Copula Estimation

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1 Introduction

In this paper, we provide a brief survey of estimation procedures for copula models. Depending on the assumptions made on copula models considered, the data generating process and an approach to inference, the procedures lead to parametric, semi-parametric and non-parametric copula inference methods for i.i.d. observations (random samples) of random vectors with dependent components and copula-based time series. We review parametric, semiparametric and nonparametric approaches to inference on copulas for random samples with dependent marginals and copula-based time series and also discuss several problems of robust estimation in these frameworks.

The survey is organized as follows. Section 2 discusses parametric (Section 2.1), semi-parametric (Section 2.2) and non-parametric (Section 2.3) estimation methods for copulas that characterize dependence among the components of random vectors: the inference procedures are based on i.i.d. vector observations. Section 3 considers the case of copula-based time series. In particular, Section 3.1 discusses copula-based characterizations of (higher-dimensional) Markov processes, and Section 3.4 reviews weak dependence

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properties of copula-based time series. Sections 3.2 and 3.3 discuss parametric, semiparametric and nonparametric estimation methods for copulas in the time series context. Section 4 discusses some further copula inference approaches motivated, in part, by robustness considerations. Section 5 reviews empirical applications of copula estimation methods and discusses pricing Collateralized Debt Obligations (CDO) using different copula models.

2 Copula estimation: Random samples with dependent marginals

2.1 Parametric models: maximum likelihood methods and inference from likelihoods for margins

Section 10.1 in [29] provides an excellent treatment and review of maximum likelihood (ML) estimation of parameters in multivariate copula models and computationally attractive parametric inference procedures for them.

Consider the key relation in copula theory given by formula (11) in [1] under absolute continuity assumptions. In the notations of Sklar's theorem in [1], the density f of the $d-$dimensional d.f. F with univariate margins $F_1, F_2, ..., F_d$ and the corresponding univariate densities $f_1, f_2, ..., f_d$ can be represented as

$$f(x_1, x_2, ..., x_d) = c(F_1(x_1), F_2(x_2), ..., F_d(x_d)) \prod_{i=1}^{d} f_i(x_i), \tag{1}$$

where $c(u_1, u_2, ..., u_d) = \frac{\partial C(u_1,u_2,...,u_d)}{\partial u_1 \partial u_2 ... \partial u_d}$ is the density of the $d-$dimensional copula $C(u_1, u_2, ..., u_d; \theta)$ in (11) in [17].

Representation (1) implies the following decomposition for the log-likelihood function $L = \sum_{j=1}^{n} \log f(x_1^{(j)}, x_2^{(j)}, ..., x_d^{(j)})$ of a random sample of (i.i.d.) vectors $x^{(j)} = (x_1^{(j)}, x_2^{(j)}, ..., x_d^{(j)})$, $j = 1, 2, ..., n$, with the density $f:$

$$L = \underbrace{L_C}_{dependence} + \underbrace{\sum_{i=1}^{d} L_i}_{marginals}, \tag{2}$$

where $L_C = \sum_{j=1}^{n} \log \ c(F_1(x_1^{(j)}), F_2(x_2^{(j)}), ..., F_d(x_d^{(j)}))$ is the log-likelihood contribution in from dependence structure in data represented by the copula C and $L_i = \sum_{j=1}^{n} \log \ f_i(x_i^{(j)})$, $i = 1, 2, ..., d$, are the log-likelihood contributions from each margin: observe that $\sum_{i=1}^{d} L_i$ in (2) is exactly the log-likelihood of the sample under the independence assumption.

Suppose that the copula C belongs to a family of copulas indexed by a (vector) parameter $\theta : C = C(u_1, u_2, ..., u_d; \theta)$ and the margins F_i and the corresponding univariate densities f_i are indexed by (vector) parameters α_i : $F_i = F_i(x_i; \alpha_i)$, $f_i = f_i(x_i; \alpha_i)$. The maximum likelihood estimator - MLE - $(\hat{\alpha}_1^{MLE}, \hat{\alpha}_2^{MLE}, ..., \hat{\alpha}_d^{MLE}, \hat{\theta}_d^{MLE})$ of the model parameters $(\alpha_1, \alpha_2, ..., \alpha_d, \theta)$ corresponds to simultaneous maximization of the log-likelihood L in (2):

