Introduction to the Reeh-Schlieder Theorem and Entanglement Entropy In QFT

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Today's Topics:

(1) The Reeh-Schlieder theorem (1961), which is the basic result, showing that entanglement is unavoidable in quantum field theory.

(2) Relative entanglement entropy in quantum field theory.

The Reeh-Schlieder Theorem:

Consider a QFT in *Minkowski* spacetime M, with a *Hilbert* space H. We assume that the vacuum state (vector) Ω, lives in H.

For a small open set $U \subset M$, there is a bounded algebra of local operators A_U supported in U acting on the vacuum vector (which is in fact the Haag-Kastler vacuum representation) Ω. This action produces state of the form:

$$\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)|\Omega\rangle = A_U\,|\Omega\rangle,$$

for all $x_i \in U$ and for any $n = 1,2,3,\ldots$

Note- $\varphi(x_i)$ are field operators defined in A_U.

The Reeh-Schlieder theorem states that <u>every</u> arbitrary state in H can be approximated by $A_U\,|\Omega\rangle$.
i.e. states $A_U\,|\Omega\rangle$ are dense in H.

The Reeh-Schlieder theorem: Every arbitrary state in H can be approximated by $A|\Omega\rangle$.

Quick Sketch of the proof:

If this statement were false (i.e. states $\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)|\Omega\rangle$ are not dense in H), then there exists a vector X in H such that it is orthogonal to $\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)|\Omega\rangle$.
i.e.

$$\langle X|\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)|\Omega\rangle = 0,$$

for all $x_i \in U$.

→ To prove the statement is true, we need to show that for all $x_i \in M$ (*Minkowski* space), $\langle X|\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)|\Omega\rangle = 0$.
So $|X\rangle$ must be 0 → is a null vector → our statement is true!

Proof of the Reeh-Schlieder Theorem:

Given that, $\langle X|\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)|\Omega\rangle = 0$, for all $x_i \in U$.

Suppose that, t is a future-pointing time-like vector and u is any real #, we can time-like shift the n-th point, $x_n \in U'$:

$$x_n \rightarrow x'_n = x_n + tu$$

Now the correlation function for x'_n is:

$$g(u) = \langle X\,|\varphi(x_1)\cdots\varphi(x_{n-1})\varphi(x_n + tu)|\Omega\rangle$$

We can also write $x'_n = e^{i\mathcal{H}u}\, x_n\, e^{-i\mathcal{H}u}$.
Where $\mathcal{H}$ is the Hamiltonian vanishing the vacuum (since $e^{-i\mathcal{H}\boldsymbol{u}}$ is a bounded operator.)

So we can write g(u) as:

$$g(u) = \langle X \,|\varphi(x_1) \cdots \varphi(x_{n-1}) e^{i\mathcal{H}u}\, \varphi\,(x_n)\, e^{-i\mathcal{H}u} |\Omega\rangle$$
$$= \langle X \,|\varphi(x_1) \cdots \varphi(x_{n-1}) e^{i\mathcal{H}u}\, \varphi\,(x_n)\, |\Omega\rangle$$

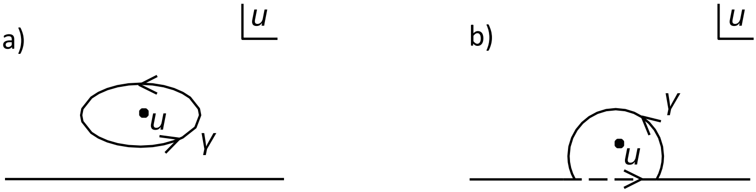
Because $\mathfrak{H}$ is bounded below by *0*, the operator $e^{i\mathcal{H}u}$ is holomorphic for *u* in the upper half of u-plane → implies, g(u) is holomorphic for *u* in the upper half of u-plane and will vanish on the real axis.

Note:

- For infinitesimally small *u*, $x'_n \rightarrow x_n$ and so *g(u)* goes to *0*.
- For *u = 0, g(u) = 0*.

