\) and \([14, \S 3]\)), the set of which is denoted by \( J(Y) \). There is a transitive \( \mathbb{Z} \)-action on \( J(Y) \), specified already in Section \( 2.3\).

3.2. Symplectic cobordisms. Let \((X, \omega)\) be a strong symplectic cobordism between (possibly disconnected or empty) closed oriented contact 3-manifolds \((Y_\pm, \lambda_\pm)\). Due to the choice of metric \( g_\pm \) on \( Y_\pm \) in Section \( 3.1\) (and following \([16, \S 4.2]\)), we do not extend \( \omega \) over \( X \) using \( d(e^s \lambda_\pm) \) on the ends \((-\infty, 0) \times Y_- \) and \([0, \infty) \times Y_+ \). Instead, we extend \( \omega \) using \( d(e^{2s} \lambda_\pm) \) as follows. Fix a smooth increasing function \( \phi_- : (-\infty, \varepsilon] \to (-\infty, \varepsilon] \) with \( \phi_-(s) = 2s \) for \( s \leq \frac{\varepsilon}{2} \) and \( \phi_- \) for \( s > \frac{\varepsilon}{2} \), and fix a smooth increasing function \( \phi_+ : [-\varepsilon, \infty) \to [-\varepsilon, \infty) \) with \( \phi_+(s) = 2s \) for \( s \geq -\frac{\varepsilon}{2} \) and \( \phi_+ \) for \( s \leq -\frac{\varepsilon}{2} \), where \( \varepsilon > 0 \) is such that \( \omega = d(e^{2s} \lambda_\pm) \) on the \( \varepsilon \)-collars of \( Y_\pm \). Then the desired extension is
\[
\tilde{\omega} := \begin{cases} 
  d(e^{\phi_-} \lambda_-) & \text{on } (-\infty, \varepsilon] \times Y_- \\
  \omega & \text{on } X \setminus \left( [(0, \varepsilon] \times Y_- \right) \cup \left( [-\varepsilon, 0] \times Y_+ \right) \\
  d(e^{\phi_+} \lambda_+) & \text{on } [-\varepsilon, \infty) \times Y_+
\end{cases}
\]
Now choose a cobordism-admissible almost complex structure \( J \) on \((X, \tilde{\omega})\). Following \([16, \S 4.2]\), we equip \( X \) with a particular metric \( g \) so that it agrees with the product metric with \( g_\pm \) on the ends \((-\infty, 0) \times Y_- \) and \([0, \infty) \times Y_+ \) and so that \( \tilde{\omega} \) is self-dual. Finally, define
\[
\hat{\tilde{\omega}} := \sqrt{2} \tilde{\omega} / |\tilde{\omega}|_g
\]
and note that \( J \) is still cobordism-admissible.

The 4-dimensional gauge-theoretic scenario is analogous to the 3-dimensional scenario. View a spin-c structure \( s \) on \( X \) as an isomorphism class of a pair \((S, \text{cl})\) consisting of a Hermitian vector bundle \( S = S_+ \oplus S_- \), where \( S_\pm \) have rank 2, and Clifford multiplication \( \text{cl} : TX \to \text{End}(S) \) such that \( \text{cl}(v) \) exchanges \( S_+ \) and \( S_- \) for each \( v \in TX \). We refer to \( S_+ \) as the positive spinor bundle and its sections as (positive) spinors. The set \( \text{Spin}^c(X) \) of spin-c structures is an affine space over \( H^2(X; \mathbb{Z}) \), and we denote by \( c_1(s) \) the first Chern class of \( S_+ = \text{det } S_- \). A spin-c connection on \( S \) is equivalent to a Hermitian connection \( A \) on \( \text{det } S_+ \) and defines a Dirac operator \( D_A : \Gamma(S_+) \to \Gamma(S_+) \).

A spin-c structure \( s \) on \( X \) restricts to a spin-c structure \( s|_{Y_\pm} \) on \( Y_\pm \) with spinor bundle \( S_{Y_\pm} := S_+|_{Y_\pm} \) and Clifford multiplication \( \text{cl}_{Y_\pm}(\cdot) := \text{cl}(v)^{-1} \text{cl}(\cdot) \), where \( v \) denotes the outward-pointing unit normal vector to \( Y_+ \) and the inward-pointing unit normal vector to \( Y_- \). There is a canonical way to extend \( s \) over \( X \), and the resulting spin-c structure is also denoted by \( s \). There is a canonical decomposition \( S_+ = E \oplus K^{-1}E \) into \( \mp 2i \) eigenbundles of \( \text{cl}(\hat{\tilde{\omega}}) \), where \( K \) is the canonical bundle of \((X, J)\) and \( \text{cl}_+ : \wedge^2_+ T^*X \to \text{End}(S_+) \) is the projection of Clifford multiplication onto \( \text{End}(S_+) \). This agrees with the decomposition of \( S_{Y_\pm} \) on the ends of \( X \).