$$
\begin{aligned}
(\hat{\alpha}_1^{MLE}, \hat{\alpha}_2^{MLE}, ..., \hat{\alpha}_d^{MLE}, \hat{\theta}_d^{MLE}) &= \\
arg\ max_{\alpha_1, \alpha_2, ..., \alpha_d, \theta}\ L(\alpha_1, \alpha_2, ..., \alpha_d, \theta) &= \\
arg\ max_{\alpha_1, \alpha_2, ..., \alpha_d, \theta}\ L_C(\alpha_1, \alpha_2, ..., \alpha_d, \theta) + \sum_{i=1}^{d} L_i(\alpha_i) &= \\
arg\ max_{\alpha_1, \alpha_2, ..., \alpha_d, \theta} \sum_{j=1}^{n} log\ c(F_1(x_1^{(j)}; \alpha_1), F_2(x_2^{(j)}; \alpha_2), ..., & \\
F_d(x_d^{(j)}; \alpha_d); \theta) + \sum_{i=1}^{d} \sum_{j=1}^{n} \log\ f_i(x_i^{(j)}; \alpha_i). & \qquad (3)
\end{aligned}
$$

Section 10.1 in [29] (see also [30] and [46]) discusses a computationally attractive alternative to the maximum likelihood (ML) estimation involving simultaneous maximization over the dependence (θ) and marginal $(\alpha_1, \alpha_2, ..., \alpha_d)$ parameters in (3). This estimation approach is motivated by decomposition (2) and is referred to as the method of inference functions for margins (IFM) in [29, Section 10.1]. In the first stage of the inference procedure, the estimators $\hat{\alpha}_i^{IFM}$ of the parameters α_i are estimated from the log-likelihood L_i of each margin in (2)-(3): $\hat{\alpha}_i^{IFM} = argmax_{\alpha_i} L_i(\alpha_i)$. That is, $(\hat{\alpha}_1^{IFM}, \hat{\alpha}_2^{IFM}, ..., \hat{\alpha}_d^{IFM})$ is defined to be the MLE of the model parameters under independence. In the second stage of the procedure, the estimator $\hat{\theta}^{IFM}$ of the copula parameter θ^{IFM} is computed by maximizing the copula likelihood contribution L_C in (2)-(3) with the marginal parameters α_i replaced by their first-stage estimators $\hat{\alpha}_i^{IFM}$: $\hat{\theta}^{IFM} = argmax_\theta\ L_C(\hat{\alpha}_1^{IFM}, \hat{\alpha}_2^{IFM}, ..., \hat{\alpha}_d^{IFM}, \theta)$. While, under regularity conditions, the MLE estimator $(\hat{\alpha}_1^{MLE}, \hat{\alpha}_2^{MLE}, ..., \hat{\alpha}_d^{MLE}, \hat{\theta}^{MLE})$ solves

$$(\partial L/\partial\alpha_1, \partial L/\partial\alpha_2, ..., \partial L/\partial\alpha_d, \partial L/\partial\theta) = 0,$$

the two-stage IFM estimator $(\hat{\alpha}_1^{IFM}, \hat{\alpha}_2^{IFM}, ..., \hat{\alpha}_d^{IFM}, \hat{\theta}^{IFM})$ solves

$$(\partial L_1/\partial\alpha_1, \partial L_2/\partial\alpha_2, ..., \partial L_d/\partial\alpha_d, \partial L/\partial\theta) = 0.$$

As discussed in [29], the MLE and IFM estimation procedures are equivalent in the special case of multivariate normal d.f.'s that have multivariate Gaussian copulas discussed in Section 6.1 in [17] and univariate normal margins. Naturally, however, this equivalence does not in general.

Similar to the MLE, the IFM estimator $(\hat{\alpha}_1^{IFM}, \hat{\alpha}_2^{IFM}, ..., \hat{\alpha}_d^{IFM}, \hat{\theta}_d^{IFM})$ is consistent and asymptotically normal under the usual regularity conditions (see [43]) for the multivariate model and for each of its margins. However, estimation of the corresponding covariance matrices is difficult both analytically and numerically due to the need to compute many derivatives, and jackknife and related methods may be used in inference (see [29]). As discussed in [29, Section 10.1], efficiency comparisons based on estimation of the asymptotic covariance matrices and Monte-Carlo simulation for different dependence models suggest that the IFM approach to inference provide a highly efficient alternative to the MLE estimation of multivariate model parameters.

