"RIEMANN–ROCH" FOR PUNCTURED CURVES VIA ANALYTIC PERTURBATION THEORY [SKETCH]

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Abstract. In [Tau96], Taubes proved the Riemann–Roch theorem for compact Riemann surfaces, as a by-product of taking clever perturbations of the Cauchy–Riemann operator in order to define his Gromov invariant for pseudoholomorphic curves. We will do the same for noncompact surfaces, i.e. we will recover the formula for the Fredholm index of a Cauchy–Riemann operator that is asymptotic to nondegenerate asymptotic operators (the formula originally appeared in the thesis of Schwarz [Sch95]).

Contents

1. Setup

Suppose that C is a connected Riemann surface with punctures, and that $E \to C$ is a holomorphic line[1] bundle with fixed trivialization τ near the punctures. Compactify C so that the neighborhood of the punctures are modeled on $[0, \infty) \times S^1$, and for each puncture denote the corresponding circle by γ. Take a Cauchy–Riemann type operator

$$\mathbf{D} : \Gamma(E) \to \Gamma(T^{0,1}C \otimes E)$$

with its asymptotic operators

$$\mathbf{A}_\gamma : \Gamma(\gamma^* E) \to \Gamma(\gamma^* E)$$

for all punctures. With respect to $\tau : \gamma^* E \xrightarrow{\cong} S^1 \times \mathbb{C}$ the asymptotic operator takes the form

$$\mathbf{A}_\gamma = i\partial_t + A_\gamma(t)$$

where $A_\gamma(t)$ is a smooth loop of symmetric matrices. Assume all asymptotic operators are nondegenerate (0 is not an eigenvalue) so that $\mathbf{D}$ is a Fredholm operator.

[1]The technique also works for higher rank bundles: take determinants and use the Splitting Lemma.

Example 1.1. In practice, C is a pseudoholomorphic curve in a 4-dimensional symplectic cobordism between contact 3-manifolds, with punctures asymptotically approaching non-degenerate Reeb orbits. Here, E is its normal bundle and $\mathbf{D}$ is the (normal) deformation operator.

Theorem 1.2 (Schwarz [Sch95])**.** *The Fredholm index of* $\mathbf{D}$ *is given by*

$$\operatorname{ind}(\mathbf{D}) = \chi(C) + 2c_1(E,\tau) + \sum_\gamma CZ_\tau(\mathbf{A}_\gamma)$$

where $CZ_\tau(\mathbf{A}_\gamma)$ *is the Conley–Zehnder index and* $c_1(E,\tau)$ *is the relative 1st Chern class.*

Here is an outline of a novel proof of the index formula, using analytic perturbation theory as in [Kat95]:

(1) Construct an "L-flat approximation" for all $\mathbf{A}_\gamma$, i.e. a suitably nice asymptotic operator which $\mathbf{A}_\gamma$ is homotopic to, such that $\operatorname{ind}(\mathbf{D})$ doesn't vary.
(2) Prove $\operatorname{ind}(\mathbf{D}) = \operatorname{ind}(\mathbf{D} + B)$ for some $B \in \Gamma(T^{0,1}C \otimes E^2)$ whose winding number along the ends of C with respect to τ satisfies $\operatorname{wind}_\tau(B_\gamma) = CZ_\tau(\mathbf{A}_\gamma)$.
(3) Prove $\operatorname{ind}(\mathbf{D} + rB) = \#B^{-1}(0)$ by taking sufficiently large $r \in \mathbb{R}_+$, this being a concentration principle.

We put these steps together, noting that

$$\#B^{-1}(0) = c_1(T^{0,1}C \otimes E^2, \tau) + \operatorname{wind}_\tau(B)$$

by definition, to get

$$\operatorname{ind}(\mathbf{D}) = c_1(T^{0,1}C,\tau) + 2c_1(E,\tau) + \sum_\gamma \operatorname{wind}_\tau(B_\gamma) = \chi(C) + 2c_1(E,\tau) + \sum_\gamma CZ_\tau(\mathbf{A}_\gamma)$$

The proof of (1) is given in [Tau10, Appendix]. The proof of (3) is a regurgitation of [Tau96, §7], because the argument is local in nature. A new contribution is (2), which did not arise in [Tau96] because there were no punctures (hence no asymptotic constraints).

