Labor Market Rigidities, Trade and Unemployment: A Dynamic Model*

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This is a technical paper which describes the extension to a dynamic environment of the two-sector Helpman and Itskhoki (2009) model with similar labor market search frictions in both sectors. The first sector produces a homogenous good, and we refer to it as the *outside* sector. The second sector produces a differentiated good and firms in this sector are heterogenous in terms of productivity. We first characterize equilibrium in the outside sector. Free entry of firms in this sector pins down equilibrium labor market tightness and hence determines the outside option of workers independently of the details of equilibrium in the differentiated sector. Then, taking the outside option of workers as given, we solve for equilibrium in the differentiated sector. We derive a sufficient statistic for labor market frictions in the differentiated sector. We show that the static model of Helpman and Itskhoki (2009) is similar to the steady state of the dynamic model developed in this note.

1 The model

We start by describing the matching technology, which is similar in both sectors. Then we characterize the equilibrium in the outside sector and proceed to characterize the equilibrium in the differentiated sector. Finally, we describe the general equilibrium of the model.

1.1 Matching Technology

In every sector firms and workers match randomly, according to a Cobb-Douglas sectoral matching function:

$$m = m(V, U) = \frac{1}{\tilde{a}^{\eta}} V^{\eta} U^{1-\eta}, \qquad 0 < \eta < 1,$$

where m is the flow rate of matches, V is the measure of posted vacancies by the firms in the sector, U is the measure of unemployed workers searching for a jobs in the sector and $\tilde{a}$ is the inverse measure of matching productivity.

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Denote by

$$x = \frac{m(V,U)}{U} = \frac{1}{\tilde{a}^{\eta}}\left(\frac{V}{U}\right)^{\eta}$$

the Poisson arrival rate of a match to an unemployed worker. We refer to x as the sectoral labor market tightness.[1] The Poisson arrival rate of a match for a vacant firm can then be written as:

$$g = \frac{m(V,U)}{V} = \frac{1}{\tilde{a}^{\eta}}\left(\frac{U}{V}\right)^{1-\eta} = \frac{1}{\tilde{a}}x^{-\alpha}, \qquad \alpha \equiv \frac{1-\eta}{\eta}.$$

A tighter labor market implies a higher probability of finding a job for unemployed workers and a lower probability of filling up vacancies for firms.

The cost of posting vacancies is denoted by γ. It is useful to introduce the following variable:

$$b = \frac{\gamma}{g} = ax^{\alpha}, \qquad a \equiv \gamma\tilde{a}, \tag{1}$$

which measures the expected cost of hiring one worker. The cost of hiring workers is increasing in the labor market tightness x and in the overall measure of labor market friction a, which depends on the cost of posting vacancies and the productivity of the matching technology.

This matching process applies to both sectors, except that we allow the measure of labor market rigidity a to differ across sectors. In what follows we denote the variables in the outside sector with a 0 subscript; for example, the labor market friction is denoted by a_0 in the outside sector and simply by a in the differentiated sector.

1.2 Equilibrium in the Outside Sector

In the outside sector a matched firm-worker pair produces a unit flow of the homogenous good per unit time until separation takes place, which happens at the rate s_0.[2] For simplicity and without loss of generality we assume that every firm hires one worker. We choose the homogenous good to be the numeraire, so that its price equals one; $p_0 = 1$. The firm pays the worker a wage rate w_0 in terms of the homogenous good and retains $1 - w_0$ as profits. When unemployed, the worker receives an unemployment benefit b_u.

The steady state values of unemployed and employed workers in the outside sector (J_0^U and J_0^E, respectively) are characterized by the following system of Bellman equations:[3]

$$rJ_0^U = b_u + x_0\big(J_0^E - J_0^U\big),$$
$$rJ_0^E = w_0 - s_0\big(J_0^E - J_0^U\big),$$

[1] A more standard definition of labor market tightness is the ratio of vacancies to unemployed, V/U. Note that the two measures are monotonic transformations of each other given the matching productivity.

[2] The overall separation rate can be decomposed into the job destruction rate s_0' and the firm destruction rate δ_0, $s_0 = s_0' + \delta_0$.

[3] Note that the values of a worker are defined in terms of the outside good and implicitly rely on the risk-neutrality of the worker. When we discuss the general equilibrium of the model, we introduce preferences which are consistent with this assumption.

where r is the interest rate. Similarly, steady state values of vacant and producing firms in the outside sector (J_0^V and J_0^F, respectively) are characterized by:

$$rJ_0^V = -\gamma + g_0\big(J_0^F - J_0^V\big),$$
$$rJ_0^F = (1 - w_0) - s_0\big(J_0^F - J_0^V\big).$$

Note that the cost of posting vacancies γ is also paid in terms of the outside good.

The free entry of firms at zero cost implies that the value of a vacant firm in the outside sector has to be zero. We write this condition as

$$rJ_0^V = -\gamma + \frac{\gamma}{b_0}\big(J_0^F - J_0^V\big) = 0,$$

where we use (1).

The wage rate is determined by bargaining between the firm and the worker. We assume Nash bargaining with equal bargaining weights so that the surplus from the match is equally divided:

$$J_0^E - J_0^U = J_0^F - J_0^V.$$

The extension to the case of non-equal bargaining weights is straightforward.

The above six conditions together with the relationship between b_0 and x_0 in (1) allow to solve for the steady state equilibrium in the outside sector (the details are provided in the Appendix). Given the surplus division rule, there is only one level of labor market tightness at which the firms break even. Specifically, it has to satisfy:

$$2(r + s_0)b_0 + x_0 b_0 = 1 - b_u, \qquad b_0 = a_0 x_0^\alpha. \tag{2}$$

These two equations pin down x_0 and b_0 as functions of the parameters (r, s_0, b_u, a_0). Note that b_0 is increasing in a_0 and x_0 is decreasing in a_0, such that $x_0 b_0$ is decreasing in a_0. And higher unemployment benefits reduce both x_0 and b_0.