2.2 Semiparametric estimation

Similar to the inference procedures discussed in Section 2.1, semi-parametric estimation methods for copula parameters are usually motivated by density representations and decompositions for the log-likelihood of dependence models as in (1) and (2). In the first stage, the univariate margins F_i are estimated non-parametrically, e.g., by the empirical d.f.'s $\hat{F}_i$ or their scaled versions. In the second stage, the copula parameters are estimated using maximization of the contribution to the log-likelihood function from the dependence structure in the data represented by a copula C of interest.

Formally, consider, as in [21], the problem of estimation of the (vector) parameter θ of a family of $d-$dimensional copulas $C(u_1, u_2, ..., u_d, \theta)$ with the density $c(u_1, u_2, ..., u_d, \theta)$. Given non-parametric estimators $\hat{F}_i$ of the univariate margins F_i, it is natural to estimate the copula parameter θ as

$$\hat{\theta} = arg\ max_\theta L_C(\theta) = arg\ max_\theta \sum_{j=1}^{n} log\ c(\hat{F}_1(x_1^{(j)}), \hat{F}_2(x_2^{(j)}), ..., \hat{F}_d(x_d^{(j)}); \theta).$$

As discussed in [21], the resulting semi-parametric estimator $\hat{\theta}$ of the dependence parameter θ is consistent and asymptotically normal under suitable regularity conditions. The authors of [21] propose a consistent estimator of the limiting variance-covariance matrix of $\hat{\theta}$. They further show that, under additional copula regularity assumptions that are satisfied for a large class of bivariate copulas, including bivariate Gaussian, Eyraud-Farlie-Gumbel-Morgenstern (EFGM), Clayton and Frank (see Section 6.1 and relations (19), (20) and (22) in [17]) families, the estimator $\hat{\theta}$ is fully efficient at independence. Numerical results presented in [21] demonstrate that efficiency of $\hat{\theta}$

compares favorably to the alternative semiparametric estimators in the Clayton family of copulas. In addition, according to the numerical results, the coverage probability of confidence intervals for θ based on asymptotic normality of $\hat{\theta}$ is close to the nominal level for random samples with Clayton copulas exhibiting small to medium range of dependence.

2.3 Nonparametric inference and empirical copula processes

Most of nonparametric estimation procedures for copulas are based on inversion formula (12) discussed in introductory survey [17]. In the procedures, an estimator $\hat{C}(u_1, u_2, ..., u_d)$ of a $d-$copula $C(u_1, u_2, ..., u_d)$ is typically given by an empirical analogue of inversion formula (12) in [17]. That is,

$$\hat{C}(u_1, u_2, ..., u_d) = \hat{F}(\hat{F}_1^{-1}(u_1), \hat{F}_2^{-1}(u_2), ..., \hat{F}_d^{-1}(u_d)), \tag{4}$$

where $\hat{F}$ is a nonparametric estimator of the $d-$dimensional d.f. F and $\hat{F}_1^{-1}, \hat{F}_2^{-1}, ..., \hat{F}_d^{-1}$ are nonparametric estimators of the pseudo-inverses $F_i^{-1}(s)$ $= \{t|F_i(t) \geq s\}$ of the univariate margins $F_1, F_2, ..., F_d$.[1] Typically, $\hat{F}$ is taken to be the empirical $d-$dimensional d.f. $\hat{F}(x_1, x_2, ..., x_d) = \frac{1}{T}\sum_{t=1}^{T} I(X_1 \leq x_1, X_2 \leq x_2, ..., X_d \leq x_d)$ and F_i^{-1} are estimated using the pseudo-inverses $\hat{F}_i^{-1}(s) = \{t|\hat{F}_i(t) \geq s\}$ of the empirical univariate d.f.'s $\hat{F}_i = \frac{1}{T}\sum_{t=1}^{T} I(X_i \leq x_i)$ or their rescaled versions (here and throughout the survey, $I(\cdot)$ denote the indicator functions).

Deheuvels [11, 12] established consistency and asymptotic normality of the empirical copula process for i.i.d. observations of random vectors with independent margins (the case of the product copula $C = \Pi$, see Section 3 in [17]). Gaenssler and Stute [20, Ch. 5] and Fermanian *et al.* [18] establish consistency and asymptotic normality of the empirical copula process for general copulas C with continuous partial derivatives (see also [19]).

