Online Appendix for:

"Matching, Sorting , and the Distributional Effects of International Trade"

by

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Part A: Motivating Evidence

We aim to provide a simple analytical framework that can shed light on the distributional implications of globalization in a world with a broad range of worker types. Our motivation comes in part from several recent findings in the empirical literature on earnings. Researchers such as Autor et al. (2008) and Kopczuk et al. (2010) have emphasized that trends in income inequality over the last decade cannot be well summarized by a single summary statistic, such as the relative wage of skilled versus unskilled workers or the college wage premium. Rather, in several countries, including the United Stages, inequality has been rising at the top end of the wage distribution, but constant or even declining at the bottom end of the distribution, generating what has been termed a "hollowing out" of the middle class. Also, Helpman et al. (2015) and Akerman et al. (2013) have documented that within-industry variation accounts for a large part of the cross-sectional evolution of wage inequality, even after controlling at a detailed level for workers' occupations. In Brazil, for example, the authors used a classification system that allows for 12 manufacturing sectors and more than 300 occupations and found that more than half of the change in wage inequality between 1986 and 1995 occurred within sectors and occupations. Together, these findings point to the need for a framework that allows for multiple worker types and that incorporates links between trade and relative wages for workers employed in the same occupation and industry.

In the evidence provided below, we draw on the set of linked employer-employee relationships that were surveyed by the Brazilian Ministry of Labor in its *Relação Anual de Informaçoẽs Sociais* (RAIS) and studied previously by Helpman et al. (2015).[19] Our purpose in re-visiting these data is not to provide a set of targets that will be explained by our theory, but rather to highlight the rich pattern of outcomes that exist in reality and to establish some stylized facts that we can use to focus attention among the several "cases" that our model can generate.

We examine distributions of wages and salaries in twelve *Brasileiro de Geografia e Estatistica* (IBGE) industry categories for the years 1986 and 1994.[20] These data represent labor-market

[19] This is a confidential data set whose property rights belong to Marc-Andreas Muendler. Marc gave us permission to use these data and he also provided the price indexes that we used to generate Figure 10. More details about these data are provided in the online supplement of Helpman et al. (2015), available at http://scholar.harvard.edu/files/helpman/files/himr_supplement_18apr15.pdf.

[20] Table 1 includes a list of the industries and their sector numbers.

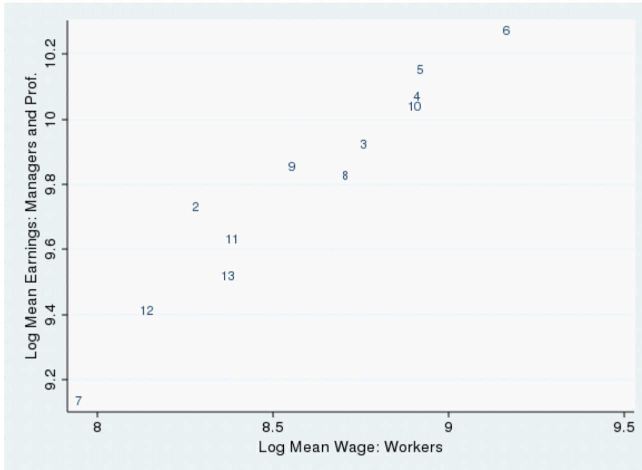

Figure 8: Variation across manufacturing industries of log mean salary of male managers and log mean wage of male workers in Brazil, 1994. Source: own calculations.

outcomes before and after the major Brazilian trade liberalization of 1991, but before the substantial stabilization program that Brazil undertook in 1994. Our model distinguishes two factors of production that we call "managers" and "workers," and so we compute earnings distributions in the Brazilian manufacturing industries separately for occupations classified in the *Classificação Brasileira de Ocupações* Category 1 (professional and managerial labor) and those in Categories 2-5 (skilled white-collar, unskilled white-collar, skilled blue-collar and unskilled-blue-collar labor).

In figure 8, we plot the log of the mean earnings for male managers and professionals in 1994 against the log of the mean wage for male workers, for each of the twelve manufacturing sectors.[21] Apparently, the correlation across sectors between the mean earnings of managers and the mean wage of workers is strongly positive. We interpret this positive correlation to suggest the greater empirical relevance of circumstances in which the more able (and thus higher paid) managers sort to the same industry as do the more able workers, as compared to circumstances in which the more able managers sort to the same industry as the less able workers. For future reference, we record

Observation 1 There is a strong positive correlation between the mean wage of male managers employed in a Brazilian industry and the mean wage of male workers employed in the industry.

Table 1 reports the Theil index of income inequality separately for male workers and male managers and professionals in 1986 and in 1994, for each of the 12 manufacturing industries.[22] In Table 2, we provide two decompositions of the these indexes for each year and for the change

[21] The plot for wages and salaries in 1986 is qualitatively similar.

[22] We compute the Theil index of inequality in group k as

$$T_k = \frac{1}{N_k} \sum_{i=1}^{N_k} \left(\frac{y_i}{\bar{y}_k} \cdot \ln \frac{y_i}{\bar{y}_k} \right)$$

where N_k is the number of individuals in group k, y_i is the income of individual i, and $\bar{y}_k$ is the mean income among all individuals in group k.

		Male Workers		Male Managers	
Industry	No.	1986	1994	1986	1994
Non-metallic mineral products	2	0.324	0.381	0.308	0.376
Metallic products	3	0.252	0.276	0.219	0.260
Machinery, equipment and instruments	4	0.240	0.266	0.226	0.224
Electrical and telecommunications equipment	5	0.261	0.294	0.203	0.207
Transport equipment	6	0.192	0.236	0.163	0.192
Wood products and furniture	7	0.238	0.331	0.392	0.423
Paper and paperboard, and publishing and printing	8	0.301	0.326	0.319	0.340
Rubber, tobacco, leather and fur	9	0.309	0.344	0.295	0.345
Chemical and pharmaceutical products	10	0.358	0.353	0.247	0.286
Apparel and textiles	11	0.275	0.309	0.347	0.393
Footwear	12	0.259	0.350	0.335	0.349
Food, beverages, and ethyl alcohol	13	0.268	0.345	0.411	0.398
All manufacturing industries		0.318	0.364	0.290	0.329

Table 1: Theil index of inequality by manufacturing industry

between them. The top part of the table shows a separate decomposition for each occupational group (i.e., workers and managers) into a component that represents dispersion within industries and one that represents dispersion between industries.[23] The bottom part of the table provides a decomposition of inequality for all male workers and managers in manufacturing taken together into components for "within occupation and industry" and "between occupation and industry." We see that, in either case, the within component accounts for the largest share of the overall inequality in each year, as well as the majority of the change that occurred during the period that spanned the trade reform. We record this finding in

Observation 2 Within-industry inequality accounts for a majority of the income inequality for male workers and for male managers in Brazil in 1986 and 1994, and for a majority of the changes in inequality between 1986 and 1994. Within-occupation-and-industry inequality

[23] For a set of groups $k = 1, \ldots, K$, the overall Theil index is

$$T = \frac{1}{N} \sum_k \sum_i \frac{y_{ik}}{\bar{y}} \ln \frac{y_{ik}}{\bar{y}}$$

where y_{ik} is the income of worker i in group k, $N = \sum_k N_k$ and $\bar{y}$ is the mean income. We compute the "within component" as

$$T_{within} = \sum_k s_k T_k$$

where $s_k = \frac{N_k \bar{y}_k}{N \bar{y}}$ is the income share of group k.The "between component" is

$$T_{between} = \sum_k s_k \ln \frac{\bar{y}_k}{\bar{y}} \ ,$$

so that $T = T_{within} + T_{between}$.

	Male Workers			Male Managers		
	1986	1994	Change	1986	1994	Change
Decomposition: Within/Between Industry						
Total inequality	0.318	0.364	0.045.	0.290	0.329	0.039
Within industry	0.272	0.309	0.038	0.262	0.291	0.029
Between industry	0.047	0.054	0.007	0.028	0.038	0.009
	Male Workers and Managers					
	1986	1994	Change			
Decomposition: Within/Between Industry and Occupation						
Total inequality	0.423	0.467	0.043			
Within occupation and industry	0.269	0.305	0.036			
Between occupation and industry	0.154	0.161	0.007			

Table 2: Decomposition of income inequality

accounts for a majority of the income inequality for male workers and managers as a group in 1986 and 1994, and for a majority of the change in inequality between 1986 and 1994.

In Figure 9, we plot the change in the Theil index for workers between 1986 and 1994 against the change in the Theil index for managers and professionals. The numbers in the figure again represent the different industries, in accordance with the labels provided in Table 6. The figure reveals a negative correlation of -0.20 between the changes in inequality for workers and that for managers; in industries where the spread in the salaries of workers increased greatly, that for managers generally increased little, or even decreased. We note

Observation 3 There is a (weak) negative correlation across industries between the change in the Theil index of earnings inequality between 1986 and 1994 for Brazilian workers and the change in the Theil index of earnings inequality for Brazilian managers.

Finally, in Figure 10, we associate these changes in inequality for each occupational group with changes in relative prices over the same period. We use wholesale price data (*Indice de Precos por Atacado*) computed by Fundação Getulio Vargas (FGV) and a concordance and aggregation to the twelve IGBE industry categories performed by Marc Muendler.[24] The top panel in the

[24] We begin with the IPA-DI series, which has been used for the Brazilian national accounts since 1944. FGV reports these prices at an FGV-specific industry level. Muendler used an internal crosswalks made available to him by IBGE to reset those data to the Nivel-100 industry level and then mapped the resulting prices to IGBE subsectors. He formed aggregates at the 12-industry level of the earnings data using sales data from the 1990 survey of manufacturing firms (PIA) that is described in Muendler (2004). Finally, we computed price indexes for 1986 and 1994 by averaging the monthly prices he gave us and constructed relative price changes by dividing the inflation in each price series by the average inflation rate. We are very grateful to Marc Muendler for his assistance in all this.