The free entry condition also implies that the value of a producing firm is $J_0^F = b_0$. From the bargaining condition this implies that the value of an unemployed worker is given by

$$rJ_0^U = b_u + x_0 b_0 = 1 - 2(r + s_0)b_0, \tag{3}$$

which is decreasing in the labor market friction a_0. Intuitively, a greater friction in the labor market requires a less tight labor market for an entering firm to break even. At the same time, lower tightness of the labor market reduces the chances of an unemployed worker to find employment and hence reduces his value. Note that the outside option of a worker in terms of the homogenous good, rJ_0^U, is pinned down as a function of exogenous parameters of the model and does not depend on the details of the equilibrium in the differentiated sector.

Finally, from the condition for the value of the employed worker, we obtain the equilibrium wage rate in the outside sector:

$$w_0 = (r + s_0)b_0 + rJ_0^U = 1 - (r + s_0)b_0, \tag{4}$$

which is decreasing in the labor market friction a_0.[4]

We summarize the key insight from the above analysis in

Proposition 1 *In a steady state of the outside sector, the sector's labor market tightness (x_0), its wage rate (w_0) and the value of an unemployed worker (J_0^U), are decreasing in the sectoral labor market friction a_0, while the hiring cost (b_0) is increasing in a_0. The steady state values of these variables are independent of the details of the rest of the economy.*

Note that the size of the outside sector, in terms of the number (measure) of workers searching for employment or the number of operating firms, remains a free variable. That is, the equilibrium characterized above is consistent with any size of the outside sector. The number of firms adjusts to the number of workers searching for employment in this sector, without affecting tightness in its labor market.

The properties of the steady state equilibrium in the outside sector, as summarized above, are qualitatively similar to the equilibrium properties of the outside sector in the static model of Helpman and Itskhoki (2009).

Finally, we show in the appendix that when the matching friction in the outside sector goes to zero (i.e., $a_0 \to 0$), its labor market becomes competitive, yet the qualitative features of the analysis remain unchanged. This justifies the use of a static model with search frictions in the differentiated sector only, as was done originally in Helpman and Itskhoki (2007, 2008).

1.3 Equilibrium in the Differentiated Sector

In this section we characterize the equilibrium in the differentiated sector, taking the outside option of workers as given. We start by describing the problem of the firm, then derive the wage rate from a bargaining game, characterize labor market tightness, the hiring cost and entry of firms. We also derive a sufficient statistic for labor market costs. To derive the wage rate, we consider the model with discrete time intervals and then take the limit as the time interval goes to zero.

1.3.1 The Firm's Problem

In the differentiated sector firms differ in terms of productivity θ. A firm employs a measure h of workers and produces $y = \theta h$ units of its distinct variety of the differentiated good. We assume that the revenue of a firm in terms of the numeraire (the homogenous good) is

$$R(\theta, h) = A^{1-\beta}(\theta h)^{\beta}.$$

[4] Since b_0 is decreasing in the unemployment benefit b_u, the wage rate is increasing in b_u. As a result, higher unemployment benefits reduce the cost of hiring, but increase equilibrium wages so that the overall effect of b_u on the firm's labor cost is ambiguous.

We derive this expression below from a familiar CES preference structure. The firm pays its workers a wage rate $w(\theta, h)$ and also bears a fixed cost of production f_d in terms of the numeraire. Therefore, the flow value to the firm is

$$\varphi(\theta, h) \equiv A^{1-\beta}(\theta h)^{\beta} - w(\theta, h)h - f_d. \tag{5}$$

Let s' be the destruction rate of matches between a firm and its workers, so that the attrition rate of the labor force is $s'h$ per unit time. Therefore, in order to go from employment h to employment h' over a time interval of length Δ the firm needs to hire $v \equiv h' - (1 - s'\Delta)h$ workers, for small values of Δ. In fact, we consider the continuous time limit ($\Delta \to 0$) and allow employment h' to be a jump variable. That is, a firm can instantly hire a positive measure of workers. In order to hire v workers, the firm has to pay a hiring cost bv, where according to (1), $b = ax^{\alpha}$ and x is the labor market tightness in the differentiated sector.

We assume that a firm dies at a Poisson rate δ, in which case all matches with its workers are destroyed. Therefore, the overall destruction rate of jobs in the differentiated sector is $s = s' + \delta$.[5] We allow s to differ from its counterpart s_0 in the outside sector.

We can now provide a Bellman equation describing the value function of a firm with productivity θ which starts with the employment level h:

$$J^F(\theta, h) = \max_{h' \geq 0} \left\{ \varphi(\theta, h)\Delta - b\big[h' - (1 - s'\Delta)h\big]^+ + e^{-(r+\delta)\Delta} J^F\big(\theta, h'\big) \right\},$$

where $[\cdot]^+ \equiv \max\{\cdot, 0\}$. Since we consider the limit $\Delta \to 0$, we make use of approximations of the type $e^{-r\Delta} \approx 1 - r\Delta$. The first order condition for the firm is

$$e^{-(r+\delta)\Delta} J_h^F(\theta, h') \leq b \qquad \text{with equality for } h' > (1 - s'\Delta)h, \tag{6}$$

where J_h^F is the derivative of the value function with respect to employment. Denote by h_θ the optimal employment level of the firm. Whenever $h \leq h_\theta/(1 - s'\Delta)$, we have $h' = h_\theta$ and the firm jumps to the optimal employment in an instant.[6]

The Envelop Theorem for the firm implies:

$$J_h^F(\theta, h) \leq \varphi_h(\theta, h)\Delta + (1 - s'\Delta)b \qquad \text{with equality for } h \leq h_\theta/(1 - s'\Delta), \tag{7}$$

where $\varphi_h(\theta, h) \equiv \partial\varphi(\theta, h)/\partial h$. In equilibrium, $h = h' = h_\theta$ and both (6) and (7) hold with

[5] More accurately, in discrete time the overall destruction rate should be defined as $(1 - s\Delta) = (1 - s'\Delta)(1 - \delta\Delta)$, so that the expression in the text is exact only as Δ goes to zero.

