ON AN EXACT CONSTANT FOR THE ROSENTHAL INEQUALITY*

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Abstract. Let \(\xi_1, \ldots, \xi_n\) be independent random variables having symmetric distribution with finite \(p\)th moment, \(2 < p < \infty\). It is shown that the precise constant \(C_p^*\) in Rosenthal's inequality

\[
\left\| \sum_{i=1}^{n} \xi_i \right\|_p \leq C_p \max \left( \left\| \sum_{i=1}^{n} \xi_i \right\|_2, \left( \sum_{i=1}^{n} \left\| \xi_i \right\|_p^p \right)^{1/p} \right)
\]

has the form

\[
C_p^* = \left( 1 + \frac{2^{p/2}}{\pi^{1/2}} \Gamma \left( \frac{p+1}{2} \right) \right)^{1/p}, \quad 2 < p < 4,
\]

where \(\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx\) and \(\xi_1, \xi_2\) are independent Poisson random variables with parameter 0.5. It is proved also that

\[
\lim_{p \to \infty} \frac{C_p^* \log_p}{p} = \frac{1}{e}.
\]

Key words. Rosenthal's inequality, random variables with symmetric distribution, Poisson random variable, moment

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Let \(\|\xi\|_s\) be the \(L_s\)-norm of the random variable \(\xi\): \(\|\xi\|_s = (\mathbb{E}|\xi|^s)^{1/s}\), where \(\mathbb{E}(\cdot)\) means the expectation operator.

Rosenthal [1] has proved the inequality

(1) \[
\left\| \sum_{i=1}^{n} \xi_i \right\|_p \leq C_p \max \left( \left\| \sum_{i=1}^{n} \xi_i \right\|_2, \left( \sum_{i=1}^{n} \left\| \xi_i \right\|_p^p \right)^{1/p} \right),
\]

where \(C_p\) is a constant depending on \(p\) only, and \(\xi_1, \ldots, \xi_n\) are independent random variables (r.v.'s.) having symmetric distribution with finite \(p\)th moment, \(2 < p < \infty\). In particular, one can take \(C_p = 2^p\) as \(C_p\).

Papers [2], [3], [5], and [9] deal with some refinements and generalizations of inequality (1).

Using Sazonov’s estimate [2] one can obtain (1) with a constant of order \(2^{p/4}\) in \(p\).

It follows from the results of [3] and [4] that one can take \(C_p = Lp\) as \(C_p\) where \(L\) is an absolute constant.

Let \(C_p^*\) be the exact constant in inequality (1), that is

\[
C_p^* = \frac{\sup \| \sum_{i=1}^{n} \xi_i \|_p}{\max(\| \sum_{i=1}^{n} \xi_i \|_2, (\sum_{i=1}^{n} \| \xi_i \|_p^p)^{1/p})}.
\]

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where sup is taken over all nondegenerate independent r.v.’s $\xi_1, \ldots, \xi_n$ with symmetric distributions having finite $p$th moment, $2 < p < \infty$.

As shown in [5],
\[
\frac{p}{2^{1/2} e \max(1, \log p)} \leq C_p^* \leq \frac{7.35 p}{\max(1, \log p)},
\]
that is $C_p^*$ behaves as $p/\log p$ if $p \to \infty$.

The problem of determining the exact value of $C_p^*$ is closely related with the problem of calculating the extreme of a convex functional defined on the class of sums of independent random variables with symmetric distributions. This problem goes back to Prokhorov’s paper [6]. It was also investigated in [7] and [8].

In the present paper we give a clear expression for $C_p^*$, $p > 2$, and find the asymptotics of $C_p^*$ as $p \to \infty$.

**Theorem 1.** The exact value of $C_p^*$ in Rosenthal’s equation has the form
\[
C_p^* = \left(1 + \frac{2^{p/2}}{\pi^{1/2}} \Gamma \left(\frac{p+1}{2}\right)\right)^{1/p}, \quad 2 < p < 4, \\
C_p^* = \|\xi_1 - \xi_2\|_p, \quad p \geq 4,
\]
where
\[
\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx, \quad \xi_1, \xi_2
\]
are independent Poisson random variables with parameter 0.5.