Remark 1.3. The Riemann–Roch formula for the case of closed Riemann surfaces follows by additivity of the Fredholm index with respect to gluing the punctured surfaces at their cylindrical ends. The Riemann–Roch formula can be recasted in the following form: the index equals twice the degree of the bundle plus the Euler characteristic of the surface. When we introduce punctures, the definition of the degree (as a relative 1st Chern class) requires some choice of trivialization of the bundle along the ends of the surface, and a different choice might change the degree. To get an invariant, a 'boundary correction' term is needed to compensate – this is the Conley–Zehnder index.

Remark 1.4. In using the Conley–Zehnder index, we suppose the circles γ are nondegenerate (possibly multiply covered) Reeb orbits.

Acknowledgements. The main idea stems from a chat with Cliff Taubes on how to extend his transversality results from closed symplectic manifolds to symplectic cobordisms, and builds off of joint work with Chris Wendl on closed symplectic manifolds [GW17]. Subsequently, a detailed version of this result (with a different style) now appears in Wendl's book on SFT [Wen, §5]. This paper forms part of the author's Ph.D. thesis.

2. L-flat approximation

The point here is to reduce everything WLOG to the case that our Fredholm operator $\mathbf{D}$ has asymptotic operators with "nice" explicit descriptions, so that we can do hands-on computations in the subsequent section. As explained in [Tau10, Appendix], there is a homotopy through nondegenerate operators (with fixed Conley–Zehnder index) from a given asymptotic operator $\mathbf{A}_\gamma$ to an *L-flat asymptotic operator*, specified below. The homotopy extends to a homotopy of Fredholm operators of constant index from $\mathbf{D}$ to a Cauchy–Riemann type operator with L-flat asymptotics, and we abuse notation by denoting the resulting operator $\mathbf{D}$ with asymptotic operators $\{\mathbf{A}_\gamma\}$.

We fix a positive real number L greater than the symplectic action of γ. We also fix the trivialization τ so that the Conley–Zehnder indices are 1 for embedded elliptic and embedded negative hyperbolic orbits, and 0 for embedded positive hyperbolic orbits. For multiply covered orbits we can pull back the asymptotic operator of the underlying embedded orbit, so we assume γ is embedded. The result of how an L-flat $A_\gamma(t)$ acts on $\eta(t) \in \Gamma(\gamma^* E)$ is given as follows (see [Tau10, Lemma 2.3]):

(1) If γ is an embedded elliptic orbit (irrational rotation number θ) then

$$\eta \mapsto \theta\, \eta$$

(2) If γ is an embedded positive hyperbolic orbit (rotation number 0) then for some sufficiently small positive constant ε

$$\eta \mapsto \varepsilon i\, \bar{\eta}$$

(3) If γ is an embedded negative hyperbolic orbit (rotation number 1) then for some sufficiently small positive constant ε

$$\eta \mapsto \frac{1}{2}\eta + \varepsilon i e^{it} \bar{\eta}$$

3. Spectral flow for the perturbed asymptotic operator

This section prescribes the asymptotic limits (with respect to the fixed trivialization τ) of the desired section $B \in \Gamma(T^{0,1}\Sigma \otimes E \otimes E)$. All asymptotic operators are assumed to be L-flat from now on. The nonzero complex numbers $\mathbb{C}^*$ will be identified as a subset of $GL_2(\mathbb{R})$ via $a + bi \mapsto \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$.

The perturbed operator

$$\mathbf{D}_r := \mathbf{D} + rB$$

will remain Fredholm of constant index for all $r \in \mathbb{R}$ if and only if each perturbed asymptotic operator

$$\mathbf{A}_{\gamma,r} := \mathbf{A}_\gamma + rB_\gamma(t)$$

remains nondegenerate. In other words, $\mathbf{A}_{\gamma,r}$ must not have any spectral flow as a function of r.