[6] When $h > h_\theta/(1 - s'\Delta)$, the firms chooses $h' = (1 - s'\Delta)h$ and its employment drifts over time towards h_θ via attrition. This however never happens in stationary equilibrium as all firms start with $h = 0$ and always keep $h = h_\theta$ immediately thereafter.

equalities. We can combine them to write:[7]

$$\varphi_h(\theta, h_\theta) = (r+s)b. \tag{8}$$

This condition characterizes the optimal employment level h_θ. Finally, note that the value of an entering firm with productivity θ and no employment equals

$$V(\theta) \equiv J^F(\theta, 0) = J^F(\theta, h_\theta) - bh_\theta = \frac{\varphi(\theta, h_\theta) - b(r+s)h_\theta}{r+\delta}. \tag{9}$$

The firm instantaneously jumps to the optimal employment h_θ and stays at this employment level throughout its life. This generates a flow value $\varphi(\theta, h_\theta)\Delta$, the labor replacement cost is $bs'\Delta h_\theta$ and the discount rate is $(r+\delta)\Delta$. Therefore, the overall lifetime hiring cost to the firm is $b + bs'/(r+\delta) = b(r+s)/(r+\delta)$ per worker.

1.3.2 Wage Bargaining and Sectoral Labor Market Equilibrium

We assume that the firm bargains bilaterally with each of its workers with equal bargaining weights, as in Stole and Zwiebel (1996a,b). The relevant range for bargaining is $h \leq h_\theta$ since the firm starts with $h = 0$, never exceeds employment h_θ, and if bargaining breaks down some workers leave which results in $h < h_\theta$. For this employment range, condition (7) holds with equality and characterizes the firm's value from a marginal employee.

Denote by $J^E(\theta, h)$ the value of a worker employed by a θ-productivity firm with current employment h. Assuming that the following instant the firm will be back to employment h_θ, the Bellman equation characterizing $J^E(\theta, h)$ is given by

$$J^E(\theta, h) = w(\theta, h)\Delta + e^{-r\Delta}\left[s\Delta J^U + (1 - s\Delta)J^E(\theta, h_\theta)\right], \tag{10}$$

where J^U is the value of an unemployed worker in the differentiated sector and as before $s = s' + \delta$ is the overall separation rate in the differentiated sector.

The bargaining solution requires $J^E(\theta, h) - J^U = J_h^F(\theta, h)$ for all $h \leq h_\theta$. That is, the firm shares the surplus equally with the marginal worker taking into account that his departure will affect bargaining with the remaining workers. At the optimal employment level h_θ, this implies

$$J^E(\theta, h_\theta) - J^U = J_h^F(\theta, h_\theta) = e^{(r+\delta)\Delta}b, \tag{11}$$

where the second equality follows from (6). Note that since the right-hand side is common across all firms, the equilibrium value of an employed worker does not depend on the type of firm he

[7] We make use here of the continuous-time approximation and replace $e^{(r+\delta)\Delta}$ with $[1 + (r+\delta)\Delta]$. Note that in the limit of continuous time, terms with Δ are higher order and both the first-order condition of the firm and the envelope theorem imply $J_h^F(\theta, h_\theta) = b$. Nevertheless, the optimality condition (8) is still correct since lower-order terms cancel. Moreover, note from the envelope theorem that in the continuous time limit $J_h^F(\theta, h_\theta) = b$ for all $h \leq h_\theta$ as the firm achieves optimal employment instantaneously. In the limit, the firm's value function is linear in h up until h_θ and concave thereafter.

is matched with. We can therefore write $J^E \equiv J^E(\theta, h_\theta)$ and $J^E = b$ in the continuous-time limit ($\Delta \to 0$).

Combining (10) and (11), we can rewrite the worker's surplus from employment as:

$$J^E(\theta, h) - J^U = w(\theta, h)\Delta - r\Delta J^U + (1 - s'\Delta)b, \tag{12}$$

where we made use of $1 - e^{-r\Delta} \approx r\Delta$ and $(1 - s\Delta)e^{\delta\Delta} \approx (1 - s\Delta)/(1 - \delta\Delta) = (1 - s'\Delta)$ which are both exact as $\Delta \to 0$. Now combining this expression with (7), the Stole and Zwiebel (1996a,b) bargaining condition becomes:

$$\varphi'(\theta, h) = w(\theta, h) - rJ^U \qquad \forall h \leq h_\theta.$$

$>$From now on we discuss only the continuous-time limit, and all equations below hold for $\Delta \to 0$. Using the definition of $\varphi(\theta, h)$ in (5) and integrating the above identity on $[0, h]$, we solve for the wage schedule:

$$w(\theta, h) = \frac{\beta}{1+\beta}\frac{R(\theta, h)}{h} + \frac{1}{2}rJ^U \qquad \forall h \leq h_\theta.$$