Fermanian *et al.* [18] further show that, under regularity conditions, asymptotic normality also holds for smoothed copula processes like $\hat{C}(u_1, u_2)$ $= \hat{F}(\hat{F}_1^{-1}(u_1), \hat{F}_2^{-1}(u_2))$ that are constructed using the nonparametric kernel estimators $\hat{F}(x_1, x_2) = \frac{1}{T}\sum_{t=1}^{T} K(\frac{x-X_t}{a_T}, \frac{y-Y_t}{a_T})$ of the joint d.f. F. Here $K(x_1, x_2) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} k(u, v)dudv$ for some bivariate kernel function k : $\mathbf{R}^2 \to \mathbf{R}$ with $\int k(x, y)dxdy = 1$, and the sequence of bandwidths $a_T > 0$ satisfies $a_T \to 0$ as $T \to \infty$ (see also [5] for the normal asymptotics of smoothed copula processes based on local linear versions of K rather than the kernel itself for dealing with the boundary bias).

[1] Here and throughout the survey, F_i are assumed to be continuous as in (12) in [17], if not stated otherwise.

3 Copula-based time series and their estimation

3.1 Copula-based characterizations for (higher-order) Markov processes

Darsow *et al.* [10] obtained the following necessary and sufficient conditions for a time series process based on bivariate copulas to be first-order Markov (see also Section 6.4 in [38]). For copulas $A, B : [0,1]^2 \to [0,1]$, set

$$(A * B)(x,y) = \int_0^1 \frac{\partial A(x,t)}{\partial t} \cdot \frac{\partial B(t,y)}{\partial t} dt.$$

Further, for copulas $A : [0,1]^m \to [0,1]$ and $B : [0,1]^n \to [0,1]$, define their $\star-$product $A \star B : [0,1]^{m+n-1} \to [0,1]$ via

$$A{\star}B(x_1,...,x_{m+n-1}) = \int_0^{x_m} \frac{\partial A(x_1,...,x_{m-1},\xi)}{\partial \xi} \cdot \frac{\partial B(\xi, x_{m+1},...,x_{m+n-1})}{\partial \xi} d\xi.$$

As shown in [10], the operators $*$ and $\star$ on copulas are distributive over convex combinations, associative and continuous in each place, but not jointly continuous.

Darsow *et al.* [10] show that the transition probabilities $P(s,x,t,A) = P(X_t \in A | X_s = x)$ of a real-valued stochastic process $\{X_t\}_{t \in T}$, $T \subseteq \mathbf{R}$, satisfy Chapman-Kolmogorov equations

$$P(s,x,t,A) = \int_{-\infty}^{\infty} P(u,\xi,t,A) P(s,x,u,d\xi)$$

for all Borel sets A, all $s < t$ in T, $u \in (s,t) \cap T$ and for almost all $x \in \mathbf{R}$ if and only if the copulas corresponding to bivariate d.f.'s of X_t are such that

$$C_{st} = C_{su} * C_{ut}$$

for all $s, u, t \in T$ such that $s < u < t$. The paper [10] also shows that a real-valued stochastic process $\{X_t\}_{t \in T}$ is a first-order Markov process if and only if the copulas corresponding to the finite-dimensional d.f.'s of $\{X_t\}$ satisfy the conditions

$$C_{t_1,...,t_n} = C_{t_1 t_2} \star C_{t_2 t_3} \star ... \star C_{t_{n-1} t_n}$$

for all $t_1, ..., t_n \in T$ such that $t_k < t_{k+1}$, $k = 1, ..., n-1$. In particular, a sequence of identically distributed r.v.'s $\{X_t\}_{t=1}^{\infty}$ is a stationary Markov process if and only if for all $n \geq 2$,

$$C_{1,...,n}(u_1,...,u_n) = \underbrace{C \star^k C \star^k ... \star^k C}_{n-k+1}(u_1,...,u_n) = C^{n-k+1}(u_1,...,u_n), \quad (5)$$

where C is bivariate copula. It is natural to refer to the processes $\{X_t\}_{t=1}^{\infty}$ constructed via (5) as stationary Markov processes based on the copula C or as stationary $C-$based Markov processes for short.

Among other results, Ibragimov [27] obtained extensions of the results in [10] that provide characterizations of higher-order Markov processes in terms of copulas corresponding to their finite-dimensional d.f.'s. The results in [27] show that a Markov process of order k is fully determined by its $(k+1)-$dimensional copulas and one-dimensional marginal cdf's. The characterizations thus provide a justification for estimation of finite-dimensional copulas of time series with higher-order Markovian dependence structure. Using the results, [27] further obtains necessary and sufficient conditions for higher-order Markov processes to exhibit several additional dependence properties, such as $m-$dependence, $r-$independence or conditional symmetry. These conditions are closely related to $U-$statistics-based characterizations for joint distributions and copulas developed in [13]. Using the obtained results, [27] also presents a study of applicability and limitations of different copula families in constructing higher-order Markov processes with the above dependence properties.