As shown in **a)** we can Taylor expand around a fixed u (in u-plane). Since g(u) is holomorphic in the upper half of u-plane, it can be computed by a Cauchy integral formula: any contour *γ* in the upper half-plane can be used to compute g(u) for u in the interior of the contour.

If g(u) is continuous on the boundary of the upper-half plane (which it is in here) then we can take *γ* to go partly on the boundary, as shown is **b)**. → this part of the contour = 0 (since g(0) = 0).



Since we saw the contour vanishes on the boundary → g(u) remains holomorphic on the boundary so we move down u into the lower half-plane without finding any singularities.
And so as we explained g(u) is identically 0.
Therefore,

$$g(u)= \langle X \,|\varphi(x_1) \cdots \varphi(x_{n-1})\varphi(x'_n)|\Omega\rangle = 0,$$

For $x_1, \ldots, x_{n-1} \in U$ with no restriction on x'_n

One can carry on doing this by introducing a new future-pointing time-like vector t' to shift the last 2 points and so on and can see that at the end:

$$g(u)= \langle X \,|\varphi(x'_1) \cdots \varphi(x'_{n-1})\varphi(x'_n)|\Omega\rangle = 0,$$

for any x_i.
Thus, we showed that for all $x_i \in M$, $\langle X \,|\varphi(x_1) \cdots \varphi(x_{n-1})\varphi(x_n)|\Omega\rangle = 0$.

→ $\varphi(x_1) \cdots \varphi(x_{n-1})\varphi(x_n)|\Omega\rangle \neq 0$,
→ $|X\rangle = 0$,
→ $\varphi(x_1) \cdots \varphi(x_{n-1})\varphi(x_n)|\Omega\rangle$ are dense in H.

Physical Interpretations of The Reeh-Schlieder Theorem:

One might expect that for a localized observable with $A_U \subset M$, states, $A_U|\Omega\rangle$ should be localized in U, i.e. states $A_U|\Omega\rangle$ should look like the vacuum in the causal complement of U.

But the Reeh-Schlieder theorem says every arbitrary state in H can be approximated by $A_U\,|\Omega\rangle$.

Looks Contradictory?!

Let's look into this:

In *Minkowski* spacetime consider a state of the universe such that on some initial time slice, it looks like the vacuum near an open set U.

Now suppose that a planet, "*Marley*", lives in region V, which is at a space-like separated distance from U.

Let's assign an operator J to be the operator creating planet "*Marley*" in region V. The expectation value of J in a state that contains the planet in region V is close to 1, and 0 otherwise.

$$\langle\Omega'|J|\Omega'\rangle \sim 1,$$
$$\langle\Omega''|J|\Omega''\rangle \sim 0.$$

The Reeh-Schlieder theorem states that there is an operator *X* in region *U* that by its action on the vacuum, *XΩ,* it can approximate the state that contains Marley in region *V*.
The "apparent contradiction" comes from that it seems like you can create a planet in region *V* by acting an operator *X* on the vacuum state in region *U.*

Note- By physical creation we mean a Unitary operator (i.e. $X^{\dagger}X = 1$)

$$\Longrightarrow \langle X\Omega|J|X\Omega\rangle = \langle \Omega|X^{\dagger}JX|\Omega\rangle.$$

Since X is supported in *U* and *J* is supported in the spacelike separated region *V*, $X^{\dagger}$ and *J* commute:

$$\Longrightarrow \langle X\Omega|J|X\Omega\rangle = \langle \Omega|JX^{\dagger}X|\Omega\rangle \sim 1.$$

$$\Longrightarrow 1 \sim 1\langle \Omega|JX^{\dagger}X|\Omega\rangle = \langle \Omega|J|\Omega\rangle \sim 0.$$

BUT:

The Reeh-Schlieder theorem does not tell us that *X* could be unitary; it just tells us that there is some *X* in region *U* that can "approximately" "create" planet "*Marley*", in a spacelike separated distant region *V* from *U*.