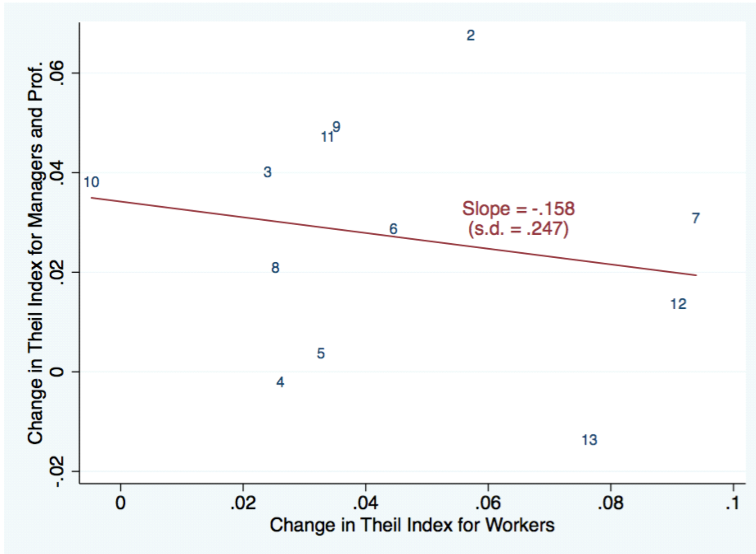

Figure 9: Correlation across industries of changes in income inequality for Brazilian workers and Brazilian Managers

figure shows that inequality among workers tended to rise in those industries that experienced an increase in relative price between 1986 and 1994. The correlation coefficient is 0.25. Meanwhile, the bottom panel in the figure depicts a negative correlation between the change in the Theil index of inequality for managers and the evolution of the industry's relative price. In this case, we compute the correlation coefficient to be -0.45. We make no claim that these correlations represent causal links between prices and inequality. Still, it is interesting that industry price changes have an opposite correlation with changes in inequality for the two factors, which we will find is a general prediction of our model.

Observation 4 The correlation across industries between the change in relative output price and the change in income inequality between 1986 and 1994 is positive for Brazilian workers and negative for Brazilian managers.

We offer these observations cautiously. For one thing, we have not attempted to isolate the influence of trade liberalization from other forces that may have impacted the wage and salary distributions in Brazil during the period under consideration. For another, we have not sought to verify that similar patterns have occurred after trade liberalization or increased exposure to trade in other countries, especially those with factor endowments similar to those in Brazil. While serious empirical analysis is beyond our scope, the data for Brazil do suggest that trade impacts differently the earnings of those in an industry and occupation who differ in skill and ability, that within industry-and-occupation redistribution is at least as important as redistribution between those in different occupations and industries, that changes in inequality among managers and workers in an industry are (weakly) negatively correlated, and that these inequality changes are (weakly) correlated with relative price movements. Finally, the data suggest that greater emphasis should

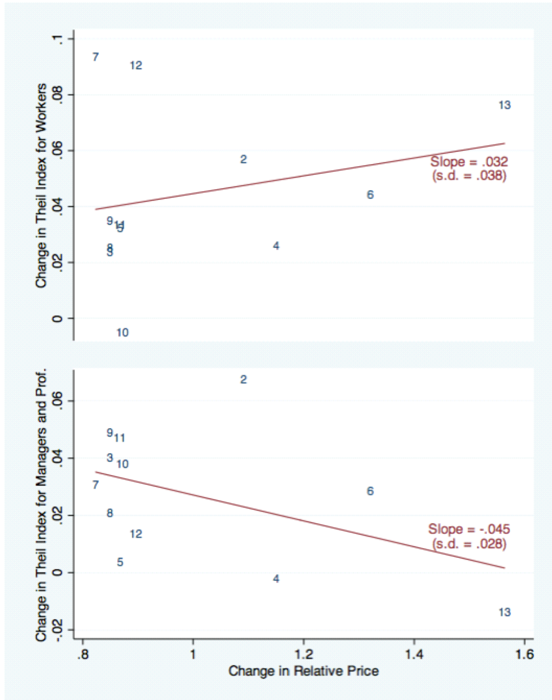

Figure 10: Correlation across industries of changes in income inequality and changes in relative prices

be placed on parameter configurations that imply sorting of the best managers and the best workers to the same sectors as compared to parameter configurations that imply otherwise.

Part B: Analytical Results

Proofs for Section 3.1

Consider Cobb-Douglas productivity according to Assumption 1. Labor demand (2) then takes the form

$$\ell\left(q_L, q_H\right) = \left[\frac{\gamma_i p_i q_H^{\beta_i} q_L^{\alpha_i}}{w\left(q_L\right)}\right]^{\frac{1}{1-\gamma_i}}. \tag{20}$$

Substituting (20) into the expression for net profits $\pi_i\left(\ell, q_L; q_H\right) - r(q_H)$ yields

$$\tilde{\pi}_i\left(q_L, q_H\right) = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \left(q_H^{\beta_i} q_L^{\alpha_i}\right)^{\frac{1}{1-\gamma_i}} w\left(q_L\right)^{-\frac{\gamma_i}{1-\gamma_i}} - r\left(q_H\right), \tag{21}$$

where $r(q_H)$ is the salary of a manager with ability q_H and $\bar{\gamma}_i \equiv \gamma_i^{\frac{\gamma_i}{1-\gamma_i}}\left(1-\gamma_i\right)$. Every firm chooses the ability of its workers and the ability of its manager so as to maximize profits, yet free entry dictates that these profits must be equal to zero in equilibrium. Let M_i be the set of all matches that maximize profits in sector i. For each pairing (q_L, q_H) in M_i,

$$r\left(q_H\right) = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \left(q_H^{\beta_i} q_L^{\alpha_i}\right)^{\frac{1}{1-\gamma_i}} w\left(q_L\right)^{-\frac{\gamma_i}{1-\gamma_i}}, \quad i = 1, 2, \tag{22}$$

by dint of the zero-profit condition. Recall from (5) the equilibrium wage schedule

$$w\left(q_L\right) = w_i q_L^{\alpha_i/\gamma_i} \quad \text{for } q_L \in Q_{Li}^{int}, \tag{23}$$

where the superscript "*int*" denotes the interior of the set. Substitutions into the previous equation establishes the salary schedule for managers

$$r\left(q_H\right) = r_i q_H^{\beta_i/(1-\gamma_i)} \quad \text{for } q_H \in Q_{Hi}^{int}, \tag{24}$$

where r_i is a "salary anchor" analogous to w_i.

As discussed in the main text, these wages and salaries leave a firm indifferent among workers and managers within a sector, and so matching between managers and workers within a sector is indeterminate. Sorting to sectors is not indeterminate, though, but higher worker types sort into sector 1 if $s_L = \alpha_1/\gamma_1 - \alpha_2/\gamma_2 > 0$ and higher firm types sort into sector 1 if $s_H = \beta_1/\left(1-\gamma_1\right) - \beta_2/\left(1-\gamma_2\right) > 0$. Let q_L^* and q_H^* denote the worker and firm type that is indifferent between sectors.

To describe the equilibrium, we invoke factor-market clearing, continuity of worker wages, continuity of managerial salaries, and the zero-profit conditions. For concreteness, let us focus on the case in which $s_H > 0$ so that the more able managers sort to industry 1; the opposite case can be handled similarly.

It proves convenient to define $e_{Hi}\left(q_H\right) = q_H^{\beta_i/(1-\gamma_i)}$ as the effective managerial input of a manager with ability q_H who works in sector i. Then the aggregate supplies of effective managerial input in sectors 1 and 2 are

$$H_1 = \bar{H} \int_{q_H^*}^{q_{H\max}} q_H^{\frac{\beta_1}{1-\gamma_1}} \phi_H(q_H)\, dq_H, \tag{25}$$

and

$$H_2 = \bar{H} \int_{q_{H\min}}^{q_H^*} q_H^{\frac{\beta_2}{1-\gamma_2}} \phi_H(q_H)\, dq_H\ , \tag{26}$$

respectively. Note that $H_1/\bar{H}$ depends only on q_H^* and is a monotonically decreasing function, and $H_2/\bar{H}$ also depends only on q_H^* and is monotonically increasing.