**Theorem 2.** $C_{2s}^* = T_{2s}^{1/(2s)}$, where $T_{2s}$ is the number of partitions of a $2s$-set (a set consisting of $2s$ elements) into parts each of which includes an even number of elements, $s \in \mathbb{N}$.

**Corollary.** $\lim_{p \to \infty} C_p^* (\log p)/p = 1/e$.

**Remark 1.** The next representation for the numbers $T_{2s}$ follows from Theorems 1 and 2. It is an analogue of Dobinski’s formula for Bell numbers:
\[
T_{2s} = \frac{2}{e} \sum_{m=1}^\infty \sum_{k=0}^\infty \frac{m^{2s}}{k!(m+k)!2^{m+2k}}.
\]

To prove Theorem 1 we need an auxiliary result which is of independent interest.

Our notation follows [8]. Let $\xi_1, \ldots, \xi_n$ be independent r.v.’s with symmetric distribution having finite $p$th moment, $2 < p < \infty$, $a_i \geq 0$, $b_i \geq 0$, $a_i^p \leq b_i$, $i = 1, \ldots, n$, $A, B, D \geq 0$. Set
\[
(\xi, n) = (\xi_1, \ldots, \xi_n), \quad (a, n) = (a_1, a_2, \ldots, a_n), \quad (b, n) = (b_1, b_2, \ldots, b_n),
\]
\[
M_1(n, a, b) = \{ (\xi, n) : \mathbf{E} \xi_i^2 = a_i, \mathbf{E} |\xi_i|^p = b_i, i = 1, \ldots, n \}, \quad M_2(n, a, b) = \{ (\xi, n) : \mathbf{E} \xi_i^2 = a_i, \mathbf{E} |\xi_i|^p = b_i, i = 1, \ldots, n \},
\]
\[
U_1(A, B) = \left\{(\xi, n) : n \geq 1, \sum_{i=1}^n \mathbf{E} \xi_i^2 = B^2, \sum_{i=1}^n \mathbf{E} |\xi_i|^p = A \right\},
\]
\[
U_2(A, B) = \left\{(\xi, n) : n \geq 1, \sum_{i=1}^n \mathbf{E} \xi_i^2 \leq B^2, \sum_{i=1}^n \mathbf{E} |\xi_i|^p \leq A \right\}.
\]
Let \( \xi_1(A, B), \xi_2(A, B) \) be independent Poisson r.v.'s with parameter \( \frac{1}{2} (B^p / A)^{2/(p-2)} \) and let \( \varepsilon_1(a_1, b_1, p), \ldots, \varepsilon_n(a_n, b_n, p) \) be independent r.v.'s with distribution \( P(\varepsilon(a, b, p) = 0) = 1 - (a^p/b^p)^{2/(p-2)} \), \( P(\varepsilon(a, b, p) = 1) = 1/2 \), \( i = 1, \ldots, n \) be independent r.v.'s with distribution \( P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = 1/2 \).

**Proposition.** If \( 2 < p < 4 \), then

\[
(2) \quad \sup_{(\xi, n) \in M_n(n,a,b)} E \left| \sum_{i=1}^n \xi_i \right|^p = \sum_{i=1}^n (b_i - a_i^p) + E \left| \sum_{i=1}^n a_i \varepsilon_i \right|^p, \quad k = 1, 2,
\]

\[
(3) \quad \sup_{(\xi, n) \in U_n(A,B)} E \left| \sum_{i=1}^n \xi_i \right|^p = A + \frac{2p/2}{\pi^{1/2}} \Gamma \left( \frac{p+1}{2} \right) B^p, \quad k = 1, 2.
\]

If \( 3 \leq p < 4 \), then

\[
(4) \quad \inf_{(\xi, n) \in M_n(n,a,b)} E \left| \sum_{i=1}^n \xi_i \right|^p = E \left| \sum_{i=1}^n \varepsilon_i(a_i, b_i, p) \right|^p,
\]