Remark:

Let U to be an open set in Minkowski spacetime, by the Reeh-Schlieder theorem:

- State $A_U \Omega$ are dense in H_0 →This is described by saying that Ω is a "cyclic" vector for the algebra A_U.
- For any nonzero local operator algebra A_U, $A_U \Omega \neq 0$ → This is described by saying that Ω is a "separating" vector of A_U.

In short:

The Reeh-Schlieder theorem and its corollary say that the vacuum is a cyclic separating vector for A_U.

Important uses of the R-S theorem:

- **Entanglement is unavoidable in QFT**:
  In the vacuum state, operators $X^{\dagger}X$ and J, which are supported in two space-like separated regions, do not commute so we get a non-zero correlation function ═► Entanglement!

Note- This always happen in QFT even in free theory so no contradiction!

- ***The state-operator correspondence in CFT:***
  is also a Reeh-Schlieder property.

2. Introduction to Entanglement Entropy in Quantum Field Theory:

The Reeh-Schlieder theorem involves the entanglement between the degrees of freedom inside an open set *U* in *Hilbert* space and its causal complement *U'*

$$\Longrightarrow H = H_U \otimes H_{U'}$$

However, this is not the case in *QFT*! The *QFT Hilbert* space does not factorize!

Fortunately, there is a mathematical machinery to analyze the entanglement in this situation: Tomita-Takesaki theory.

Relative entropy of two states of a von Neumann algebra is defined in terms of the relative modular operator.
The strict positivity, lower semi-continuity, convexity and monotonicity of relative entropy are proved. [Araki in 1970's]
A relative entropy (also called relative information) is a useful tool in the study of equilibrium states of lattice systems. The relative entropy for (normal faithful) positive linear functional of *Ψ and Φ,* of a von Neumann algebra is defined as,

$$\mathcal{S}_{\Psi/\Phi}(U) = -\langle \Psi | \log \Delta_{\Psi/\Phi} | \Psi \rangle$$

where $\Delta_{\Psi/\Phi}$ is the relative modular operator of cyclic and separating vector representatives *Ψ and Φ.*
The definition coincides with usual definition of,

$$\mathcal{S}(\rho_{\Psi}/\rho_{\Phi}) = \mathrm{tr}\,(\rho_{\Psi} \log \rho_{\Psi}) - \mathrm{tr}\,(\rho_{\Phi} \log \rho_{\Phi}),$$

in finite dimensional algebra and ρ_{Ψ} and ρ_{Φ} are density matrices for *Ψ and Φ.*

Quick Recap EE in QFT:

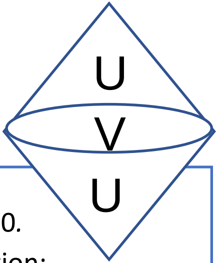
Consider a cyclic separating vector Ψ in an open set $U \subset H$, with a given local operator algebra A_U :

- Tomita operator is defined as an antilinear operator, $S_\Psi : H \rightarrow H$:
  $S_\Psi a|\Psi\rangle = a^\dagger|\Psi\rangle$ with $a \in A_U$
- The modular operator is a linear, self-adjoint operator defined by, $\Delta_\Psi = S_\Psi^\dagger S_\Psi$.
- The relative Tomita operator: $S_{\Psi|\Phi}\, a\, |\Psi\rangle = a^\dagger|\Phi\rangle$.
- The relative modular operator: $\Delta_{\Psi|\Phi} = S_{\Psi|\Phi}^\dagger S_{\Psi|\Phi}$.
- The relative entanglement entropy: $\mathcal{S}_{\Psi|\Phi}(U) = -\langle\Psi|\log \Delta_{\Psi|\Phi}|\Psi\rangle$.

Remark:

For V $\subset$ *U, we* defined the Tomita operator as,

$$S_U\, a|\Psi\rangle = a^\dagger|\Psi\rangle \; with \; a \in A_U$$
$$S_V\, a|\Psi\rangle = a^\dagger|\Psi\rangle \; with \; a \in A_V$$



Important Properties:

- Positivity of relative entanglement entropy: $\mathcal{S}_{\Psi/\Phi} \geq 0$.
- Monotonicity of relative EE under an increasing region:
  $-\langle\Psi|\log \Delta_V|\Psi\rangle \leq -\langle\Psi|\log \Delta_U|\Psi\rangle$.