The results in [10] and [27] provide a copula-based approach to the analysis of higher-order Markov processes which is alternative to the conventional one based on transition probabilities. The advantage of the approach based on copulas is that it allows one to separate the study of dependence properties (e.g., $r-$independence, $m-$dependence or conditional symmetry) of the stochastic processes in consideration from the analysis of the effects of marginal distributions (say, unconditional heavy-tailedness or skewness). In particular, the results provide methods for construction of higher-order Markov processes with arbitrary one-dimensional margins that, possibly, satisfy additional dependence assumptions. These processes can be used, for instance, in the analysis of the robustness of statistical and econometric procedures to weak dependence. In addition, they provide examples of non-Markovian processes that nevertheless satisfy Chapman-Kolmogorov stochastic equations. Higher-order Markov processes with prescribed dependence properties can be constructed, for instance, using inversion inversion formula (12) in [17] for finite-dimensional cdf's of known examples of dependent time series (see the discussion in [10] and [27]).

3.2 Parametric and semiparametric copula estimation methods for Markov processes

As discussed in Section 10.4 in [29], under suitable regularity conditions, asymptotic results for ML estimation under i.i.d. observations (see Section 2.1) also hold for stationary Markov processes.

Chen and Fan [6] propose semiparametric estimation procedures for copula-based Markov processes. These procedures generalize the two-stage semiparametric inference approaches discussed in Section 2.2 to the time series framework.

Consider a stationary Markov process based on a bivariate copula $C = C(u_1, u_2; \theta)$ (see Section 3). Similar to Section 2.2, the semiparametric estimator $\hat{\theta}$ of the copula parameter in [6] is defined as

$$\hat{\theta} = arg\ max_\theta L_C(\theta) = arg\ max_\theta \sum_{j=1}^{n} log\ c(\hat{F}(x_1^{(j)}), \hat{F}(x_2^{(j)}); \theta),$$

where $\hat{F}(x)$ is a nonparametric estimator of the univariate margin of X_t : e.g., the rescaled empirical d.f. $\hat{F}(x) = \frac{1}{T+1} \sum_{t=1}^{T} I\{X_t \leq x\}$ as in [6]. Chen and Fan [6] show that the semiparametric estimator $\hat{\theta}$ is consistent and asymptotically normal under suitable regularity assumptions. These assumptions include a condition that the process $\{X_t\}$ is $\beta-$mixing with polynomial decay rate. Chen and Fan [6] provide copula-based sufficient conditions for the latter weak dependence assumption and further verify the assumptions implying consistency and asymptotic normality of $\hat{\theta}$ for Markov processes based on Gaussian, Clayton and Frank copulas. As discussed in [6], the asymptotic variance of $\hat{\theta}$ can be estimated using heteroskedasticity autocorrelation consistent (HAC) estimators (see, for instance, [1], [39] and Ch. 10 in [24]). Alternatively, the asymptotic distributions of the estimator can be approximated using bootstrap (see Section 4.3 in [6]).

In a recent paper, Chen, Wu and Yi [7] consider efficient sieve ML estimation methods for copula-based stationary Markov processes. The authors show that sieve MLE's of any smooth functionals of the copula parameter and marginal d.f. and are root$-n$ consistent, asymptotically normal and efficient; and that their sieve likelihood ratio statistics are asymptotically chi-square distributed. The numerical results in [7] further indicate that the sieve MLEs of copula parameters, the margins and the conditional quantiles all perform very well in finite samples even for Markov models based on tail dependent copulas, such as Clayton, Gumbel-Hougaard or (see Section 6.1 and relations (18) and (19) in [17]). In addition, in the case of Markov models generated via tail dependent copulas, the sieve MLEs have much smaller biases and smaller variances than the two-step semiparametric estimation procedures in [6] discussed above.