Consider now the supply and demand for effective labor in sector 1, where we define $e_{Li}(q_L) = q_L^{\alpha_i/\gamma_i}$ as the effective labor provided by a worker of ability q_L in sector i. From the labor demand equation (20), a firm in sector 1 combines a manager with e_{Hi} units of effective managerial input with $e_{Hi}(\gamma_i p_i/w_i)^{1/(1-\gamma_i)}$ units of effective labor. Therefore, the H_1 units of effective managerial input that are hired into sector 1 are combined with $H_1(\gamma_1 p_1/w_1)^{1/(1-\gamma_1)}$ units of effective labor. Noting the definition of H_1 and equating the demand for effective labor in sector 1 with the supply of effective labor among those with ability above q_L^*, we have

$$\bar{H}\left(\frac{\gamma_1 p_1}{w_1}\right)^{\frac{1}{1-\gamma_1}} \int_{q_H^*}^{q_{H\max}} q_H^{\frac{\beta_1}{1-\gamma_1}} \phi_H(q_H)\, dq_H = \bar{L} \int_{q_L^*}^{q_{L\max}} q_L^{\frac{\alpha_1}{\gamma_1}} \phi_L dq_L\ . \tag{27}$$

A similar condition applies in sector 2, where labor-market clearing requires

$$\bar{H}\left(\frac{\gamma_2 p_2}{w_2}\right)^{\frac{1}{1-\gamma_2}} \int_{q_{H\min}}^{q_H^*} q_H^{\frac{\beta_2}{1-\gamma_2}} \phi_H(q_H)\, dq_H = \bar{L} \int_{q_{L\min}}^{q_L^*} q_L^{\frac{\alpha_2}{\gamma_2}} \phi_L dq_L\ . \tag{28}$$

Continuity of the wage schedule at q_L^* requires that

$$w_1 (q_L^*)^{\frac{\alpha_1}{\gamma_1}} = w_2 (q_L^*)^{\frac{\alpha_2}{\gamma_2}}\ . \tag{29}$$

The salary function for managers must also be continuous and firms that hire managers with ability q_H^* must earn zero profits in either sector. Together, these considerations imply

$$\bar{\gamma}_1 p_1^{\frac{1}{1-\gamma_1}} w_1^{-\frac{\gamma_1}{1-\gamma_1}} (q_H^*)^{\frac{\beta_1}{1-\gamma_1}} = \bar{\gamma}_2 p_2^{\frac{1}{1-\gamma_2}} w_2^{-\frac{\gamma_2}{1-\gamma_2}} (q_H^*)^{\frac{\beta_2}{1-\gamma_2}}\ . \tag{30}$$

Equations (27)-(30) comprise four equations that can be used to solve for the two wage anchors, w_1 and w_2, and the two cutoffs, q_L^* and q_H^*. The effective supply of managers in sectors 1 and 2, H_1 and H_2, can then be solved from (25) and (26). Finally, the salary anchors for the managers can be computed from the zero-profit conditions, which imply

$$r_i = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} w_i^{-\frac{\gamma_i}{1-\gamma_i}} \quad \text{for } i = 1, 2. \tag{31}$$

This completes our characterization of the supply-side equilibrium for an economy that faces prices p_1 and p_2.

Proofs for Section 3.2

Consider strictly log-supermodular productivity, i.e., Assumption 1' holds. Substituting the optimal labor supply (2) into the gross profit function $\pi_i(\ell, q_L; q_H)$ and deducting manager salaries yields net profit

$$\tilde{\pi}_i(q_H, q_L) = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \psi_i(q_H, q_L)^{\frac{1}{1-\gamma_i}} w(q_L)^{-\frac{\gamma_i}{1-\gamma_i}} - r(q_H), \text{ where } \bar{\gamma}_i \equiv \gamma_i^{\frac{\gamma_i}{1-\gamma_i}} (1-\gamma_i). \tag{32}$$

The firm identifies the most suitable workers to combine with the manager, taking the continuous and strictly increasing wage schedule as given.[25] This yields a profit function,

$$\Pi_i(q_H) = \max_{q_L \in S_L} \tilde{\pi}_i(q_H, q_L), \tag{33}$$

for $q_H \in S_H$, $i = 1, 2$. This profit function describes a firm's profits per manager when it hires mangers of ability q_H and optimizes the choice of workers. Finally, the firm selects q_H to maximize $\Pi_i(q_H)$, given the continuous and strictly increasing salary schedule, $r(q_H)$.[26] In equilibrium

$$\max_{q_H \in Q_{Hi}} \Pi_i(q_H) = 0 \text{ and } \max_{q_H \in S_H} \Pi_i(q_H) \le 0, \tag{34}$$

where Q_{Hi} is the set of types of managers that sort into sector i. Firms break even when among the mangers that sort into their sector they hire those that bring about the highest profits, and their profits cannot be raised by hiring mangers from the other sector.

Denote by $m_i(q_H)$ the solution set to problem (33). Because S_L and S_H are compact, $m_i(q_H)$ is upper hemicontinuous (because $\tilde{\pi}_i(q_L, q_H)$ is a continuous function), $m_i(q_H)$ is closed-valued, and the graph

$$G_i = [\{q_H, q_L\} \mid q_L \in m_i(q_H) \text{ for all } q_H \in S_H]$$

is closed. The matching correspondence satisfies

$$m(q_H) = \begin{cases} m_1(q_H) & \text{for } q_H \in Q_{H1}, \\ m_2(q_H) & \text{for } q_H \in Q_{H2}, \end{cases}$$

and the equilibrium allocation graph in sector i is

$$M_i = [\{q_H, q_L\} \mid q_L \in m_i(q_H) \text{ for all } q_H \in Q_{Hi}] \subseteq G_i.$$

[25] The strict monotonicity of the wage function follows from the strict monotonicity of the productivity functions $\psi_i(q_H, q_L)$; if wages were declining over some range of abilities, all firms would prefer to hire the most able workers in this range. The continuity of the wage function follows from the continuity of the productivity function; if wages were to jump at some q'_L, firms would strictly prefer workers with ability a shade below q'_L to workers with ability a shade above q'_L, because the former would be only slightly less productive but would cost discretely less. Below we also prove that the wage function must be differentiable in the interior of the ability range employed by an industry.

[26] The salary schedule must be continuous and strictly increasing for the same reason that the wage schedule must be continuous and strictly increasing.

Since $Q_{Hi} \subseteq S_H$, the graph M_i is also closed.

Now consider a connected subset $M_i^n \subseteq M_i$:

$$M_i^n = [\{q_H, q_L\} \mid q_L \in m_i(q_H) \text{ for all } q_H \in [q_{H1}, q_{H2}] \subseteq Q_{Hi}].$$

Since M_i is a closed graph, such a subset exists and there exists an interval $[q_{L1}, q_{L2}]$, $q_{L2} > q_{L1}$, that satisfies both (i) $m_i(q_H) \in [q_{L1}, q_{L2}]$ for all $q_H \in [q_{H1}, q_{H2}]$ and (ii) for every point $q_L \in [q_{L1}, q_{L2}]$ there exists a managerial ability level $q_H \in [q_{H1}, q_{H2}]$ satisfying $q_L \in m_i(q_H)$. This means that, in M_i^n, workers of ability $[q_{L1}, q_{L2}]$ are matched with managers of ability $[q_{H1}, q_{H2}]$ and all workers and managers have matches. Then, as Eeckhout and Kircher (2012) have shown, strict log supermodularity of $\psi_i(\cdot)$ ensures strict positive assortative matching (PAM) between the factors allocated to sector i. It follows that $m_i(q_H)$ is a continuous and strictly increasing function in the interior of $[q_{H1}, q_{H2}]$. M_i consists of a union of connected sets, $M_i = \cup_{n \in N_i} M_i^n$, such that $m_i(q_H)$ is continuous and strictly increasing in each such set and $m_i(q_H)$ jumps upwards between them.

We next establish the differentiability of $w(\cdot)$ in $M_i^{n,int}$.[27] Let $m^{-1}(\cdot)$ be the inverse of the sectoral matching function in $M_i^{n,int}$. Since $m(\cdot)$ is continuous and strictly increasing in $M_i^{n,int}$, this inverse exists. Now consider an interval $[q_L', q_L' + dq_L) \in M_i^{n,int}$. The zero-profit condition (8) implies

$$w\left(q_L'\right) = \bar{\gamma}_i^{\frac{1-\gamma_i}{\gamma_i}} p_i^{\frac{1}{\gamma_i}} \psi_i \left[m^{-1}\left(q_L'\right), q_L'\right]^{\frac{1}{\gamma_i}} r\left[m^{-1}\left(q_L'\right)\right]^{-\frac{1-\gamma_i}{\gamma_i}}$$

and profit maximization implies

$$w\left(q_L' + dq_L\right) \geq \bar{\gamma}_i^{\frac{1-\gamma_i}{\gamma_i}} p_i^{\frac{1}{\gamma_i}} \psi_i \left[m^{-1}\left(q_L'\right), q_L' + dq_L\right]^{\frac{1}{\gamma_i}} r\left[m^{-1}\left(q_L'\right)\right]^{-\frac{1-\gamma_i}{\gamma_i}}.$$

Together, these expressions imply

$$w\left(q_L' + dq_L\right) \geq w\left(q_L'\right) \left\{ \frac{\psi_i\left[m^{-1}\left(q_L'\right), q_L' + dq_L\right]}{\psi_i\left[m^{-1}\left(q_L'\right), q_L'\right]} \right\}^{\frac{1}{\gamma_i}}. \tag{35}$$

Similarly, (8) implies

$$w\left(q_L' + dq_L\right) = \bar{\gamma}_i^{\frac{1-\gamma_i}{\gamma_i}} p_i^{\frac{1}{\gamma_i}} \psi_i \left[m^{-1}\left(q_L' + dq_L\right), q_L' + dq_L\right]^{\frac{1}{\gamma_i}} r\left[m^{-1}\left(q_L' + dq_L\right)\right]^{-\frac{1-\gamma_i}{\gamma_i}}$$

and profit maximization implies

$$w\left(q_L'\right) \geq \bar{\gamma}_i^{\frac{1-\gamma_i}{\gamma_i}} p_i^{\frac{1}{\gamma_i}} \psi_i \left[m^{-1}\left(q_L' + dq_L\right), q_L'\right]^{\frac{1}{\gamma_i}} r\left[m^{-1}\left(q_L' + dq_L\right)\right]^{-\frac{1-\gamma_i}{\gamma_i}}.$$

[27] This proof is similar to the proof of differentiability of the wage function in Sampson (2014).