If \( p \geq 4 \), then

\[
(5) \quad \sup_{(\xi, n) \in M_n(n,a,b)} E \left| \sum_{i=1}^n \xi_i \right|^p = E \left| \sum_{i=1}^n \varepsilon_i(a_i, b_i, p) \right|^p, \quad k = 1, 2,
\]

\[
(6) \quad \sup_{(\xi, n) \in U_n(A,B)} E \left| \sum_{i=1}^n \xi_i \right|^p = \left( \frac{A}{B^2} \right)^{p/(p-2)} E \left| \xi_1(A, B) - \xi_2(A, B) \right|^p, \quad k = 1, 2,
\]

\[
(7) \quad \inf_{(\xi, n) \in M_n(n,a,b)} E \left| \sum_{i=1}^n \xi_i \right|^p = \sum_{i=1}^n (b_i - a_i^p) + E \left| \sum_{i=1}^n a_i \varepsilon_i \right|^p.
\]

**Remark 2.** Relations (5)-(7) constitute the statement of Theorem 5 in [8].

**Remark 3.** Relations (5) and (7) mean that for \( p \geq 4 \)

\[
\sup_{(\xi, n) \in M_n(n,a,b)} E \left| \sum_{i=1}^n \xi_i \right|^p, \quad k = 1, 2,
\]

is attained at three-point and

\[
\inf_{(\xi, n) \in M_n(n,a,b)} E \left| \sum_{i=1}^n \xi_i \right|^p
\]

at two-point distributions. Relations (2) and (4) show that in the case \( 3 \leq p < 4 \) one should permute the answers above.

**Remark 4.** Letting \( k = 1, n = 2, p = 3 \) in (2), we see that for independent random variables \( X \) and \( Y \) with common symmetric distribution and finite third moment the following inequalities hold true:

\[
E |X + Y|^3 \leq 2E |X|^3 + 2(EX^2)^{3/2},
\]

\[
E |X - Y|^3 \leq 2E |X|^3 + 2(EX^2)^{3/2},
\]
in which the constants are optimal. Provided \( EX = 0 \) these inequalities have been proved by Esseen [9].

Before proceeding to demonstrate relations (2)–(4), we establish several lemmas.

Let \( \varepsilon \) be a r.v. with distribution \( P\{ \varepsilon = 1 \} = P\{ \varepsilon = -1 \} = \frac{1}{2} \). Following [8] we set

\[
H(x, p, z) = E|\varepsilon x^{1/2} + z|^p - x^{p/2} - z^p,
R(x, p, z) = x^{-2/(p-2)}\left( E|\varepsilon x^{1/(p-2)} + z|^p - z^p \right), \quad x > 0, \ z > 0.
\]

**Lemma 1.** The function \( H(x, p, z) \) is non-negative and concave in \( x \) for \( p \in (2, 4) \).

*Proof.* Let \( p \in (2, 4) \) and \( f(x) = E|\varepsilon x + z|^p - x^p - z^p \). It is not difficult to check that

\[
f'(x) = p \left( \frac{1}{2} ( (x + z)^{p-1} + |x - z|^{p-2}(x - z)) - x^{p-1} \right) \geq 0
\]

for all \( x \geq 0, \ z \geq 0 \). Since \( f(0) = 0 \), the function \( H(x, p, z) \) is non-negative.

We show that \( H(x, p, z) \) is concave in \( x \) for \( p \in (2, 4) \). Since

\[
\frac{\partial^2 H}{\partial x^2} = 0.25 p x^{p/2-2} \left( (p-2) E|\varepsilon + u|^{p-2} - E|\varepsilon + u|^{p-2} \varepsilon u - (p-2) \right),
\]

where \( u = x^{-1/2} z \), it suffices to demonstrate that

\[
g(u) = (p-2) E|\varepsilon + u|^{p-2} - E|\varepsilon + u|^{p-2} \varepsilon u - (p-2) \leq 0
\]

for \( u \geq 0 \) and \( p \in (2, 4) \).