3.3 Nonparametric copula inference for time series

Doukhan, Fermanian and Lang [16] discuss extensions of the results on copula processes in Section 2.3 to the time series framework. The results in [16] imply, in particular, that empirical copula process (4) is asymptotically Gaussian for weakly dependent vector-valued processes $\{X_t\}$, including the case of strongly mixing and $\beta-$mixing sequences with polynomial decay rates. More generally, asymptotic normality of the copula processes holds for random sequences that satisfy multivariate functional central limit theorem for empirical processes (see [15]). Applied to the vector-valued process (X_t, X_{t+1}) for a $C-$based stationary Markov sequence $\{X_t\}$ satisfying mixing conditions (see Section 3.1), these results imply asymptotic normality of the empirical copula process $\hat{C}(u_1, u_2) = \hat{G}(\hat{F}^{-1}(u_1), \hat{F}^{-1}(u_2))$, where $\hat{F}(x)$ is a nonparametric estimator of the univariate margin of X_t and $\hat{G}$ is a nonparametric estimator of the bivariate d.f. of (X_t, X_{t+1}) : e.g., in the case of empirical d.f.'s, $\hat{F}(x) = \frac{1}{T}\sum_{t=1}^{T} I(X_t \leq x)$, and $\hat{G}(x) = \frac{1}{T}\sum_{t=1}^{T} I(X_t \leq x)I(X_{t+1} \leq x)$. Doukhan, Fermanian and Lang [16] further establish asymptotic normality of the smoothed copula process with kernel estimates of the multivariate d.f. and univariate margins in (4) for weakly dependent vector-valued sequences. Similar asymptotic results are also shown to hold for smoothed copula densities (see also, among others, [23] for the analysis of the asymptotics of kernel estimators of the copula density for i.i.d. vector observations with dependent components).

3.4 Dependence properties of copula-based time series

As discussed in Sections 3.2 and 3.3, consistency and asymptotic normality of estimators of copula functions and their parameters are obtained under assumptions of weak dependence in the time series considered. Among other results, Beare [2], Chen, Wu and Yi [7] and Lentzas and Ibragimov [35] provide a study of persistence properties of of stationary copula-based Markov processes. Lentzas and Ibragimov [35] show via simulations that stationary Markov processes based on tail-dependent Clayton copulas (see (19) in [17]) can behave as long memory time series on the level of copulas exhibiting high persistence important for financial and economic applications. This long memory-like behavior is captured by an extremely slow decay of copula-based dependence measures between lagged values of the processes for commonly used lag numbers. The theoretical results in [35] further show that, in contrast, Gaussian and EFGM copulas (see Section 6.1 and (22) in [17]) always produce short memory stationary Markov processes. Beare [2] shows that a $C-$based stationary Markov process exhibits weak dependence properties, including $\alpha-$ and $\beta-$mixing with exponential decay rates, if C is a symmetric absolutely continuous copula with a square integrable density and the maximal correlation coefficient of C is less than one. These results imply, in particular, that stationary Markov processes based on Gaussian and EFGM copulas are weakly dependent and mixing. Beare [2] also provides numerical results that suggest exponential decay in $\beta-$mixing coefficients and, thus, also in $\alpha-$mixing coefficients of Clayton copula-based stationary Markov processes. Chen, Wu and Yi [7] obtain theoretical results that show that tail-dependent Clayton, survival Clayton, Gumbel and $t-$copulas always generate Markov processes that are geometric ergodic and hence geometric $\beta-$mixing. The conclusions in [7] imply that, although, according to the numerical results in [35], Clayton copula-based Markov processes can behave like long memory time series on copula levels exhibiting high persistence for commonly used lag numbers, they are in fact weakly dependent and short memory in terms of mixing properties.

4 Further copula inference methods

Naturally, applications of the copula inference methods described in Sections 2.1-3.3 require consistent estimation of the asymptotic covariance matrices in the Gaussian convergence results. As discussed in [29, Section 10] and Section 2.1 in this paper, this is a difficult task both analytically and numerically even in the case of i.i.d. observations of vectors with dependent margins. At the same time, the conclusions and numerical results in [2, 7] and [35] reviewed in Section 3.4 indicate importance of development of robust methods for differentiating short and long memory in copula-based time series and robust inference approaches for copula models with dependent observations. As discussed in Chen and Fan [6] (see Section 3.2), in the case of time series, inference on the asymptotic covariance matrices may be based on HAC estimators and computationally expensive bootstrap procedures. As discussed in a number of works (see [33] and references therein), however, tests based on HAC estimators may have substantial size distortions in finite samples. Motivated, in part, by these results, Kiefer, Vogelsang and Bunzel [33] show that asymptotically justified inference in a linear time series regression may be based on inconsistent analogues of HAC estimators with a nondegenerate limiting distribution (see also [31, 32] for extensions of the approach). It would be interesting to use inconsistent analogues of HAC estimators similar to those considered in [31, 32, 33] in semiparametric tests on copula parameters in the time series framework (see Section 3.2).

