The Distribution of Multipliers in a Networked Economy and Topology-Induced Negative Multipliers

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Abstract

We have a collection of N agents and an outside entity that is interested in the population's aggregate action. The outside entity would like to enact a policy with the intention of increasing the aggregate action; for example, if the outside entity were a national government, it might be interested in enacting a policy that increases aggregate output and stimulates economic growth. The policy of the outside entity allocates $\epsilon > 0$ units of additional wealth to $n \leq N$ agents, funded either by internal transfer or by an external source. Our outside entity would like to know the exact effect of its planned policy on the aggregate action, and it would like to know the corresponding economic multiplier, that is, the change in the aggregate action from the ϵ shock. Now, even though the setting is simple, the effect is complicated: the population of agents is networked; agents' actions are interdependent, so depending on which subset of agents actually receives the positive shock, the change in the aggregate action and the corresponding economic multiplier can both widely differ. In this work, we consider three broad settings with network-based interaction: (1) networked environments with strategic complements and substitutes, (2) networked environments with coordination and anti-coordination, and (3) networked environments with production. We show how there is an entire distribution of possible aggregate actions and economic multipliers associated with a particular policy: given n, for each environment, we map the topology of agents' interaction network to the distribution of possible resulting aggregate actions and economic multipliers. The mathematics is the same across all three environments. We can compute the features of these distributions in closed-form, including the maximum and minimum possible aggregate actions and economic multipliers for a particular network topology. We can also rank networks so that the outside entity's policy is more effective the higher ranked the network. We show how non-trivial network topologies generate negative multipliers. Across all three settings, there is a non-zero probability that the enacted policy will reduce the aggregate action below its no-intervention level, and we can compute this probability in closed form.

JEL Classification: D85, E60, H30, C40, C60

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1 Introduction

There are many economic settings in which agents are networked and the actions that agents take are interdependent. The complexities of such network-based agent interactions can make it difficult to ascertain in advance the effects of a planned policy on agents' aggregate behavior. Given that it is difficult to ascertain the aggregate action in advance, it is also difficult to predict the policy-specific economic multiplier; this latter quantity measures the change in the aggregate action arising from implementation of the planned policy. The present work shows how to compute the effects of a planned policy on the aggregate action and how to determine the policy's economic multiplier. For each policy, given agents' decision-making behavior and the topology of agents' interaction network, there are entire probability distributions of possible aggregate actions and economic multipliers. This work develops and applies a set of theoretical tools so that we can explicitly map the planned policy to the corresponding probability distributions of possible aggregate actions and economic multipliers.

In this work, we focus on a population of N networked agents, each of whom chooses an action, and an outside actor. The outside actor is interested in the population's aggregate action. In particular, the outside actor would like to adjust the aggregate action and therefore chooses a policy with that intention. We can imagine that this actor would like to increase the aggregate action above its no-intervention level. There are many different real-world settings that parallel the theoretical setting of the present work, with its N networked agents and outside actor interested in increasing the aggregate action. The outside actor might be a government interested in jump-starting its economy during a recession; the government would like to provide stimulus to a set of firms organized on a production network with the intention of increasing aggregate output. The outside actor might alternatively be a nonprofit organization, such as a cancer research foundation; the foundation would like to increase the amount of innovative activity in cancer research. To achieve this goal, it allocates funds to research groups who are organized on both formal and informal R&D networks; the linkages of these networks capture both collaboration and competition among research groups.

There are many different classes of policies that the outside actor can implement. In this work, we focus on one type of policy. Here, when a policy gets enacted, the outside actor transmits a positive shock of ϵ magnitude to $n \leq N$ agents. Such a shock can be a positive wealth shock, in which the outside actor provides $\epsilon > 0$ units of additional wealth to n agents. In general, though, the exact interpretation of the shock depends on the environment. There are two different ways that the outside actor can finance its policy. The outside actor can either gather funds from the other networked agents in the population, or

it can receive funds from agents who are outside of the system. We refer to the former case as a setting with internally provided transfers, and we refer to the latter case as a setting with externally provided stimulus. The internal transfers might be implemented via taxation, while the external stimulus might originate from issuance of debt or donation. In the setting with transfers, the financing agents receive a negative shock, so the net adjustment that the policy initially induces for all agents in the population is zero.

When issuing a policy, the outside actor chooses both n and the method of financing. Therefore, given a particular policy, we can introduce a binary-valued attribute that identifies which agents have received a positive shock; we assign the attribute's unit value to the n agents targeted by the policy, while we assign the attribute's zero value to the other $N - n$ agents. This assignment of positive shocks to n agents represents a particular configuration of positive shocks; specifically, the configuration identifies the indices for the subset of n agents who have the attribute's unit value. Given that a policy targets n agents, there are $\binom{N}{n}$, or combinatorially many, total possible configurations of positive shocks. For each configuration, a different group of agents receives the positive shock. Agents' actions are interdependent, so the aggregate action and economic multiplier can vary depending on which group of agents actually receives the positive shock. Accordingly, holding both n and the method of financing fixed, we can construct an entire distribution of possible aggregate actions and an entire distribution of possible economic multipliers.

When the outside actor is planning to issue a policy targeting n agents, the actor is interested in the full range of possible outcomes. As a result, both the distribution of possible aggregate actions and the distribution of possible economic multipliers are the relevant theoretical objects of interest. We can alternatively imagine that the outside actor knows the set of agents that it would like to target, but it does not know which nodes on the network these agents occupy. In such circumstances, the distribution of possible aggregate actions and the distribution of possible economic multipliers constructed from all feasible configurations are also the appropriate objects of interest. We can separately imagine that the outside actor selects agents at random for receipt of a positive shock; the outside actor would then like to know the distribution of possible aggregate actions and the distribution of possible economic multipliers as well. Across all scenarios, the outside actor has a sense of the topology for the underlying agent interaction network. If the outside actor does not quite know the underlying network's topology, it can carry out sensitivity analysis, perturbing different features of the underlying network and examining how these perturbations shift the policy-induced distributions of possible aggregate actions and economic multipliers.

In this work, we develop a large body of theoretical results. These results explicitly show how the outside actor's policy generates the probability distributions of possible ag-

gregate actions and economic multipliers. We are able to map the outside actor's policy to two simple probability distributions even though agents' interactions are complex. Given agents' decision-making behavior, we can characterize these probability distributions for every feasible policy, that is, for every feasible number of agents being targeted by the policy, n, and for both methods of financing. The shapes of these probability distributions fundamentally depend on the population size of networked agents, N, the number of agents being targeted by the policy, n, and the topology of agents' interaction network; our theoretical results explicitly show how each of these quantities impacts the statistical features for our distributions.

We characterize in closed form the main statistical features as well as the CDFs for both the distribution of aggregate actions and the distribution of economic multipliers. We solve for the first two moments of these distributions. We find that, in settings with transfers, the mean aggregate action is always equal to its no-intervention level and the mean economic multiplier is always equal to zero. On average, the outside actor's policy has no effect. In settings with stimulus, however, the mean aggregate action can deviate from its no-intervention level, and the mean economic multiplier can deviate from zero. We also present closed-form expressions for the second moments of these distributions. These second moments determine how much risk is entailed in enacting a particular policy. We find that the second moments for these distributions depend on the variance of average weighted in-degrees for a network that is a mathematical transformation of the original agent interaction structure. Beyond these first two moments, we provide closed-form expressions for the lower and upper bounds on the supports of the distributions of aggregate actions and economic multipliers. Given a particular policy, we are able to determine both the worst and the best possible outcomes, and we can show how these values depend on the topology of the underlying network. We develop a theoretical result that essentially allows us to draw the CDFs for the distributions of aggregate actions and economic multipliers. From this result, we can see how the topology of the mathematically transformed network generates properties of skewness and/or heavy-tailedness in the distributions of aggregate actions and economic multipliers. If these distributions are heavy-tailed, extreme values for the aggregate action and economic multiplier can be more likely than the outside actor might have otherwise thought. Meanwhile, in the setting with transfers, skewness in the distribution of economic multipliers might mean that the probability of a negative multiplier is greater than the probability of a positive multiplier, which can make implementation of the policy less attractive. We can characterize the distributions of aggregate actions and economic multipliers in the limit as $N \to \infty$ as well. All of these theoretical results equip the policy-making actor with the tools that it might need to evaluate the effects and pitfalls

of implementing particular policies in networked environments.

We develop and present all of these theoretical results for a general networked environment. We then proceed by studying three specific environments with network-based interaction: (1) networked environments with strategic complements and strategic substitutes, (2) networked environments with coordination and anti-coordination, and (3) networked environments with production. All of the results from the general environment apply to these three broad classes of environments. When we developed the general set of results, we introduced two free parameters into the theory: a matrix and a scalar quantity. For each of the three networked environments, specific expressions for these two parameters naturally emerge. The general theoretical results therefore nest the specific theoretical results for each of the three networked environments, and the mathematics ends up being the same. For each specific environment, we develop additional theoretical results. We identify the topologies of those networks for which the distributions of aggregate actions and economic multipliers are exactly degenerate; then, both the aggregate action and the economic multiplier are invariant to configuration. When possible, we identify the network structures that deliver the highest feasible economic multiplier and the lowest feasible economic multiplier. We moreover rank networks so that the distributions of aggregate actions and economic multipliers for relatively higher-ranked networks first-order stochastically dominate the distributions of aggregate actions and economic multipliers for relatively lower-ranked networks. Consequently, the higher-ranked the network, the more effective the policy.

Quite importantly, for both the general networked environment and the three specific networked environments, we show how to compute in closed form the probability that the aggregate action ends up being below its no-intervention level and the economic multiplier ends up being negative. The outside actor chooses to implement a policy with the intention of increasing the aggregate action, but depending on the topology of agents' interaction network, there can be a non-negligible probability that the policy ends up being harmful. Indeed, in settings with transfers, provided that agents' weights are not all equal, there is *always* a positive probability of a negative multiplier for every level n; agents' weights here are equal to the average weighted in-degrees for the mathematically transformed network. Whenever agents' interaction network is non-trivial, negative multipliers emerge naturally, especially in settings with transfers but also in settings with stimulus. By being able to compute the probability of a negative multiplier in closed form, the outside actor can better assess the risks inherent in enacting a particular policy.

1.1 Relation to the Literature

This work interfaces with four different areas of the literature: research on (1) networks, (2) economic complexity, (3) economic multipliers, and (4) fiscal stimulus. Throughout the present work, agent interaction is organized on networks. In particular, for environments featuring strategic complements and strategic substitutes, and environments featuring coordination and anti-coordination, agents choose actions by playing games on networks. The recent literature on network games includes work by Ballester, Calvó-Armengol, and Zenou (2006), Galeotti et al. (2010), Jackson and Yariv (2011), Jackson and Zenou (2015), and Jackson, Rogers, and Zenou (2017). The present work uses the structure of network games and the flexibility that they offer to study aggregate actions in a variety of environments and characterize how aggregate actions adjust once the behavior of some subset of agents is perturbed. The present work also studies environments with production networks. Recent literature on production networks includes Acemoglu et al. (2012), Chaney (2014), Acemoglu, Akcigit, and Kerr (2016), Barrot and Sauvagnat (2016), Boehm, Flaaen, and Pandalai-Nayar (2017), and Oberfield (2018). Several of these papers study the transmission of shocks across production networks. In the present paper, shocks are demand-originating; rather than the value of the shock be stochastic, as in other work, shocks in the present work are fixed in magnitude and they target a subset of firms and/or sectors. Randomness here arises in the particular configuration of firms and/or sectors actually receiving a positive shock given that the implemented policy exactly targets n firms and/or sectors. Our object of interest is the *probability distribution* of possible levels of aggregate output that result once a fixed number of firms and/or sectors are targeted via a demand channel.

The theoretical environment of this work is complex. The present work therefore engages strongly with past research on economic complexity, some of which includes Topa (2001), Brock and Durlauf (2001a), and Brock and Durlauf (2001b). These past papers all develop interactions-based models, with interactions either being local or global social interactions. The present work focuses on local interactions; it examines agent decision-making when interactions are defined locally by an underlying network structure. Complexity in the present work emerges when we look at the behavior of the aggregate action and the corresponding economic multiplier after a policy targets n agents. Network-based interactions cause the aggregate action to adjust more or less than the aggregate action would absent any network-based interaction. The networked system moreover exhibits a degree of nonlinearity. Given that a group of n agents has received an initial positive shock, as we incrementally increase the additional agents receiving that positive shock, the sequence of changes in the aggregate action is, in general, very much nonlinear. The present work also departs from past work in the area of economic complexity by studying probability distributions. Rather

than identifying a unique equilibrium, this work characterizes probability distributions of possible equilibria, focusing on aggregate actions in the economy and corresponding economic multipliers. We can study how the structure of agents' interaction network shapes the distribution of aggregate actions and the distribution of economic multipliers. As in Schlossberger (2018), this work condenses the complexities of agent-based interactions into a probability distribution. Schlossberger (2018) operates in a general setting; it maps networks and agent actions to a probability distribution of possible outcomes for the economy, given the global prevalence of some binary-valued attribute within the population. The present work extends and applies the theoretical results from Schlossberger (2018), mapping networks and agents' decision-making behavior to a distribution of possible aggregate actions and economic multipliers; the binary-valued attribute here denotes an agent's receipt of a positive transfer or positive stimulus.

The two main objects studied in the present work are the aggregate action and the corresponding economic multiplier. We define the economic multiplier as the increase in the aggregate action that results when n agents each receive an additional unit of a positive transfer or positive stimulus. The economic multiplier therefore varies with configuration, and given n, we can construct the entire distribution of possible economic multipliers. Recent research concerning economic multipliers focuses on social multipliers and network multipliers. Glaeser, Sacerdote, and Scheinkman (2003) and Calvó-Armengol and Zenou (2004) identify social multipliers. They study the disconnect between aggregate-level behavior and individual-level behavior, attributing that disconnect to social interactions; these two works therefore study the social multiplier effects that exist on individual decisions. Baqaee (2013), Carvalho (2014), and Acemoglu, Akcigit, and Kerr (2016) meanwhile identify network multipliers. Carvalho (2014) and Acemoglu, Akcigit, and Kerr (2016) study how the network structure of the economy amplifies sector-specific volatility. Baqaee (2013) introduces an employment multiplier that studies adjustments to equilibrium employment following sector-specific shocks; the employment multiplier depends on the underlying structure of the network. The present work enriches the existing literature on multipliers by studying and characterizing entire distributions of multipliers whose values depend on agents' interaction structure and the particular class of agent actions. This work additionally introduces closed-form expressions that allow us to compute the probability that an economic multiplier is negative.