For \( u \geq 1, \ 0 \leq \alpha < 1 \) we have

\[
(u + \alpha) (u - 1)^\alpha \leq (u + \alpha) \left( 1 - \frac{\alpha}{u} + \frac{\alpha(\alpha - 1)}{2u^2} \right) u^\alpha \\
\leq (u - \alpha) \left( 1 - \frac{\alpha}{u} + \frac{\alpha(\alpha - 1)}{2u^2} \right) u^\alpha \leq (u - \alpha) (u + 1)^\alpha,
\]

that is \( \alpha E|\varepsilon + u|^\alpha \leq E|\varepsilon + u|^\alpha \varepsilon u \). Consequently, \( g(u) \leq 0 \) for \( p \in (2, 3), \ u \geq 1 \) and \( g'(u) \leq 0 \) for \( p \in [3, 4], \ u \geq 1 \).

We know that for \( 0 \leq u < 1 \) and \( 0 \leq \alpha < 1 \)

\[
E|\varepsilon + u|^\alpha \leq 1, \quad (\alpha - u) (1 + u)^\alpha \leq (u + \alpha) (1 - u)^\alpha;
\]

hence, \( g(u) \leq 0 \) for \( p \in (2, 3), \ 0 \leq u < 1 \) and \( g'(u) \leq 0 \) for \( p \in [3, 4], \ 0 \leq u < 1 \).

Combining these estimates and recalling that \( g(0) = 0 \), we conclude that \( g(u) \leq 0 \) for \( p \in (2, 4) \) and \( u \geq 0 \). Lemma 1 is proved.

**Lemma 2.** For all \( p \in (2, 4), \ 0 \leq a_1 \leq a_2, \ 0 \leq b_1 \leq b_2, \ z \geq 0 \) the following inequality is valid:

\[
b_1 - a_1^p + E|a_1 \varepsilon + z|^p \leq b_2 - a_2^p + E|a_2 \varepsilon + z|^p.
\]

*Proof.* While checking the non-negativity of \( H(x, p, z) \) we have established that the function \( E|\varepsilon x + z|^p - x^p \) is nondecreasing in \( x \geq 0 \) for \( z \geq 0 \). Hence, the statement of Lemma 2 follows easily.

**Lemma 3.** The function \( R(x, p, z) \) is non-negative and convex in \( x \) for \( p \in [3, 4] \).
Proof. By Jensen’s inequality \( R(x, p, z) \) is non-negative for \( p \geq 1 \). To demonstrate the convexity of \( R(x, p, z) \) in \( x \) for \( p \in [3, 4) \) it suffices to establish the validity of the inequality (see [8, the proof of Lemma 3.3]):

\[
M(u) = (p - 3) E|\varepsilon u + 1|^{p-3}(\varepsilon u + 1) - (p - 1) E|\varepsilon u + 1|^{p-2} + 2 \leq 0, \quad u \geq 0.
\]

It is not difficult to show that the following inequalities are valid:

\[
\alpha u < (\alpha u + 1)^\alpha - (\alpha u + 1)(u - 1)^\alpha \leq 0
\]

for \( u \geq 1, 0 \leq \alpha < 1 \),

\[
(\alpha u + 1) (1 - u)^\alpha \leq (\alpha u + 1) \left( 1 - \alpha u + \frac{\alpha(\alpha - 1)u^2}{2} \right) \leq (1 - \alpha u) (1 - \alpha u) (u + 1)^\alpha
\]

for \( 0 \leq u < 1, 0 \leq \alpha < 1 \). Since

\[
M'(u) = \frac{p - 1}{2} \left[ ((p - 3) u - 1) (u + 1)^{p-3} - ((p - 3) u + 1) (u - 1)^{p-3} \right]
\]

for \( u \geq 1 \) and

\[
M'(u) = \frac{p - 1}{2} \left[ ((p - 3) u + 1) (1 - u)^{p-3} - (1 - (p - 3) u) (u + 1)^{p-3} \right]
\]

for \( 0 \leq u < 1 \), it follows that \( M'(u) \leq 0 \) for \( u \geq 0 \). On account of \( M(0) = 0 \) we obtain the statement of Lemma 3.

Lemmas 4–7 below are analogues of the lemmas proved in [8].