This implies that equilibrium flow value of the firm equals

$$\varphi(\theta, h) = \frac{1}{1+\beta}R(\theta, h) - \frac{1}{2}rJ^U h - f_d. \tag{13}$$

Combining (11) and (12), the equilibrium wage rate—common to all firms–is:

$$w = w(\theta, h_\theta) = rJ^U + (r + s)b. \tag{14}$$

$>$From the Bellman equation for the steady state value of an unemployed worker in the differentiated sector, we obtain:[8]

$$rJ^U = b_u + xb. \tag{15}$$

The worker's indifference condition between the two sectors is $rJ^U = rJ_0^U$, where rJ_0^U is given in (3). Combining (3) and (15), we can express the worker's indifference condition as

$$x_0 b_0 = xb. \tag{16}$$

Recalling that $b = ax^\alpha$, this implies

$$x = x_0\left(\frac{a_0}{a}\right)^{\frac{1}{1+\alpha}} \qquad \text{and} \qquad b = b_0\left(\frac{a}{a_0}\right)^{\frac{\alpha}{1+\alpha}}.$$

That is, $x = x_0$ and $b = b_0$ whenever $a = a_0$. When the friction in the differentiated sector is larger than in the homogeneous sector ($a > a_0$), the labor market is less tight in the differentiated sector ($x < x_0$) and its cost of hiring is larger ($b > b_0$). Holding a_0 constant, x decreases and b increases

[8] The Bellman equation for J^U is $rJ^U = b_u + x\left[J^E - J^U\right]$ and from (11) $J^E - J^U = b$ as $\Delta \to 0$.

in a. Finally, combining (14) and (3), we can write the equilibrium wage rate in the differentiated sector as:

$$w = 1 + (r+s)b - 2(r+s_0)b_0. \tag{17}$$

In particular, when $s = s_0$ and $b = b_0$, $w = w_0$, and when $s > s_0$ or $b > b_0$ (which happens for $a > a_0$), the wage rate in the differentiated sector is higher, i.e., $w > w_0$. These are the properties used in Helpman and Itskhoki (2009; see in particular their expressions 17).

1.3.3 Sectoral Equilibrium

Using (13) we can rewrite the optimality condition (8) for the firm as:

$$\frac{1}{1+\beta}R'(\theta,h_\theta) = \frac{\beta}{1+\beta}A^{1-\beta}\theta^\beta h_\theta^{\beta-1} = \frac{1}{2}\big[1+2(r+s)b-2(r+s_0)b_0\big],$$

where we made use of the fact that $rJ^U = rJ_0^U = 1 - 2(r+s_0)b_0$. Denote by

$$\phi \equiv \big[1+2(r+s)b-2(r+s_0)b_0\big]$$

the sufficient statistic for relative labor market rigidities in the two sectors. The effective cost of labor to a firm in the differentiated sector is proportional to ϕ, which equals 1 when $s = s_0$ and $b = b_0$, and is greater than 1 for $s > s_0$ or $b > b_0$.[9]

Using the optimality condition above we can solve for equilibrium firm size:

$$h_\theta = \left(\frac{2\beta}{1+\beta}\right)^{\frac{1}{1-\beta}} \phi^{\frac{-1}{1-\beta}} A\theta^{\frac{\beta}{1-\beta}}.$$

More productive firms are larger and larger ϕ reduces the equilibrium size of all firms proportionally. Substituting the optimal employment into the value of an entering firm with productivity θ, given in (9), we have

$$V(\theta) = \frac{1}{r+\delta}\left[\frac{1-\beta}{1+\beta}\left(\frac{2\beta}{1+\beta}\right)^{\frac{\beta}{1-\beta}} \phi^{\frac{-\beta}{1-\beta}} A\theta^{\frac{\beta}{1-\beta}} - f_d\right]. \tag{18}$$

Note that the value of the firm is increasing in its productivity θ and for all firms in the sector the value increases proportionally with the demand level A and decreases with the labor market frictions parameter ϕ.

Now we can characterize equilibrium in the differentiated sector. To enter the sector firms pay an entry cost f_e in terms of the numeraire. Upon entry they draw their productivity θ from a known distribution $F(\theta)$. If a firm draws a high enough productivity so that $V(\theta) \geq 0$, it stays in the industry and produces. The threshold productivity θ_d is defined by

$$V(\theta_d) = 0. \tag{19}$$

[9] Two special cases are: (i) $s = s_0$, so that $\phi = [1+2(r+s)(b-b_0)]$; and (ii) $a = a_0$, so that $b = b_0$ and $\phi = [1+2(s-s_0)b]$.

Firms with productivity $\theta < \theta_d$ exit immediately after entry. Finally, the free entry condition implies that the expected value from entry has to equal the fixed entry cost:

$$f_e = \int_{\theta_d}^{\infty} V(\theta) \mathrm{d}F(\theta) = \frac{f_d}{r+\delta} \int_{\theta_d}^{\infty} \left[\left(\frac{\theta}{\theta_d} \right)^{\frac{\beta}{1-\beta}} - 1 \right] \mathrm{d}F(\theta), \tag{20}$$

where the second equality uses (18)-(19) which imply $V(\theta) = f_d/(r+\delta)\big[(\theta/\theta_d)^{\beta/(1-\beta)} - 1\big]$.

Given ϕ, conditions (18)-(20) allow us to solve for the productivity cutoff θ_d and the equilibrium demand level A. Note that ϕ is a sufficient statistic for labor market frictions in the differentiated sector for the analysis of the sector's equilibrium. It measure labor market friction in the differentiated sector relative to the homogenous sector. Specifically, when $a = a_0$ and $s = s_0$, we have $\phi = 1$ and the overall level of labor market rigidity does not affect equilibrium in the differentiated sector.