Now, the present work studies an outside entity whose policy provides positive transfers or positive stimulus to n agents in the system. If this outside entity is indeed a government, then the types of policies that we are examining are fiscal policies, and the positive shocks to the n agents in the system represent fiscal stimulus. Research in the area of fiscal

stimulus includes: Christiano, Eichenbaum, and Rebelo (2011), Woodford (2011), Eggertsson (2011), Ilzetzki, Mendoza, and Végh (2013), Nakamura and Steinsson (2014), Farhi and Werning (2016), Chodorow-Reich (2018), and Hagedorn, Manovskii, and Mitman (2018). The Great Recession and the handicapping of monetary policy during the zero-lower-bound environment of the time sparked renewed interest in fiscal multipliers. On the theory side, research has focused on mapping macroeconomic models with particular underlying assumptions and features to corresponding expressions for the government expenditure multiplier. Such models include variants of neoclassical models, New Keynesian models, open-economy models, closed-economy models, representative agent models, and heterogeneous agent models. Depending on the particular environment and how government spending is financed, the magnitude of the fiscal multiplier differs. On the empirical side, recent work has sought to estimate the value of the government spending multiplier in different environments; Nakamura and Steinsson (2014) estimate an open-economy multiplier, and Chodorow-Reich (2018) bounds the national government expenditure multiplier from estimates of cross-sectional fiscal spending multipliers. These multipliers need not always be positive. Ilzetzki, Mendoza, and Végh (2013) empirically estimate negative fiscal multipliers in countries with high public debt ratios. Hagedorn, Manovskii, and Mitman (2018) show theoretically that tax-financed fiscal stimulus generates multipliers whose magnitudes are smaller than those of fiscal multipliers financed externally via issuance of debt. When the outside actor in our work is a government, the economic multipliers that we study end up being fiscal multipliers. For a given level of fiscal stimulus, which is either financed internally by tax or transfer or financed externally, we can compute the entire non-degenerate distribution of possible fiscal multipliers. The particular configuration of fiscal stimulus among economic agents fundamentally matters. Through this work, we also introduce a new channel by which fiscal multipliers can be negative: network-based interactions among agents in the economy. We can illustrate how negative fiscal multipliers arise from natural patterns of agent interaction, and we can quantify the probability that a particular level of fiscal stimulus decreases aggregate output or the aggregate action below its no-intervention level.

1.2 Outline of Paper

Section 2 begins by introducing notation and definitions. It then proceeds to develop a high-level unifying theoretical framework for the study of policy-induced distributions of aggregate actions and economic multipliers in general networked settings. The theoretical framework of Section 2 nests the specific networked environments of Sections 3, 4, and 5. Section 3 focuses on distributions of aggregate actions and economic multipliers in networked

environments with strategic complements and strategic substitutes. Section 4 characterizes the policy-induced distributions of aggregate actions and economic multipliers in networked environments with coordination and anti-coordination. Section 5 studies the distributions of aggregate output and economic multipliers in networked environments with production. Section 6 concludes.

2 Theoretical Framework

2.1 Notation and Definitions

The cardinality of a set $\mathcal{X}$ is $|\mathcal{X}|$. A *multiset* is an object similar to a set, but it allows for multiple instances of each of its elements. Vector $\mathbf{x}$ is a column vector by default. The ith element of vector $\mathbf{x}$ is x_i or $[\mathbf{x}]_i$. The ij^{th} element of matrix $\mathbf{X}$ is $[\mathbf{X}]_{ij}$, the ith row of $\mathbf{X}$ is $[\mathbf{X}]_{i*}$ and the jth column of $\mathbf{X}$ is $[\mathbf{X}]_{*j}$. $\mathbf{x}' \geq \mathbf{x}$ for vectors $\mathbf{x}, \mathbf{x}'$ if $[\mathbf{x}']_i \geq [\mathbf{x}]_i$ element-wise; meanwhile, $\mathbf{x}' > \mathbf{x}$ if $[\mathbf{x}']_i \geq [\mathbf{x}]_i$ element-wise with at least one integer i for which $[\mathbf{x}']_i > [\mathbf{x}]_i$. $\mathbf{X}' \geq \mathbf{X}$ for matrices $\mathbf{X}, \mathbf{X}'$ if $[\mathbf{X}']_{ij} \geq [\mathbf{X}]_{ij}$ for all pairs (i, j); meanwhile, $\mathbf{X}' > \mathbf{X}$ if $[\mathbf{X}']_{ij} \geq [\mathbf{X}]_{ij}$ element-wise with at least one pair (i, j) for which $[\mathbf{X}']_{ij} > [\mathbf{X}]_{ij}$. The identity matrix is $\mathbf{I}$ and the column vector whose elements all equal 1 is $\mathbf{1}$. Depending on the particular context, $\mathbf{0}$ is either a column vector whose elements all equal 0, or a matrix whose elements all equal 0. $\mathbb{Z}_+$ is the set of all non-negative integers.

Matrix $\mathbf{X}$ is *row-stochastic* if $\mathbf{X1} = \mathbf{1}$ and all matrix elements of $\mathbf{X}$ are non-negative. Matrix $\mathbf{X}$ is *column-stochastic* if $\mathbf{X}^T\mathbf{1} = \mathbf{1}$ and all matrix elements of $\mathbf{X}$ are non-negative. Matrix $\mathbf{X}$ is *doubly stochastic* if it is both row-stochastic and column-stochastic. The Hadamard product of matrices $\mathbf{X}$ and $\mathbf{Y}$, $\mathbf{X} \circ \mathbf{Y}$, is their element-wise multiplication: $[\mathbf{X} \circ \mathbf{Y}]_{ij} = [\mathbf{X}]_{ij} [\mathbf{Y}]_{ij}$. Non-negative matrix $\mathbf{X}$ is *primitive* if there exists an integer $q \geq 1$ such that $[\mathbf{X}^q]_{ij} > 0$ for all matrix elements in $\mathbf{X}^q$. Matrix $\mathbf{X}$ is *semi-convergent* if the limit $\lim_{q\to\infty} \mathbf{X}^q$ exists. $(\lambda, \mathbf{w})$ is a left eigenpair of matrix $\mathbf{X}$ if $\mathbf{w}^T\mathbf{X} = \lambda\mathbf{w}^T$; $(\lambda, \mathbf{w})$ is the *dominant* left eigenpair of $\mathbf{X}$ when the magnitude $|\lambda|$ weakly exceeds that of all other eigenvalues of $\mathbf{X}$. For permutation matrix $\mathbf{P}$, $\mathbf{PX}$ permutes the rows of $\mathbf{X}$ and $\mathbf{XP}$ permutes the columns of $\mathbf{X}$. The *spectral radius* $r(\mathbf{X})$ of a matrix $\mathbf{X}$ is the largest absolute value among the eigenvalues of $\mathbf{X}$. Real random variable $X' \succeq X$ *in the usual stochastic order* if $\Pr[X' > t] \geq \Pr[X > t]$ for all $t \in (-\infty, \infty)$; random variable X' then *first-order stochastically dominates* random variable X.

Graph[1] $\mathcal{G}$ is an ordered pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consisting of a set of vertices (nodes) $\mathcal{V}$ and a set of edges $\mathcal{E}$. $(i, j, e_{i,j}) \in \mathcal{E}$ is an edge between nodes i and j with weight $e_{i,j}$. If the graph

[1] We will use the terms *graph* and *network* interchangeably.

is directed, the edge is oriented from node i to node j; otherwise, the edge is not oriented. If we define a weighted adjacency matrix $\mathbf{X}$ whose ij^{th} element $[\mathbf{X}]_{ij} = e_{i,j}$ denotes the edge weight between nodes i and j, then $\mathcal{G}(\mathbf{X}) = (\mathcal{V}(\mathbf{X}), \mathcal{E}(\mathbf{X}))$ is the corresponding weighted graph.

2.2 Theoretical Preliminaries

In this work, we study three different environments featuring N networked agents: (1) networked environments with strategic complements and strategic substitutes, (2) networked environments with coordination and anti-coordination, and (3) networked environments with production. In each environment, an outside actor prescribes a policy that exogenously delivers a positive shock of a fixed magnitude to the actions of $n \leq N$ agents. The policy targets n agents, but it does not specify the identities of those agents. The policy is either financed internally by the other agents in the population or it is financed externally by agents who are outside of the system. We therefore have two classes of policies; we refer to the former class of policies as transfers and the latter class of policies as stimulus. Given a prescribed policy, we are interested in the resulting distributions of possible aggregate actions and corresponding economic multipliers. We would like to understand how these distributions and their statistical properties depend on the implemented policy and the topology of the network.

We find that the mathematics is the same across all three environments. Therefore, in the present section, we write out general expressions for the aggregate action and the corresponding economic multiplier for each of our two classes of policies. These general expressions for the aggregate action and the economic multiplier depend on a matrix $\mathbf{Z}$ and parameter γ_1. In later sections, we identify the specific expressions for $\mathbf{Z}$ and γ_1 for each of the three networked environments. We substitute those quantities into the general expressions for the aggregate action and the economic multiplier provided in Section 2 to obtain environment-specific formulae.

In this section, we solve for all distributional features of the aggregate action and the economic multiplier using the general expressions. Later on, we can solve for the distributional features of the aggregate action and the economic multiplier in specific environments by plugging in the relevant values of $\mathbf{Z}$ and γ_1. These results that characterize the distributional features of the aggregate action and the corresponding economic multiplier build on a theoretical framework and a core set of mathematics introduced and developed in Schlossberger (2018). We adapt the theoretical framework and the core set of mathematical results for the present paper. The present subsection develops this core set of mathematical re-

sults as a collection of lemmata. All of the general results characterizing the distributions of aggregate actions and economic multipliers, which build on the set of lemmata, are then presented in the next subsection.

We have a population of N agents organized on a network $\mathcal{G}(\mathbf{Z}) = (\mathcal{V}(\mathbf{Z}), \mathcal{E}(\mathbf{Z}))$ with weighted adjacency matrix $\mathbf{Z}$. There are no constraints on the matrix elements of $\mathbf{Z}$, other than that each element is a real number. Now, every agent i in the population has a binary-valued attribute, b_i. This binary-valued attribute identifies which agents in the population are recipients of positive funds; depending on the policy, these funds are either financed by transfer or stimulus. Specifically, attribute $b_i = 1$ if agent i is the recipient of positive funds, and otherwise $b_i = 0$. There are $n \leq N$ agents receiving positive funds and therefore $n \leq N$ agents with the attribute's unit value. Given n, there is a particular configuration, or arrangement, of this binary-valued attribute among agents in the population. A *configuration* is defined as follows:

Definition 1 *A* configuration $\mathbf{b} \equiv \mathbf{b}(N, n)$ *of a binary-valued attribute in a population of* N *agents is an allocation of this attribute so that* $b_i \in \{0, 1\}$ *for all agents* $i \in \{1, \ldots, N\}$ *and* $\mathbf{b}^T\mathbf{1} = n$.

A configuration $\mathbf{b} \equiv \mathbf{b}(N, n)$ is an allocation of the attribute's unit value to exactly n agents in a total population of N agents. Vector $\mathbf{b}$ stacks each agent's binary-valued attribute and identifies the indices of those agents that have the attribute's unit value. Two configurations $\mathbf{b}, \mathbf{b}'$ are distinct when $\mathbf{b} \neq \mathbf{b}'$ because the subsets of agents with the attribute's unit value differ across these two configurations. Given a population of size N and n agents with the attribute's unit value, there are many different possible configurations of the attribute. The set of all possible configurations is $\mathcal{B}(N, n)$, and the cardinality of this set is combinatorial: $|\mathcal{B}(N, n)| = \binom{N}{n}$.

For each agent positioned on the network, we compute the local relative frequency of the attribute's unit value. Since the attribute's unit value denotes the receipt of positive funds, we are therefore essentially computing the local prevalence of this positive shock for every agent in his or her network neighborhood. We compute this quantity for each agent by taking a weighted sum of the values of the binary attribute for the agent's out-neighbors. The weights that we use in this sum are the edge weights that link the agent to these out-neighbors. We accordingly define $\widehat{\mathbf{f}}(\mathbf{Z}, \mathbf{b}, N, n) = \mathbf{Z}\mathbf{b}(N, n)$ as the $N \times 1$ population vector of local relative frequencies of the attribute; the local relative frequency of the attribute for each agent depends on the topology of the network, $\mathcal{G}(\mathbf{Z})$, and it depends on which subset of n agents actually has the attribute's unit value, $\mathbf{b}(N, n)$. The local relative frequency of the attribute for agent i, $\widehat{f}_i(\mathbf{Z}, \mathbf{b}, N, n) = [\mathbf{Z}]_{i*}\mathbf{b}(N, n)$, can take values outside of the $[0, 1]$

interval because $[\mathbf{Z}]_{i*}$ is not constrained to have all non-negative elements and $[\mathbf{Z}]_{i*}\mathbf{1}$ is not constrained to sum to 1. In Schlossberger (2018), matrix $\mathbf{Z} \equiv \bar{\mathbf{A}}$ is row-stochastic, so all of its elements are non-negative and the elements in each row sum to 1. As a result, for all agents $i \in \{1, \ldots, N\}$, $\left[\bar{\mathbf{A}}\right]_{i*}\mathbf{b}(N,n) \in [0,1]$, and we can exactly interpret $\left[\bar{\mathbf{A}}\right]_{i*}\mathbf{b}(N,n)$ as the local relative frequency of the attribute for agent i. To be consistent with the nomenclature of Schlossberger (2018), we also refer to $\widehat{f}_i(\mathbf{Z},\mathbf{b},N,n) = [\mathbf{Z}]_{i*}\mathbf{b}(N,n)$ as the local relative frequency of the attribute for agent i; if we want to be more precise, we can think of this quantity as a *scaled* local relative frequency of the attribute.

We are interested in the population-averaged local relative frequency of the attribute's unit value, $\widehat{f}_{avg}(\mathbf{Z},\mathbf{b},N,n)$. We are essentially computing the average local prevalence of the positive shock given n. We calculate this quantity as follows:

$$\widehat{f}_{avg}(\mathbf{Z},\mathbf{b},N,n) = \frac{1}{N}\mathbf{1}^T\widehat{\mathbf{f}}(\mathbf{Z},\mathbf{b},N,n) = \frac{1}{N}\mathbf{1}^T\mathbf{Z}\mathbf{b}(N,n) = \left[\mathbf{d}_w^-(\mathbf{Z})\right]^T\mathbf{b}(N,n),$$

where $\mathbf{d}_w^-(\mathbf{Z}) = \frac{1}{N}\mathbf{Z}^T\mathbf{1}$ is the vector of average weighted in-degrees for graph $\mathcal{G}(\mathbf{Z})$. To determine the average local relative frequency of the attribute for a particular configuration, we derive the vector of agent weights, $\mathbf{d}_w^-(\mathbf{Z})$, from the underlying graph $\mathcal{G}(\mathbf{Z})$. We then multiply $\mathbf{d}_w^-(\mathbf{Z})$ by the configuration vector $\mathbf{b}(N,n)$, and we obtain the configuration-specific average local relative frequency of the attribute's unit value. The higher an agent's weight, the more that agent contributes to the average local relative frequency of the attribute provided that he or she has the attribute's unit value. The sum of agents' weights is k: $\mathbf{1}^T\mathbf{d}_w^-(\mathbf{Z}) = k$. Without restrictions on $\mathbf{Z}$, the average local relative frequency of the attribute is not constrained to the interval $[0,1]$; in Schlossberger (2018), $\widehat{f}_{avg}(\mathbf{Z},\mathbf{b},N,n) \in [0,1]$ because $\mathbf{Z} \equiv \bar{\mathbf{A}}$ is row-stochastic. To be consistent with the nomenclature of Schlossberger (2018), we refer to $\widehat{f}_{avg}(\mathbf{Z},\mathbf{b},N,n)$ as the average local relative frequency of the attribute; if we want to be more precise, we can think of this quantity $\widehat{f}_{avg}(\mathbf{Z},\mathbf{b},N,n)$ as a *scaled* average local relative frequency of the attribute.