Together, these expressions imply

$$w\left(q_L'\right) \geq w\left(q_L' + dq_L\right) \left\{ \frac{\psi_i \left[m^{-1}\left(q_L' + dq_L\right), q_L'\right]}{\psi_i \left[m^{-1}\left(q_L' + dq_L\right), q_L' + dq_L\right]} \right\}^{\frac{1}{\gamma_i}}. \tag{36}$$

Inequalities (35) and (36) jointly imply

$$\frac{w\left(q_L'\right)}{\psi_i \left[m^{-1}\left(q_L'\right), q_L'\right]^{\frac{1}{\gamma_i}}} \left[\frac{\psi_i \left[m^{-1}\left(q_L'\right), q_L' + dq_L\right]^{\frac{1}{\gamma_i}} - \psi_i \left[m^{-1}\left(q_L'\right), q_L'\right]^{\frac{1}{\gamma_i}}}{dq_L} \right] \leq \frac{w\left(q_L' + dq_L\right) - w\left(q_L'\right)}{dq_L}$$

$$\leq \frac{w\left(q_L'\right)}{\psi_i \left[m^{-1}\left(q_L' + dq_L\right), q_L'\right]^{\frac{1}{\gamma_i}}} \left[\frac{\psi_i \left[m^{-1}\left(q_L' + dq_L\right), q_L' + dq_L\right]^{\frac{1}{\gamma_i}} - \psi_i \left[m^{-1}\left(q_L' + dq_L\right), q_L'\right]^{\frac{1}{\gamma_i}}}{dq_L} \right].$$

Since the productivity function is continuous, strictly increasing, and differentiable, and since the inverse of the sectoral matching function is continuous and strictly increasing in this range, taking the limit as $dq_L \to 0$ implies that the derivative of $w(\cdot)$ at q_L' exists and

$$\frac{dw\left(q_L'\right)}{dq_L} = \frac{w\left(q_L'\right)}{\psi_i \left[m^{-1}\left(q_L'\right), q_L'\right]^{\frac{1}{\gamma_i}}} \frac{\partial \psi_i \left[m^{-1}\left(q_L'\right), q_L'\right]^{\frac{1}{\gamma_i}}}{\partial q_L}.$$

Similar arguments can be used to show that the salary function is differentiable.

We now prove Proposition 3 by contradiction. (Proposition 2 can be proved similarly.) To this end, suppose that the inequality condition holds, but the equilibrium is such that there are managers employed in sector j who have greater ability than some managers employed in sector i. In such circumstances, there exists an ability level $\tilde{q}_H$ at one of the boundaries between Q_{Hi} and Q_{Hj} such that managers with ability in $(\tilde{q}_H - \varepsilon_i, \tilde{q}_H) \subset Q_{Hi}^{int}$ are employed in sector i and managers with ability $(\tilde{q}_H, \tilde{q}_H + \varepsilon_j) \subset Q_{Hj}^{int}$ are employed in sector j, for $\varepsilon_i > 0$ and $\varepsilon_j > 0$ small enough. Moreover, the equilibrium conditions (7)-(10) are satisfied, the matching function $m(q_H)$ is continuous at Q_{Hi}^{int} and Q_{Hj}^{int} close to $\tilde{q}_H$ (but can be discontinuous at the boundary point between these sets), the wage function $w(q_L)$ is continuous and increasing in S_L and differentiable in Q_{Li}^{int} and Q_{Lj}^{int}, and the salary function $r(q_H)$ is continuous and increasing in S_H and differentiable in Q_{Hi}^{int} and Q_{Hj}^{int}.

Now recall the continuous profit function $\Pi_i(q_H)$ defined in (33). In equilibrium, $\Pi_i(q_H) = 0$ for all $q_H \in Q_{Hi}$, but the maximal profits $\Pi_i(q_H)$ may differ from zero for $q_H \notin Q_{Hi}$. Therefore $\Pi_i(q_H) = 0$ for all $q_H \in (\tilde{q}_H - \varepsilon_i, \tilde{q}_H)$ and, by continuity, $\lim_{q_H \nearrow \tilde{q}_H} \Pi_i(q_H) = 0$.

Next consider the profits that would accrue to an entrepreneur that hires a manager with ability $\tilde{q}_H + \varepsilon$ in order to produce good i, where $\varepsilon < \varepsilon_j$. Choosing workers so as to maximize profits, this entrepreneur earns $\Pi_i(\tilde{q}_H + \varepsilon) \geq \tilde{\pi}_i \left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]$, where $m\left(\tilde{q}_H^-\right) = \lim_{\varepsilon \searrow 0} m(\tilde{q}_H - \varepsilon)$ and $\lim_{\varepsilon \searrow 0} \Pi_i(\tilde{q}_H + \varepsilon) = \lim_{\varepsilon \searrow 0} \tilde{\pi}_i \left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right] = 0$. The first-order approximation to $\tilde{\pi}_i \left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]$ is

$$\tilde{\pi}_i \left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right] \approx \varepsilon \tilde{\pi}_{iH} \left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right],$$

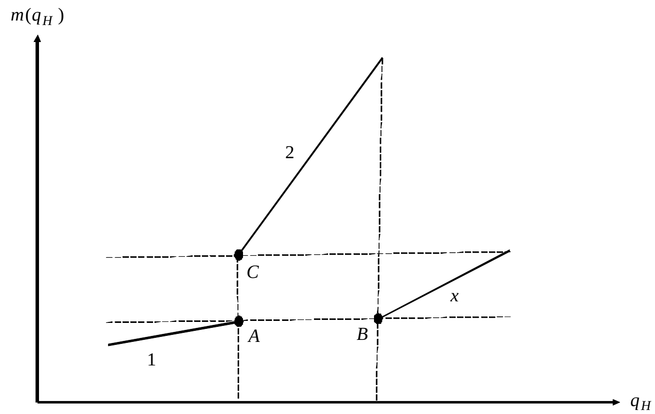

Figure 11: Matching function with discontinuity

where $\tilde{\pi}_{iH}(\cdot)$ is the partial derivative of $\tilde{\pi}_i(\cdot)$ with respect to q_H. This derivative exists because the salary function is differentiable in Q_{Hj}^{int}, and

$$
\begin{aligned}
&\tilde{\pi}_{iH}\left[\tilde{q}_H+\varepsilon, m\left(\tilde{q}_H^-\right)\right] \\
&=\bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \psi_i\left[\tilde{q}_H+\varepsilon, m\left(\tilde{q}_H^-\right)\right]^{\frac{1}{1-\gamma_i}} w\left[m\left(\tilde{q}_H^-\right)\right]^{-\frac{\gamma_i}{1-\gamma_i}} \frac{\psi_{iH}\left[\tilde{q}_H+\varepsilon, m\left(\tilde{q}_H^-\right)\right]}{\left(1-\gamma_i\right)\psi_i\left[\tilde{q}_H+\varepsilon, m\left(\tilde{q}_H^-\right)\right]}-r'\left(\tilde{q}_H+\varepsilon\right) \\
&=\left\{\frac{\psi_i\left[\tilde{q}_H+\varepsilon, m\left(\tilde{q}_H^-\right)\right]}{\psi_i\left[\tilde{q}_H, m\left(\tilde{q}_H^-\right)\right]}\right\}^{\frac{1}{1-\gamma_i}} r\left(\tilde{q}_H^-\right) \frac{\psi_{iH}\left[\tilde{q}_H+\varepsilon, m\left(\tilde{q}_H^-\right)\right]}{\left(1-\gamma_i\right)\psi_i\left[\tilde{q}_H+\varepsilon, m\left(\tilde{q}_H^-\right)\right]}-r'\left(\tilde{q}_H+\varepsilon\right),
\end{aligned}
$$

where the last equality uses the free-entry condition (8), which applies to sector 1 at points in Q_{Hi}^{int} in the conjectured equilibrium, and $r\left(\tilde{q}_H^-\right)=r\left(\tilde{q}_H\right)$ due to the continuity of the salary function. Since $\tilde{q}_H+\varepsilon \in Q_{Hj}^{int}$, condition (9) implies

$$
\lim_{\varepsilon \searrow 0} \tilde{\pi}_{iH}\left[\tilde{q}_H+\varepsilon, m_i\left(\tilde{q}_H^-\right)\right]=r\left(\tilde{q}_H\right)\left\{\frac{\psi_{iH}\left[q_H, m_i\left(\tilde{q}_H^-\right)\right]}{\left(1-\gamma_i\right)\psi_i\left[q_H, m_i\left(\tilde{q}_H^-\right)\right]}-\frac{\psi_{jH}\left[q_H, m\left(\tilde{q}_H^+\right)\right]}{\left(1-\gamma_j\right)\psi_j\left[q_H, m\left(\tilde{q}_H^+\right)\right]}\right\},
$$

where $m\left(\tilde{q}_H^+\right)=\lim_{\varepsilon \searrow 0} m\left(\tilde{q}_H+\varepsilon\right)$. It now follows from the supposition of Proposition 3 that the right-hand side of this equation is strictly positive irrespective of the values of $m_i\left(\tilde{q}_H^-\right)$ and $m\left(\tilde{q}_H^+\right)$, and therefore that $\tilde{\pi}_{iH}\left[\tilde{q}_H+\varepsilon, m_i\left(\tilde{q}_H^-\right)\right]>0$ for ε small enough, which contradicts the zero-profit condition as profits rise above zero. This contradicts the supposition that in equilibrium there are managers employed in sector j who are more able than some managers employed in sector i. Consequently, every manager in sector i has greater ability than any manager employed in sector j. This completes the proof.