---

\(^5\)This convention is opposite to that used in \([18]\).
The symplectic form \( \omega \) picks out the canonical spin-c structure \( s_\omega = (s_\omega, \text{cl}) \), namely that for which \( E \) is trivial, and the \( H^2(X; \mathbb{Z}) \)-action on \( \text{Spin}^c(X) \) becomes a canonical isomorphism. There is a unique connection \( A_{K^{-1}} \) on \( K^{-1} \) such that its Dirac operator annihilates the spinor \( (1, 0) \in \Gamma((s_\omega)_+) \), and we henceforth identify a spin-c connection with a Hermitian connection \( A \) on \( E \).

In this paper, a configuration \( \mathfrak{d} \) refers to a gauge-equivalence class of a pair \( (A, \Psi) \) under the gauge group \( C^\infty(X, S^1) \)-action. A connection \( A \) on \( \text{det} \mathbb{S}_+ \) is in temporal gauge on the ends of \( X \) if

\[
\nabla_A = \frac{\partial}{\partial s} + \nabla_{A(s)}
\]

on \((-\infty, 0] \times Y_- \) and \([0, \infty) \), where \( A(s) \) is a connection on \( \text{det} \mathbb{S}_Y \) depending on \( s \). Connections can be placed into temporal gauge by an appropriate gauge transformation.

Fix suitably generic exact 2-forms \( \mu_\pm \in \Omega^2(Y_\pm) \) as in Section 3.1, a suitably generic exact 2-form \( \mu \in \Omega^2(X) \) that agrees with \( \mu_\pm \) on the ends of \( X \) (with \( \mu_\pm \) denoting its self-dual part), and a positive real number \( r \in \mathbb{R} \). Taubes’ perturbed Seiberg-Witten equations for a configuration \( \mathfrak{d} \) are

\[
(D_A \Psi = 0, \quad F_A^+ = \frac{r}{2}(\rho(\Psi) - i\omega) - \frac{1}{2}F_{A_{K^{-1}}} + i\mu_+)
\]

where \( F_A^+ \) is the self-dual part of the curvature of \( A \) and \( \rho : \mathbb{S}_+ \to \bigwedge^2_+ T^* X \) is the quadratic bundle map

\[
\rho(\Psi)(\cdot, \cdot) = -\frac{1}{2} \langle [\text{cl}(\cdot)], \text{cl}(\cdot)] \Psi, \Psi \rangle
\]

Similarly to the 3-dimensional equations, there are additional “abstract tame perturbations” which have been suppressed (see [18, §24.1]). Given monopoles \( c_\pm \) on \( Y_\pm \), denote by \( \mathcal{M}(c_- \times X, c_+; s) \) the set of solutions to (3.2) which are asymptotic to \( c_\pm \) (in temporal gauge on the ends of \( X \)), called SW instantons.

Similarly to ECH, an “index” is associated with each SW instanton, namely the local expected dimension of the moduli space of SW instantons. Denote by \( \mathcal{M}_k(c_- \times X, c_+; s) \) the subset of elements in \( \mathcal{M}(c_- \times X, c_+; s) \) that have index \( k \).

3.3. Taubes’ isomorphisms. With \( \mathbb{Z}/2\mathbb{Z} \) coefficients, there is a canonical isomorphism of relatively graded modules

\[
ECH_*(Y, \xi, \Gamma) \cong \hat{HM}^{-j}(Y, s_\xi + \text{PD}(\Gamma))
\]

which also preserves the absolute gradings by homotopy classes of oriented 2-plane fields

\[
ECH_j(Y, \xi) \cong \hat{HM}^j(Y)
\]

where \( j \in J(Y) \). This isomorphism is constructed on the \( L \)-filtered chain level.

**Theorem 3.2** ([28, Theorem 4.2]). Fix \( L > 0 \) and a generic \( L \)-flat pair \( (\lambda, J) \) on \( (Y, \xi) \). Then for \( r \) sufficiently large and \( \Gamma \in H_1(Y; \mathbb{Z}) \), there is a canonical bijection from the set of generators of \( \text{ECH}_L^*(Y, \lambda, \Gamma, J) \) to the set of generators of \( \text{CM}_L^*(Y, \lambda, s_\xi + \text{PD}(\Gamma), J, r) \).