1.4 General Equilibrium

In this section we characterize the general equilibrium in a closed economy, using the results on sectoral equilibrium in the homogenous and differentiated sectors. We first introduce preferences and characterize the demand functions which yield A for the differentiated sector. Next we characterize general equilibrium conditions in the product and labor markets.

1.4.1 Preferences and Demand

For simplicity we assume a quasi-linear intratemporal utility function between homogenous and differentiated goods:[10]

$$\mathcal{U} = q_0 + \frac{1}{\zeta} Q^{\zeta}, \qquad 0 < \zeta < 1,$$

where q_0 and Q are the consumption levels of homogenous and differentiated goods, respectively. The differentiated good is assumed to be a CES aggregator of individual varieties:

$$Q = \left[\int_{\omega \in \Omega} q(\omega)^{\beta} \mathrm{d}\omega \right]^{\beta}, \qquad \zeta < \beta < 1,$$

where ω represents variety, Ω is the set of varieties available for consumption, $q(\omega)$ is consumption of variety ω, and β controls the elasticity of substitution between varieties.[11] We denote by P and $p(\omega)$ the ideal price index for the differentiated good and the price of variety ω, respectively, both in terms of the numeraire (recall that $p_0 = 1$).

With this utility specification, intratemporal optimization results in the following demand pa-

[10] A similar analysis is possible for a CES utility over the homogenous and differentiated good (see Helpman and Itskhoki, 2007).

[11] Restriction $\beta > \zeta$ implies that differentiated goods are better substitutes for each other than for the homogenous good.

rameter (see Helpman and Itskhoki, 2009):

$$A = Q^{-\frac{\beta-\zeta}{1-\beta}}. \tag{21}$$

An increase in the consumption level of the differentiated good Q is associated with a reduction in the price index P. Lower P results either from lower prices of individual varieties or from an increase in the number of available varieties. A lower price index implies tighter competition in the product market and reduces demand and revenues for every producer of a differentiated variety (through lower demand A). Additionally, the total expenditure on differentiated varieties is equal to

$$PQ = Q^{\zeta}$$

and the indirect intratemporal utility is

$$\mathcal{V} = E + \frac{1-\zeta}{\zeta} Q^{\zeta}, \tag{22}$$

where E is aggregate expenditure on consumption and the second term is consumer surplus from the differentiated good.[12]

We assume that a representative agent is risk-neutral with life-time utility given by

$$\mathbb{U}_0 = \int_0^{\infty} e^{-\rho t} \mathcal{U}_t \mathrm{d}t,$$

where ρ is the time preference rate. For simplicity we assume a family interpretation so that each worker is a part of a big family which perfectly shares risk among its members. Workers maximize family utility by participating in the labor market. The income of every family is big enough to ensure positive consumption of the homogenous good. This effectively implies that all workers are risk-neutral, consistent with our earlier analysis, and general equilibrium requires $r = \rho$.[13]

1.4.2 Product Market in the Differentiated Sector

Consider the product market in the differentiated sector. Taking into account (21), the zero profit cutoff condition (19) can be written as:

$$\phi^{-\frac{1}{1-\beta}} Q^{-\frac{\beta-\zeta}{1-\beta}} \theta_d^{\frac{\beta}{1-\beta}} = f_d \frac{1+\beta}{1-\beta} \left(\frac{1+\beta}{2\beta} \right)^{\frac{\beta}{1-\beta}}. \tag{23}$$

[12] This characterization is accurate as long as the consumption of the homogenous good is positive which at the aggregate requires $E > PQ = Q^{\zeta}$. This condition is always satisfied as long as the total labor force in the economy L is large enough.

[13] In Helpman, Itskhoki and Redding (2009) we dispense with the family interpretation and extend the static analysis to the case of risk-aversion. A similar extension is possible in the dynamic framework with the extra complication of optimal savings and inter-temporal consumption substitution. The family interpretation and risk-neutrality imply no need for savings in the steady state equilibrium. Finally, note that the worker's indifference condition between the two sectors (16) has to be modified in the case of risk-aversion.

Note that the cutoff productivity θ_d below which firms exit is pinned down by the free entry condition (20) as a function of entry and fixed production costs. Cutoff θ_d does not depend on equilibrium labor market tightness or competitiveness of the product market. This is because the product and labor market environments affect the value of all firms proportionally, while the free entry condition requires that the expected value of the firm equals the entry cost.

Given the productivity cutoff, (23) determines the consumption level in the differentiated sector Q as a function of the sufficient statistic for the labor market ϕ. Specifically, Q is decreasing in ϕ. Lower ϕ implies lower labor market costs to the firms which increases profits and leads to more firm entry. Eventually, in equilibrium competition in the product market becomes tight enough (via higher Q and lower A) to ensure zero expected profits to the marginal entrant.

Once we determined Q, we know the equilibrium expenditure on the differentiated good which is equal to Q^ζ. In equilibrium, total expenditure in the sector equals total revenues of all firms serving the demand in this sector. In the closed economy, this implies that

$$Q^\zeta = \frac{M_e}{\delta}\int_{\theta_d}^{\infty} R(\theta, h_\theta)\mathrm{d}F(\theta),$$

where M_e is the steady state entry rate of new firms. Therefore, the total number of producing firms in steady state is $M_e\big[1-F(\theta_d)\big]/\delta$, which die at rate δ and $\big[1-F(\theta_d)\big]$ is the fraction of remaining firms upon entry. Using the definition of revenues, optimal firm size and the zero profit condition, (18)-(19), we can rewrite this equilibrium requirement as

$$Q^\zeta = f_d\frac{1+\beta}{1-\beta}\frac{M_e}{\delta}\int_{\theta_d}^{\infty}\left(\frac{\theta}{\theta_d}\right)^{\frac{\beta}{1-\beta}}\mathrm{d}F(\theta). \tag{24}$$

Given θ_d and Q, this condition determines M_e.