Theorem 3.1. *$\mathbf{A}_{\gamma,r}$ has no spectral flow if and only if $\mathrm{wind}_\tau(B_\gamma) = CZ_\tau(\mathbf{A}_\gamma)$, with the additional constraint that $B_\gamma(0)$ is not purely complex whenever γ is a positive hyperbolic orbit or an even cover of a negative hyperbolic orbit.*

Proof. As the focus is now on a single asymptotic operator, the subscript γ will be dropped. The Reeb orbit γ is an m-fold cover of some embedded orbit α, and is parametrized by $t \in \mathbb{R}/2\pi m\mathbb{Z}$. Note that the perturbed asymptotic operators act on complex-valued sections $\eta(t) \in L^2_1(\mathbb{R}/2\pi m\mathbb{Z}, \mathbb{C})$, and the study of A_γ reduces to the study of A_α by pull-back via $\mathbb{R}/2\pi m\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$.

Suppose α is **elliptic** with rotation number $0 < \theta < 1$. Then $CZ_\tau(\gamma) = 2\lfloor m\theta \rfloor + 1$ and

$$A(t) \cdot \eta(t) = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} \eta(t) = \theta\, \eta(t).$$

We may assume that B satisfies $\mathrm{wind}_\tau(B_\gamma) = n \in \mathbb{Z}$ and takes the form

$$B(t) \cdot \eta(t) = B_0 e^{int/m} \bar{\eta}(t), \quad B_0 \in \mathbb{C}^*.$$

We now look for solutions to $\mathbf{A}_r \eta = 0$. This first-order complex differential equation is

$$i\frac{d\eta}{dt} + \theta\eta + rB_0 e^{int/m}\bar{\eta} = 0.$$

Use the Fourier expansion $\eta(t) = \sum_{k\in\mathbb{Z}} a_k e^{ikt/m}$ and compare modes to obtain the recurrence relation

$$(\theta - \frac{k}{m})a_k + rB_0\bar{a}_{n-k} = 0.$$

Swapping $k \to n-k$ gives another recurrence relation (concerning the same coefficients a_k and a_{n-k}), and these two relations combine to give the constraint

$$r^2 = \frac{1}{|B_0|^2}(\theta - \frac{k}{m})(\theta - \frac{n-k}{m}).$$

If $r^2 > 0$ then there is spectral flow, and if $r^2 < 0$ then there is no spectral flow. Indexing over $l \in \{0, \dots, m-1\}$, suppose $\frac{l}{m} < \theta < \frac{l+1}{m}$, so that $CZ_\tau(\gamma) = 2l+1$. If $n \geq 2l+2$ then there is spectral flow (choosing $k = l+1$) because $(\theta - \frac{k}{m})$ and $(\theta - \frac{n-k}{m})$ are both negative. If $n \leq 2l$ then there is spectral flow (choosing $k = l$) because $(\theta - \frac{k}{m})$ and $(\theta - \frac{n-k}{m})$ are both positive. If $n = 2l+1$ then there is no spectral flow because $(\theta - \frac{k}{m})$ and $(\theta - \frac{n-k}{m})$ are of opposite sign for any k (if $k \leq l$ then $\frac{n-k}{m} \geq \frac{l+1}{m}$, and if $k \geq l+1$ then $\frac{n-k}{m} \leq \frac{l}{m}$). We're done.

Suppose α is **positive hyperbolic** with rotation number 0. Then $CZ_\tau(\gamma) = 0$ and

$$A(t) \cdot \eta(t) = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} \eta(t) = \varepsilon i\, \bar{\eta}(t)$$

for $\varepsilon \in \mathbb{R}_+$ small. Again we may assume that B satisfies $\mathrm{wind}_\tau(B_\gamma) = n \in \mathbb{Z}$ and takes the form

$$B(t) \cdot \eta(t) = B_0 e^{int/m} \bar{\eta}(t), \quad B_0 \in \mathbb{C}^*.$$

The first-order complex differential equation to solve is now

$$i\frac{d\eta}{dt} + (rB_0 e^{int/m} + \varepsilon i)\bar{\eta} = 0.$$

Use the Fourier expansion $\eta(t) = \sum_{k\in\mathbb{Z}} a_k e^{ikt/m}$ to obtain the recurrence relation

$$\frac{-k}{m}a_k + rB_0\bar{a}_{n-k} + \varepsilon i\bar{a}_{-k} = 0.$$

There is spectral flow when $n \neq 0$ (see Appendix 5), but when $n = 0$ the recurrence relation contradicts itself (unless B_0 is purely complex). Indeed,

$$\frac{-k}{m}a_k + (rB_0 + \varepsilon i)\bar{a}_{-k} = 0$$

which by swapping $k \to -k$ gives another recurrence relation (concerning the same coefficients a_k and a_{-k}), and these two relations combine to give the constraint

$$|B_0|^2r^2 + (\bar{B}_0 - B_0)\varepsilon ir + \varepsilon^2 = -\frac{k^2}{m^2}.$$

This is never satisfied unless $B_0 = b_0 i$ (where $b_0 \in \mathbb{R}^*$) and $k = 0$, in which case $r = -\frac{\varepsilon}{b_0}$ (hence $\eta(t) = 1$ is in the kernel). We're done.