Ibragimov and Mueller [28] develop a general strategy for conducting inference about a scalar parameter with potentially heterogenous and correlated data, when relatively little is known about the precise property of the correlations. Assume that the data can be classified in a finite number q of groups that allow asymptotically independent normal inference about the scalar parameter of interest θ (e.g., a scalar component of the vector copula parameter as in the settings discussed in Sections 2.1, 2.2 and 3.2) That is, there exist estimators $\hat{\theta}_j$ calculated for groups $j = 1, ..., q$, that are asymptotically normal: $\sqrt{T}(\hat{\theta}_j - \theta) \Rightarrow \mathcal{N}(\theta, \sigma_j)$, and $\hat{\theta}_j$ is asymptotically independent of $\hat{\theta}_i$ for $j \neq i$. The asymptotic normality of $\sqrt{T}(\hat{\theta}_j - \theta)$, $j = 1, ..., q$, typically follows from the same reasoning as the asymptotic normality of the full sample estimator $\hat{\theta}$: as discussed in Sections 2.1-3.3, asymptotic Gaussianity is typically a standard result for many estimators of copula model parameters.

As follows from [28], one can perform an asymptotically valid test of level α, $\alpha \leq 0.05$ of $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ by rejecting H_0 when $|t_\theta|$ exceeds the $(1-\alpha/2)$ percentile of the Student$-t$ distribution with $q-1$ degrees of freedom, where t_θ is the usual $t-$statistic

$$t_\theta = \sqrt{q}\frac{\overline{\hat{\theta}} - \theta_0}{s_{\hat{\theta}}} \tag{6}$$

with $\overline{\hat{\theta}} = q^{-1}\sum_{j=1}^{q}\hat{\theta}_j$, the sample mean of the group estimators $\hat{\theta}_j$, $j = 1, ..., q$, and $s_{\hat{\theta}}^2 = (q-1)^{-1}\sum_{j=1}^{q}(\hat{\theta}_j - \overline{\hat{\theta}})^2$, the sample variance of $\hat{\theta}_j$, $j = 1, ..., q$.

In other words, in general settings considered in [28], the usual $t-$tests can be used in the presence of asymptotic heteroskedasticity in group estimators as long the level of the tests is not greater than the typically used 5% threshold. Similar to applications considered in [28], the $t-$statistic-based approach to robust inference can be applied in statistical analysis of copula models with autocorrelated vector observations. As discussed in [28], the $t-$statistic-based approach provides a number of important advantages over the existing methods of inference in time series, panel, clustered and spatially correlated data. In particular, the approach can be employed when data are potentially heterogeneous and dependent in a largely unknown way, as is typically the case for copula models. In addition, the approach is simple to implement and does not need new tables of critical values. The assumptions of asymptotic normality for group estimators in the approach are explicit and easy to interpret, in contrast to conditions that imply validity of alternative procedures in many settings. Furthermore, as shown in [28], the $t-$statistic based approach to robust inference efficiently exploits the information contained in these regularity assumptions, both in the small sample settings (uniformly most powerful scale invariant test against a benchmark alternative with equal variances) and also in the asymptotic frameworks. The numerical results presented in [28] demonstrate that, for many dependence and heterogeneity settings considered in the literature and typically encountered in practice for time series, panel, clustered and spatially correlated data, the applications of the approach lead to robust tests with attractive finite sample performance. The analysis of the performance of the approach in inference on copula parameters and, more generally, dependence measures and other characteristics, appears to be an interesting problem that is left for further research.

5 Empirical Applications

Many works in the literature have focuses on empirical applications of copulas and related concepts in different fields, including economics, finance, risk management and insurance (see, among others, the review and discussion in [8], [13] and [37]). Hu [26] focuses on mixed copula modeling of dependence among financial variables, with the parameters estimated using semiparametric procedures described in Section 2.2 applied to the residuals in GARCH models fitted to data. Patton [40] applies copula structures for modeling asymmetric dependence in foreign exchange markets and, among other results, finds a dramatic decrease in the tail dependence parameters in Clayton family models for the Deutsche mark and the yen exchange rates following the introduction of the euro in January 1999. Giacomini, Härdle and Spokoiny[22] focus on modeling dependence of financial returns using copulas with time-varying adaptively estimated parameters. Lee and Long [34] consider extensions and empirical applications of multivariate GARCH models with copula-based specifications of dependence among the vectors components. Among other results, Caillault and Guégan [4] provide nonparametric estimates of tail dependence parameters for Asian financial indices. Smith [44, 45] discusses applications of copulas in sample selection and regime switching models.