The image of an admissible orbit set \( \Theta \) under this bijection will be denoted by \( c_\Theta \), and is an irreducible SW monopole that solves Taubes’ perturbed Seiberg-Witten equations (3.1).
4. Floer Theory

Fix an asymptotically standard near-symplectic form $\omega$ on $X^*$ having $N \geq 0$ untwisted zero-circles and $N^\sigma \geq 0$ twisted zero-circles; remember that $N$ is even. Let $X^o$ denote the compact submanifold with $S^3$ boundary such that $\omega$ is standard on $X^* - X^o$ (view $X^o$ as the manifold obtained from $X$ by removing a small standard 4-ball); note that $\omega|_{\partial X^o} = \omega_{\text{std}}|_{S^3} = d\lambda_{\text{std}}$ with $\lambda_{\text{std}} = y_2dx_2 - y_1dx_1$. Let $N$ denote the union of arbitrarily small tubular neighborhoods of all components of $\omega^{-1}(0) \subset X^*$, so it is diffeomorphic to the disjoint union of $N + N^\sigma$ copies of $S^1 \times B^3$.

Consider the relative homology class

$$A_1 \in H_2(X^*,\omega^{-1}(0);\mathbb{Z}) \cong H_1(\omega^{-1}(0);\mathbb{Z})$$

uniquely specified by

$$\partial A_1 = 1 := (1, \ldots, 1) \in \mathbb{Z}^{N+N^\sigma} \cong H_1(\omega^{-1}(0);\mathbb{Z})$$

under the long exact sequence of the pair $(X^*,\omega^{-1}(0))$. Equivalently, view $A_1$ as the corresponding relative class

$$A_1 \in H_2(X^o - N, \partial N;\mathbb{Z}) \cong H_1(\partial N;\mathbb{Z})$$

uniquely specified by

$$\partial A_1 = 1 := (1, \ldots, 1) \in \mathbb{Z}^{N+N^\sigma} \cong H_1(\partial N;\mathbb{Z})$$

under the long exact sequence for the pair $(X^o - N, \partial N)$\footnote{The map $H_2(\partial N;\mathbb{Z}) \to H_2(X^o - N;\mathbb{Z})$ in the long exact sequence is an isomorphism, as seen using the Mayer-Vietoris sequence for the union $(X^o - N) \cup N = X^o$.}.

**Lemma 4.1.** The tubular neighborhood $N$ may be chosen in such a way that $(X^o - N, \omega)$ is a strong symplectic cobordism from $(S^3, \lambda_{\text{std}})$ to $N$ copies of $(S^1 \times S^2, \lambda_{\text{ns}})$ and $N^\sigma$ copies of $(S^1 \times S^2, \lambda^\sigma_{\text{ns}})$. Here, $\lambda_{\text{ns}}$ and $\lambda^\sigma_{\text{ns}}$ are overtwisted contact forms (with different contact structures) whose orbits of symplectic action less than $\rho(A_1)$ are all $\rho(A_1)$-flat and are either hyperbolic or $\rho(A_1)$-positive elliptic.

We have yet to define the quantity $\rho(A_1) \in \mathbb{R}$ and the adjective “$\rho(A_1)$-positive” (but the notion of “flatness” was clarified in Section 2.4). In order to obtain well-defined counts of pseudoholomorphic curves in $X^o - N$ which represent the relative class $A_1$, we need to ensure a uniform bound on their energy as well as a bound on the symplectic action of their orbit sets, and we need to guarantee transversality of the relevant moduli spaces of curves (specifically, to rule out negative ECH index curves). The quantity $\rho(A_1)$ provides the bounds, and the adjective “$\rho(A_1)$-positive” ultimately ensures transversality – we will not define this adjective here (but see \cite{Chernov} §3.2).