Next we prove Proposition 4. Suppose that the inequality conditions in Proposition 4 hold but the equilibrium is such that there exist managers in sector 2 who are more able than some managers in sector 1. In such circumstances, there exists an ability $\tilde{q}_H$ at one of the boundary points between Q_{H1} and Q_{H2} such that managers of ability $\tilde{q}_H - \varepsilon_1$ are employed in sector 1 and managers of ability $\tilde{q}_H + \varepsilon_2$ are employed in sector 2 for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ small enough. Let $m(\tilde{q}_H^-) = \lim_{q_H \nearrow \tilde{q}_H} m(q_H)$ and $m(\tilde{q}_H^+) = \lim_{q_H \searrow \tilde{q}_H} m(q_H)$ Then

$$\lim_{\varepsilon \to 0} \tilde{\pi}_{iH} \left[\tilde{q}_H + \varepsilon, m(q_H^-)\right] = r\left(\tilde{q}_H\right) \left[\frac{\psi_{1H}\left[\tilde{q}_H, m(\tilde{q}_H^-)\right]}{\left(1-\gamma_1\right)\psi_1\left[\tilde{q}_H, m(\tilde{q}_H^-)\right]} - \frac{\psi_{2H}\left[\tilde{q}_H, m\left(\tilde{q}_H^+\right)\right]}{\left(1-\gamma_2\right)\psi_2\left[\tilde{q}_H, m\left(\tilde{q}_H^+\right)\right]}\right], \tag{37}$$

which we derive in the same way as in the proof of Proposition 3. Under the supposition that the managers to the left of $\tilde{q}_H$ sort into sector 1 and those to the right of $\tilde{q}_H$ sort into sector 2 the partial derivative in (37) cannot be positive and therefore

$$\frac{\psi_{1H}\left[\tilde{q}_H, m(\tilde{q}_H^-)\right]}{\left(1-\gamma_1\right)\psi_1\left[\tilde{q}_H, m(\tilde{q}_H^-)\right]} \leq \frac{\psi_{2H}\left[\tilde{q}_H, m\left(\tilde{q}_H^+\right)\right]}{\left(1-\gamma_2\right)\psi_2\left[\tilde{q}_H, m\left(\tilde{q}_H^+\right)\right]}.$$

In view of the first inequality in Proposition 4 and the strict log supermodularity of the productivity function, this inequality implies $m\left(\tilde{q}_H^+\right) > m\left(\tilde{q}_H^-\right)$. That is, the matching function is discontinuous at $\tilde{q}_H$ and it jumps upwards there. As a result, there must exist an ability level for workers $\check{q}_L \in \left[m\left(\tilde{q}_H^-\right), m\left(\tilde{q}_H^+\right)\right]$ such that workers in the range $(\check{q}_L - \check{\varepsilon}_1, \check{q}_L)$ are employed in sector 1 and workers in the range $(\check{q}_L, \check{q}_L + \check{\varepsilon}_2)$ are employed in sector 2, for $\check{\varepsilon}_1$ and $\check{\varepsilon}_2$ small enough. Due to the upward jump of the matching function and due to PAM in each sector, in this range of worker types the ability of managers matched with workers in sector 1 must be strictly greater than the ability of managers matched with workers in sector 2. This is illustrated in Figure 11. At point A, we have $q_H = \tilde{q}_H$ and the matching function exhibits an upward jump from point A to C. The supposition is that managers to the left of A sort into sector 1 and managers to the right of A sort into sector 2, as illustrated in the figure. Clearly, workers with ability between points A and C must be matched with managers in some sector. Segment x illustrates a possible matching of these workers with high-ability managers. It is not possible for x to be sector 2, however, because this would imply non-monotonic matching in this sector, which is ruled out by the strict log supermodularity of the productivity function there. So x must be sector 1. In this case, $\check{q}_L$ is the ability of workers at point C. Workers with ability just below C are employed in sector 1 and workers with ability just above C are employed in sector 2. Evidently, the ability of managers with whom these workers are matched in sector 1 is higher than the ability of managers with whom their slightly better peers are matched in sector 2. It can be seen from the figure that a similar outcome obtains if the matching along x is to the left of point A, except that in this case x stands for sector 2 and $\check{q}_L$ is the ability of workers at point A. Evidently, in this case too, at points around $\check{q}_L$ the ability of managers matched with workers in sector 1 is higher than the ability of managers matched with workers in sector 2.

In short, consider the inverse function $m_1^{-1}(q_L)$ for $q_L \in (\check{q}_L - \check{\varepsilon}_1, \check{q}_L)$; this inverse exists in the specified range because $m_1(q_H)$ is continuous and strictly increasing at points in $(\tilde{q}_H - \varepsilon, \tilde{q}_H)$ for ε small enough. Similarly, consider the inverse function $m_2^{-1}(q_L)$ for $q_L \in (\check{q}_L, \check{q}_L + \check{\varepsilon}_2)$; this inverse also exists in the specified range because $m_2(q_H)$ is continuous and strictly increasing at points in $(\tilde{q}_H, \tilde{q}_H + \varepsilon)$ for ε small enough. Moreover, under the supposition of our sorting pattern $m^{-1}(q_L) = m_1^{-1}(q_L)$ for $q_L \in (\check{q}_L - \check{\varepsilon}_1, \check{q}_L)$ and $m^{-1}(q_L) = m_2^{-1}(q_L)$ for $q_L \in (\check{q}_L, \check{q}_L + \check{\varepsilon}_2)$ and the argument in the previous paragraph showed that $m^{-1}(q_L) = m_1^{-1}(q_L) > m^{-1}(q'_L) = m_2^{-1}(q'_L)$ for $q_L \in (\check{q}_L - \check{\varepsilon}_1, \check{q}_L)$ and $q'_L \in (\check{q}_L, \check{q}_L + \check{\varepsilon}_2)$. Taking limits as $\check{\varepsilon}_1, \check{\varepsilon}_2 \searrow 0$, this implies that $m^{-1}\left(\check{q}_L^-\right) > m^{-1}\left(\check{q}_L^+\right)$.

Next, following steps similar to those used in the proof of Proposition 3, which considered the response of profits to variations in the ability of managers at points around $\tilde{q}_H$, an analysis of the response of profits to variations in the ability of workers at points around $\check{q}_L$ establishes that a necessary condition for optimality is

$$\frac{\psi_{1L}\left[m^{-1}\left(\check{q}_L^-\right), \check{q}_L\right]}{\gamma_1 \psi_1\left[m^{-1}\left(\check{q}_L^-\right), \check{q}_L\right]} \leq \frac{\psi_{2L}\left[m^{-1}\left(\check{q}_L^+\right), \check{q}_L\right]}{\gamma_2 \psi_2\left[m^{-1}\left(\check{q}_L^+\right), \check{q}_L\right]}.$$

In view of the second inequality in Proposition 4 and the strict log supermodularity of the productivity function, this inequality implies $m^{-1}\left(\check{q}_L^+\right) = m_2^{-1}\left(\check{q}_L^+\right) > m_1^{-1}\left(\check{q}_L^-\right) = m^{-1}\left(\check{q}_L^-\right)$, which contradicts the above established result that $m_1^{-1}\left(\check{q}_L^-\right) > m_2^{-1}\left(\check{q}_L^+\right)$. It follows that the best managers sort into sector 1. By symmetrical arguments the best workers also sort into sector 1.

Proofs for Section 5.1

Consider the two-sector economy for the case of Cobb-Douglas productivity under Assumption 1, and adopt the label for the two sectors such that $s_H = \beta_1/(1-\gamma_1) - \beta_2/(1-\gamma_2) > 0$. We establish

Proposition 9 *Suppose that $s_H \approx 0$. When $\hat{p}_1 > 0$, (i) $\hat{w}_1 > \hat{w}_2$; (ii) if $\gamma_1 \approx \gamma_2$, then $\hat{w}_1 > \hat{p}_1 > \hat{r}_1 \approx \hat{r}_2 > 0 > \hat{w}_2$; (iii) if $\gamma_1 > \gamma_2$ and $s_L \approx 0$, then $\hat{w}_1 \approx \hat{w}_2 > \hat{p}_1 > 0 > \hat{r}_1 \approx \hat{r}_2$; (iv) if $\gamma_1 < \gamma_2$ and $s_L \approx 0$, then $\hat{r}_1 \approx \hat{r}_2 > \hat{p}_1 > 0 > \hat{w}_1 \approx \hat{w}_2$.*

Proof. Differentiating the equilibrium system (27)-(30), we obtain

$$\begin{pmatrix} 1 & -1 & s_L & 0 \\ -\frac{\gamma_1}{1-\gamma_1} & \frac{\gamma_2}{1-\gamma_2} & 0 & s_H \\ 0 & \frac{E_2}{1-\gamma_2} & \Lambda_2 & -\Theta_2 \\ \frac{E_1}{1-\gamma_1} & 0 & -\Lambda_1 & \Theta_1 \end{pmatrix} \begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{q}_L^* \\ \hat{q}_H^* \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{1-\gamma_1} \\ 0 \\ \frac{E_1}{1-\gamma_1} \end{pmatrix} \hat{p}_1,$$

where E_i is effective labor in sector i, defined as

$$E_1 = \bar{H}\left(\frac{\gamma_1 p_1}{w_1}\right)^{\frac{1}{1-\gamma_1}} \int_{q_H^*}^{q_{H\max}} q_H^{\frac{\beta_1}{1-\gamma_1}} \phi_H(q_H)\, dq_H,$$

$$E_2 = \bar{H}\left(\frac{\gamma_2 p_2}{w_2}\right)^{\frac{1}{1-\gamma_2}} \int_{q_{H\min}}^{q_H^*} q_H^{\frac{\beta_2}{1-\gamma_2}} \phi_H(q_H)\, dq_H,$$

and

$$\Lambda_1 = \bar{L}\left(q_L^*\right)^{\frac{\alpha_1}{\gamma_1}+1}\phi_L\left(q_L^*\right),$$

$$\Lambda_2 = \bar{L}\left(q_L^*\right)^{\frac{\alpha_2}{\gamma_2}+1}\phi_L\left(q_L^*\right),$$

$$\Theta_1 = \bar{H}\left(\frac{\gamma_1 p_1}{w_1}\right)^{\frac{1}{1-\gamma_1}}\left(q_H^*\right)^{\frac{\beta_1}{1-\gamma_1}+1}\phi_H\left(q_H^*\right),$$

$$\Theta_2 = \bar{H}\left(\frac{\gamma_2 p_2}{w_2}\right)^{\frac{1}{1-\gamma_2}}\left(q_H^*\right)^{\frac{\beta_2}{1-\gamma_2}+1}\phi_H\left(q_H^*\right).$$