Holding n fixed, that is, holding fixed the total number of agents receiving positive funds, we can imagine that $\widehat{f}_{avg}(\mathbf{Z},\mathbf{b},N,n)$ varies with configuration. Depending on which subset of agents receives the positive shock, we can have variation in its average local prevalence. We would like to determine the distribution of possible average local relative frequencies of the positive shock given N, n, and $\mathbf{Z}$. We therefore introduce random variable $\widehat{F}_{avg}(\mathbf{Z},N,n)$. This random variable has configuration-specific realizations $\widehat{f}_{avg}(\mathbf{Z},\mathbf{b},N,n)$. The CDF of $\widehat{F}_{avg}(\mathbf{Z},N,n)$ is $G_{\widehat{F}_{avg}(\mathbf{Z},N,n)}(t)$, and we are interested in the distributional features of $\widehat{F}_{avg}(\mathbf{Z},N,n)$. We assume that each configuration is equally likely, although this is an assumption that we can relax (see Schlossberger (2018)). By characterizing the distribu-

tional features of $\widehat{F}_{avg}(\mathbf{Z}, N, n)$, we are able to later compute in closed form the distributional features of the aggregate action and corresponding economic multiplier across all three networked environments.

We now present a set of lemmata that characterizes the distributional features of $\widehat{F}_{avg}(\mathbf{Z}, N, n)$. We begin with the first moment of the distribution:

Lemma 1 $E\widehat{F}_{avg}(\mathbf{Z}, N, n) = \frac{kn}{N}$.

The average local relative frequency of the attribute can vary with configuration, but across all possible configurations, this quantity is equal to $\frac{kn}{N}$ on average. The value of constant k depends on the topology of network $\mathcal{G}(\mathbf{Z})$.

We would also like to know how much the average local relative frequency of the attribute can vary with configuration. By computing this second moment, we can determine how much the aggregate action in the economy and the accompanying economic multiplier vary with configuration for a given policy. To compute this second moment, $\operatorname{Var}\widehat{F}_{avg}(\mathbf{Z}, N, n)$, we introduce one more piece of notation. We define random variable $D_w^-(\mathbf{Z})$ whose realizations are agent weights $[\mathbf{d}_w^-(\mathbf{Z})]_i$. By introducing random variable $D_w^-(\mathbf{Z})$, we can compactly express population moments for the set of agent weights; for example, $ED_w^-(\mathbf{Z}) = \frac{1}{N}\sum_{i=1}^{N}[\mathbf{d}_w^-(\mathbf{Z})]_i = \frac{k}{N}$ and $\operatorname{Var} D_w^-(\mathbf{Z}) = \frac{1}{N}\sum_{i=1}^{N}\left([\mathbf{d}_w^-(\mathbf{Z})]_i - \frac{k}{N}\right)^2$. The closed-form expression for $\operatorname{Var}\widehat{F}_{avg}(\mathbf{Z}, N, n)$ is then as follows:

Lemma 2 $\operatorname{Var}\widehat{F}_{avg}(\mathbf{Z}, N, n) = \frac{n}{N}\left(1 - \frac{n}{N}\right)\frac{N}{N-1}\left(N \operatorname{Var} D_w^-(\mathbf{Z})\right)$.

The variance of $\widehat{F}_{avg}(\mathbf{Z}, N, n)$ fundamentally depends on the variance of agents' weights. The greater the heterogeneity in agents' weights, the greater the variation in the average local relative frequency of the attribute because this quantity then depends more strongly on which subset of agents actually has the attribute's unit value.

Lemma 3 shows how to compute the lower and upper bounds on the support of $\widehat{F}_{avg}(\mathbf{Z}, N, n)$. Given that n agents have received a positive shock, we compute both the lowest and the highest possible average local relative frequencies of the attribute. We later use this result to compute the lowest and highest possible aggregate actions and economic multipliers consistent with a particular policy.

Lemma 3 *Construct the ordered multiset $\{\widetilde{w}_i\}_{i=1}^{N}$ from the elements of $\mathbf{d}_w^-(\mathbf{Z})$ so that $\widetilde{w}_i \leq \widetilde{w}_{i'}$ whenever $i \leq i'$. The lower and upper bounds on the support of $\widehat{F}_{avg}(\mathbf{Z}, N, n)$ are respectively:*

$$\min \operatorname{supp} \widehat{F}_{avg}(\mathbf{Z}, N, n) = \sum_{i=1}^{n} \widetilde{w}_i \quad \textit{and} \quad \max \operatorname{supp} \widehat{F}_{avg}(\mathbf{Z}, N, n) = \sum_{i=N-n+1}^{N} \widetilde{w}_i.$$

We attain the lower bound on the support of $\widehat{F}_{avg}(\mathbf{Z}, N, n)$ when the n agents with the smallest weight have the attribute's unit value. Meanwhile, we attain the upper bound on the support of $\widehat{F}_{avg}(\mathbf{Z}, N, n)$ when the n agents with the largest weight have the attribute's unit value.

We would moreover like to identify those network topologies and those vectors of agent weights, $\mathbf{d}_w^-(\mathbf{Z})$, for which $\widehat{f}_{avg}(\mathbf{Z}, \mathbf{b}, N, n)$ is *invariant to configuration*:

Definition 2 $\widehat{f}_{avg}(\mathbf{Z}, \mathbf{b}, N, n)$ *is* invariant to configuration *when* $\widehat{f}_{avg}(\mathbf{Z}, \mathbf{b}, N, n) = \widehat{f}_{avg}(\mathbf{Z}, \mathbf{b}', N, n)$ *for all configurations* $\mathbf{b}(N, n), \mathbf{b}'(N, n) \in \mathcal{B}(N, n)$*, and this property holds for all feasible* n*.*

When $\widehat{f}_{avg}(\mathbf{Z}, \mathbf{b}, N, n)$ is invariant to configuration, the distribution $G_{\widehat{F}_{avg}(\mathbf{Z},N,n)}(t)$ is degenerate. The particular configuration of positive shocks among agents is irrelevant; regardless of the configuration, the average local relative frequency of the attribute is the same, which makes the distributions of aggregate output and economic multipliers likewise degenerate. In the next result, we identify the necessary values for agents' weights so that $\widehat{f}_{avg}(\mathbf{Z}, \mathbf{b}, N, n)$ is invariant to configuration:

Lemma 4 $\widehat{f}_{avg}(\mathbf{Z}, \mathbf{b}, N, n) = [\mathbf{d}_w^-(\mathbf{Z})]^T \mathbf{b}(N, n)$ *is invariant to configuration if and only if* $[\mathbf{d}_w^-(\mathbf{Z})]_i = \frac{k}{N}$ *for all* $i \in \{1, \ldots, N\}$*. When* $\widehat{f}_{avg}(\mathbf{Z}, \mathbf{b}, N, n)$ *is invariant to configuration,* $\widehat{f}_{avg}(\mathbf{Z}, \mathbf{b}, N, n) = \frac{kn}{N}$*.*

Provided that every agent has the same weight, and in particular, the same average weighted in-degree, $\widehat{F}_{avg}(\mathbf{Z}, N, n) = \frac{kn}{N}$ with probability 1. Any network $\mathcal{G}(\mathbf{Z})$ structured so that each agent has the same weighted in-degree yields this null case in which configuration is irrelevant. The behavior of the economy here only depends on k, N, and n, and not on the underlying configuration $\mathbf{b}(N, n)$.

We proceed by returning to the original general setting in which the distribution $G_{\widehat{F}_{avg}(\mathbf{Z},N,n)}(t)$ is non-degenerate. In addition to presenting closed-form expressions for the statistical features of $\widehat{F}_{avg}(\mathbf{Z}, N, n)$, we are interested in characterizing its CDF, $G_{\widehat{F}_{avg}(\mathbf{Z},N,n)}(t)$. The next lemma identifies a closed-form expression that essentially allows us to draw the CDF of $\widehat{F}_{avg}(\mathbf{Z}, N, n)$ for all feasible network structures, population sizes, and number of agents being targeted by the policy. We first introduce the function $J(\mathbf{Z}, N, n, t)$:

$$J(\mathbf{Z}, N, n, t) = \Phi(t) - H_2(t)\,\phi(t)\,C_1 \sum_{i=1}^{N} \widehat{w}_i^3 - H_3(t)\,\phi(t)\left[C_2\left(\sum_{i=1}^{N} \widehat{w}_i^4 - \frac{3}{N}\right) - \frac{1}{4N}\right] - H_5(t)\,\phi(t)\,C_3\left(\sum_{i=1}^{N} \widehat{w}_i^3\right)^2,$$

where $\widehat{w}_i = \frac{\left[\mathbf{d}_w^-(\mathbf{Z})\right]i - ED_w^-(\mathbf{Z})}{\sqrt{N \operatorname{Var} D_w^-(\mathbf{Z})}}$, $C_1 = \frac{1-\frac{2n}{N}}{6\left(\frac{n}{N}\left(1-\frac{n}{N}\right)\right)^{1/2}}$, $C_2 = \frac{1-6\left(\frac{n}{N}\right)\left(1-\frac{n}{N}\right)}{24\left(\frac{n}{N}\right)\left(1-\frac{n}{N}\right)}$, $C_3 = \frac{\left(1-\frac{2n}{N}\right)^2}{72\left(\frac{n}{N}\right)\left(1-\frac{n}{N}\right)}$, $\phi(t) = \Phi'(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$, and $H_i(t)\,\phi(t) = (-1)^i \frac{d^i}{dt^i}\phi(t)$. We then approximate CDF $G_{\frac{\widehat{F}_{avg}(\mathbf{Z},N,n) - E\widehat{F}_{avg}(\mathbf{Z},N,n)}{\left(\operatorname{Var}\widehat{F}_{avg}(\mathbf{Z},N,n)\right)^{1/2}}}(t)$ by the function $J(\mathbf{Z},N,n,t)$:

Lemma 5 *Provided that condition (c) holds,*

$$\left| G_{\frac{\widehat{F}_{avg}(\mathbf{Z},N,n) - E\widehat{F}_{avg}(\mathbf{Z},N,n)}{\left(\operatorname{Var}\widehat{F}_{avg}(\mathbf{Z},N,n)\right)^{1/2}}}(t) - J(\mathbf{Z},N,n,t) \right| < C_4 \times \sum_{i=1}^{N} \left|\widehat{w}_i\right|^5$$

for all t, where C_4 is only a function of $\frac{n}{N}$. Condition (c) is as follows:

Condition (c) (Robinson (1978)) *Given $C' > 0$, there exist $\epsilon' > 0$, $C > 0$, and $\delta > 0$ not depending on N such that, for any fixed t, the number of indices j, for which $|\widehat{w}_j\,\widehat{x} - t - 2\widehat{r}\pi| > \epsilon'$, for all $\widehat{x} \in \left(C'\left[\max_i |\,\widehat{w}_i\,|\,\right]^{-1}, C\left[\sum_{i=1}^{N} |\,\widehat{w}_i\,|^5\right]^{-1}\right)$ and all $\widehat{r} = 0, \pm 1, \pm 2, \ldots,$ is greater than δN, for all N.*

Condition (c) requires that the multiset of agent weights, $\{[\mathbf{d}_w^-(\mathbf{Z})]_i\}_{i=1}^N$ not be clustered around two few values. Given Lemma 5, we can very strongly approximate the distribution, $G_{\widehat{F}_{avg}(\mathbf{Z},N,n)}(t)$:

$$G_{\widehat{F}_{avg}(\mathbf{Z},N,n)}(t) \approx J\left(\mathbf{Z}, N, n, \frac{t - E\widehat{F}_{avg}(\mathbf{Z},N,n)}{\left(\operatorname{Var}\widehat{F}_{avg}(\mathbf{Z},N,n)\right)^{1/2}}\right).$$

Note that $\sum_{i=1}^{N} \widehat{w}_i^3 = N^{-1/2} \operatorname{Skew} D_w^-(\mathbf{Z})$ and $\sum_{i=1}^{N} \widehat{w}_i^4 - \frac{3}{N} = N^{-1} \times (\operatorname{Excess Kurtosis} D_w^-(\mathbf{Z}))$. We can therefore re-write the approximating function $J(\mathbf{Z},N,n,t)$ in terms of the higher-order moments of $D_w^-(\mathbf{Z})$. The asymptotic expansion $J(\mathbf{Z},N,n,t)$ is to order $1/N$.

Lastly, we characterize the limiting behavior of $G_{\widehat{F}_{avg}(\mathbf{Z},N,n)}(t)$ as $N \to \infty$. We define the quantity

$$\kappa_N(\epsilon') = \frac{1}{\sum_{i=1}^{N}\left(\left[\mathbf{d}_{N,w}^-(\mathbf{Z})\right]_i - \frac{k}{N}\right)^2} \sum_{\substack{j \in \{1,\ldots,N\} \text{ s.t.} \\ \left|\left[\mathbf{d}_{N,w}^-(\mathbf{Z})\right]_j - \frac{k}{N}\right| > \epsilon' \sigma_N}} \left(\left[\mathbf{d}_{N,w}^-(\mathbf{Z})\right]_j - \frac{k}{N}\right)^2$$

where $\sigma_N = \left(\frac{n}{N}\left(1-\frac{n}{N}\right)\sum_{i=1}^{N}\left(\left[\mathbf{d}_{N,w}^-(\mathbf{Z})\right]_i - \frac{k}{N}\right)^2\right)^{1/2}$. We make the population size, N, explicit for the $N \times 1$ vector of agent weights (i.e., we rewrite $\mathbf{d}_w^-(\mathbf{Z})$ as $\mathbf{d}_{N,w}^-(\mathbf{Z})$) because

we wish to characterize the distribution of $\widehat{F}_{avg}(\mathbf{Z}, N, n)$ as N increases. Our central limit theorem-type result is the following:

Lemma 6 *If* $\lim_{N\to\infty} \kappa_N(\epsilon') = 0$ *for any* $\epsilon' > 0$*, then* $\lim_{N\to\infty} G_{\frac{\widehat{F}_{avg}(\mathbf{Z},N,n)-\frac{kn}{N}}{\sigma_N}}(t) = \Phi(t)$ *for all real* t*, where* $\Phi(\cdot)$ *is the standard normal CDF.*

The requirement that $\lim_{N\to\infty} \kappa_N(\epsilon') = 0$ for any $\epsilon' > 0$ is a Lindeberg-type condition. When this condition holds, we informally have that $\lim_{N\to\infty} G_{\widehat{F}_{avg}(\mathbf{Z},N,n)}(t) \approx \Phi\left(\frac{t-\frac{kn}{N}}{\sigma_N}\right)$. The distribution of $\widehat{F}_{avg}(\mathbf{Z}, N, n)$ is asymptotically normal, with a mean of $\frac{kn}{N}$ and a variance that collapses to zero as the population size increases provided that the Lindeberg-type condition holds. Given this set of lemmata, we can characterize in closed form the distributions of possible aggregate actions and economic multipliers for any feasible network structure, population size, and policy targeting n agents.