**Definition 4.2.**

$$\rho(A_1) := \int_{\Sigma} \omega - \int_{\partial \Sigma \cap S^3} \lambda_{\text{std}} + \sum_{k=1}^N \int_{\partial \Sigma \cap (S^1 \times S^2)_k} \lambda_{\text{ns}} + \sum_{j=1}^{N^\sigma} \int_{\partial \Sigma \cap (S^1 \times S^2)} \lambda^\sigma_{\text{ns}}$$

where $u : \Sigma \to X^o - N$ is any smooth map representing $A_1 \in H_2(X^o - N, \partial N;\mathbb{Z})$, such that $\Sigma$ is a compact oriented smooth surface with boundary satisfying $u(\partial \Sigma) \subset \partial(X^o - N)$.\footnote{The map $H_2(\partial N;\mathbb{Z}) \to H_2(X^o - N;\mathbb{Z})$ in the long exact sequence is an isomorphism, as seen using the Mayer-Vietoris sequence for the union $(X^o - N) \cup N = X^o$.}
Note that \( \rho(A_1) \) need not be 0 even though \( \omega \) is an exact 2-form on \( X^o - N \), because a primitive 1-form \( \nu \) (such that \( \omega = d\nu \)) need not agree with our contact forms on any copy of \( S^1 \times S^2 \). We can only arrange that \( \nu|_{S^3} = \lambda_{std} \) since \( H^1(S^3; \mathbb{R}) = 0 \). In the literature, the cobordism \( (X^o - N, \omega) \) is called \textit{weakly exact}.

**Assumption.** \( N^\sigma = 0 \), there are no twisted zero-circles. This is only for convenience to declutter statements; it is possible to include them. Although the near-symplectic Gromov invariants in [11] were not constructed for contractible twisted zero-circles (see the explanation in [11, Appendix]), the construction can be modified to include them using Bao-Honda’s “supersimple” perturbations of \( \lambda_{ns}^\sigma \). The key fact is that multiple covers of holomorphic planes can no longer arise, hence no negative ECH index curves.

Fix a cobordism-admissible almost complex structure \( J \) on the completion \( (X^o - N, \omega) \), as specified in Section 2.2. We now present counts of \( J \)-holomorphic curves which assemble into a well-defined element of tensor products of copies of \( ECH^*(S^1 \times S^2, \xi_{ns}, 1) \). We recall from Section 2.3 that the set \( M_0(\emptyset; X^o - N, \Theta) \) consists of \( J \)-holomorphic currents which have negative ends asymptotic to \( \Theta \) (and no positive ends). Let \( M_0(\emptyset; A_1) \) denote the subset of elements in \( M_0(\emptyset; X^o - N, \Theta) \) which represent the class \( A_1 \). Define the chain

\[
\sum_\Theta \#M_0(\emptyset; \Theta_1; A) \cdot \Theta \in \bigotimes_{k=1}^N ECH^*(S^1 \times S^2, \lambda_{ns}, 1)
\]

where \( \Theta \) indexes over the admissible orbit sets (representing \( 1 \)) – the implicit fact that each moduli space \( M_0(\emptyset; \Theta_1; A) \) is a finite set of points is subsumed in the following theorem.

**Theorem 4.3.** For generic \( J \), the chain (4.1) induces a well-defined element

\[
Gr_{X,\omega} \in \bigotimes_{k=1}^N ECH^*(S^1 \times S^2, \xi_{ns}, 1)
\]

concentrated in a single absolute grading.

**Proof.** The fact that the chain is a cycle follows [11] verbatim, using the conditions granted by Lemma 4.1. The reason we may copy the arguments in [11] is that there are no positive ends of the relevant pseudoholomorphic curves, so we may treat \( (X^o - N, \omega) \) as if it were a symplectic cobordism with only negative boundary components. Note that we do not take a \textit{weighted} count of elements of any \( M_0(\emptyset; \Theta_1; A) \) when defining \( Gr_{X,\omega} \) because there are no non-constant \textit{closed} pseudoholomorphic curves in \( X^o - N \). \( \square \)

**Remark 4.4.** As shown in [11], \( ECH^*(S^1 \times S^2, \xi_{ns}, \Gamma) = 0 \) for \( \Gamma \neq 1 \). This is why we only consider the relative class \( A_1 \) satisfying \( \partial A_1|_{S^1 \times S^2} = 1 \).

**Remark 4.5.** We may define \( Gr_{X,\omega} \) over \( \mathbb{Z} \) by introducing orientations, but see Remark 6.3.

We would hope that \( Gr_{X,\omega} \) only depends on \( X \). In particular, part of this project would involve showing that \( Gr_{X,\omega} \) does not depend on the choice of near-symplectic form, by analyzing the moduli spaces of pseudoholomorphic curves as \( \omega \) deforms and \( X^* - \omega^{-1}(0) \) changes topological type. In this regard, if \( \omega \) is symplectic (i.e. \( N = N^\sigma = 0 \)) then \( Gr_{X,\omega} \) lives in \( ECH_0(\emptyset, 0) \cong \mathbb{Z}/2\mathbb{Z} \) generated by the empty orbit set, or alternatively we can
remove a Darboux ball $D$ from $X^*$ to view $Gr_{X,\omega}$ in $ECH_0(S^3,\xi_{\text{std}}) \cong \mathbb{Z}/2\mathbb{Z}$ generated by the empty orbit set. Then $Gr_{X,\omega} = 1$ because only the empty curve exists, representing $0 \in H_2(X^* - D, S^3; \mathbb{Z}) \cong H_2(X^*; \mathbb{Z}) \cong 0$.