The determinant of the matrix on the left-hand side of this system, D_{CD}, satisfies

$$\begin{aligned}(1-\gamma_2)(1-\gamma_1)(-D_{CD}) &= (\Theta_1\Lambda_2 - \Theta_2\Lambda_1)(\gamma_1-\gamma_2) + s_H\left[\Lambda_1 E_2(1-\gamma_1) + \Lambda_2 E_1(1-\gamma_2)\right] \\ &\quad + s_L(\Theta_1\gamma_1 E_2 + \Theta_2\gamma_2 E_1) + E_1 E_2 s_H s_L.\end{aligned}$$

Using the equilibrium conditions (29) and (30), we find that

$$(\Theta_1\Lambda_2 - \Theta_2\Lambda_1)(\gamma_1-\gamma_2) = \Theta_2\Lambda_1\frac{(\gamma_1-\gamma_2)^2}{\gamma_2(1-\gamma_1)} > 0.$$

Therefore $D_{CD} < 0$. We also compute

$$\begin{aligned}\hat{w}_1(1-\gamma_2)(1-\gamma_1)(-D_{CD}) &= (\Theta_1\Lambda_2 - \Theta_2\Lambda_1 + \Lambda_2 E_1 s_H)(1-\gamma_2)\hat{p}_1 \\ &\quad + \left[(\Theta_1 E_2 + \Theta_2\gamma_2 E_1)s_L + E_1 E_2 s_H s_L\right]\hat{p}_1, \\ \hat{w}_2(1-\gamma_1)(-D_{CD}) &= (\Theta_1\Lambda_2 - \Theta_2\Lambda_1 + \Lambda_2 E_1 s_H - \Theta_2 E_1 s_L)\hat{p}_1.\end{aligned}$$

Therefore,

$$(\hat{w}_1 - \hat{w}_2)(1-\gamma_1)(1-\gamma_2)(-D_{CD}) = \left[(\Theta_1 E_2 + \Theta_2\gamma_2 E_1)s_L + E_1 E_2 s_H s_L + \Theta_2 E_1 s_L(1-\gamma_2)\right]\hat{p}_1.$$

Since $D_{CD} < 0$, it follows that an increase in the price of good 1 results in $\hat{w}_1 > \hat{w}_2$, which proves part (i) of Proposition 9.

Next, consider the case in which $s_H \approx 0$ and $\gamma_1 \approx \gamma_2$. In this case,

$$(1-\gamma_2)(1-\gamma_1)(-D_{CD}) \approx s_L(\Theta_1\gamma_1 E_2 + \Theta_2\gamma_2 E_1).$$

Then

$$\hat{w}_1 - \hat{w}_2 \approx \frac{\Theta_1 E_2 + \Theta_2 E_1}{\Theta_1\gamma_1 E_2 + \Theta_2\gamma_2 E_1}\hat{p}_1,$$

because $\gamma_1 \approx \gamma_2$ implies $\Theta_1\Lambda_2 - \Theta_2\Lambda_1 \approx 0$. Evidently, in this case, $\hat{w}_1 > \hat{p}_1 > 0 > \hat{w}_2$. To complete the proof of part (ii) of Proposition 9, we need to calculate the response of the anchors r_1 and r_2 for the managers' salaries. When p_1 rises, (31) yields $\hat{r}_1 = (1-\gamma_1)^{-1}\hat{p}_1 - \gamma_1(1-\gamma_1)^{-1}\hat{w}_1$ and

$\hat{r}_2 = -\gamma_2 (1-\gamma_2)^{-1} \hat{w}_2$. In case (ii) of Proposition 9, with $s_H \approx 0$ and $\gamma_1 \approx \gamma_2$, these imply

$$\hat{r}_1 - \hat{r}_2 \approx \frac{\Theta_2 \gamma_2 E_1}{\Theta_1 \gamma_1 E_2 + \Theta_2 \gamma_2 E_1} \hat{p}_1.$$

It follows that $\hat{p}_1 > \hat{r}_1 \approx \hat{r}_2 > 0$. So, part (ii) of the proposition is proved.

We turn now to parts (iii) and (iv) of Proposition 9. The antecedents $s_H \approx 0$ and $s_L \approx 0$ imply

$$\begin{aligned} (1-\gamma_2)(1-\gamma_1)(-D_{CD}) &\approx (\Theta_1 \Lambda_2 - \Theta_2 \Lambda_1)(\gamma_1 - \gamma_2), \\ \hat{w}_1 (1-\gamma_1)(-D_{CD}) &\approx (\Theta_1 \Lambda_2 - \Theta_2 \Lambda_1) \hat{p}_1, \\ \hat{w}_2 (1-\gamma_1)(-D_{CD}) &\approx (\Theta_1 \Lambda_2 - \Theta_2 \Lambda_1) \hat{p}_1. \end{aligned}$$

It follows that

$$\hat{w}_1 - \hat{w}_2 \approx \frac{1-\gamma_2}{\gamma_1 - \gamma_2} \hat{p}_1,$$

which implies that $\hat{w}_1 \approx \hat{w}_2 > \hat{p}_1 > 0$ for $\gamma_1 > \gamma_2$ and $\hat{w}_1 \approx \hat{w}_2 < 0 < \hat{p}_1$ for $\gamma_1 < \gamma_2$. Moreover, since $\hat{r}_1 = (1-\gamma_1)^{-1} \hat{p}_1 - \gamma_1 (1-\gamma_1)^{-1} \hat{w}_1$ and $\hat{r}_2 = -\gamma_2 (1-\gamma_2)^{-1} \hat{w}_2$, we have

$$\hat{r}_1 - \hat{r}_2 \approx -\frac{\gamma_2}{\gamma_1 - \gamma_2} \hat{p}_1.$$

Evidently, in this case, $\hat{r}_1 \approx \hat{r}_2 < 0 < \hat{p}_1$ when $\gamma_1 > \gamma_2$ and $\hat{r}_1 \approx \hat{r}_2 > \hat{p}_1 > 0$ when $\gamma_1 < \gamma_2$. This completes the proof of Proposition 9. ∎

Proofs for Section 5.2

Consider a two-sector economy with strictly log-supermodular productivity under Assumption 1'. We first prove the result for an HH/LL equilibrium and then for an HL/LH equilibrium. We label sectors such that the best workers sort into sector 1.

HH/LL Equilibrium

In an HH/LL equilibrium the cutoffs $\{q_H^*, q_L^*\}$ satisfy:

$$w_1(q_L^*) = w_2(q_L^*), \tag{38}$$

$$r_1(q_H^*) = r_2(q_H^*), \tag{39}$$

where $[w_i(\cdot), r_i(\cdot), m_i(\cdot)]$ is a solution to the single-sector differential equations (13) and (18) for $i = 1, 2$ with the boundary conditions

$$m_2(q_{H\min}) = q_{L\min}, \quad m_2(q_H^*) = q_L^*, \tag{40}$$

$$m_1(q_H^*) = q_L^*, \quad m_1(q_{H\max}) = q_{L\max}. \tag{41}$$

Clearly, the solutions for the wage function, the salary function, and the matching functions depend on the parameters of the model, such as prices and factor endowments, as do the equilibrium cutoffs $\{q_H^*, q_L^*\}$. We denote by $dw_i(q_L)/dp_i$ the derivative of the wage function in sector i with respect to price p_i, where this derivative accounts for the endogenous adjustments of all three functions. This derivative contrasts with $w_i'(q_L)$, which is the slope of the wage function for given parameters. We use similar notation to represent derivatives of the salary function.

For now, we are interested in $\eta = p_2/p_1$ and we shall use the following elasticities

$$\varepsilon_{w_i,\eta}^* = \frac{dw_i(q_L)}{d(p_2/p_1)} \cdot \frac{p_2/p_1}{q_L}\bigg|_{q_L=q_L^*}, \quad \varepsilon_{r_i,\eta}^* = \frac{dr_i(q_H)}{d(p_2/p_1)} \cdot \frac{p_2/p_1}{q_H}\bigg|_{q_H=q_H^*}.$$

Differentiating (38)-(39) with respect to $\eta \equiv p_2/p_1$ yields

$$\left[\frac{w_1'(q_L^*)}{w_1(q_L^*)} - \frac{w_2'(q_L^*)}{w_2(q_L^*)}\right] dq_L^* = \varepsilon_{w_2,\eta}^* - \varepsilon_{w_1,\eta}^*, \tag{42}$$

$$\left[\frac{r_1'(q_H^*)}{r_1(q_H^*)} - \frac{r_2'(q_H^*)}{r_2(q_H^*)}\right] dq_H^* = \varepsilon_{r_2,\eta}^* - \varepsilon_{r_1,\eta}^*. \tag{43}$$

The assumptions that the equilibrium is of the HH/LL type and that the best workers and managers sort into sector 1 imply that the expressions in the square brackets are positive in both equations; that is, at the boundary $\{q_H^*, q_L^*\}$ between the two sectors the slopes of the wage and salary functions have to be steeper in sector 1 into which the more able employees sort. It follows that q_L^* rises in response to an increase in the relative price of sector 2 if and only if $\varepsilon_{w_2,\eta}^* > \varepsilon_{w_1,\eta}^*$ and the cutoff q_H^* rises if and only if $\varepsilon_{r_2,\eta}^* > \varepsilon_{r_1,\eta}^*$.