2.3 General Environment

In this subsection, we develop the general theoretical environment. Our general environment nests the three classes of networked environments that we later consider: (1) networked environments with strategic complements and strategic substitutes, (2) networked environments with coordination and anti-coordination, and (3) networked environments with production. Now, in the general environment, we have a population of N agents who are organized on the network $\mathcal{G}(\mathbf{Z}') = (\mathcal{V}(\mathbf{Z}'), \mathcal{E}(\mathbf{Z}'))$. $\mathcal{G}(\mathbf{Z}')$ is the naturally occurring network in the environment being studied; in the environment with production, for example, it is the production network. Given the theoretical environment and implemented policy, we are interested in the resulting aggregate action and the corresponding economic multiplier. To compute these two quantities, a transformed version of $\mathcal{G}(\mathbf{Z}')$ becomes the relevant network. We refer to this new network as $\mathcal{G}(\mathbf{Z})$ with weighted adjacency matrix $\mathbf{Z}$. For each of the specific environments that we later study, we explicitly identify both $\mathcal{G}(\mathbf{Z}')$ and $\mathcal{G}(\mathbf{Z})$.

In the general environment, an outside actor enacts a policy. This policy exogenously delivers a positive shock of a fixed magnitude to a subset of $n \leq N$ agents. We introduce the $N \times 1$ vector $\boldsymbol{\rho}$ to capture the policy-induced shock. If the policy is financed by transfers, we set $[\boldsymbol{\rho}]_i = \epsilon > 0$ if agent i is receiving the positive shock, and otherwise we set $[\boldsymbol{\rho}]_i = -\frac{n\epsilon}{N-n}$. Note that $\mathbf{1}^T\boldsymbol{\rho} = 0$ when the policy is financed by transfers. Meanwhile, if the policy is financed by stimulus, we set $[\boldsymbol{\rho}]_i = \epsilon$ if agent i is receiving the positive shock, and otherwise we set $[\boldsymbol{\rho}]_i = 0$. How the shock exactly impacts agent decision-making behavior depends on the specific environment. Given $\boldsymbol{\rho}$, we can construct the configuration vector $\mathbf{b}(N, n)$: $[\mathbf{b}(N, n)]_i = 1$ if $[\boldsymbol{\rho}]_i = \epsilon$ and otherwise $[\mathbf{b}(N, n)]_i = 0$ with $\mathbf{1}^T\mathbf{b}(N, n) = n$.

Given that N agents are organized on the network $\mathcal{G}(\mathbf{Z})$ and the outside actor has enacted a policy providing a positive shock to n agents, we are interested in the resulting aggregate action and economic multiplier. In the general environment, the configuration-specific aggregate action, $y_{agg}(\mathbf{Z}, \mathbf{b}, N, n, \ell)$, takes the following form:

$$y_{agg}(\mathbf{Z}, \mathbf{b}, N, n, \ell) = y_{agg}^{no} + \gamma_1 N \left[\mathbf{d}_w^-(\mathbf{Z})\right]^T \boldsymbol{\rho}; \tag{1}$$

y_{agg}^{no} is the aggregate action for all agents in the population in the absence of any policy, γ_1 is an environment-specific constant, $\mathbf{1}^T \mathbf{d}_w^-(\mathbf{Z}) = k$, and $\ell \in \{0, 1\}$ is an argument that indicates whether the enacted policy is being financed by transfers or external stimulus. $\ell = 0$ denotes a policy financed by transfers and $\ell = 1$ denotes a policy financed by stimulus; the elements of vector $\boldsymbol{\rho}$ adjust depending on the particular configuration and whether $\ell = 0$ or $\ell = 1$. We compute the corresponding configuration-specific economic multiplier as follows: $m(\mathbf{Z}, \mathbf{b}, N, n, \ell) = \frac{dy_{agg}(\mathbf{Z}, \mathbf{b}, N, n, \ell)}{d\epsilon}$. The economic multiplier captures the change in the aggregate action given that a particular configuration of n agents is receiving ϵ units of a positive shock.

The aggregate action is, in essence, a weighted sum of shocks; the relevant weights here are agents' average weighted in-degrees, $\mathbf{d}_w^-(\mathbf{Z})$, for the network $\mathcal{G}(\mathbf{Z})$. The higher an agent's weight, as determined by the structure of $\mathbf{Z}$, the greater the effect that the agent has on the aggregate action if he or she is the recipient of a positive shock. In general, in networked environments, the interdependencies of agents' actions can be complicated; the network $\mathcal{G}(\mathbf{Z})$ disentangles these complexities. It transforms the original network $\mathcal{G}(\mathbf{Z}')$ into a new structure $\mathcal{G}(\mathbf{Z})$ whose average weighted in-degrees simply determine the effect that each agent's action has on the aggregate action.

We now examine how to express the aggregate action when we have a particular configuration of positive shocks. Let's suppose that agents $1, \ldots, n$ are the recipients of a positive shock. We then have $b_i = 1$ for $i \in \{1, \ldots, n\}$ and otherwise $b_i = 0$. If the policy is financed by transfers, the aggregate action is as follows:

$$\begin{aligned} y_{agg}(\mathbf{Z}, \mathbf{b}, N, n, 0) = y_{agg}^{no} + \gamma_1 N \epsilon \Big[\Big(\left[\mathbf{d}_w^-(\mathbf{Z})\right]_1 + \cdots + \left[\mathbf{d}_w^-(\mathbf{Z})\right]_n \Big) \\ - \frac{n}{N-n} \Big(\left[\mathbf{d}_w^-(\mathbf{Z})\right]_{n+1} + \cdots + \left[\mathbf{d}_w^-(\mathbf{Z})\right]_N \Big) \Big]. \end{aligned}$$

If the policy is instead financed by stimulus, the aggregate action is as follows:

$$y_{agg}(\mathbf{Z}, \mathbf{b}, N, n, 1) = y_{agg}^{no} + \gamma_1 N \epsilon \Big(\left[\mathbf{d}_w^-(\mathbf{Z})\right]_1 + \cdots + \left[\mathbf{d}_w^-(\mathbf{Z})\right]_n \Big).$$

These expressions for $y_{agg}(\mathbf{Z}, \mathbf{b}, N, n, 0)$ and $y_{agg}(\mathbf{Z}, \mathbf{b}, N, n, 1)$ immediately follow from Equation 1 after substituting in the appropriate vector $\boldsymbol{\rho}$.

Now that we have studied the aggregate action for a particular configuration of positive shocks, we introduce the accompanying random variable. We define random variable $Y_{agg}(\mathbf{Z}, N, n, 0)$ as the aggregate action in a setting with transfers and random variable $Y_{agg}(\mathbf{Z}, N, n, 1)$ as the aggregate action in a setting with stimulus. The realizations of these random variables are configuration-specific realizations of the aggregate action when a certain subset of the population receives the positive shock. The accompanying CDFs are $G_{Y_{agg}(\mathbf{Z},N,n,0)}(t)$ and $G_{Y_{agg}(\mathbf{Z},N,n,1)}(t)$, and in constructing these CDFs, we assume that every configuration is equally likely. We also define the random variables for the corresponding economic multipliers. Random variable $M(\mathbf{Z}, N, n, 0) = \frac{dY_{agg}(\mathbf{Z},N,n,0)}{d\epsilon}$ is the economic multiplier in a setting with transfers and random variable $M(\mathbf{Z}, N, n, 1) = \frac{dY_{agg}(\mathbf{Z},N,n,1)}{d\epsilon}$ is the economic multiplier in a setting with stimulus. The accompanying CDFs are $G_{M(\mathbf{Z},N,n,0)}(t)$ and $G_{M(\mathbf{Z},N,n,1)}(t)$; in constructing these CDFs, we assume as well that each configuration is equally likely.

Given Equation 1, we can write the random variables for aggregate output and the corresponding economic multiplier in a setting with transfers as follows:

$$Y_{agg}(\mathbf{Z}, N, n, 0) = y_{agg}^{no} + \gamma_1 \frac{N^2\epsilon}{N-n} \times \left(\widehat{F}_{avg}(\mathbf{Z}, N, n) - \frac{kn}{N}\right) \text{ and} \tag{2}$$

$$M(\mathbf{Z}, N, n, 0) = \gamma_1 \frac{N^2}{N-n} \times \left(\widehat{F}_{avg}(\mathbf{Z}, N, n) - \frac{kn}{N}\right). \tag{3}$$

Meanwhile, the random variables for aggregate output and the corresponding economic multiplier in a setting with externally funded stimulus are as follows:

$$Y_{agg}(\mathbf{Z}, N, n, 1) = y_{agg}^{no} + \gamma_1 N\epsilon \times \widehat{F}_{avg}(\mathbf{Z}, N, n) \text{ and} \tag{4}$$

$$M(\mathbf{Z}, N, n, 1) = \gamma_1 N \times \widehat{F}_{avg}(\mathbf{Z}, N, n). \tag{5}$$

Depending on the particular configuration of the positive shock among agents in the population, we can have variation in its average local relative frequency. The higher the average local relative frequency of the positive shock, the higher the aggregate action and the higher the economic multiplier.

We proceed by building on the set of lemmata from the previous subsection. Here, we present theoretical results in which we characterize the distributional features of the

aggregate action and the corresponding economic multiplier in settings with transfers and in settings with stimulus. We first compute in closed form the first moment for the distributions of possible aggregate actions and economic multipliers:

Proposition 1 *The first moments for the aggregate action and economic multiplier are:*

$$EY_{agg}\left(\mathbf{Z}, N, n, 0\right) = y_{agg}^{no}, \qquad EM\left(\mathbf{Z}, N, n, 0\right) = 0,$$

$$EY_{agg}\left(\mathbf{Z}, N, n, 1\right) = y_{agg}^{no} + \gamma_1 kn\epsilon, \quad \textit{and} \quad EM\left(\mathbf{Z}, N, n, 1\right) = \gamma_1 kn.$$

In settings with transfers, the mean aggregate action is equal to its no-intervention level, y_{agg}^{no}, and the mean economic multiplier is equal to zero. When the outside actor finances its positive shock to a subset of agents by requesting a transfer of funds from the other agents in the population, the policy has no aggregate effect on average. This result holds for every feasible agent interaction structure, $\mathcal{G}\left(\mathbf{Z}\right)$, population size, N, and number of agents being targeted for a positive shock, n. When the policy is instead financed by stimulus, the mean value of the aggregate action can deviate from its no-intervention level and the corresponding mean economic multiplier can also deviate from zero. The values of these first moments in settings with stimulus depend both on k, which we derive from $\mathbf{Z}$, and γ_1 which depends on the particular networked environment. When the network and environment are such that $k = 1$ and $\gamma_1 = 1$, then the increase in the mean aggregate action above its no-intervention level is equal to the aggregate amount of stimulus, $n\epsilon$.

The next proposition computes in closed form the second moment for the distributions of aggregate actions and economic multipliers:

Proposition 2 *The second moments for the aggregate action and economic multiplier are:*

$$\begin{aligned}
\operatorname{Var} Y_{agg}\left(\mathbf{Z}, N, n, 0\right) &= \left(\gamma_1 \frac{N^2\epsilon}{N-n}\right)^2 \frac{n}{N}\left(1-\frac{n}{N}\right)\frac{N}{N-1}\left(N \operatorname{Var} D_w^-\left(\mathbf{Z}\right)\right), \\
\operatorname{Var} M\left(\mathbf{Z}, N, n, 0\right) &= \left(\gamma_1 \frac{N^2}{N-n}\right)^2 \frac{n}{N}\left(1-\frac{n}{N}\right)\frac{N}{N-1}\left(N \operatorname{Var} D_w^-\left(\mathbf{Z}\right)\right), \\
\operatorname{Var} Y_{agg}\left(\mathbf{Z}, N, n, 1\right) &= \left(\gamma_1 N\epsilon\right)^2 \frac{n}{N}\left(1-\frac{n}{N}\right)\frac{N}{N-1}\left(N \operatorname{Var} D_w^-\left(\mathbf{Z}\right)\right), \ \textit{and} \\
\operatorname{Var} M\left(\mathbf{Z}, N, n, 1\right) &= \left(\gamma_1 N\right)^2 \frac{n}{N}\left(1-\frac{n}{N}\right)\frac{N}{N-1}\left(N \operatorname{Var} D_w^-\left(\mathbf{Z}\right)\right).
\end{aligned}$$

These second moments depend on the environment-specific constant, γ_1, the fraction of agents receiving a positive shock, $\frac{n}{N}$, the total population size, N, and the distribution of agents' weights, $G_{D_w^-(\mathbf{Z})}\left(t\right)$. Each agent's weight determines that agent's effect on the aggregate action if he or she is the recipient of a positive shock or, in the setting with

transfers, the recipient of a negative shock. When there is a large amount of heterogeneity in agents' weights, the overall effect on the aggregate action strongly varies depending on which configuration of agents is receiving a positive shock. Therefore, the variance of the aggregate action and the variance of the economic multiplier directly depend in the variance of agents' weights.