Suppose α is **negative hyperbolic** with rotation number 1. Then $CZ_\tau(\gamma) = m$ and

$$A(t)\cdot\eta(t) = \frac{1}{2}\eta(t) + \varepsilon ie^{it}\bar{\eta}(t)$$

for $\varepsilon \in \mathbb{R}_+$ small. Again we may assume that B satisfies $\mathrm{wind}_\tau(B_\gamma) = n \in \mathbb{Z}$ and takes the form

$$B(t)\cdot\eta(t) = B_0 e^{int/m}\bar{\eta}(t), \quad B_0 \in \mathbb{C}^*.$$

The first-order complex differential equation to solve is now

$$i\frac{d\eta}{dt} + \frac{1}{2}\eta + (rB_0 e^{int/m} + \varepsilon ie^{it})\bar{\eta} = 0.$$

Use the Fourier expansion $\eta(t) = \sum_{k\in\mathbb{Z}} a_k e^{ikt/m}$ to obtain the recurrence relation

$$(\frac{1}{2} - \frac{k}{m})a_k + rB_0\bar{a}_{n-k} + \varepsilon i\bar{a}_{m-k} = 0.$$

There is spectral flow when $n \neq m$ (see Appendix 5), but when $n = m$ the recurrence relation contradicts itself (unless B_0 is purely complex and m is even). Indeed,

$$(\frac{1}{2} - \frac{k}{m})a_k + (rB_0 + \varepsilon i)\bar{a}_{m-k} = 0$$

which by swapping $k \to m - k$ gives another recurrence relation (concerning the same coefficients a_k and a_{m-k}), and these two relations combine to give the constraint

$$|B_0|^2r^2 + (\bar{B}_0 - B_0)\varepsilon ir + \varepsilon^2 = -(\frac{1}{2} - \frac{k}{m})^2.$$

This is never satisfied unless $B_0 = b_0 i$ (where $b_0 \in \mathbb{R}^*$) and $k = \frac{m}{2}$ with m even, in which case $r = -\frac{\varepsilon}{b_0}$ (hence $\eta(t) = e^{it/2}$ is in the kernel). We're done. □

4. Localization argument

Using homotopy theory (i.e. obstruction theory), we can extend the choices of B_γ given in Theorem 3.1 to a global $B \in \Gamma(T^{0,1}C \otimes E^2)$ having nondegenerate zeros. Note that its zeros are bounded away from the punctures due to the existence of the trivialization τ.

Theorem 4.1. *For $r \gg 0$ and B as above,* $\operatorname{ind}(\mathbf{D} + rB) = \#B^{-1}(0)$.

Proof. Using a sequence of cutoff functions to invoke a noncompact version of Stokes' theorem, there is a Bochner–Weitzenböck formula for $||\mathbf{D}\eta + rB\bar{\eta}||^2$ in terms of the quantities $r^2 \int_C |B\bar{\eta}|^2$ and $r\Re \int_C \partial B \cdot \bar{\eta}^2$ (here, $\Re$ means the real part). The same localization argument as in [Tau96, §7] shows that the positive zeros of B contribute to the kernel of $\mathbf{D} + rB$ for sufficiently large $r \in \mathbb{R}_+$, while the negative zeros of B contribute to the cokernel. Roughly speaking, the support of any sequence $\eta_r \in \operatorname{Ker}(\mathbf{D} + rB)$ must concentrate near the (positive) zeros of B. The result follows. □

Remark 4.2. This proof hinges on the asymptotic behavior of the perturbation object B. If B had compact support or exponentially decayed to zero at the punctures of C, we could have a sequence $\{(\eta_r, r)\} \in \Gamma(E) \oplus \mathbb{R}_+$ with $\eta_r \in \operatorname{Ker}(\mathbf{D} + rB)$ and $r \to \infty$ such that $\sup_C(\eta_r)$ "runs away" towards the punctures.