On hindsight it turns out that $Gr_{X,\omega}$ does not depend on $X$ (let alone $\omega$ and $J$), because ECH is related to Seiberg-Witten theory. The rest of this paper will clarify what this means.

5. Relative Seiberg-Witten

We digress to study Seiberg-Witten theory on $X$ and $X^*$. Here, we note that the relative class $A_\sharp$ corresponds to a unique spin-c structure $\mathfrak{s}_0$ on $X^* - \mathcal{N}$ which agrees with the unique spin-c structure on the positive end $[0, \infty) \times S^3$, and it extends over $\mathcal{N}$ as the unique spin-c structure $\mathfrak{s}_0$ on $X^*$.

Consider more generally a closed connected oriented Riemannian 4-manifold $(M,g)$. We can recover Seiberg-Witten theory from Section 3.2 by taking $(Y,\lambda_\pm) = (\varnothing,0)$, ignoring the appearance of $\omega$, and thus ignoring the canonical decomposition of $S_\pm$. Then the set of spin-c structures is only an $H^2(M;\mathbb{Z})$-torsor. A configuration $[A,\Psi]$ solves the (perturbed) Seiberg-Witten equations when

$$D_A \Psi = 0, \quad F_A^+ = \frac{1}{2} \rho(\Psi) + i\mu$$

where $\mu \in \Omega^2_+(M)$ is a self-dual 2-form. The moduli space of such solutions is denoted by $\mathfrak{M}^\mu(M,\mathfrak{s})$, and its (virtual) dimension is given by

$$d(\mathfrak{s}) := \frac{1}{4} \left( c_1(\mathfrak{s})^2 - 2\chi(M) - 3\sigma(M) \right)$$

where $c_1(\mathfrak{s})$ denotes the first Chern class of the spin-c structure’s positive spinor bundle, $\chi(M)$ denotes the Euler characteristic of $M$, and $\sigma(M)$ denotes the signature of $M$.

**Example 5.1.** For $M$ a homotopy 4-sphere there is a unique spin-c structure $\mathfrak{s}_0$, and $d(\mathfrak{s}_0) = -1$. When $M = S^4$ equipped with the round metric, the unperturbed moduli space $\mathfrak{M}^\emptyset(M,\mathfrak{s}_0)$ consists of a single point, the unique reducible Seiberg-Witten solution.

Now instead of a closed 4-manifold, consider our scenario $(X^*,\omega,\mathfrak{s}_0)$ equipped with a metric which has positive scalar curvature on the positive end $[0,\infty) \times S^3$. Then we may formally apply Section 3.2 to the cobordism $(X^0,\omega) : (S^3,\lambda_{\text{std}}) \to (\varnothing,0)$ while ignoring the canonical decomposition of $S_\pm$ on $X^0$, but remembering the decomposition on the positive end $[0,\infty) \times S^3$. Then a configuration $[A,\Psi]$ on $X^0 = X^*$ with respect to $\mathfrak{s}_0$, which is asymptotic to $c_{\omega} \in \mathfrak{M}(S^3,\mathfrak{s}_0)$, solves the $\tilde{\omega}$-perturbed Seiberg-Witten equations when

$$D_A \Psi = 0, \quad F_A^+ = r(\rho(\Psi) - i\tilde{\omega}) + 2i\mu_\star$$

The moduli space of such solutions is denoted by $\mathfrak{M}^{\tilde{\omega}}(X^*,c_{\omega};\mathfrak{s}_0)$. This moduli space is the one which is used to define the relative Seiberg-Witten invariant of $(X^*,\xi_{\text{std}})$ in [17].

We take a moment to elaborate.