Copulas have become standard tools in credit risk modelling, see [42]. Unfortunately, the current global financial crises has shown that industry models, that are using copulas to evaluate risk, can be very inadequate and should be treated with caution. However, copulas are powerful and flexible tools and there is still a lot of space for improvement.

In this section, we present applications of copula functions in pricing Collateralized Debt Obligation (CDO). A CDO is a structured financial product that enables securitization of a large portfolio of assets. The portfolio's risk is sliced into tranches and transfered to investors with different risk preferences. Each CDO tranche is defined by the detachment and attachment points which are the percentages of the portfolio losses. The losses are caused by defaults of the reference entities. The tranches of the iTraxx index are created by following levels: 0%, 3%, 6%, 9%, 12%, 22%, 100%. Investors agree to cover the losses that might appear in the range of a particular tranche in exchange for a fee. During the life of the contract investors are periodically paid a premium, usually once per quarter, which mainly depends on the type of the tranche and the value of the accumulated losses. The most risky tranche, called equity, bears first 3% losses of the portfolio nominal but pays the highest spread. If the losses exceed 3%, the next are absorbed by mezzanine tranches and after by senior tranches. The losses over 22% are allocated to the most senior tranche, often called super super senior. With the increase of seniority, the value of the offered premium decreases. For a survey of the CDO construction and pricing we refer to [3].

The values of the spreads of the CDO tranches depend on the joint behavior and joint distribution of the assets in the underlying pool and on their tendency to default simultaneously. The idea of modelling the joint distribution of defaults with copula functions was introduced by Li [36]. In this method a default is defined by a random variable called a time-until-default whose distribution is derived from market data. The joint distribution of default times is specified with a one parameter Gaussian copula. The Li model has been seen until now as the industry standard in the CDO valuation. However, during the financial meltdown it has been strongly criticized for its incapability of modelling joint extreme events due to tail independence of the underlying Gaussian copulas. Because of the drawbacks of the one-factor Gaussian copula model numerous new approaches have been proposed. Due to the high dimension nature of the problem most of the CDO models are fully parametric (see Section 2.1). Semi-parametric and non-parametric calibration procedures were proposed in [14], [41] and [42].

We present below the results of the empirical study conducted with the iTraxx Euro index series 8 with a maturity of 5 years. The series 8 was issued on 20 September 2007 and expires on 20 December 2012. For computations we consider the market values of the first $J = 5$ tranches, see table 1, observed on 3rd August 2009. We also use spreads of 125 credit default swaps (CDS) that constitute the portfolio of iTraxx. The default dependency structure of the credits is specified with the one-parameter Gumbel and the one-parameter survival Clayton copula. The results are compared with the Gaussian copula model. The CDO valuations based on copulas of many parameters are studied in [9] and [25]. The first step in CDO pricing consists of estimating the risk of the underlyings. The default times of portfolio credits are assumed to be exponentially distributed with parameters implied from the market quotes of CDS contracts. We assume the constant interest rate of 3% and the constant recovery rate of 40%. The fit of the models to market data is archived by minimizing the following function:

$$D(t_0) \stackrel{\text{def}}{=} \sum_{j=1}^{J} \frac{|s_j^c(t_0) - s_j^m(t_0)|}{s_j^m(t_0)} \to \min, \tag{7}$$

which sums the relative deviations of the model upfront fee s_1^c and the spreads s_j^c, $j = 2, \ldots, J$, from the market values s_j^m, $j = 1, \ldots, J$, observed at time t_0. The detailed description of the calibration algorithm is provided in [9].

Table 1 presents the obtained results. The values of measure D show that the market prices are better described by the Gumbel and by the survival Clayton copula than by the Gaussian copula. The improvement is achieved as the Gumbel and the survival Clayton copula exhibit the upper tail dependence which allows to quantify the risk that many obligors default at the same time. The Gumbel copula gives the most accurate fit to the market tranche quotes.

Table 1 Results of CDO calibration on 3rd August 2009.

Model	Parameter	D	0%-3%	3%-6%	6%-9%	9%-12%	12%-22%
Market			39.593	503.750	407.500	165.228	84.727
Gauss	0.2209	3.2620	28.957	1284.324	622.758	313.985	86.025
Surv.Clayton	1.1215	2.6420	15.573	526.393	364.179	279.925	185.547
Gumbel	1.2000	1.1554	39.473	896.437	317.965	179.453	90.405

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