Next we can solve for labor demand in the differentiated sector:

$$H = \frac{M_e}{\delta}\int_{\theta_d}^{\infty} h_\theta \mathrm{d}F(\theta) = \frac{2\beta}{1+\beta}\frac{Q^\zeta}{\phi}, \tag{25}$$

where we have used $\phi h_\theta = 2\beta R(\theta, h_\theta)/(1+\beta)$, which follows from the firm's first order condition. The demand for labor in the differentiated sector H increases in the sectoral expenditure level Q^ζ and decreases in the cost of labor ϕ. Since in equilibrium Q decreases in ϕ, H also decreases in ϕ.

1.4.3 Labor Market Equilibrium

We now characterize equilibrium in the labor market. We denote by H and U the number (measure) of employed and unemployed workers in the differentiated sector. These are stock variables. $N = H+U$ denotes the total number of workers attached to the differentiated sector. Similarly, H_0, U_0 and N_0 are the the numbers of employed, unemployed and attached workers in the homogeneous sector. The economy-wide labor supply is given by L, so that $N + N_0 = L$.

Labor dynamics in the steady state requires that flows into and out of unemployment be equal;

that is $sH = xU$, and similarly in the outside sector. Therefore, the unemployment rate in the differentiated sector is given by

$$u \equiv \frac{U}{N} = \frac{s}{s+x}, \tag{26}$$

and similarly for the outside sector. As a result, the sectoral unemployment rate is increasing in the sectoral separation rate and decreasing in sectoral labor market tightness (that is, increasing in the sectoral matching friction). The aggregate unemployment rate is then

$$\mathbf{u} = \frac{U_0 + U}{L} = \frac{1}{L}\left[u_0 N_0 + uN\right] = u_0 + \left(u - u_0\right)\frac{N}{L},$$

where

$$u_0 \equiv \frac{U_0}{N_0} = \frac{s_0}{s_0 + x_0}.$$

Aggregate unemployment increases in sectoral unemployment, but also depends on the composition of labor across sectors. A shift of labor toward a higher-unemployment rate sector increases aggregate unemployment.

To characterize equilibrium in the labor market, we make use of labor demand in the differentiated sector (25). From (26), the employment rate is $H/N = x/(s+x)$. Therefore,

$$N = \frac{s+x}{x}\frac{2\beta}{1+\beta}\frac{Q^{\zeta}}{\phi}. \tag{27}$$

Similar to H, N is decreasing in ϕ because Q is decreasing in ϕ. In addition, N increases in s and decreases in x (increases in a). Therefore, the labor market friction parameters s and a have two opposing effects on total labor allocated to the differentiated sector N. When s or a rise, this reduces both labor demand H and the employment rate $x/(s+x)$ in the differentiated sector. The former acts to reduce N while the latter acts to increase N.

The residual labor $N_0 = L - N$ is allocated to the outside sector. This generates employment

$$H_0 = \frac{x_0}{s_0 + x_0} N_0.$$

Recall that output in the outside sector is also equal to H_0 and the number of firms occupying the outside sector can be recovered from H_0 and the sector's labor market tightness x_0.

Finally, we solve for the total labor market income in the economy. Employed workers in each sector receive wages, while unemployed workers receive unemployment benefits. Therefore, total labor income in the differentiated sector is

$$\left[\frac{x}{s+x}w + \frac{s}{s+x}b_u\right]N$$

and similarly in the outside sector. In order to finance unemployment benefits, the government levies a lump-sum tax on all families. As a result, aggregate after-tax income is

$$E = \frac{xw}{s+x}N + \frac{x_0 w_0}{s_0 + x_0}N_0. \tag{28}$$

Unemployment benefits affect aggregate income only indirectly by affecting the equilibrium labor market tightness (x and x_0) and wage rates (w and w_0). In the special case $s = s_0$ and $a = a_0$, this condition simplifies to $E/L = x_0 w_0/(s_0 + x_0)$. This condition allows us to characterize welfare (using (22)), which completes the description of the closed economy equilibrium.

2 Open Economy

In this section we characterize the open economy equilibrium and discuss some of its properties. Consider a world of two possibly asymmetric countries. The countries can costlessly trade the homogenous good, while trade in differentiated goods is associated with both fixed and variable (iceberg) trade costs. Specifically, exporting of each variety requires a flow fixed cost of f_x in terms of the numeraire and in order to deliver one unit of the differentiated variety abroad, the firm needs to send $\tau > 1$ units.

In the open economy, the characterization of equilibrium in the outside sector remains unchanged. Specifically, x_0, b_0, w_0 and rJ_0^U are still determined by the labor market parameters in the outside sector and do not respond to trade. In the differentiated sector, the firms now have the option to export. This requires paying a fixed cost f_x. It can be shown that access to the foreign market expands the revenues of every firm by a factor $\Upsilon_x^{1-\beta}$, where $\Upsilon_x = 1 + \tau^{\frac{-\beta}{1-\beta}} A^*/A$ and an asterisk denotes foreign variables (see Helpman and Itskhoki, 2009). Therefore, we can rewrite the flow value to the firm as

$$\varphi(\theta, h, I_x) = \left[1 + I_x \tau^{-\frac{\beta}{1-\beta}} \frac{A^*}{A}\right]^{1-\beta} A^{1-\beta}(\theta h)^\beta - w(\theta, h)h - I_x f_x - f_d,$$

where I_x is the indicator of export status, i.e., it equals one when the firm exports and zero otherwise. The firm optimizes with respect to h and $I_x \in \{0, 1\}$, but the rest of the equilibrium characterization in the differentiated sector remains unchanged. Specifically, x, b, w and ϕ are still characterized by the closed-economy conditions and depend solely on the labor market parameters in the two sectors.