---

[7See 27, §4 for a discussion about 17 and 18. In particular, the positive end of $X^*$ is “AFAK” and the equations (5.1) prescribing $\mathfrak{M}^{\tilde{\omega}}(X^*,c_{\omega};\mathfrak{s}_0)$ are those used in 17 by an appropriate change of variables.]
Consider more generally a connected oriented 4-manifold $M$ with nonempty positive boundary, equipped with an oriented contact structure $\xi \to \partial M$ that is compatible with the boundary orientation of $M$. Then the relative Seiberg-Witten invariant

$$SW_{M,\xi}(s) \in \mathbb{Z}$$

is defined for each spin-c structure $s \in \text{Spin}^c(M)$ that extends the canonical spin-c structure $s_\xi$ on $\partial M$, given a choice of “homology orientation” of $(M, \xi)$. We will not review the construction here, but suffice to say that

$$SW_{X^0,\xi_{\text{std}}}(s_0) := \# \mathcal{M}\hat{\omega}(X^*, c_\xi; s_0)$$

because

$$\dim \mathcal{M}\hat{\omega}(X^*, c_\xi; s_0) = 0$$

by [17] Corollary 3.12, Theorem 3.3, Theorem 2.4]. Here, we have made use of the fact that $X^0$ admits a global $\xi_{\text{std}}$-compatible almost complex structure (in particular, it has a canonical homology orientation [17 Appendix]).

**Theorem 5.2.** Given $(X, \omega, s_0)$ as above with the canonical homology orientation of $(X^0, \xi_{\text{std}})$, the signed count of points in $\mathcal{M}\hat{\omega}(X^*, c_\xi; s_0)$ is one. In other words,

$$SW_{X^0,\xi_{\text{std}}}(s_0) = SW_{\mathbb{R}^4,\xi_{\text{std}}}(s_0) = 1$$

and so the relative Seiberg-Witten invariant cannot distinguish homotopy 4-spheres.

**Proof.** Add a compactly supported perturbation to the $\hat{\omega}$-perturbed Seiberg-Witten equations (5.1) on $X^*$, so that the self-dual 2-form $\hat{\omega}$ (defined in Section 3.2) shifts to $\hat{\omega} + \eta$, where

$$\eta :=
\begin{cases}
-\hat{\omega}, & \text{on } X^0 \cup \left([0, \frac{\varepsilon}{2}] \times S^3\right) \\
0, & \text{on } [\varepsilon, \infty) \times S^3 \\
\text{interpolate, on } \left[\frac{\varepsilon}{2}, \varepsilon\right] \times S^3
\end{cases}$$

Then the signed count of points of $\mathcal{M}\hat{\omega}(X^*, c_\xi; s_0)$ is equal to that of the moduli space $\mathcal{M}\hat{\omega}^{\eta}(X^*, c_\xi; s_0)$ (this is a standard result, detailed in [17 §3]). The reason we consider this newly perturbed moduli space is to demonstrate a gluing formula below.

View $X^*$ as a connected sum

$$X^* = X^0 \cup_{S^3} (X^* - X^0) = X \# \mathbb{R}^4$$

Then take a sequence of Riemannian metrics $\{g^k\}_{k \in \mathbb{N}}$ on $X^*$ which “pinch the neck” along $\partial X^0 = S^3$ as $k \to \infty$, and take a corresponding sequence of small perturbations $\{\mu^k\}_{k \in \mathbb{N}}$ appearing in the $(\hat{\omega} + \eta)$-perturbed Seiberg-Witten equations (5.1) which vanish on a small neighborhood of $\partial X^0$ and are independent of $k$ on a slightly larger neighborhood of $\partial X^0$.

The a priori estimates for SW solutions and a removable singularities theorem imply that a sequence of SW solutions on $(X^*, g^k)$ has a subsequence converging away from the neck to SW solutions over $X$ and $\mathbb{R}^4$, yielding the gluing formula

$$\mathcal{M}\hat{\omega}^{\eta}(X^*, c_\xi; s_0) = \mathcal{M}\hat{\omega}^{\infty}(X, s_0) \times \mathcal{M}\hat{\omega}_{\text{std}}(\mathbb{R}^4, c_\xi; s_0)$$

It is not known whether $X$ admits metrics of positive scalar curvature, so we cannot rule out the existence of irreducible unperturbed SW solutions in $\mathcal{M}\hat{\omega}(X, s_0)$. But the virtual dimension of $\mathcal{M}\hat{\omega}^{\infty}(X, s_0)$ is $d(s_0) = -1$, so the irreducible unperturbed SW solutions do
not persist (under the small perturbation $\mu^\infty$) and hence $\mathcal{M}_{\mu^\infty}(X, s_0)$ consists of a single point (corresponding to the unique reducible unperturbed SW solution). Thus

$$\# \mathcal{M}_{\mu^\infty}(X^*, c_\partial; s_0) = \# \mathcal{M}_{\mu^\infty}(\mathbb{R}^4, c_\partial; s_0)$$

The conclusions of the theorem now follow from the fact that $\mathcal{M}_{\mu^\infty}(\mathbb{R}^4, c_\partial; s_0)$ consists of the unique finite-energy SW solution on $\mathbb{R}^4$ asymptotic to the canonical SW solution on $S^3$. This fact also appears as [17, Theorem 1.1].