In Proposition 3, we compute the lowest and the highest possible aggregate actions and the lowest and the highest possible economic multipliers given that n agents in a total population of N agents are receiving a positive shock:

Proposition 3 *Construct the ordered multiset* $\{\widetilde{w}_i\}_{i=1}^N$ *from the elements of* $\mathbf{d}_w^-(\mathbf{Z})$ *so that* $\widetilde{w}_i \leq \widetilde{w}_{i'}$ *whenever* $i \leq i'$. *Given Equations 2-5, we compute* $\min \operatorname{supp} Y_{agg}(\mathbf{Z}, N, n, 0)$, $\min \operatorname{supp} M(\mathbf{Z}, N, n, 0)$, $\min \operatorname{supp} Y_{agg}(\mathbf{Z}, N, n, 1)$, *and* $\min \operatorname{supp} M(\mathbf{Z}, N, n, 1)$ *by setting* $\widehat{F}_{avg}(\mathbf{Z}, N, n) = \sum_{i=1}^n \widetilde{w}_i$, *and we compute* $\max \operatorname{supp} Y_{agg}(\mathbf{Z}, N, n, 0)$, $\max \operatorname{supp} M(\mathbf{Z}, N, n, 0)$, $\max \operatorname{supp} Y_{agg}(\mathbf{Z}, N, n, 1)$, *and* $\max \operatorname{supp} M(\mathbf{Z}, N, n, 1)$ *by setting* $\widehat{F}_{avg}(\mathbf{Z}, N, n) = \sum_{i=N-n+1}^N \widetilde{w}_i$.

The lower and upper bounds on the support of the aggregate action and the economic multiplier directly depend on the topology of the network $\mathcal{G}(\mathbf{Z})$. We compute the lower bound by supposing that the n agents with the smallest average weighted in-degrees receive a positive shock. Meanwhile, we compute the upper bound by supposing that the n agents with the largest average weighted in-degrees receive a positive shock. The extent to which the lower and upper bounds on the support of the aggregate action and the support of the economic multiplier differ from each other depend on the extent to which the smallest and largest average weighted in-degrees for the graph $\mathcal{G}(\mathbf{Z})$ differ from each other.

When we have heterogeneity in agents' weights, $\operatorname{Var} D_w^-(\mathbf{Z})$ is non-zero, which makes the distributions of possible aggregate actions and economic multipliers also have positive variance. We refer to these distributions of aggregate actions and economic multipliers as being non-degenerate. Depending on the particular configuration of positive shocks, we experience variation in both the aggregate action and the economic multiplier. The next result focuses on the null case, in which the distributions of aggregate actions and economic multipliers are degenerate; regardless of the particular configuration of positive shocks among agents in the population, holding n fixed, both the aggregate action and the economic multiplier remain unchanged. This result identifies the necessary condition for degeneracy and the resulting values for the aggregate action and economic multiplier:

Proposition 4 $y_{agg}(\mathbf{Z}, \mathbf{b}, N, n, 0)$, $m(\mathbf{Z}, \mathbf{b}, N, n, 0)$, $y_{agg}(\mathbf{Z}, \mathbf{b}, N, n, 1)$, *and* $m(\mathbf{Z}, \mathbf{b}, N, n, 1)$ *are all invariant to configuration if and only if* $[\mathbf{d}_w^-(\mathbf{Z})]_i = \frac{k}{N}$ *for all* $i \in \{1, \ldots, N\}$. *When these four quantities are invariant to configuration,* $Y_{agg}(\mathbf{Z}, N, n, 0) = y_{agg}^{no}$, $M(\mathbf{Z}, N, n, 0) = 0$, $Y_{agg}(\mathbf{Z}, N, n, 1) = y_{agg}^{no} + \gamma_1 kn\epsilon$, *and* $M(\mathbf{Z}, N, n, 1) = \gamma_1 kn$, *all with probability 1.*

When every agent has the same weight, that is, when $[\mathbf{d}_w^-(\mathbf{Z})]_i = \frac{k}{N}$ for all $i \in \{1, \ldots, N\}$, the distributions of the aggregate action and economic multiplier are degenerate for all values n. In a setting with transfers, the aggregate action is equal to its no-intervention level and the economic multiplier is equal to zero with probability 1. Therefore, any policy that the outside actor implements via transfer is ineffective; there is no possibility for adjustment to the aggregate action. However, in a setting with stimulus, the aggregate action can deviate from its no-intervention level and the economic multiplier can deviate from zero; the effect of the policy on both the aggregate action and the economic multiplier in a setting with stimulus indeed does depend on the values of k and γ_1, but regardless of the configuration, holding n fixed, the effect is always the same.

We return to the original setting in which the distributions of possible aggregate actions and economic multipliers are non-degenerate. Thus far, we have been presenting results that characterize certain features of these distributions, namely, their first and second moments and the bounds on their supports. The next result shows us how to actually draw the CDFs for these distributions:

Proposition 5 *For $\ell \in \{0, 1\}$, provided that condition (c) of Lemma 5 holds,*

$$\left| G_{\frac{Y_{agg}(\mathbf{Z},N,n,\ell) - EY_{agg}(\mathbf{Z},N,n,\ell)}{(\operatorname{Var} Y_{agg}(\mathbf{Z},N,n,\ell))^{1/2}}}(t) - J(\mathbf{Z}, N, n, t) \right| < C_4 \times \sum_{i=1}^{N} |\widehat{w}_i|^5 \quad \textit{and}$$

$$\left| G_{\frac{M(\mathbf{Z},N,n,\ell) - EM(\mathbf{Z},N,n,\ell)}{(\operatorname{Var} M(\mathbf{Z},N,n,\ell))^{1/2}}}(t) - J(\mathbf{Z}, N, n, t) \right| < C_4 \times \sum_{i=1}^{N} |\widehat{w}_i|^5$$

for all t, where C_4 is only a function of $\frac{n}{N}$ and $\widehat{w}_i = \frac{\left[\mathbf{d}_w^-(\mathbf{Z})\right]_i - ED_w^-(\mathbf{Z})}{\sqrt{N \operatorname{Var} D_w^-(\mathbf{Z})}}$.

Given this result, for $\ell \in \{0, 1\}$, we can strongly approximate the CDFs for the aggregate action and the corresponding economic multiplier as follows:

$$G_{Y_{agg}(\mathbf{Z},N,n,\ell)}(t) \approx J\left(\mathbf{Z}, N, n, \frac{t - EY_{agg}(\mathbf{Z}, N, n, \ell)}{(\operatorname{Var} Y_{agg}(\mathbf{Z}, N, n, \ell))^{1/2}}\right) \text{ and}$$

$$G_{M(\mathbf{Z},N,n,\ell)}(t) \approx J\left(\mathbf{Z}, N, n, \frac{t - EM(\mathbf{Z}, N, n, \ell)}{(\operatorname{Var} M(\mathbf{Z}, N, n, \ell))^{1/2}}\right).$$

We observe that these approximations depend on the function $J(\mathbf{Z}, N, n, t)$. The function $J(\mathbf{Z}, N, n, t)$ is an asymptotic expansion whose first term is the normal distribution and whose other terms represent deviations away from the normal distribution. The extent to which these other terms are non-zero depends on the extent to which the distribution of

agent weights, $G_{D_w^-(\mathbf{Z})}(t)$, has non-zero skewness and/or non-zero excess kurtosis. When the distribution of agent weights has these non-zero higher-order moments, then the CDFs $G_{Y_{agg}(\mathbf{Z},N,n,\ell)}(t)$ and $G_{M(\mathbf{Z},N,n,\ell)}(t)$, for $\ell \in \{0,1\}$, deviate from distributions that are normal. It is ultimately the topology of $\mathcal{G}(\mathbf{Z})$ that shapes the lower-order and higher-order distributional features of $G_{Y_{agg}(\mathbf{Z},N,n,\ell)}(t)$ and $G_{M(\mathbf{Z},N,n,\ell)}(t)$, for $\ell \in \{0,1\}$. The topology of the network can generate distributions with properties of skewness and/or heavy-tailedness based on the statistical features of its accompanying network-derived vector of agent weights, $\mathbf{d}_w^-(\mathbf{Z})$. Given Proposition 5, we can draw the CDFs of aggregate actions and economic multipliers for any feasible network structure, $\mathcal{G}(\mathbf{Z})$, population size, N, and number of agents receiving a positive shock, n. For any policy, we can draw, via closed-form expressions, the resulting distribution of possible aggregate actions and the resulting distribution of possible economic multipliers.

We proceed to characterize limiting distributions for the aggregate action and economic multiplier as $N \to \infty$:

Proposition 6 *If* $\lim_{N\to\infty} \kappa_N(\epsilon') = 0$ *for any* $\epsilon' > 0$, *then* $\lim_{N\to\infty} G_{\frac{Y_{agg}(\mathbf{Z},N,n,\ell)-EY_{agg}(\mathbf{Z},N,n,\ell)}{(\mathrm{Var}\, Y_{agg}(\mathbf{Z},N,n,\ell))^{1/2}}}(t) = \Phi(t)$ *and* $\lim_{N\to\infty} G_{\frac{M(\mathbf{Z},N,n,\ell)-EM(\mathbf{Z},N,n,\ell)}{(\mathrm{Var}\, M(\mathbf{Z},N,n,\ell))^{1/2}}}(t) = \Phi(t)$ *for* $\ell \in \{0,1\}$ *and for all real* t.

When the Lindeberg-type condition is satisfied, as $N \to \infty$, the aggregate action and economic multiplier become normally distributed. Informally, $\lim_{N\to\infty} G_{Y_{agg}(\mathbf{Z},N,n,\ell)}(t) \approx \Phi\left(\frac{t-EY_{agg}(\mathbf{Z},N,n,\ell)}{(\mathrm{Var}\, Y_{agg}(\mathbf{Z},N,n,\ell))^{1/2}}\right)$, and $\lim_{N\to\infty} G_{M(\mathbf{Z},N,n,\ell)}(t) \approx \Phi\left(\frac{t-EM(\mathbf{Z},N,n,\ell)}{(\mathrm{Var}\, M(\mathbf{Z},N,n,\ell))^{1/2}}\right)$. Even though we are studying the limiting case in which $N \to \infty$, in this particular setting, $\mathrm{Var}\, Y_{agg}(\mathbf{Z},N,n,\ell)$ and $\mathrm{Var}\, M(\mathbf{Z},N,n,\ell)$ for $\ell \in \{0,1\}$ do not generally tend to zero. To see this, let us examine $\mathrm{Var}\, Y_{agg}(\mathbf{Z},N,n,0)$. From Proposition 2,

$$\mathrm{Var}\, Y_{agg}(\mathbf{Z},N,n,0) = \left(\gamma_1 \frac{N}{N-n} N\epsilon\right)^2 \frac{n}{N}\left(1-\frac{n}{N}\right)\frac{N}{N-1}\sum_{i=1}^{N}\left(\left[\mathbf{d}_w^-(\mathbf{Z})\right]_i - \frac{k}{N}\right)^2.$$

Let us hold $\frac{n}{N}$ fixed as N grows. Then:

$$\mathrm{Var}\, Y_{agg}(\mathbf{Z},N,n,0) \propto N^2\left[\sum_{i=1}^{N}\left(\left[\mathbf{d}_w^-(\mathbf{Z})\right]_i - \frac{k}{N}\right)^2\right],$$

so the behavior of $\mathrm{Var}\, Y_{agg}(\mathbf{Z},N,n,0)$ depends on how $\mathbf{d}_w^-(\mathbf{Z})$ and $\frac{k}{N}$ evolve as the population grows; we can imagine that there are many scenarios in which $\mathrm{Var}\, Y_{agg}(\mathbf{Z},N,n,0)$ either does not tend toward zero or tends to zero very slowly. As a result, the particular configuration of positive shocks remains relevant for all population sizes; even for large N, there will still be variation in both the aggregate action and the economic multiplier across configurations.

Care must be taken in determining whether the Lindeberg-type condition actually gets satisfied, that is, $\lim_{N\to\infty}\kappa_N(\epsilon')=0$ for any $\epsilon'>0$. When $\mathbf{Z}$ is row-stochastic, the Lindeberg-type condition is generally satisfied. We have $k=1$ for all population sizes N; moreover, agents' weights, $[\mathbf{d}_w^-(\mathbf{Z})]_i$, are non-negative and constrained to sum to 1, so as N increases, agents' weights generally tend toward zero, and they become increasingly closer to the average agent weight. However, there exist many types of matrices $\mathbf{Z}$ for which the Lindeberg-type condition does not get satisfied. We can imagine that there exist classes of matrices $\mathbf{Z}$ for which k changes as N grows; for example, there exist growing matrices $\mathbf{Z}$ for which $\frac{k}{N}=1$ for all N. Then, in such settings, agents' weights, $[\mathbf{d}_w^-(\mathbf{Z})]_i$, need not move closer to the average agent weight, $\frac{k}{N}$, as $N\to\infty$.

In Proposition 7, we compute in closed form the probability that a policy targeting n agents leads to an aggregate action below its no-intervention level and a negative multiplier:

Proposition 7 *The probability of an aggregate action below its no-intervention level and a negative multiplier are as follows:*

$$\Pr\left[Y_{agg}(\mathbf{Z},N,n,0)<y_{agg}^{no}\right]=\Pr\left[M(\mathbf{Z},N,n,0)<0\right]=\Pr\left[\widehat{F}_{avg}(\mathbf{Z},N,n)<\frac{kn}{N}\right],\ \textit{and}$$

$$\Pr\left[Y_{agg}(\mathbf{Z},N,n,1)<y_{agg}^{no}\right]=\Pr\left[M(\mathbf{Z},N,n,1)<0\right]=\Pr\left[\widehat{F}_{avg}(\mathbf{Z},N,n)<0\right].$$

Provided that condition (c) of Lemma 5 holds, $\Pr\left[\widehat{F}_{avg}(\mathbf{Z},N,n)<\frac{kn}{N}\right]\approx J(\mathbf{Z},N,n,0)$ *and* $\Pr\left[\widehat{F}_{avg}(\mathbf{Z},N,n)<0\right]\approx J\left(\mathbf{Z},N,n,-\frac{E\widehat{F}_{avg}(\mathbf{Z},N,n)}{\left(\operatorname{Var}\widehat{F}_{avg}(\mathbf{Z},N,n)\right)^{1/2}}\right)$, *with* $\widehat{w}_i=\frac{\left[\mathbf{d}_w^-(\mathbf{Z})\right]_i-ED_w^-(\mathbf{Z})}{\sqrt{N\operatorname{Var}D_w^-(\mathbf{Z})}}$.

Provided that condition (c) of Lemma 5 holds, we can compute in closed form the probability that a policy targeting n agents lowers the aggregate action and generates a negative economic multiplier. We can compute this probability in both settings with transfers and settings with stimulus. We can moreover compute this probability for any feasible network structure, $\mathcal{G}(\mathbf{Z})$, population size, N, and number of agents, n, being targeted by the outside actor's policy. When the outside actor's policy is financed by transfers and the network structure, $\mathcal{G}(\mathbf{Z})$, is such that the distribution of agent weights, $G_{D_w^-(\mathbf{Z})}(t)$, has zero skewness and zero excess kurtosis, $J(\mathbf{Z},N,n,0)=0.50$; the probability that the policy targeting n agents leads to a negative multiplier and a reduction in the aggregate action below its no-intervention level is equal to 50 percent. The topological features of the network $\mathcal{G}(\mathbf{Z})$ shape the distributional features of $D_w^-(\mathbf{Z})$ and thereby determine the probability that a policy targeting n agents generates a reduction in the aggregate action and a negative economic multiplier.