The algebras are not the same: A_U is bigger than A_V, thus S_U is defined on more states than S_V.

$$\Longrightarrow \Delta_V \geq \Delta_U\,,$$

$$\Longrightarrow \log \Delta_V \geq \log \Delta_U\,,$$

$$\Longrightarrow -\langle\Psi|\log \Delta_V|\Psi\rangle \leq -\langle\Psi|\log \Delta_U|\Psi\rangle.$$

References:

1. Notes On Some Entanglement Properties Of Quantum Field Theory
   Edward Witten: https://arxiv.org/pdf/1803.04993.pdf

2. TASI Lectures on the Emergence of Bulk Physics in AdS/CFT:
   https://arxiv.org/pdf/1802.01040.pdf

3. On the Renormalization of Entanglement Entropy:

- https://arxiv.org/pdf/1709.03205.pdf

- 4. For more traditional proofs:
- R. Haag and B. Schroer, “Postulates of Quantum Field Theory,”

Tomita Operator Definition:

The Tomita operator for a state Ψ in region U of a Hilbert space, with a given algebra A_U, is defined as an antilinear operator S_Ψ, of the form,

$$S_\Psi : H \to H,$$
$$S_\Psi \, a|\Psi\rangle = a^\dagger|\Psi\rangle,$$

for $a \in A_U$, whenever $|\Psi\rangle$ is a cyclic separating vector for A_U.

Note:

- Separating property → The state $a|\Psi\rangle$ is nonzero for all nonzero $a \in A_U$.
- Cyclic property → Defines S_Ψ on a dense set of states in H.
- From Tomita operator definition → $S_\Psi^2 = 1$ → S_Ψ is invertible.

  [Assumption for simplicity: $\langle\Psi|\Psi\rangle = \langle\Phi|\Phi\rangle = 1$.]

- Tomita operator definition is leading us to an unbounded operator.

Modular Operator Definition:

Since S_Ψ is invertible, it has a unique polar decomposition and so it can be written as,

$$S_\Psi = J_\Psi \Delta_\Psi^{1/2},$$

where J_Ψ is antiunitary (modular conjugate) and $\Delta_\Psi^{1/2}$ (modular operator) is Hermitian and positive-definite.

So this implies that the modular operator is a linear, self-adjoint operator defined by,

$$\Delta_\Psi = S_\Psi^\dagger S_\Psi.$$

Let the $|\Psi\rangle$ be a cyclic separating state and $|\Phi\rangle$ be any other state, we define:

The relative Tomita operator, $S_{\Psi|\Phi}$ as an anti-linear operator defined by,

$$S_{\Psi|\Phi}\, a\, |\Psi\rangle = a^{\dagger}|\Phi\rangle.$$

- The relative Tomita operator only depends on the cyclic separating nature of $|\Psi\rangle$, and not on any property of $|\Phi\rangle$.

- If Φ is cyclic separating, then we can define, $S_{\Phi|\Psi}\, a\, |\Phi\rangle = a^{\dagger}|\Psi\rangle$.
  ⇒ $S_{\Psi|\Phi} S_{\Phi|\Psi} = 1$ and so $S_{\Psi|\Phi}$ is invertible.

The relative modular operator is a linear, self-adjoint operator defined by,

$$\Delta_{\Psi|\Phi} = S_{\Psi|\Phi}^{\dagger} S_{\Psi|\Phi}.$$

Note:

If $\Phi = \Psi$, then the definitions would reduce to the previous results:

$$S_{\Psi/\Psi} = S_{\Psi} \quad \text{and} \quad \Delta_{\Psi/\Psi} = \Delta_{\Psi}.$$

Relative Entropy in QFT:

Consider an open set *U,* small enough such that it is spacelike separated from some other open set in Hilbert space.