**Lemma 4.** Let \( \xi \) be a random variable with symmetric distribution and let \( \xi, \eta, \varepsilon \) be independent random variables with \( E\xi^2 = a^2 \), \( E|\xi|^p < \infty \), \( E|\eta|^p < \infty \), \( p \in (2, 4) \). Then

\[
E|\xi + \eta|^p \leq E|\xi|^p + E|\varepsilon\alpha + \eta|^p - a^p.
\]

**Proof.** From the proof of Lemma 7.4 in [8] it follows that to demonstrate (8) it suffices to check that

\[
H(x^2, p, z) \lambda + H(y^2, p, z) \mu \leq H(a^2)
\]

for \( \lambda + \mu \leq 1, \lambda, \mu \geq 0, \lambda x^2 + \mu y^2 = a^2 \). By Lemma 1

\[
H(x^2, p, z) \lambda + H(y^2, p, z) \mu = (\lambda + \mu) \left( \frac{H(x^2, p, z)}{\lambda + \mu} + \frac{H(y^2, p, z)}{\lambda + \mu} \right) \leq (\lambda + \mu) H \left( \frac{a^2}{\lambda + \mu} \right) + (1 - \lambda - \mu) H(0) \leq H(a^2).
\]

Lemma 4 is proved.

**Lemma 5.** Let \( \xi \) be a random variable with symmetric distribution and let \( \xi, \eta, \varepsilon \) be independent random variables with \( E|\eta|^p < \infty \), \( E\xi^2 \leq a^2 \), \( E|\xi|^p \leq b \) and \( a^p \leq b \), \( a, b \geq 0, p \in (2, 4) \). Then

\[
E|\xi + \eta|^p \leq b + E|\varepsilon\alpha + \eta|^p - a^p.
\]

**Proof.** Lemma 5 follows from Lemmas 2 and 4.
If η and ε are independent random variables with $E|\eta|^p < \infty$ and $a^p \leq b$, $a, b \geq 0$, $p \in (2, 4)$, then

$$\sup \xi E|\xi + \eta|^p = b + E|\varepsilon \alpha + \eta|^p - a^p,$$

where $\sup$ is taken over all random variables $\xi$ with symmetric distributions being independent of $\eta$ and satisfying the conditions $E\xi^2 \leq a^2$, $E|\xi|^p \leq b$.

Proof. Lemma 6 easily follows from Lemma 5 and the proof of Lemma 7.5 in [8].

Lemma 7. Let $\xi$ be a random variable with symmetric distribution, and let $\xi, \eta, \varepsilon(a, b, p)$ be independent random variables with

$$E|\eta|^p < \infty, \quad E\xi^2 = a^2, \quad E|\xi|^p = b, \quad a^p \leq b, \quad a, b \geq 0, \quad p \in [3, 4).$$

Then

$$E|\xi + \eta|^p \geq E|\varepsilon(a, b, p) + \eta|^p.$$  

Proof. The proof of Lemma 7 follows the pattern of Lemma 7.1 in [8]. According to Lemma 3 it is necessary only to reverse the inequality signs in (7.2) and (7.3).

Proof of the proposition. Inequalities (2) and (4) easily follow from Lemmas 6 and 7 by induction.

We prove (3). Let

$$D_1^0(A, B) = \left\{(a, n), (b, n): a_i \geq 0, b_i \geq 0, a_i^p \leq b_i, \quad i = 1, \ldots, n, \right\}$$

$$D_2^0(A, B) = \left\{(a, n), (b, n): a_i \geq 0, b_i \geq 0, a_i^p \leq b_i, \quad i = 1, \ldots, n, \right\}$$

$$D_1(A, B) = \bigcup_{n=1}^{\infty} D_1^n(A, B), \quad D_2(A, B) = \bigcup_{n=1}^{\infty} D_2^n(A, B).$$