**Remark 5.3.** The “neck pinching” argument in the proof of Theorem 5.2 also proves that the usual Seiberg-Witten invariants of a closed 4-manifold $M$ with $b_2^+(M) > 1$ are equal to those of $X \# M$ for any homotopy 4-sphere $X$. This folklore result was certainly known to experts; see for example [29].

### 6. Final result and outlook

The following proposition shows that $Gr_{X, \omega}$ equivalently counts SW instantons on a completion of $X^o - N$ and may be viewed as an element of $\bigotimes_{k=1}^N \mathcal{H}^j(S^1 \times S^2, s_{\xi_{\text{std}}} + 1)$. In accord with [11,12] we normalize the $\mathbb{Z}$-grading on each Floer homology factor so that

$$ECH_j(S^1 \times S^2, \xi_{\text{ins}}, 1) \cong \mathcal{H}^j(S^1 \times S^2, s_{\xi_{\text{ins}}} + 1) \cong \begin{cases} \mathbb{Z} / 2\mathbb{Z}, & \text{if } j \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

The theorem proceeding the proposition then computes $Gr_{X, \omega}$.

**Proposition 6.1.** For generic $J$, sufficiently large $r$, and an admissible orbit set $\Theta$ on $(\partial N, \xi_{\text{ins}})$ with action less than $\rho(A_1)$, there is a multi-valued bijection

$$\mathcal{M}_0(\emptyset, \Theta; A_1) \rightarrow \mathcal{M}_0(c_\partial, X^o - N, c_\partial; s_0)$$

such that

$$\# \mathcal{M}_0(\emptyset, \Theta; A_1) = \# \mathcal{M}_0(c_\partial, X^o - N, c_\partial; s_0) \in \mathbb{Z} / 2\mathbb{Z}$$

**Proof.** For the same reason as in the proof of Theorem 4.3, we may follow [12] verbatim. □

**Theorem 6.2.** Given a homotopy 4-sphere $X$ and asymptotically standard near-symplectic form $\omega$ on $X^*$,

$$Gr_{X, \omega} = SW_{X^o, \xi_{\text{std}}}(s_0) \in \mathbb{Z} / 2\mathbb{Z}$$

and so the near-symplectic Gromov invariant $Gr_{X, \omega}$ cannot distinguish homotopy 4-spheres.

**Proof.** From the decomposition $X^o = (X^o - N) \cup N$ and Proposition 6.1 we can repeat the “neck stretching” argument in [12] along $\partial N$, applied to $\mathcal{M}_{\mu^\infty}(X^*, c_\partial; s_0)$, to conclude

$$Gr_{X, \omega} = \# \mathcal{M}_{\mu^\infty}(X^*, c_\partial; s_0) \in \bigotimes_{k=1}^N ECH_0(S^1 \times S^2, \xi_{\text{ins}}, 1) \cong \bigotimes_{k=1}^N \mathcal{H}^0(S^1 \times S^2, s_{\xi_{\text{ins}}} + 1) \cong \mathbb{Z} / 2\mathbb{Z}$$

That is, $Gr_{X, \omega}$ lives in the absolute grading of Floer homology for which it may be identified with an integer modulo 2, and this number is equal to $SW_{X^o, \xi_{\text{std}}}(s_0)$ via Theorem 5.2. □

**Remark 6.3.** Although we work over $\mathbb{Z} / 2\mathbb{Z}$ when relating $Gr_{X, \omega}$ to Seiberg-Witten theory, we expect the same argument to apply over $\mathbb{Z}$ once we figure out how the multi-valued bijections (6.1) intertwine the “coherent orientations” of the moduli spaces (see [12]).
6.1. Related idea. Take \( N = 2 \) for simplicity of discussion. The invariant \( Gr_{X,\omega} \) only counted pseudoholomorphic curves in \((X^0 - \mathcal{N}, \omega)\) which had no positive ends, and so it may be viewed as the image of the generator \([\varnothing] \in ECH_0(S^3, \xi_{\text{std}}, 0) \cong \mathbb{Z}/2\mathbb{Z}\) under the ECH cobordism map

\[
\Phi_0 : ECH_0(S^3, \xi_{\text{std}}, 0) \to ECH_0(S^1 \times S^2, \xi_{\text{ns}}, 1) \otimes ECH_0(S^1 \times S^2, \xi_{\text{ns}}, 1)
\]