In equilibrium there are two productivity cutoffs now: θ_d and $\theta_x > \theta_d$. Firms with productivity below θ_d exit, firms with productivity in $[\theta_d, \theta_x)$ serve the domestic market and firms with productivity above θ_x serve the domestic market and export. The condition for θ_d is still given by (23).

It can be shown that the condition for θ_x is:[14]

$$\tau^{-\frac{\beta}{1-\beta}} \frac{A^*}{A} \left(\frac{\theta_x}{\theta_d}\right)^{\frac{\beta}{1-\beta}} = \frac{f_x}{f_d} \tag{29}$$

and the free entry condition can be written as

$$(r+\delta) f_e = f_d \int_{\theta_d}^{\infty} \left[\left(\frac{\theta}{\theta_d}\right)^{\frac{\beta}{1-\beta}} - 1\right] \mathrm{d}F(\theta) + f_x \int_{\theta_x}^{\infty} \left[\left(\frac{\theta}{\theta_x}\right)^{\frac{\beta}{1-\beta}} - 1\right] \mathrm{d}F(\theta). \tag{30}$$

Given θ_d, (29) determines the exporting cutoff θ_x; it is increasing in the fixed and variable costs of trade. Free entry condition (30) introduces a negative relationship between the two cutoffs θ_d and θ_x. A lower θ_x implies more profitable exporting opportunities, but in equilibrium it should be offset by a lower expected profits from domestic sales (which requires higher θ_d) in order to satisfy zero expected profits for a marginal entrant.

For a given A^*, conditions (23), (29) and (30) allow to solve for θ_d, θ_x and A. Symmetric conditions hold for the trade partner. These six conditions together allow to solve for the four cutoffs and two demand levels simultaneously. Note that this block of the equilibrium system is equivalent to the same block in the static model of Helpman and Itskhoki (2009; equations 19-20) with ϕ replacing b. In the open economy the relationship (21) between A and Q still holds. As a result, it follows from Proposition 1 in Helpman and Itskhoki (2009) that a reduction in ϕ increases Q and reduces Q^*, while a reduction in τ increases both Q and Q^*.

The requirement that total expenditure in the differentiated sector equals total revenues of the firms serving the demand in this sector now takes the form:

$$Q^{\zeta} = f_d \frac{1+\beta}{1-\beta} \frac{M_e}{\delta} \int_{\theta_d}^{\infty} \left(\frac{\theta}{\theta_d}\right)^{\frac{\beta}{1-\beta}} \mathrm{d}F(\theta) + f_x \frac{1+\beta}{1-\beta} \frac{M_e^*}{\delta} \int_{\theta_x^*}^{\infty} \left(\frac{\theta}{\theta_x^*}\right)^{\frac{\beta}{1-\beta}} \mathrm{d}F(\theta). \tag{31}$$

The first term on the right-hand side is the revenues of the domestic firms serving the domestic market, while the second term is the revenues of the foreign firms serving the domestic market. This equation together with its counterpart for the foreign country allows to solve for the number of entrants (M_e, M_e^*) in both markets.

Following the closed-economy logic, the demand for labor in the differentiated sector can be written as

$$H = \phi^{-1} \frac{2\beta}{1-\beta} \frac{M_e}{\delta} \left[f_d \int_{\theta_d}^{\infty} \left(\frac{\theta}{\theta_d}\right)^{\frac{\beta}{1-\beta}} \mathrm{d}F(\theta) + f_x \int_{\theta_x}^{\infty} \left(\frac{\theta}{\theta_x}\right)^{\frac{\beta}{1-\beta}} \mathrm{d}F(\theta) \right], \tag{32}$$

where the first term on the right-hand side represents labor demand for domestic production, while

[14] Equation (18) now characterizes the value from domestic sales. A similar equation characterizes the value from exports with $\tau^{-\frac{\beta}{1-\beta}} A^*$ replacing A and f_x replacing f_d. Denote it $V_x(\theta)$. Then the exporting cutoff is defined by $V_x(\theta_x) = 0$. Combining it with $V(\theta_d) = 0$ yields (29). The total value of the firm is $\max\{V(\theta), 0\} + \max\{V_x(\theta), 0\}$. Taking the expectation over θ and using the cutoff conditions results in (30).

the second term is labor demand for exporting activities. Given M_e and (θ_d, θ_x), this condition determines labor demand in the differentiated sector. Further equilibrium conditions in the labor market are the same as in the closed economy. Specifically, the labor allocated to the differentiated sector is $N = H(x+s)/x$. This completes the description of the open economy equilibrium.

Note one special case, when countries are symmetric in terms of the labor costs in the differentiated sector, $\phi = \phi^*$. Then $(\theta_d, \theta_x, Q, M_e, H)$ are the same in both countries. Combining (31) and (32) in this case yields:

$$H = \frac{2\beta}{1+\beta}\frac{Q^\zeta}{\phi},$$

the same condition as (25) in the closed economy. With symmetric countries, lower τ leads to higher Q, M_e, H and N. When labor markets are symmetric in both sectors, this will have no effect on the unemployment rate, but will increase welfare through the effect on Q. When $u > u_0$ (either because $s > s_0$ or $a > a_0$), trade increases the unemployment rate by shifting labor towards a high-unemployment sector. Equivalent results hold in the static model of Helpman and Itskhoki (2009).