To understand the elasticities $\varepsilon_{w_i,\eta}^*$ and $\varepsilon_{r_i,\eta}^*$, note that a shift in p_2/p_1 impacts wages and salaries through two channels. First, there is the direct effect described in part (ii) of Lemma 1, which means that wages and salaries grow proportionally to the price within the sector when boundaries remain unchanged. But wages cannot increase everywhere by more in sector 2 than in sector 1, since in equilibrium the wages at the cutoff type q_L^* have to equalize across sectors. Therefore, re-matching in each sector is necessary, which impacts in turn the wage and salary functions, as implied by Lemmas 3-6 and Corollaries 1 and 2 to Lemma 6. In other words, the impact effect of a rise in the relative price of sector 2 increases the cutoffs for both workers and managers, but we also have to account for the induced change in matching in order to obtain the full effect. To this end, we now express the elasticities $\varepsilon_{w_i,\eta}^*$ and $\varepsilon_{r_i,\eta}^*$ as follows:

$$\varepsilon_{w_i,\eta}^* = \hat{\eta} + \varepsilon_{w_iL}^* \hat{q}_L^* + \varepsilon_{w_iH}^* \hat{q}_H^*, \quad i = 1, 2, \tag{44}$$

$$\varepsilon_{r_i,\eta}^* = \hat{\eta} + \varepsilon_{r_iL}^* \hat{q}_L^* + \varepsilon_{r_iH}^* \hat{q}_H^*, \quad i = 1, 2, \tag{45}$$

where the first term captures the direct effect from part (ii) of Lemma 1, $\varepsilon_{w_iL}^*$ is the elasticity of $w_i(\cdot)$ with respect to the boundary q_L^* through the induced re-matching (evaluated at q_L^*), and

$\varepsilon^*_{w_iH}$ is the elasticity of $w_i(\cdot)$ with respect to the boundary q^*_H through the induced re-matching (evaluated at q^*_L). From (12) and (13) we also have

$$\varepsilon^*_{r_iF} = -\frac{\gamma_i}{1-\gamma_i}\varepsilon^*_{w_iF}, \quad F = H, L; \quad i = 1, 2. \tag{46}$$

Now substitute these equations into (42) and (43) to obtain

$$M_h^{HH/LL}\begin{pmatrix} \hat{q}^*_L \\ \hat{q}^*_H \end{pmatrix} = \begin{pmatrix} \hat{p}_2 - \hat{p}_1 \\ \hat{p}_2 - \hat{p}_1 \end{pmatrix}, \tag{47}$$

where

$$M_h^{HH/LL} = \begin{pmatrix} q^*_L\left[\frac{w_1'(q^*_L)}{w_1(q^*_L)} - \frac{w_2'(q^*_L)}{w_2(q^*_L)}\right] + \varepsilon^*_{w_1L} - \varepsilon^*_{w_2L} & \varepsilon^*_{w_1H} - \varepsilon^*_{w_2H} \\ \frac{\gamma_2\varepsilon^*_{w_2L}}{1-\gamma_2} - \frac{\gamma_1\varepsilon^*_{w_1L}}{1-\gamma_1} & q^*_H\left[\frac{r_1'(q^*_H)}{r_1(q^*_H)} - \frac{r_2'(q^*_H)}{r_2(q^*_H)}\right] + \frac{\gamma_2\varepsilon^*_{w_2H}}{1-\gamma_2} - \frac{\gamma_1\varepsilon^*_{w_1H}}{1-\gamma_1} \end{pmatrix}.$$

From Lemmas 3-6 we have

$$\varepsilon^*_{w_1L} > 0, \quad \varepsilon^*_{w_2L} < 0, \quad \varepsilon^*_{w_1H} < 0, \quad \varepsilon^*_{w_2H} > 0.$$

These equations provide a solution to $\hat{q}^*_L$ and $\hat{q}^*_H$.

The determinant of the matrix $M_h^{HH/LL}$ is

$$\begin{aligned} D_{M_h^{HH/LL}} = & \left\{q^*_L\left[\frac{w_1'(q^*_L)}{w_1(q^*_L)} - \frac{w_2'(q^*_L)}{w_2(q^*_L)}\right] + \varepsilon^*_{w_1L} - \varepsilon^*_{w_2L}\right\} q^*_H\left[\frac{r_1'(q^*_H)}{r_1(q^*_H)} - \frac{r_2'(q^*_H)}{r_2(q^*_H)}\right] \\ & + \left(\frac{\gamma_2\varepsilon^*_{w_2H}}{1-\gamma_2} - \frac{\gamma_1\varepsilon^*_{w_1H}}{1-\gamma_1}\right) q^*_L\left[\frac{w_1'(q^*_L)}{w_1(q^*_L)} - \frac{w_2'(q^*_L)}{w_2(q^*_L)}\right] - \frac{\gamma_1-\gamma_2}{(1-\gamma_1)(1-\gamma_2)}\left(\varepsilon^*_{w_2H}\varepsilon^*_{w_1L} - \varepsilon^*_{w_1H}\varepsilon^*_{w_2L}\right). \end{aligned}$$

The first two terms on the right-hand side are positive. We now show that the third term also is positive. To this end, note from Lemma 2 that if we change a single boundary and the new boundary is on the original matching function then the new matching function coincides with the old one in the overlapping range of abilities. Therefore, if we choose $dq^*_L = m_i'(q^*_H)\,dq^*_H$, where $m_i(\cdot)$ is the solution of matching in sector i, then a change in the boundary (dq^*_H, dq^*_L) does not change the wage $w_i(q^*_L)$. In other words,

$$\varepsilon^*_{w_iH} + \varepsilon^*_{w_iL}\varepsilon^*_{m_i} = 0,$$

where $\varepsilon^*_{m_i}$ is the elasticity of $m_i(\cdot)$ evaluated at q^*_H. On the other hand, (14) implies for the HH/LL case that

$$\varepsilon^*_{m_i} = \frac{\kappa_m\gamma_i}{1-\gamma_i},$$

where

$$\kappa_m = \frac{\bar{H} r\left(q_H^*\right) \phi_H\left(q_H^*\right) q_H^*}{\bar{L} w\left(q_L^*\right) \phi_L\left(q_L^*\right) q_L^*}.$$

Therefore,

$$\varepsilon_{w_i H}^* = -\frac{\kappa_m \gamma_i}{1-\gamma_i} \varepsilon_{w_i L}^*.$$

Using this expression, we obtain

$$-\frac{\gamma_1-\gamma_2}{\left(1-\gamma_1\right)\left(1-\gamma_2\right)}\left(\varepsilon_{w_2 H}^* \varepsilon_{w_1 L}^* - \varepsilon_{w_1 H}^* \varepsilon_{w_2 L}^*\right) = -\frac{\left(\gamma_1-\gamma_2\right)^2 \kappa_m \varepsilon_{w_1 L}^* \varepsilon_{w_2 L}^*}{\left(1-\gamma_1\right)^2\left(1-\gamma_2\right)^2} > 0,$$

which proves that $D_{M_h^{HH/LL}} > 0$.

Therefore, solving (47) implies that $\hat{q}_L^* > 0$ and $\hat{q}_H^* > 0$ if and only if $\hat{\eta} > 0$. In other words, a rise in $\bar{H}/\bar{L}$ increases both cutoffs if and only if the relative price in sector 2 increases. That is, an increase in the relative price of good 2 raises both cutoffs and therefore raises output in sector 2 and reduces that in sector 1.

Next consider further implications of a rise in the price of good 2 on re-matching. Since the most able workers and the most able mangers sort into sector 1, we can use the differential equations (13) and (18) for $i = 1, 2$ with the boundary conditions (40) and (41) to characterize the solution to the matching functions $m_i\left(q_H\right)$ for $i = 1, 2$, given the equilibrium cutoffs $\left(q_H^*, q_L^*\right)$. An increase in p_2 shifts both cutoffs up, and this shift in boundary changes the matching functions in each sector. Also note that the solution to the differential equations extends beyond the range of abilities of workers and mangers who sort into a sector, so that $m_i\left(q_H\right)$ can be extended to abilities that are not employed in sector i.