5. Appendix

We supply the missing computation in Section 3 but only in the case that γ is an embedded positive hyperbolic orbit and the perturbation is $B_\gamma(t) = e^{it}$ (so $n = m = B_0 = 1$), because the other cases are handled in the same way. Solving for the kernels of the perturbed asymptotic operators $\mathbf{A}_{\gamma,r}$ is equivalent to solving the following 1st order complex differential equations for smooth functions $\eta : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{C} - \{0\}$ defined on the circle,

$$i\frac{\partial \eta}{\partial t} + (re^{it} + \varepsilon i)\bar{\eta} = 0$$

where $\varepsilon \in \mathbb{R}_+$ is sufficiently small and fixed and $r \in \mathbb{R}_+$ is a nonzero positive real parameter. We would like to find the set of all such r for which $\operatorname{Ker} \mathbf{A}_{\gamma,r} \neq \varnothing$, as well as $\dim_{\mathbb{R}} \operatorname{Ker} \mathbf{A}_{\gamma,r}$, though it suffices to prove the existence of one such r.

There are at least three ways to do this. The result is a unique solution to $\mathbf{A}_{\gamma,r}\eta = 0$ which occurs around $r \approx \sqrt{\varepsilon}$. Two methods are given in a MathOverflow post [htt].

5.1. **Alternate solution:** When γ is a (cover of a) hyperbolic orbit, the perturbed asymptotic operator $\mathbf{A}_{\gamma,r}$ has trivial kernel when $r = 0$ and $r = r_* \gg 0$ (use the Bochner–Weitzenböck formula to see this). The Conley–Zehnder index is a $\mathbb{Z}$-valued invariant of homotopy classes of such operators, and the *net* spectral flow of $r \in [0, r_*] \mapsto \mathbf{A}_{\gamma,r}$ is

$$CZ_\tau(\mathbf{A}_{\gamma,r_*}) - CZ_\tau(\mathbf{A}_\gamma) = \operatorname{wind}_\tau(B_\gamma) - CZ_\tau(\gamma)$$

Thus when $\operatorname{wind}_\tau(B_\gamma) \neq CZ_\tau(\gamma)$ there must exist at least one $r > 0$ with $\operatorname{Ker} \mathbf{A}_{\gamma,r} \neq 0$.

5.2. **Previous attempt:** My original attempt was to Fourier expand $\eta(t) = \sum_{k\in\mathbb{Z}} a_k e^{ikt}$ and obtain the recurrence relation

$$-ka_k + r\bar{a}_{1-k} + \varepsilon i\bar{a}_{-k} = 0$$

I need a specific collection $\{a_k\} \subset \mathbb{C}$ which solves this (for some $r > 0$), but I get stuck. What I get at the least is $r = \varepsilon i \frac{a_0}{a_1}$ (and subsequently $\frac{a_0}{a_1} \in i\mathbb{R}_-$). The coefficients a_k will "blow up" as $k \to \infty$ if not chosen carefully. The recurrence relation combines with another relation (obtained by replacing $k \mapsto 1-k$) to yield

$$a_k = \frac{1}{r}\Big(i\varepsilon a_{k-1} - (k-1)\bar{a}_{1-k}\Big) = \frac{a_1}{a_0}\Big(a_{k-1} + \frac{i(k-1)}{\varepsilon}\bar{a}_{1-k}\Big) \tag{5.1}$$

$$\bar{a}_{-k} = \frac{1}{i\varepsilon}\Big(i\varepsilon a_{k-1} - (r^2 + k(k-1))\bar{a}_{1-k}\Big) = a_{k-1} + i\Big(\frac{k(k-1)}{\varepsilon} - \varepsilon\frac{a_0^2}{a_1^2}\Big)\bar{a}_{1-k} \tag{5.2}$$

This is too complicated for me to predict the next coefficient: We are iteratively adding and conjugating coefficients, flipping their real and imaginary parts via multiplication by i, and putting small ε in both numerators and denominators. I cannot parse whether these are helpful or harmful, and I have no indication what the values a_0 and a_1 should be to have sufficient decay in the norms of the coefficients.

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