Now let $|\Psi\rangle$ be any cyclic separating vector for A_U, and $|\Phi\rangle$ any other vector. The <u>relative entropy</u> between these two states, for any measurements done in region *U* is defined by,

$$\mathcal{S}_{\Psi|\Phi}(U) = -\langle\Psi|\log\Delta_{\Psi|\Phi}|\Psi\rangle$$

[Araki in the 1970's]

What we'd want to explore here is to discuss some the properties of the EE given above.

Positivity of Relative Entropy: Mathematical Aside:

For a positive real number λ,

$$-\log \lambda \geq 1 - \lambda,$$

⟹ we can use this inequality for an operator, Δ:

$$-\log \Delta \geq 1 - \Delta.$$

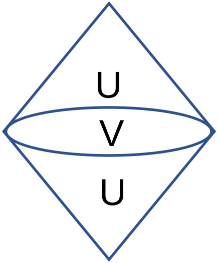
Now consider a completely general state Φ:

$$S_{\Psi/\Phi}(U) = -\langle \Psi \,/\, \log \Delta_{\Psi/\Phi} | \Psi \rangle \geq \langle \Psi \,/\, (1 - \Delta_{\Psi/\Phi}) \,| \Psi \rangle$$
$$= \langle \Psi \,/\, \Psi \rangle - \langle \Psi \,/\, S^{\dagger}{}_{\Psi|\Phi} \, S_{\Psi/\Phi} \,| \Psi \rangle$$
$$= \langle \Psi \,/\, \Psi \rangle - \langle \Phi / \Phi \rangle \;= 0.$$

$$\Longrightarrow S_{\Psi/\Phi} \geq 0.$$

Monotonicity of Relative Entropy under an increasing region:

- Let us consider an open set V, such that $V \subset U$, where U is a bigger open set.



- The two different algebras, A_U and A_V are defined in regions U and V, respectively.

- We denote the relative Tomita operators as, $S_{\Psi|\Phi,U}$ and $S_{\Psi|\Phi,V}$ and their relative modular operators as $\Delta_{\Psi|\Phi;U}$ and $\Delta_{\Psi|\Phi;V}$.

Remark:

We defined the tomato operator as,

$$S_U \, a|\Psi\rangle = a^\dagger|\Psi\rangle \; with \; a \in A_U$$
$$S_V \, a|\Psi\rangle = a^\dagger|\Psi\rangle \; with \; a \in A_V$$

Naively, one might think since they look the same, S_U and S_V are essentially the same!

The algebras are not the same: A_U is bigger than A_V, thus S_U is defined on more states than S_V.

If we have unbounded operators, they are not defined on all Hilbert space (only, at most, on a dense subspace).

Proper statement: S_U is an extension of S_V ═► *Only on the states that both S_U and S_V are defined, they coincide.*

- The relative entropy for measurements in U is:

  - $S_{\Psi|\Phi}(U) = -\langle\Psi|\log \Delta_{\Psi|\Phi;U}|\Psi\rangle.$

- Similarly for measurements in V,

  - $S_{\Psi|\Phi}(V) = -\langle\Psi|\log \Delta_{\Psi|\Phi;V}|\Psi\rangle.$

Monotonicity of Relative Entropy with increasing region

Notation:

We will be keeping Ψ and Φ fixed, so for the sake of notation, we'll drop the subscript of Ψ and Φ from the operators throughout this proof.

We want to show that the relative modular operator Δ_V is bounded from below by the other relative modular operator Δ_U defined in *U* :

$$\Delta_V \geq \Delta_U ,$$

$$\Longrightarrow \log \Delta_V \geq \log \Delta_U ,$$

$$\Longrightarrow -\langle\Psi \mid \log \Delta_V|\Psi\rangle \leq -\langle\Psi \mid \log \Delta_U \mid\Psi\rangle.$$

Note: We only showed the monotonicity of entropy special to the case of increasing the size of a region considered in spacetime.