Since (see [10])

$$E \left| \sum_{i=1}^{n} a_i \xi_i \right|^p \leq \frac{2^p}{\pi^{1/2}} \Gamma \left( \frac{p+1}{2} \right) B^p$$

for $\sum_{i=1}^{n} a_i^2 = B^2$, it follows that

$$\sup_{(\xi, n) \in U(A, B)} E \left| \sum_{i=1}^{n} \xi_i \right|^p = \sup_{(a, n), (b, n) \in D_1(A, B)} \sup_{(\xi, n) \in M_1(n, a, b)} E \left| \sum_{i=1}^{n} \xi_i \right|^p$$

$$= \sup_{(a, n), (b, n) \in D_1(A, B)} \left\{ \sum_{i=1}^{n} (b_i - a_i^p) + E \left| \sum_{i=1}^{n} a_i \xi_i \right|^p \right\}$$

$$\leq A + \frac{2^p}{\pi^{1/2}} \Gamma \left( \frac{p+1}{2} \right) B^p.$$  

We know that

$$A + \sup_{n} B^p \left( E \left| \sum_{i=1}^{n} \frac{\varepsilon_i}{n^{1/2}} \right|^p - n^{1-p/2} \right) = A + \frac{2^p}{\pi^{1/2}} \Gamma \left( \frac{p+1}{2} \right) B^p;$$
hence,
\[
\sup_{(\xi,n) \in U_k(A,B)} E \left| \sum_{i=1}^{n} \xi_i \right|^p = A + \frac{\alpha^{p/2}}{\pi^{1/2}} \Gamma\left(\frac{p+1}{2}\right) B^p.
\]

**Proof of Theorem 1.**

\[
\sup_{(\xi,n) \in U_1(D,D^{1/p})} E \left| \sum_{i=1}^{n} \xi_i \right|^p \leq \sup_{(\xi,n) \in U(D)} E \left| \sum_{i=1}^{n} \xi_i \right|^p \leq \sup_{(\xi,n) \in U_2(D,D^{1/p})} E \left| \sum_{i=1}^{n} \xi_i \right|^p
\]

and relations (3) and (6) we obtain

\[
\sup_{(\xi,n) \in U(D)} E \left| \sum_{i=1}^{n} \xi_i \right|^p = \left(1 + \frac{2^{p/2}}{\pi^{1/2}} \Gamma\left(\frac{p+1}{2}\right)\right) D, \quad 2 < p < 4,
\]

\[
\sup_{(\xi,n) \in U(D)} E \left| \sum_{i=1}^{n} \xi_i \right|^p = E |\theta_1 - \theta_2|^p D, \quad p \geq 4.
\]

This fact and the obvious relation

\[
C_p^* = \sup_{D>0} \left( \frac{1}{D} \right) \left( \frac{1}{D} E \left| \sum_{i=1}^{n} \xi_i \right|^p \right)^{1/p}
\]

yield the statement of the theorem.

Before proving Theorem 2 we formulate the following lemma.

**Lemma 8 ([7]).** For \( p = 2s, \ s \in \mathbb{N}, \) the following relation is valid:

\[
\sup_{(\xi,n) \in U_1(A,B)} E \left| \sum_{i=1}^{n} \xi_i \right|^p = \sum_{j=1}^{s} \Gamma_{j,s}(A^{s-j}B^{2s(j-1)})^{1/(s-1)},
\]

where

\[
\Gamma_{j,s} = (2s)! \sum_{j=1}^{s} \sum_{r=1}^{j} \prod_{k=1}^{r} \frac{((2m_k)!)^{-j_k}}{j_k!},
\]

and the internal sum is ranged over all positive integers \( m_1 > \cdots > m_r \) and \( j_1, \ldots, j_r, \)

meeting the conditions \( m_1 j_1 + \cdots + m_r j_r = s, \ j_1 + \cdots + j_r = j. \)

**Proof of Theorem 2.**

\[
\sup_{(\xi,n) \in U_k(D,D^{1/p})} E \left| \sum_{i=1}^{n} \xi_i \right|^p = \left( \sum_{j=1}^{s} \Gamma_{j,s} \right) D = T_{2s} D, \quad k = 1, 2,
\]

where \( T_{2s} \) is the number of partitions of a \((2s)\)-set into parts each of which contains an even number of elements (see [11, p. 280–281]). Hence we deduce, following the pattern of demonstrating Theorem 1, that \( C_{2s}^* = T_{2s}^{1/(2s)}. \) Theorem 2 is proved.