Since \( ECH_{2k}(S^3, \xi_{\text{std}}, 0) \cong \mathbb{Z}/2\mathbb{Z} \) for all \( k \geq 0 \) and vanishes otherwise, we may define similar ECH (or monopole Floer) cobordism maps by counting index 0 pseudoholomorphic curves (or SW instantons) with certain positive and negative ends,

\[
\Phi_{2k} : ECH_{2k}(S^3, \xi_{\text{std}}, 0) \to \bigoplus_{i+j=2k} ECH_i(S^1 \times S^2, \xi_{\text{ns}}, 1) \otimes ECH_j(S^1 \times S^2, \xi_{\text{ns}}, 1)
\]

Take \( k = 1 \) for example, so that \( \Phi_2(\text{generator}) \in (\mathbb{Z}/2\mathbb{Z})^3 \) in the only nontrivial gradings \((i, j) \in \{(0, 2), (1, 1), (2, 0)\}\). We would hope that the values in these gradings contain more information than our invariant \( Gr_{X,\omega} = \Phi_0(\text{generator}) \in \mathbb{Z}/2\mathbb{Z} \) in grading \((0, 0)\). Unfortunately, they do not:

\[
\Phi_2(\text{generator}) = (Gr_{X,\omega}, 0, Gr_{X,\omega}) = (1, 0, 1) \in (\mathbb{Z}/2\mathbb{Z})^3
\]

The \((0, 2)\) and \((2, 0)\) gradings reduce to the \((0, 0)\) grading thanks to the U-maps. The U-maps are degree \(-2\) maps \( U : ECH_j(S^1 \times S^2, \xi_{\text{ns}}, 1) \to ECH_{j-2}(S^1 \times S^2, \xi_{\text{ns}}, 1) \) and \( U : ECH_j(S^3, \xi_{\text{std}}, 0) \to ECH_{j-2}(S^3, \xi_{\text{std}}, 0) \), which are isomorphisms for \( j \geq 2 \). We can compose the U-maps on either side of the cobordism

\[
\Phi_{2k-2} \circ U = (U \otimes I) \circ \Phi_{2k} = (I \otimes U) \circ \Phi_{2k}
\]

as explained in \[18\] §3.4 and \[15\] §3.8.

The \((1, 1)\) grading vanishes thanks to the loop-maps. Each loop-map is a degree \(-1\) map \( \Delta_{\gamma} : ECH_j(S^1 \times S^2, \xi_{\text{ns}}, 1) \to ECH_{j-1}(S^1 \times S^2, \xi_{\text{ns}}, 1) \) defined using a generator \( \gamma \in H_1(S^1 \times S^2; \mathbb{Z}) \), which is an isomorphism for odd \( j \geq 1 \) (and satisfies \( \Delta_{\gamma} \circ \Delta_{\gamma'} = 0 \)). We can compose the loop-maps to obtain

\[
(\Delta_{\gamma} \otimes \Delta_{\gamma'}) \circ \Phi_{2k} = 0
\]

because the generators \((\gamma, \gamma') \in H_1(S^1 \times S^2; \mathbb{Z}) \oplus H_1(S^1 \times S^2; \mathbb{Z}) \) become homologous in the cobordism \( X^0 - \mathcal{N} \), as explained in \[18\] §3.4 and \[15\] §3.8.

6.2. Symplectic field theory. We could not build a version of \( Gr_{X,\omega} \) using SFT because the contact homology of any overtwisted contact 3-manifold is trivial \[3, 30\]. That is, a tentative SFT-type invariant would use moduli spaces of curves in the SFT framework and subsequently represent an element of a contact homology \( CH_\ast(\bigcup_{k=1}^N S^1 \times S^2, \xi_{\text{ns}}) = 0 \).

References


C. Gerig, Taming the pseudoholomorphic beasts in $\mathbb{R} \times (\mathbb{S}^1 \times \mathbb{S}^2)$, preprint arXiv:1711.02069.


C. Taubes, Broken dreams seminar: “A cancellation lemma and other dreams about almost symplectic forms”. MSRI (October 22, 2009).