The first thing to note is that due to the continuity of the wage and salary functions (14) implies:

$$\frac{\gamma_i/\left(1-\gamma_i\right)}{\gamma_j/\left(1-\gamma_j\right)} = \frac{m_i'\left(q_H^*\right)}{m_j'\left(q_H^*\right)}.$$

Therefore the matching function is steeper in the labor intensive sector at the cutoff q_H^*, and if labor intensity is the same in both sectors then $m_1'\left(q_H^*\right) = m_2'\left(q_H^*\right)$. Next note that if dq_L^* and dq_H^* are the changes in the boundaries in response to $\hat{p}_2 > 0$, then:

$$\frac{dq_L^*}{dp_2} = m_i'\left(q_H^*\right)\frac{dq_H^*}{dp_2} + \frac{\partial m_i\left(q_H^*\right)}{\partial p_2} \quad \text{for } i = 1, 2, \tag{48}$$

where $\partial m_i\left(q_H^*\right)/\partial p_2$ is the change in matching of a manager of ability q_H^* in response to the price rise. Evidently, if $\gamma_1 = \gamma_2$, in which case $m_1'\left(q_H^*\right) = m_2'\left(q_H^*\right)$, this equation implies

$$\frac{\partial m_1\left(q_H^*\right)}{\partial p_2} = \frac{\partial m_2\left(q_H^*\right)}{\partial p_2}.$$

This implies that in Figure 3 the matching functions in both sectors shift from point b either to the right or to the left. Therefore small changes $\hat{p}_2 > 0$ cannot lead to a shift in matching of the

ab_3c type. Also note from (48) that:

$$\frac{\partial m_i\left(q_H^*\right)}{\partial p_2}=\frac{\partial m_j\left(q_H^*\right)}{\partial p_2}+\left[m_j'\left(q_H^*\right)-m_i'\left(q_H^*\right)\right]\frac{dq_H^*}{dp_2}.$$

It follows that $\partial m_i\left(q_H^*\right)/\partial p_2 > \partial m_j\left(q_H^*\right)/\partial p_2$ if an only if $m_j'\left(q_H^*\right) > m_i'\left(q_H^*\right)$, or if and only if $\gamma_j > \gamma_i$. This implies that if the matching function for $q_H = q_H^*$ shifts in this figure to the right in one sector and to the left in the other, the leftward shift has to be in the labor intensive sector. Finally, note that Lemma 4 implies that if $\partial m_i\left(q_H^*\right)/\partial p_2 > 0$ then $\partial m_i\left(q_H\right)/\partial p_2 > 0$ for all ability levels q_H between q_H^* and the other end point ($q_{H\min}$ for sector 2 and $q_{H\max}$ for sector 1). And if $\partial m_i\left(q_H^*\right)/\partial p_2 < 0$ then $\partial m_i\left(q_H\right)/\partial p_2 < 0$ for all ability levels q_H between q_H^* and the other end point.

HL/LH Equilibrium

In an HL/LH equilibrium, the cutoffs $\{q_H^*, q_L^*\}$ also satisfy the continuity conditions (38) and (39), but the boundary conditions are different. Assuming as before that the best workers sort into sector 1, this means that in an HL/LH equilibrium the best managers sort into sector 2 and the boundary conditions are

$$m_1\left(q_{H\min}\right)=q_L^*,\quad m_1\left(q_H^*\right)=q_{L\max},$$

$$m_2\left(q_H^*\right)=q_{L\min},\quad m_2\left(q_{H\max}\right)=q_L^*.$$

Figure 6 depicts the pattern of sorting and matching in this type of equilibrium. The more able workers sort into sector 1 only if

$$\frac{w_1'\left(q_L^*\right)}{w_1\left(q_L^*\right)}>\frac{w_2'\left(q_L^*\right)}{w_2\left(q_L^*\right)}$$

and the more-able managers sort into sector 2 only if

$$\frac{r_1'\left(q_H^*\right)}{r_1\left(q_H^*\right)}<\frac{r_2'\left(q_H^*\right)}{r_2\left(q_H^*\right)}.$$

To derive the comparative statics, we use as before conditions (42) and (43), which apply in this case too. We also can use the decomposition of elasticities (44) and (45), which still apply. Now, however, the relationship between the elasticities of the salary and wage functions, as described by (46), does not apply, because workers of ability q_L^* do not pair with managers of ability q_H^*, as is evident from Figure 6. Instead, from (12) and (13) we now obtain

$$\varepsilon_{r_1F}^*=-\frac{\gamma_1}{1-\gamma_1}\varepsilon_{w_1F}^{\max},\quad F=H,L,$$

$$\varepsilon_{r_2F}^*=-\frac{\gamma_2}{1-\gamma_2}\varepsilon_{w_2F}^{\min},\quad F=H,L,$$

where $\varepsilon_{r_iF}^*$ is defined in the same way as before, $\varepsilon_{w_1F}^{\max}$ is the elasticity of $w_1(\cdot)$ with respect to the boundary q_F^* through the induced re-matching in sector 1 (evaluated at $q_{L\max}$) and $\varepsilon_{w_2F}^{\min}$ is the elasticity of $w_2(\cdot)$ with respect to the boundary q_F^* through the induced re-matching in sector 2 (evaluated at $q_{L\min}$). Using these results the system of equations (47) is replaced by

$$M_h^{HL/LH}\begin{pmatrix}\hat{q}_L^*\\ \hat{q}_H^*\end{pmatrix}=\begin{pmatrix}\hat{p}_2-\hat{p}_1\\ \hat{p}_2-\hat{p}_1\end{pmatrix}, \tag{49}$$

where

$$M_h^{HL/LH}=\begin{pmatrix} q_L^*\left[\frac{w_1'(q_L^*)}{w_1(q_L^*)}-\frac{w_2'(q_L^*)}{w_2(q_L^*)}\right]+\varepsilon_{w_1L}^*-\varepsilon_{w_2L}^* & \varepsilon_{w_1H}^*-\varepsilon_{w_2H}^* \\ \frac{\gamma_2\varepsilon_{w_2L}^{\min}}{1-\gamma_2}-\frac{\gamma_1\varepsilon_{w_1L}^{\max}}{1-\gamma_1} & q_H^*\left[\frac{r_1'(q_H^*)}{r_1(q_H^*)}-\frac{r_2'(q_H^*)}{r_2(q_H^*)}\right]+\frac{\gamma_2\varepsilon_{w_2H}^{\min}}{1-\gamma_2}-\frac{\gamma_1\varepsilon_{w_1H}^{\max}}{1-\gamma_1}\end{pmatrix}. \tag{50}$$

From Lemmas 3-6, we have $\varepsilon_{w_1L}^*>0>\varepsilon_{w_2L}^*$, $\varepsilon_{w_1H}^*>0>\varepsilon_{w_2H}^*$, $\varepsilon_{r_1H}^*<0<\varepsilon_{r_2H}^*$, $\varepsilon_{r_1L}^*<0<\varepsilon_{r_2L}^*$. This implies that both entries in the top row in (50) are strictly positive and both entries in the bottom row are strictly negative.

The previous observations imply that a positive term $\hat{p}_2-\hat{p}_1$ either raises q_L^* and reduces q_H^*, or it reduces q_L^* and raises q_H^*. The cutoffs cannot both move in the same direction, because the effect in the top row on the left hand side of (49) would then be opposite to those in the bottom row, whereas on the right hand side both effects have the same sign. We will show that only a rise in q_L^* and a reduction q_H^* can be associated with equilibrium responses, which implies that the determinant of $M_h^{HL/LH}$ must be negative ($D_{M_h^{HL/LH}}<0$). To prove this, consider an increase in the price p_2 to $p_2'>p_2$ while the price p_1 stays constant. Let X_1 and X_2 denote the output in each sector prior to the price change, and let X_1' and X_2' denote the corresponding output after the price change. Since only prices have changed (and not endowments), under each set of prices both the outputs (X_1,X_2) and (X_1',X_2') are feasible. Since the competitive equilibrium is efficient, the value of output is maximized given prices, which implies that

$$\begin{aligned} p_1X_1+p_2X_2 &\geq p_1X_1'+p_2X_2', \\ p_1X_1+p_2'X_2 &\leq p_1X_1'+p_2'X_2', \end{aligned}$$

where the first inequality states that prior to the price change the value of output is higher under production bundle (X_1,X_2) than under (X_1',X_2'), while the opposite holds after the price change. Subtracting and rearranging gives

$$(p_2-p_2')(X_2-X_2')\geq 0,$$

which implies that $X_2\leq X_2'$. An increase in output in sector two cannot be achieved with a fall in q_L^* and a rise q_H^*, because in this case there would be less worker types and less manager types in sector 2. Therefore, an increase in the relative price of good 2 leads to a rise in q_L^* and a reduction q_H^*.

References

[1] Akerman, Anders, Helpman, Elhanan, Itskhoki, Oleg, Muendler, Marc-Andreas and Redding, Stephen J., 2013. "Sources of Wage Inequality," *A.E.R. Papers and Proc. 103(3)*: 214-219.

[2] Autor, David H., Katz, Lawrence F., and Kearney, Melissa S., 2008. "Trends in U.S. Wage Inequality: Revising the Revisionists," *Rev. of Econ. and Stat. 90(2)*: 300-23.

[3] Eeckhout, Jan and Kircher, Philipp, 2012. "Assortative Matching with Large Firms: Span of Control over More versus Better Workers." Downloadable at http://homepages.econ.ed.ac.uk/~pkircher/Papers/Sorting-and-Factor-Intensity.pdf.

[4] Helpman, Elhanan, Itskhoki, Oleg, Muendler, Marc-Andreas and Redding, Stephen J., 2015. "Trade and Inequality: From Theory to Estimation," Downloadable at https://www.princeton.edu/~itskhoki/papers/TradeInequalityEvidence.pdf.

[5] Kopczuk, Wojciech, Saez, Emmanuel, and Song, Jae, 2010. "Earnings Inequality and Mobility in the United States: Evidence from Social Security Data Since 1937," *Q.J.E. 125*(1): 91-128.

[6] Muendler, Marc-Andreas, 2004. "Trade, Technology, and Productivity: A Study of Brazilian Manufacturers, 1986-1998", CESifo Working Paper No. 1148.

[7] Sampson, Thomas, 2014. "Selection into Trade and Wage Inequality," *A.E.J. Macro 6(3): 157-202*.