Negative multipliers emerge in settings with transfers when agent weights are not all equal:

Proposition 8 *For every* $n \in \{1, \ldots, N-1\}$*, provided that* $\mathbf{d}_w^-(\mathbf{Z}) \neq \frac{k}{N}\mathbf{1}$*,*
$\Pr\left[Y_{agg}(\mathbf{Z}, N, n, 0) < y_{agg}^{no}\right] = \Pr\left[M(\mathbf{Z}, N, n, 0) < 0\right] > 0.$

Given any policy that targets n agents and is financed by transfers, there is a positive probability of a negative multiplier and a positive probability that the aggregate action can be less than its no-intervention level. Practically every network structure $\mathcal{G}(\mathbf{Z})$ generates negative economic multipliers in settings with transfers. The only class of networks for which there is zero probability of a negative multiplier is the one for which $\mathbf{1}^T\mathbf{Z} = k\mathbf{1}^T$. For this particular class, the configuration of positive shocks is irrelevant, so $M(\mathbf{Z}, N, n, 0) = 0$ with probability 1. Once configuration becomes relevant, negative economic multipliers naturally emerge.

Now that we have finished characterizing policy-induced distributions of possible aggregate actions and economic multipliers in a general networked environment, we transition towards studying policy-induced distributions of aggregate actions and economic multipliers in three specific networked environments.

3 Networked Environments with Strategic Complements and Strategic Substitutes

We transition to our first setting of network-based interaction among agents. The environment that we study in this section is one of strategic complementarities and strategic substitutabilities. In this environment, the action of an agent can potentially tilt away from its autarkic level depending on the network of linkages and the extent to which other agents' actions act as complements or substitutes. We are able to characterize the distribution of possible aggregate actions and the distribution of possible multipliers in such a setting when a random subset of networked agents in the economy receives either a transfer of wealth or stimulus. Our model builds on work by Ballester, Calvó-Armengol, and Zenou (2006), which studies network games with linear-quadratic payoffs.

Let's consider an economy with a population of N agents. Each agent $i \in \{1, \ldots, N\}$ chooses an action $y_i \geq 0$ and receives a payoff

$$u_i(y_1, \ldots, y_N) = \alpha_i y_i + \frac{1}{2}\sigma y_i^2 + \sum_{\substack{j=1 \\ j \neq i}}^{N} \sigma_{ij} y_i y_j,$$

with $\alpha_i > 0$. Agents' payoffs are strictly concave in their own individual actions: $\frac{\partial^2 u_i}{\partial y_i^2} = \sigma < 0$. Bilateral influences on agent i's payoff are the quantities: $\frac{\partial^2 u_i}{\partial y_i \partial y_j} = \sigma_{ij}$. When $\sigma_{ij} > 0$, agent j's action is a strategic complement to agent i's action. When $\sigma_{ij} < 0$, agent j's action is a strategic substitute to agent i's action. In the absence of bilateral influences, agent i's autarkic action is $y_i^* = -\frac{\alpha_i}{\sigma}$. Actions that are strategic complements push agent i's action above its autarkic level, while actions that are strategic substitutes push agent i's action below its autarkic level. We capture the interdependencies of agents' behavior with the matrix $\mathbf{Z}' = \mathbf{\Sigma}$ whose diagonal elements are σ and off-diagonal elements are σ_{ij}. $\mathbf{\Sigma}$ is the weighted adjacency matrix that corresponds to network $\mathcal{G}(\mathbf{\Sigma})$. Parameter α_i, the marginal benefit accrued from an additional unit of action by agent i, can vary across individuals. Its value depends on each agent's wealth, that is, $\alpha_i = \psi\omega_i$ for $\psi > 0$ and wealth ω_i. Greater wealth increases agent i's action. The optimization problem for each agent i is therefore:

$$\max_{y_i} \psi\omega_i y_i + \frac{1}{2}\sigma y_i^2 + \sum_{\substack{j=1 \\ j\neq i}}^{N} \sigma_{ij} y_i y_j.$$

We map this theoretical environment to two different real-world settings. The first setting concerns a population of students and the amount of effort that they exert towards their education. This amount of effort depends on the behavior of their peers; see, for example, Calvó-Armengol, Patacchini, and Zenou (2009), Sacerdote (2011), and Epple and Romano (2011) for research on peer effects in education. The social network $\mathcal{G}(\mathbf{\Sigma})$ captures these peer effects. In this model, the amount of effort also depends on the wealth of the student's family. Björklund and Salvanes (2011) documents the positive relationship between family income and educational attainment; we might imagine that student effort strongly correlates with educational attainment. Family wealth raises the marginal utility of a student's effort through a variety of channels: for example, the family can afford to live in a neighborhood with higher-quality public schools that better motivate students to perform; the family can pay for enrichment activities that make learning more exciting and therefore more rewarding for the student; and the family can pay for tutoring, which increases the return that the student receives on every unit of effort. Here, the aggregate action is the aggregate effort of all students; greater aggregate effort is positively correlated with greater aggregate earning potential.

The second setting concerns the R&D divisions of firms and the amount of effort that they each allocate towards innovation. Among different R&D divisions, there is an underlying network of collaborators and competitors (Goyal and Moraga-Gonzalez (2001), König, Liu, and Zenou (2018)). The network $\mathcal{G}(\mathbf{\Sigma})$ captures these relationships as well as the

extent to which firms' R&D efforts acts as strategic complements or strategic substitutes to each other. Within formal R&D alliances, knowledge spillovers can boost the productivity for the R&D divisions of linked firms and thereby serve as a strategic complement; alternatively, they can reduce the incentives for a linked firm to engage in R&D activity (see D'Aspremont and Jacquemin (1988) and Suzumura (1992)). Meanwhile, R&D activity by competitors in a firm's product space can spark either positive or negative adjustments to the amount of effort that a firm exerts for its own R&D. Effort towards R&D is costly. The marginal benefit of effort for the R&D division is the reward of innovation. For every additional unit of effort, the reward of innovation depends on the likelihood of innovation. We can imagine that this likelihood of innovation scales with the firm's wealth. The greater the firm's wealth, the more productive and innovative are the employees that the firm hires, and therefore the more likely innovation will take place. Here, the aggregate action is aggregate R&D effort across all firms; greater effort generally leads to greater innovation.

We now return to the original setup of our model and define the unique, interior Nash equilibrium in this setting. Consistent with Ballester, Calvó-Armengol, and Zenou (2006), we introduce some additional notation. We set $\underline{\sigma} = \min\{\sigma_{ij} | i \neq j\}$, $\overline{\sigma} = \max\{\sigma_{ij} | i \neq j\}$, and $\gamma = -\min\{\underline{\sigma}, 0\} \geq 0$. We assume that $\sigma < \min\{\underline{\sigma}, 0\}$. We set $\lambda = \overline{\sigma} + \gamma$, which we take to be positive. We then define the zero-diagonal non-negative square matrix $\mathbf{G}$ whose off-diagonal elements are $[\mathbf{G}]_{ij} = \frac{\sigma_{ij}+\gamma}{\lambda} \in [0, 1]$. Constant $\beta = -\gamma - \sigma > 0$ and $r(\mathbf{G})$ is the spectral radius of $\mathbf{G}$. Our unique equilibrium is as follows:

Proposition 9 *Provided that $\beta > \lambda r(\mathbf{G})$, the unique interior Nash equilibrium in pure strategies is $\mathbf{y}^* = -\psi \boldsymbol{\Sigma}^{-1} \boldsymbol{\omega}$.*

Vector $\boldsymbol{\omega}$ denotes agents' wealth prior to any transfers or receipt of stimulus. Given agents' equilibrium behavior, we can compute the aggregate action, y_{agg}^{no}, for all agents in the population in the absence of any transfers or stimulus:

$$y_{agg}^{no} = \mathbf{1}^T \mathbf{y}^* = \psi N \left[\mathbf{d}_w^- \left(-\boldsymbol{\Sigma}^{-1}\right)\right]^T \boldsymbol{\omega},$$

where $\mathbf{d}_w^- \left(-\boldsymbol{\Sigma}^{-1}\right) = \frac{1}{N} \left(-\boldsymbol{\Sigma}^{-1}\right)^T \mathbf{1}$. The aggregate action crucially depends on the structure of agents' interaction network. As the topology of agents' interaction network changes, agents' individual actions as well as the aggregate action adjust. We essentially have two relevant networks: (1) the original agent interaction structure, $\mathcal{G}(\mathbf{Z}') = \mathcal{G}(\boldsymbol{\Sigma})$, that captures strategic complementarities and substitutabilities between agents, and (2) the network, $\mathcal{G}(\mathbf{Z}) = \mathcal{G}\left(-\boldsymbol{\Sigma}^{-1}\right)$, that determines each agent's effective weight in the population. The vector that captures agents' weights is the vector of average weighted in-degrees for the graph $\mathcal{G}\left(-\boldsymbol{\Sigma}^{-1}\right)$: $\mathbf{d}_w^- \left(-\boldsymbol{\Sigma}^{-1}\right)$. Agents' weights sum to k: $\mathbf{1}^T \left[\mathbf{d}_w^- \left(-\boldsymbol{\Sigma}^{-1}\right)\right] = k$.

Agents' weights, $\mathbf{d}_w^-\left(-\boldsymbol{\Sigma}^{-1}\right)$, determine how much of an effect targeted stimulus or a targeted transfer has on the aggregate action. We have this complex web of interactions. A positive shock to an agent's wealth increases that agent's autarkic action. However, actions are not decided by agents in isolation. A shock to an agent's wealth adjusts the actions of his or her neighbors, which then adjusts the actions of the neighbors of that agent's original set of neighbors, etc. The vector of agent weights, $\mathbf{d}_w^-\left(-\boldsymbol{\Sigma}^{-1}\right)$, condenses all of these effects; the larger an agent's weight, the greater the effect on the aggregate action following a shock to that agent's wealth.

We can examine what happens to the aggregate action for a particular configuration of transfers or stimulus, holding fixed agents' interaction structure. The configuration vector $\mathbf{b}\left(N,n\right)\in\mathcal{B}\left(N,n\right)$ identifies which subset of $n\leq N$ agents is receiving a positive adjustment to wealth. Element $b_i=1$ if agent i is receiving a positive transfer or stimulus, and otherwise $b_i=0$. Agents' wealth following either a transfer or stimulus changes from $\boldsymbol{\omega}$ to $\boldsymbol{\omega}+\boldsymbol{\rho}$. In a setting with transfers, $\left[\boldsymbol{\rho}\right]_i=\epsilon$ if $b_i=1$ and $\left[\boldsymbol{\rho}\right]_i=-\frac{n\epsilon}{N-n}$ if $b_i=0$. In a setting with stimulus, $\left[\boldsymbol{\rho}\right]_i=\epsilon$ if $b_i=1$ and $\left[\boldsymbol{\rho}\right]_i=0$ if $b_i=0$.

We can characterize the aggregate action and the economic multiplier on the aggregate action when agents $1,\ldots,n$ receive a positive transfer of wealth and agents $n+1,\ldots,N$ receive a negative transfer of wealth so that there is a zero net transfer. In this setting, $b_i=1$ for $i\in\{1,\ldots,n\}$:

$$\begin{aligned}y_{agg}\left(-\boldsymbol{\Sigma}^{-1},\mathbf{b},N,n,0\right)=y_{agg}^{no}+\psi N\epsilon\Big[&\left(\left[\mathbf{d}_w^-\left(-\boldsymbol{\Sigma}^{-1}\right)\right]_1+\cdots+\left[\mathbf{d}_w^-\left(-\boldsymbol{\Sigma}^{-1}\right)\right]_n\right)\\&-\frac{n}{N-n}\left(\left[\mathbf{d}_w^-\left(-\boldsymbol{\Sigma}^{-1}\right)\right]_{n+1}+\cdots+\left[\mathbf{d}_w^-\left(-\boldsymbol{\Sigma}^{-1}\right)\right]_N\right)\Big]\end{aligned}$$

The fifth argument of $y_{agg}\left(-\boldsymbol{\Sigma}^{-1},\mathbf{b},N,n,0\right)$, that is, the 0, denotes the setting in which there is a transfer of wealth. The multiplier is $m\left(-\boldsymbol{\Sigma}^{-1},\mathbf{b},N,n,0\right)=\frac{dy_{agg}\left(-\boldsymbol{\Sigma}^{-1},\mathbf{b},N,n,0\right)}{d\epsilon}$. We can also characterize the aggregate action and the economic multiplier on the aggregate action when agents $1,\ldots,n$ receive positive stimulus while agents $n+1,\ldots,N$ receive zero adjustment to wealth. In this setting with stimulus, $b_i=1$ for $i\in\{1,\ldots,n\}$:

$$y_{agg}\left(-\boldsymbol{\Sigma}^{-1},\mathbf{b},N,n,1\right)=y_{agg}^{no}+\psi N\epsilon\left(\left[\mathbf{d}_w^-\left(-\boldsymbol{\Sigma}^{-1}\right)\right]_1+\cdots+\left[\mathbf{d}_w^-\left(-\boldsymbol{\Sigma}^{-1}\right)\right]_n\right).$$

The fifth argument of $y_{agg}\left(-\boldsymbol{\Sigma}^{-1},\mathbf{b},N,n,1\right)$, that is, the 1, denotes the setting in which stimulus is externally funded. The multiplier is $m\left(-\boldsymbol{\Sigma}^{-1},\mathbf{b},N,n,1\right)=\frac{dy_{agg}\left(-\boldsymbol{\Sigma}^{-1},\mathbf{b},N,n,1\right)}{d\epsilon}$.

We have computed the aggregate action and the corresponding economic multiplier given a particular configuration of transfers and a particular configuration of stimulus. We are interested in all possible aggregate actions and all possible economic multipliers when

n agents each receive $\epsilon > 0$ units of additional wealth. We therefore introduce random variables that allow us to characterize the distribution of possible aggregate actions as well as the distribution of possible economic multipliers given n:

Proposition 10 *In a setting with transfers, the aggregate action and the corresponding economic multiplier are:*

$$Y_{agg}\left(-\mathbf{\Sigma}^{-1}, N, n, 0\right) = y_{agg}^{no} + \frac{\psi N^2 \epsilon}{N-n}\left[\widehat{F}_{avg}\left(-\mathbf{\Sigma}^{-1}, N, n\right) - \frac{kn}{N}\right] \text{ and}$$
$$M\left(-\mathbf{\Sigma}^{-1}, N, n, 0\right) = \frac{\psi N^2}{N-n}\left[\widehat{F}_{avg}\left(-\mathbf{\Sigma}^{-1}, N, n\right) - \frac{kn}{N}\right].$$

In a setting with stimulus, the aggregate action and the corresponding economic multiplier are:

$$Y_{agg}\left(-\mathbf{\Sigma}^{-1}, N, n, 1\right) = y_{agg}^{no} + \psi N \epsilon \widehat{F}_{avg}\left(-\mathbf{\Sigma}^{-1}, N, n\right) \text{ and}$$
$$M\left(-\mathbf{\Sigma}^{-1}, N, n, 1\right) = \psi N \widehat{F}_{avg}\left(-\mathbf{\Sigma}^{-1}, N, n\right).$$

The expressions for $Y_{agg}\left(-\mathbf{\Sigma}^{-1}, N, n, 0\right)$, $M\left(-\mathbf{\Sigma}^{-1}, N, n, 0\right)$, $Y_{agg}\left(-\mathbf{\Sigma}^{-1}, N, n, 1\right)$, and $M\left(-\mathbf{\Sigma}^{-1}, N, n, 1\right)$ in Proposition 10 directly map to Equations 2-5 in Section 2, where we set $\mathbf{Z} = -\mathbf{\Sigma}^{-1}$ and $\gamma_1 = \psi$.