**Proof of the corollary.** We establish a stronger result:

\[
C_p^* = \frac{p}{e \log p} \left( 1 + o\left( \frac{\log^2 \log p}{\log p} \right) \right).
\]

Theorem 1 implies

\[
(C_p^*)^p = \frac{2}{e} \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \frac{m^p}{k!(m+k)! 2^{m+2k}}.
\]
First we observe that

\begin{equation}
\frac{2}{e} \sum_{m=1}^{\infty} 2^m m! \leq (C_p^*)^p \leq \frac{2}{e} \sum_{m=1}^{\infty} 2^m m! \sum_{k=0}^{\infty} \frac{1}{2^k k!} \leq \frac{32}{15e} \sum_{m=1}^{\infty} 2^m m!.
\end{equation}

Set

\[
S = \sum_{m=1}^{\infty} \frac{m^p}{2^m m!}.
\]

We find the asymptotics of $S$ in a manner similar to the procedure for calculating the asymptotics of Bell's numbers [11, Chapter V, section 8].

We write $S$ as a sum of three summands

\[
S = S_1 + S_2 + S_3 = \sum_{m=1}^{\mu-1} + \sum_{m=\mu}^{\nu} + \sum_{m=\nu+1}^{\infty},
\]

where $\mu, \nu = [e^r/2 + Ap^{1/2}]$, $r$ is a unique solution of the equation $re^r = 2p$ and $A$ is a positive constant.

By Stirling's formula we find (here all the quantities $o(1)$ are of exponential order)

\[
S_2 = \frac{1}{(2\pi)^{1/2}} \sum_{-Ap^{1/2}}^{Ap^{1/2}} \left( m + \left[ \frac{r}{2} \right] \right) \frac{p^m - e^{-r/2} - 1}{2} \left( \frac{e}{2} \right)^{r/2} (1 + o(1))
\]

\[
= \frac{1}{(2\pi(r+1))^{1/2}} \left( \frac{e^r}{2} \right)^{-r/2} \left( \frac{e}{2} \right)^{r/2} \times \sum_{-Ap^{1/2}}^{Ap^{1/2}} \exp \left( -m^2 r (r+1) - \frac{r}{2p} \left( \frac{r}{2} \right) \right) \left( \frac{r}{m} \right)^{1/2} (1 + o(1))
\]

\[
= \frac{1}{(2\pi(r+1))^{1/2} 2^p} \exp \left( p \left( r + \frac{1}{r} - 1 \right) \right) \int_{-A(p+1)^{1/2}}^{A(p+1)^{1/2}} \exp \left( -y^2/2 \right) dy (1 + o(1))
\]

\[
= \frac{1}{(r+1)^{1/2} 2^p} \exp \left( p \left( r + \frac{1}{r} - 1 \right) \right) (1 + o(1)).
\]

By an analogy with [11] one can show that for some $A$

\[
\frac{S_1}{S_2} = O\left( \frac{1}{r} \left( \frac{r}{p} \right)^{(r+1)/2} \right), \quad \frac{S_3}{S_2} = O\left( \frac{1}{r} \left( \frac{r}{p} \right)^{(r+1)/2} \right).
\]

Therefore,

\[
S = \frac{1}{(r+1)^{1/2} 2^p} \exp \left( p \left( r + \frac{1}{r} - 1 \right) \right) (1 + o(1)),
\]

where $r$ is a unique solution of the equation $re^r = 2p$.

Taking into account the relation

\[
r = \log 2p - \log \log 2p + O\left( \frac{\log \log p}{\log p} \right),
\]

which is valid for large $p$, we deduce from (9) that

\[
C_p^* = S^{1/p} \left( 1 + o\left( \frac{1}{p} \right) \right) = 0.5 \exp \left( r + \frac{1}{r} - 1 \right) \left( 1 + o\left( \frac{\log p}{p} \right) \right)
\]
\[ p = \frac{e \log p}{\log p} \left( 1 + o \left( \frac{\log^2 \log p}{\log p} \right) \right). \]

The corollary is proved.

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REFERENCES