Finally, there are two effects on welfare. First, trade increases the consumer surplus from the differentiated good by raising Q. Second, it leads to a change in the aggregate income E (given in (28)). While the first effect unambiguously increases welfare, the second effect will increase welfare if and only if $wx/(s+x) > w_0 x_0/(s_0 + x_0)$. When the two sectors are symmetric in terms of their labor markets, only the first effect is present and trade unambiguously raises welfare in both countries.[15]

[15] Additionally, we can characterize welfare outcomes for the unemployed. Recall that the value of an unemployed worker equals $rJ_0^U = 1 - 2(r+s_0)b_0$ in terms of the numeraire. It is not affected by trade liberalization which however reduces the price level (by increasing Q and reducing P). Therefore, unemployed workers unambiguously gain from trade liberalization.

A Appendix

A.1 Equilibrium in the Outside Sector

Define the surplus from the employment relationship by $S_0 \equiv \left(J_0^E - J_0^U\right) + \left(J_0^F - J_0^V\right)$. Equal division of surplus implies

$$J_0^E - J_0^U = J_0^F - J_0^V = \frac{1}{2}S_0.$$

Manipulating the Bellman equations for J_0^E and J_0^F yields:

$$\begin{aligned}(r+s_0)\big(J_0^E - J_0^U\big) &= w_0 - rJ_0^U,\\ (r+s_0)\big(J_0^F - J_0^V\big) &= (1-w_0) - rJ_0^V.\end{aligned}$$

Making use of the definition of S_0, this implies

$$(r+s_0)S_0 = 1 - rJ_0^U - rJ_0^V.$$

Intuitively, the flow surplus from the match is equal to the flow output of 1 minus the flow values of unemployment to the worker and the flow value of being vacant to the firm.

Substituting the surplus division rule into the Bellman equations for J_0^U and J_0^V, we have:

$$\begin{aligned}rJ_0^U &= b_u + x_0\frac{S_0}{2},\\ rJ_0^V &= -\gamma + \frac{\gamma}{b_0}\frac{S_0}{2}.\end{aligned}$$

Using the above three equations we can solve for the equilibrium surplus:

$$S_0 = \frac{2[1+\gamma-b_u]}{2(r+s_0)+x_0+\gamma/b_0}.$$

Note that the surplus from the match depends on exogenous parameters of the model and equilibrium labor market tightness and hiring costs, which are linked by (1).

The knowledge of S_0 allows us to recover the equilibrium values of all other variables. From the free entry of firms ($J_0^V = 0$), we have $S_0/2 = b_0$. From the expression for J_0^U and the surplus division, this immediately implies

$$\begin{aligned}rJ_0^U &= b_u + x_0 b_0,\\ J_0^F &= J_0^E - J_0^U = b_0.\end{aligned}$$

Making use of the solution for S_0, we obtain:

$$\frac{1+\gamma-b_u}{2(r+s_0)+x_0+\gamma/b_0} = b_0,$$

which after simple manipulations yields (2) in the text and allows to solve for equilibrium x_0 and b_0 (taking into account (1): $b_0 = a_0 x_0^{\alpha}$). This condition also implies $b_u + x_0 b_0 = 1 - 2(r + s_0) b_0$, which allows us to derive (3) in the text. Finally, the wage rate that implements the surplus division has to equal (see the expression for $J_0^E - J_0^U$):

$$w_0 = rJ_0^U + (r + s_0)\frac{S_0}{2} = rJ_0^U + (r + s_0)b_0,$$

which together with (3) yields (4) in the text.

A.2 Competitive Labor Market in the Outside Sector

In this section we consider a special limiting case of the model when $a_0 \to 0$, so that the labor market in the outside sector is competitive. Since this is a special case, the equilibrium characterization in the previous sections is still accurate; however, additional results emerge. From (2) it follows that $a_0 \to 0$ implies $x_0 \to \infty$ and $b_0 \to 0$ so that $x_0 b_0 \to 1 - b_u$. It further implies that the outside option of workers (rJ_0^U) and the wage rate in the outside sector both converges to 1. In words, as the matching friction in the outside sector becomes trivial, the labor market approaches a competitive equilibrium with full employment and wages equal to labor productivity.

Turning to the equilibrium in the differentiated sector, the worker's indifference condition between sectors (16) becomes

$$xb = 1 - b_u$$

which together with $b = ax^{\alpha}$ results in

$$x = \left(\frac{1 - b_u}{a}\right)^{\frac{1}{1+\alpha}} \qquad \text{and} \qquad b = (1 - b_u)^{\frac{\alpha}{1+\alpha}} a^{\frac{1}{1+\alpha}}.$$

Finally, the sufficient statistic for labor market costs in the differentiated sector simplifies to $\phi = 1 + 2(r + s)b$ and depends only on the extent of search frictions in the differentiated sector.

There is no unemployment in the outside sector. Therefore, the aggregate unemployment rate is

$$\mathbf{u} = \frac{s}{s + x}\frac{N}{L}.$$

Since trade does not affect labor market tightness in the differentiated sector, it raises unemployment if and only if it increases the fraction of labor allocated to the differentiated sector.

Finally, aggregate income in this limiting case is given by

$$E = \frac{xw}{s + x}N + N_0 = L + \left[\frac{xw}{s + x} - 1\right]N.$$

In our case we have $w = 1 + (r + s)b$ so that $wx = x + (r + s)(1 - b_u)$. This implies

$$E = L + \frac{(r + s)(1 - b_u) - s}{s + x}N.$$

Aggregate income E increases in N as long as unemployment benefits are not too high. This is the condition for welfare to increase with trade. This result is similar to the result in the static model in Helpman and Itskhoki (2008).

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