Therefore, all of the results from Section 2 that characterize the distributions of aggregate actions and economic multipliers map to the present setting. For example, we can analytically compute the moments of these distributions. Given the topology of agents' interaction network, we can identify the lowest possible economic multiplier and the highest possible economic multiplier on the aggregate action that is consistent with a particular fraction of the population receiving positive wealth transfers or externally financed stimulus. Moreover, we can analytically determine the probability that a wealth transfer leads to a negative economic multiplier and the probability that externally financed stimulus leads to a negative economic multiplier; when the multiplier is negative, we have a reduction in the aggregate action below the level y_{agg}^{no}.

There do indeed exist interaction structures $\mathbf{\Sigma}$ for which the multiplier on agents' aggregate action is negative, even when every agent receives a non-negative shock to wealth. In such settings, negative multipliers emerge from the strategic substitutability of agents' actions. In general, we need there to be some level of strategic substitutability to generate a negative economic multiplier:

Proposition 11 *In environments without strategic substitutes, provided that* $-\sigma > r\left(\mathbf{\Sigma} - \sigma\mathbf{I}\right)$, $\Pr\left[Y_{agg}\left(-\mathbf{\Sigma}^{-1}, N, n, 1\right) \geq y_{agg}^{no}\right] = 1$ *and* $\Pr\left[M\left(-\mathbf{\Sigma}^{-1}, N, n, 1\right) \geq 0\right] = 1$.

The requirement that $-\sigma > r(\mathbf{\Sigma} - \sigma\mathbf{I})$ ensures the non-singularity of $\mathbf{\Sigma}$ and hence its invertibility: $\mathbf{\Sigma}^{-1}$ exists and we can construct $\mathbf{d}_w^-(-\mathbf{\Sigma}^{-1})$. As demonstrated in the proof for Proposition 11, provided that $-\sigma > r(\mathbf{\Sigma} - \sigma\mathbf{I})$, every agent has a non-negative weight in environments without strategic substitutes: $\mathbf{d}_w^-(-\mathbf{\Sigma}^{-1}) \geq \mathbf{0}$. From the expressions for $Y_{agg}(-\mathbf{\Sigma}^{-1}, N, n, 1)$ and $M(-\mathbf{\Sigma}^{-1}, N, n, 1)$ in Proposition 10, we know that any configuration of stimulus leads to a weak increase in the aggregate action and a non-negative economic multiplier whenever $\mathbf{d}_w^-(-\mathbf{\Sigma}^{-1}) \geq \mathbf{0}$. Therefore, in environments without strategic substitutes, we can never have a reduction in the aggregate action following stimulus. However, in settings with transfers, there is generally still a positive probability of negative economic multipliers, even in environments with only strategic complements. Negative multipliers emerge in such settings due to the mixture of positive and negative shocks to wealth that agents are experiencing and the heterogeneity in agents' effective weights, as captured by $\mathbf{d}_w^-(-\mathbf{\Sigma}^{-1})$.

We would like to characterize both the aggregate action and the corresponding economic multiplier in two different null settings. For the first null setting, there is no underlying network:

Proposition 12 *In the absence of any network-based interaction, that is,* $\mathbf{\Sigma} = \sigma\mathbf{I}$,

$$Y_{agg}\left(-\mathbf{\Sigma}^{-1}, N, n, 0\right) = y_{agg}^{no} \quad \textit{and} \quad M\left(-\mathbf{\Sigma}^{-1}, N, n, 0\right) = 0$$

with probability 1, and

$$Y_{agg}\left(-\mathbf{\Sigma}^{-1}, N, n, 1\right) = y_{agg}^{no} - \frac{\psi\epsilon n}{\sigma} \quad \textit{and} \quad M\left(-\mathbf{\Sigma}^{-1}, N, n, 1\right) = -\frac{\psi n}{\sigma}$$

with probability 1.

When agents' actions no longer serve as strategic complements or strategic substitutes for each other, $\mathbf{\Sigma} = \sigma\mathbf{I}$ and each agent chooses an autarkic action. Transfers of wealth across agents have no effect on the aggregate action; the economic multiplier is zero with probability 1. Externally funded stimulus does lead to an increase in the aggregate action; note that $-\frac{\psi n}{\sigma} > 0$ because $\sigma < 0$. The more agents that receive $\epsilon > 0$ units of stimulus, the higher the aggregate action. The existence of a non-trivial network structure causes the aggregate action and the corresponding economic multiplier to deviate in either direction away from these autarkic values.

For the second null setting, the aggregate action is invariant to the particular configuration of transfers or stimulus. We identify the necessary restrictions on $\mathbf{\Sigma}$ that make the

aggregate action invariant to configuration, and we then solve for the resulting values of the aggregate action and the corresponding economic multiplier:

Proposition 13 *If $\mathbf{1}^T\mathbf{\Sigma} = \delta\mathbf{1}^T$ for some $\delta \in \mathbb{R}$, both the aggregate action and the economic multiplier are invariant to configuration:*

$$Y_{agg}\left(-\mathbf{\Sigma}^{-1}, N, n, 0\right) = y_{agg}^{no} \quad \textit{and} \quad M\left(-\mathbf{\Sigma}^{-1}, N, n, 0\right) = 0$$

with probability 1, and

$$Y_{agg}\left(-\mathbf{\Sigma}^{-1}, N, n, 1\right) = y_{agg}^{no} - \frac{\psi\epsilon n}{\delta} \quad \textit{and} \quad M\left(-\mathbf{\Sigma}^{-1}, N, n, 1\right) = -\frac{\psi n}{\delta}$$

with probability 1.

Proposition 13 nests the setting of Proposition 12 if we set $\delta = \sigma$. When $\mathbf{1}^T\mathbf{\Sigma} = \delta\mathbf{1}^T$, it turns out that $\mathbf{d}_w^-\left(-\mathbf{\Sigma}^{-1}\right) = -\frac{1}{N\delta}\mathbf{1}$, and every agent has the same weight. As a result, regardless of which subset of agents receives a positive monetary transfer or stimulus, the aggregate action remains the same. The distribution of aggregate actions and the distribution of economic multipliers are both degenerate. Agents' interaction structure needs to deviate from this null setting in order to obtain non-degenerate distributions. For null interaction structures, $M\left(-\mathbf{\Sigma}^{-1}, N, n, 0\right) = 0$, so any deviation from this class of network topologies leads to the emergence of negative economic multipliers in a setting with transfers.

For the remainder of this section, we provide a set of results that allows us to rank networks. A higher-ranked network generates distributions of aggregate actions and/or distributions of multipliers that first-order stochastically dominate those generated by a lower-ranked network. For every possible configuration of transfers or stimulus, we find that the higher-ranked network generates a higher level of aggregate actions and/or a larger economic multiplier than a lower-ranked network. Transfers and stimulus are therefore relatively more effective for the higher-ranked network. Our first result focuses on aggregate actions. It ranks networks according to their corresponding distributions of aggregate actions:

Proposition 14 *Provided that $\beta > \lambda r\left(\mathbf{G}\right)$, $\beta' > \lambda' r\left(\mathbf{G}'\right)$, and $\gamma' = 0$, when $\mathbf{\Sigma}' > \mathbf{\Sigma}$, $Y_{agg}\left(-\left(\mathbf{\Sigma}'\right)^{-1}, N, n, 0\right) \succeq Y_{agg}\left(-\mathbf{\Sigma}^{-1}, N, n, 0\right)$ and $Y_{agg}\left(-\left(\mathbf{\Sigma}'\right)^{-1}, N, n, 1\right) \succeq Y_{agg}\left(-\mathbf{\Sigma}^{-1}, N, n, 1\right)$ for all $n \in \{1, \ldots, N-1\}$.*

Proposition 14 requires that $\gamma' = 0$, which means that there can be no strategic substitutes in the $\mathbf{\Sigma}'$ environment. The $\mathbf{\Sigma}$ environment can admit both strategic complements and strategic substitutes. In general, when $\mathbf{\Sigma}' > \mathbf{\Sigma}$ and $\gamma' = 0$, the distribution of possible aggregate actions in the $\mathbf{\Sigma}'$ environment first-order stochastically dominates the distribution

of possible aggregate actions in the $\mathbf{\Sigma}$ environment. This result separately holds in settings with transfers and in settings with stimulus. To prove this result, we demonstrate that the aggregate action in the $\mathbf{\Sigma}'$ environment exceeds the aggregate action in the $\mathbf{\Sigma}$ environment for any wealth vector $\boldsymbol{\omega}+\boldsymbol{\rho}$; this wealth vector can represent the wealth of agents in settings with transfers and it can represent the wealth of agents in settings with stimulus.

Our second result focuses on economic multipliers. In settings with transfers, we are unable to rank networks so that the distribution of multipliers for the higher-ranked network first-order stochastically dominates the distribution of multipliers for the lower-ranked network. This is because $EM\left(-\mathbf{\Sigma}^{-1}, N, n, 0\right) = 0$; regardless of the underlying network the mean multiplier is always the same. However, in settings with stimulus, we can rank networks according to their corresponding distributions of economic multipliers:

Proposition 15 *Provided that $\beta > \lambda r\left(\mathbf{G}\right)$, $\beta' > \lambda' r\left(\mathbf{G}'\right)$, and $\gamma' = 0$, $M\left(-\left(\mathbf{\Sigma}'\right)^{-1}, N, n, 1\right) \succeq M\left(-\mathbf{\Sigma}^{-1}, N, n, 1\right)$ for all $n \in \{1, \ldots, N\}$ when $\mathbf{\Sigma}' > \mathbf{\Sigma}$ for symmetric $\mathbf{\Sigma}, \mathbf{\Sigma}'$.*

The $\mathbf{\Sigma}'$ environment only admits strategic complements, while the $\mathbf{\Sigma}$ environment can admit both strategic complements and strategic substitutes. Under these assumptions, $\mathbf{d}_w^-\left(-\left(\mathbf{\Sigma}'\right)^{-1}\right) > \mathbf{d}_w^-\left(-\mathbf{\Sigma}^{-1}\right)$, which makes the distribution of multipliers in the $\mathbf{\Sigma}'$ environment first-order stochastically dominate the distribution of multipliers in the $\mathbf{\Sigma}$ environment. Given that n agents each receive $\epsilon > 0$ units of stimulus, the effect on the aggregate action is relatively more positive for the higher-ranked network.

4 Networked Environments with Coordination and Anti-Coordination

We proceed to our second environment with network-based interaction. The environment that we focus on in this section is a dynamic one in which agents engage in a mixture of coordinating and anti-coordinating behavior with other agents in the population. We are interested in the aggregate action for the population and its dynamic evolution.

We have a population of N agents. Each agent chooses an action that somehow depends on other agents' past behaviors. In choosing this action, each agent essentially segments the population into two groups: (1) a group with whom the agent seeks to choose a coordinating action and (2) a group with whom the agent seeks to choose an anti-coordinating action. Each agent moreover decides how much weight to accord to every other agent in the population. The $N \times N$ matrix $\mathbf{T}$ captures each agent's desire for coordination or anti-coordination; it is a matrix of linkage types. Similar to Eger (2016a), there are two types

of linkages: $[\mathbf{T}]_{ij} \in \{\mathcal{F}, \mathcal{D}\}$ $\forall i, j \in \{1, \ldots, N\}$. $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ is the *follow* linkage, while $\mathcal{D} : \mathbb{R} \to \mathbb{R}$ is the *deviation* linkage. Given the past action of an agent j, function $\mathcal{F}$ or $\mathcal{D}$ transforms that past action into the present desired responding action for agent i. When $[\mathbf{T}]_{ij} = \mathcal{F}$, agent i seeks to follow the past action of agent j, so we have myopic coordination, while when $[\mathbf{T}]_{ij} = \mathcal{D}$, agent i seeks to deviate from the past action of agent j, so we have myopic anti-coordination. Later on, we will be introducing a matrix $\mathbf{O}$, with $[\mathbf{O}]_{ij} = 1$ if $[\mathbf{T}]_{ij} = \mathcal{F}$ and $[\mathbf{O}]_{ij} = -1$ if $[\mathbf{T}]_{ij} = \mathcal{D}$. Meanwhile, the $N \times N$ row-stochastic matrix $\bar{\mathbf{A}}$ captures the weight that agents assign to other agents. The ij^{th} element of $\bar{\mathbf{A}}$ represents the non-negative weight that agent i allocates to agent j. For every agent $i \in \{1, \ldots, N\}$, the sum of the weights that each agent i accords to every other agent j sums to 1, that is, $\left[\bar{\mathbf{A}}\right]_{i*} \mathbf{1} = 1$, and unless otherwise specified, $\left[\bar{\mathbf{A}}\right]_{ii} = 0$. In our environment with coordinating and anti-coordinating behavior, agents are therefore organized on the network $\mathcal{G}\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)$ with corresponding weighted adjacency matrix $\bar{\mathbf{A}} \circ \mathbf{O}$.

Every period q, agent i chooses an action, $y_{i,q}$, that maximizes his period-q utility:

$$\max_{y_{i,q}} u_{i,q} = \max_{y_{i,q}} \; -\sum_{j=1}^{N} \left[\bar{\mathbf{A}}\right]_{ij} \left(y_{i,q} - [\mathbf{T}]_{ij} \left(y_{j,q-1}\right)\right)^2 .$$

We define $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ to be an identity function; $\mathcal{F}\left(y_{j,q-1}\right) = y_{j,q-1}$, so when $[\mathbf{T}]_{ij} = \mathcal{F}$, agent i seeks to choose an action that follows agent j's past action. We define $\mathcal{D} : \mathbb{R} \to \mathbb{R}$ as $\mathcal{D}\left(y_{j,q-1}\right) = y - \left(y_{j,q-1} - y\right)$. When $[\mathbf{T}]_{ij} = \mathcal{D}$, agent i seeks to choose an action that deviates from the past action of agent j. In particular, agent i wishes to choose an action that deviates in a direction opposite to the previous action of agent j; for example, if the past action of agent j is less than a benchmark action y, then agent i seeks to choose an action that is greater than y, and vice versa. We assume that desired deviating behavior takes the same form across all agents. In this setting, coordination and anti-coordination occur on past actions, so agents' behavior is myopic.