Generally, monotonicity of relative entropy under partial trace, implies the **strong subadditivity** of entropy (recently, it has had numbers of applications in QFT).

A1. The Proof of Monotonicity under Increasing Region:

- The relative entropy for measurements in U is:
  - $S_{\Psi|\Phi}(U) = -\langle\Psi|\log\Delta_{\Psi|\Phi;U}|\Psi\rangle.$
- Similarly for measurements in V,
  - $S_{\Psi|\Phi}(V) = -\langle\Psi|\log\Delta_{\Psi|\Phi;V}|\Psi\rangle.$

Our goal:
We want to prove that the relative entropy is monotonic under increasing the region considered.

A.1. *Monotonicity of Relative Entropy with increasing region:*

Notation:

We will be keeping Ψ and Φ fixed, so for the sake of notation, we'll drop the subscript of Ψ and Φ from the operators throughout this proof.

We want to show that the relative modular operator Δ_V is bounded from below by the other relative modular operator Δ_U defined in U:

$$\Delta_V \geq \Delta_U\,,$$

$$\Longrightarrow \log \Delta_V \geq \log \Delta_U\,,$$

$$\Longrightarrow -\langle \Psi \mid \log \Delta_V | \Psi \rangle \leq -\langle \Psi \mid \log \Delta_U | \Psi \rangle.$$

We need to show: If P and Q are positive self-adjoint operators such that: $P \geq Q$, then, $\log P \geq \log Q$.

A.1. Proof for Bounded positive self-adjoint operators:

If P and Q are bounded operators such that $P \geq Q$, we can say:

$$P - Q \geq 0.$$

For any real s > 0, $\frac{1}{s+P} \leq \frac{1}{s+Q}$

Define:

$$R(t) = tP + (1-t)Q \,,$$

as a family of operators P and Q.

$$\Rightarrow \frac{dR}{dt} = P - Q \geq 0$$

$$\frac{d}{dt}\frac{1}{s+R(t)} = -\frac{1}{s+R(t)}\frac{dR}{dt}\frac{1}{s+R(t)}$$

$$\Rightarrow \frac{d}{dt}\frac{1}{s+R(t)} \leq 0$$

A.1. Proof for Bounded positive self-adjoint operators:

$$\Longrightarrow \int_0^1 dt \, \left(\frac{1}{s+R(t)}\right) = \frac{1}{s+R(1)} - \frac{1}{s+R(0)} \leq 0$$

$$\Longrightarrow \frac{1}{s+R(1)} \leq \frac{1}{s+R(0)}$$

We describe this result by saying that, $\frac{1}{s+R}$ is indeed a decreasing function of R so this implies that:

$$\frac{1}{s+P} \leq \frac{1}{s+Q}$$

$$\Longrightarrow log\,(P) \geq log\,(Q)$$

A.1. Proof for Unbounded positive self-adjoint operators:

If P and Q are *unbounded operators, then generally they are defined on different dense subspaces.*
Since it is difficult to deal with unbounded operators, we define s>0, so that $\frac{1}{s+P}$ and $\frac{1}{s+Q}$ are bounded.
As before we define: R(t) = tP + (1-t)Q.

$$\Longrightarrow \log R(t) = \int_0^{\infty} ds \left(\frac{1}{s+1} - \frac{1}{s+R}\right)$$

$$\Longrightarrow \frac{d}{dt} \log R(t) =$$

$$\int_0^{\infty} ds \left(\frac{1}{s+R}\frac{dR}{dt}\frac{1}{s+R}\right)$$

Since $\frac{1}{s+R}$ *and* $\frac{dR}{dt} = P - Q$ *are both positive, the integral is positive.*

$$\frac{d}{dt} \log R(t) \geq 0.$$

$$\Longrightarrow \log(P) \geq \log(Q)$$

$$\Longrightarrow -\langle\Psi \mid \log \Delta_V|\Psi\rangle \leq -\langle\Psi \mid \log \Delta_U|\Psi\rangle.$$