There are different ways that we can think about this theoretical environment and how it maps to realistic settings. Here, I focus on one particular mapping. We can imagine that there is a population of N agents who choose an action every period. The magnitude of the action that each agent chooses depends on the prior actions of other agents. In particular, each agent has a set of role models and anti-role models. Matrix element $[\mathbf{O}]_{ij} = 1$ if agent j is a role model for agent i, and matrix element $[\mathbf{O}]_{ij} = -1$ if agent j is instead an anti-role model for agent i. Matrix $\bar{\mathbf{A}}$ then captures the weight that each agent accords to his or her role models and anti-role models. In a setting with role models and anti-role models, agents respectively engage in myopic coordination and anti-coordination. In general, agents do not

communicate with their role models and anti-role models. Rather, they observe the past actions of these agents, and they then seek to choose an action that imitates the past actions of their role models and deviates from the past actions of their anti-role models. A setting with role models and anti-role models is therefore a natural setting for myopic coordination and anti-coordination. Now, there is a wide range of possible actions that these agents can take. Let's assume that agents are engaging in prosocial behavior, such as volunteering or providing a public good. These agents are selecting the amount of time that they engage in this activity, with the amount of time dependent on the past actions of other agents. The outside observer to this system is interested in the aggregate action, which is the total amount of time spent on the activity.

We next return to our model and solve the optimization problem for each agent in this environment, that is, the optimal choice of a period-specific action:

Proposition 16 *For each agent $i \in \{1, \ldots, N\}$, $y^*_{i,q} = \sum_{j=1}^{N} \left[\bar{\mathbf{A}}\right]_{ij} [\mathbf{T}]_{ij} (y_{j,q-1})$, and therefore $\mathbf{y}^*_q = \left(\bar{\mathbf{A}} \circ \mathbf{T}\right)^q \mathbf{y}_0$.*

Depending on the structure of $[\mathbf{T}]_{i*}$, agent i engages in a mixture of coordinating and anti-coordinating behavior. Agent i chooses an action that is a weighted sum of other agents' past actions, for those agents that agent i seeks to follow, and a weighted sum of desired deviating actions, for those agents from whom agent i seeks to deviate. Agents coordinate and anti-coordinate on past actions. They have myopic best-response functions.

We assume that all agents prior to time period zero choose action y. This action is an optimal action; given that every agent chooses y in period $q-1$, every agent will continue to choose y in period q:

$$y^*_{i,q} = \sum_{j=1}^{N} \left[\bar{\mathbf{A}}\right]_{ij} [\mathbf{T}]_{ij} (y_{j,q-1})$$
$$y^*_{i,q} = \sum_{\substack{j \in \{1,\ldots,N\} \\ \text{s.t. } [\mathbf{T}]_{ij} = \mathcal{F}}} \left[\bar{\mathbf{A}}\right]_{ij} y + \sum_{\substack{j \in \{1,\ldots,N\} \\ \text{s.t. } [\mathbf{T}]_{ij} = \mathcal{D}}} \left[\bar{\mathbf{A}}\right]_{ij} (2y - y) = y.$$

When time period 0 arrives, an outside entity adjusts agents' actions: $\mathbf{y}^*_0 = y\mathbf{1} + \boldsymbol{\rho}$; the $\boldsymbol{\rho}$ vector captures that adjustment. In settings with transfers, the outside entity increases the actions of n agents by $\epsilon > 0$ units, and decreases the actions of the remaining $N - n$ agents by $\frac{n\epsilon}{N-n}$ units. Therefore, if the ith agent receives a positive transfer, $[\boldsymbol{\rho}]_i = \epsilon$, while if the ith agent receives a negative transfer, $[\boldsymbol{\rho}]_i = -\frac{n\epsilon}{N-n}$. In settings with stimulus, the outside entity only increases the actions of n agents by $\epsilon > 0$ units; it does not adjust the period-0 actions of the remaining $N - n$ agents. Therefore, if the ith agent receives positive stimulus, $[\boldsymbol{\rho}]_i = \epsilon$,

while if the ith agent does not receive positive stimulus, then $[\boldsymbol{\rho}]_i = 0$. Configuration vector $\mathbf{b}(N,n) \in \mathcal{B}(N,n)$ determines which agents receive that positive period-0 shock in both settings with transfers and settings with stimulus. The next result allows us to analytically trace the population vector of optimal agent actions for all periods q given the period-0 shock $\boldsymbol{\rho}$:

Proposition 17 *With $\mathbf{y}_q^* = y\mathbf{1}$ for $q < 0$, the population vector of agent actions is $\mathbf{y}_q^* = y\mathbf{1} + \left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q \boldsymbol{\rho}$ for all $q \in \mathbb{Z}_+$.*

We can now compute the aggregate action in period q:

$$y_{agg,q}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, \mathbf{b}, N, n, \ell\right) = \mathbf{1}^T \mathbf{y}_q^* = y_{agg}^{no} + N\left[\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)\right]^T \boldsymbol{\rho}$$

for $\ell \in \{0,1\}$.[2] Quantity $y_{agg}^{no} = Ny$ is the aggregate action absent transfers or stimulus, that is, when $\boldsymbol{\rho} = \mathbf{0}$. $\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)$ is the vector of average weighted in-degrees for graph $\mathcal{G}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)$. We set $\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right) = \frac{1}{N}\left[\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right]^T \mathbf{1}$. The sum of agents' weights in period q is k_q: $\left[\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)\right]^T \mathbf{1} = k_q$.

In this particular environment, the aggregate action depends on the structure of agents' interaction network. We essentially have two relevant networks: (1) the original agent interaction structure, $\mathcal{G}(\mathbf{Z}') = \mathcal{G}\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)$, that captures agents' myopic coordinating and anti-coordinating behavior, and (2) the network, $\mathcal{G}(\mathbf{Z}) = \mathcal{G}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)$, that determines each agent's effective weight in the population. Each agent's weight identifies how much of an effect targeted stimulus or a targeted transfer towards that particular agent has on the aggregate action. We have this complicated mixture of coordinating and anti-coordinating behavior among agents that increases in complexity as time evolves. The period-specific vector of agent weights, $\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)$, summarizes the net amount of coordination or anti-coordination that the entire population of agents undertakes given the action of every agent.

For a particular configuration of transfers or stimulus, we can compute the aggregate action, the corresponding dynamic multiplier (i.e., the period-specific economic multiplier), and the impulse response. First, let's suppose that agents $1, \ldots, n$ receive a positive transfer. The aggregate action in period q is then:

$$\begin{aligned} y_{agg,q}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, \mathbf{b}, N, n, 0\right) = y_{agg}^{no} &+ N\epsilon\left(\left[\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)\right]_1 + \cdots + \left[\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)\right]_n\right) \\ &- N\frac{n\epsilon}{N-n}\left(\left[\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)\right]_{n+1} + \cdots + \left[\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)\right]_N\right), \end{aligned}$$

[2] The notation in this section deviates slightly from the notation introduced in Schlossberger (2018). To be consistent with the other sections in the present work, we use $\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)$ instead of $\mathbf{d}_w^{-(q)}\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)$, and later on, we use $\widehat{F}_{avg}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n\right)$ instead of $\widehat{F}_{avg}^{(q)}\left(\bar{\mathbf{A}} \circ \mathbf{O}, N, n\right)$.

The period-specific dynamic multiplier is $m_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,\mathbf{b},N,n,0\right)=\frac{dy_{agg,q}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,\mathbf{b},N,n,0\right)}{d\epsilon}$, and the impulse response function is $irf_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,\mathbf{b},N,n,0\right)=y_{agg,q}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,\mathbf{b},N,n,0\right)-y_{agg}^{no}$. Next, let's suppose that agents $1,\ldots,n$ instead receive positive stimulus. The aggregate action in period q is then:

$$y_{agg,q}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,\mathbf{b},N,n,1\right)=y_{agg}^{no}+N\epsilon\left(\left[\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q\right)\right]_1+\cdots+\left[\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q\right)\right]_n\right)$$

The period-specific dynamic multiplier is $m_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,\mathbf{b},N,n,1\right)=\frac{dy_{agg,q}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,\mathbf{b},N,n,1\right)}{d\epsilon}$, and the impulse response function is $irf_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,\mathbf{b},N,n,1\right)=y_{agg,q}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,\mathbf{b},N,n,1\right)-y_{agg}^{no}$.

Depending on which subset of agents receives a positive transfer or stimulus, we can have wide variation in the aggregate action, the economic multiplier, and the impulse response for each period. In the next proposition, we define the random variables that allow us to construct these distributions of possible values for the aggregate action, the economic multiplier, and the impulse response for every period q:

Proposition 18 *In a setting with transfers, the aggregate action, the dynamic multiplier, and the impulse response are:*

$$\begin{aligned}
Y_{agg,q}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,N,n,0\right)&=y_{agg}^{no}+\frac{N^2\epsilon}{N-n}\left(\widehat{F}_{avg}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,N,n\right)-\frac{k_qn}{N}\right),\\
M_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,N,n,0\right)&=\frac{N^2}{N-n}\left(\widehat{F}_{avg}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,N,n\right)-\frac{k_qn}{N}\right),\ \textit{and}\\
IRF_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,N,n,0\right)&=\frac{N^2\epsilon}{N-n}\left(\widehat{F}_{avg}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,N,n\right)-\frac{k_qn}{N}\right).
\end{aligned}$$

In a setting with stimulus, the aggregate action, the dynamic multiplier, and the impulse response are:

$$\begin{aligned}
Y_{agg,q}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,N,n,1\right)&=y_{agg}^{ns}+N\epsilon\widehat{F}_{avg}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,N,n\right),\\
M_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,N,n,1\right)&=N\widehat{F}_{avg}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,N,n\right),\ \textit{and}\\
IRF_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,N,n,1\right)&=N\epsilon\widehat{F}_{avg}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,N,n\right).
\end{aligned}$$

The random variables for the impulse response function, $IRF_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,N,n,0\right)$ and $IRF_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,N,n,1\right)$, are defined in a similar manner to the other random variables in Proposition 18.[3]

The expressions for $Y_{agg,q}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,N,n,0\right)$, $M_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q,N,n,0\right)$,

[3]Provided that each period-0 configuration of transfers or stimulus is equally likely, the CDF for

$Y_{agg,q}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^{q},N,n,1\right)$, and $M_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^{q},N,n,1\right)$ in Proposition 18 directly map to Equations 2-5 in Section 2, where we set $\mathbf{Z}=\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^{q}$ and $\gamma_1=1$. Accordingly, we can map all of the results from Section 2 to the present theoretical environment. We can characterize the period-specific distributions of aggregate actions, dynamic economic multipliers, and impulse responses. We can solve for the moments of these distributions in closed form. We can determine the lowest possible aggregate action and the highest possible aggregate action for a particular level of transfers or stimulus, and we can demonstrate how those values depend on the topology of agents' interaction network. Based on the asymptotic expansions that approximate these distributions, we can determine the extent to which agents' interaction topology leads to skewness and/or heavy-tailedness in the distributions of possible aggregate actions, dynamic economic multipliers, and impulse responses.

We can also importantly compute both the probability that the dynamic multiplier is negative and the probability that the aggregate action drops below its no-intervention level, y_{agg}^{no}, for finite q and in the limiting case as $q\to\infty$. In this environment with myopic coordination and anti-coordination, there are different reasons why negative multipliers can arise. We now illustrate a couple of these pathways. Let's first consider a setting with transfers. The aggregate action can decline if an agent receives a positive shock and other agents seek to anti-coordinate with that particular agent. The aggregate action can separately decline if an agent receives a negative shock and other agents seek to coordinate with that particular agent. Let's next consider a setting with stimulus. Every agent is receiving a non-negative shock at period zero. If we only have coordinating behavior, then there is zero probability that the multiplier can be negative. However, if we have anti-coordinating behavior, the aggregate action can decline if the desire to anti-coordinate with an agent receiving stimulus is sufficiently strong. For the mechanisms just described, agents are adjusting their actions upward or downward based on the shocks that their immediate neighbors receive. However, as time evolves, even though agents are continuing to myopically coordinate and anti-coordinate with their network neighbors, they are indirectly coordinating and anti-coordinating with agents whose distance on the network exceeds 1.

We would like to characterize the period-specific aggregate action, dynamic multiplier, and impulse response in two different null settings. For the first null setting, there is no underlying network:

$IRF_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^{q},N,n,\ell\right)$ is

$$G_{IRF_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^{q},N,n,\ell\right)}(t)=\frac{1}{\left|\mathcal{B}(N,n)\right|}\sum_{\mathbf{b}(N,n)\in\mathcal{B}(N,n)}\mathbb{1}_{irf_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^{q},\mathbf{b},N,n,\ell\right)\leq t}\quad\text{for }\ell\in\{0,1\}.$$

Proposition 19 *In the absence of network-based interaction,* $Y_{agg,q}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q, N, n, 0\right) = y_{agg}^{no}$*,* $M_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q, N, n, 0\right) = 0$*, and* $IRF_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q, N, n, 0\right) = 0$ *with probability 1 for all* $q \in \mathbb{Z}_+$*, and* $Y_{agg,q}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q, N, n, 1\right) = y_{agg}^{no} + n\epsilon$*,* $M_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q, N, n, 1\right) = n$*, and* $IRF_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q, N, n, 1\right) = n\epsilon$ *with probability 1 for all* $q \in \mathbb{Z}_+$*.*

When there is no network, $\bar{\mathbf{A}}\circ\mathbf{O} = \mathbf{I}$, so agents coordinate on their past actions. In settings with transfers, the aggregate action equals its no-intervention level for all possible initial configurations and all periods $q \in \mathbb{Z}_+$. In settings with stimulus, the aggregate action exceeds its no-intervention level by the total amount of stimulus, $n\epsilon$, and the aggregate action maintains this value for all possible initial configurations of stimulus and all periods $q \in \mathbb{Z}_+$. Non-trivial network-based interaction enables us to have a non-degenerate distribution for the aggregate action; in such an environment, the aggregate action can deviate in either direction away from its no-intervention level.

For the second null setting, the aggregate action is invariant to the particular configuration of transfers or stimulus. We identify the necessary restrictions on $\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q$ that make the period-q aggregate action invariant to configuration, and we then solve for the resulting values of the aggregate action, the corresponding economic multiplier, and the impulse response:

Proposition 20 *If* $\mathbf{1}^T\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q = k_q\mathbf{1}^T$*, the period-q aggregate action, dynamic multiplier, and impulse response are invariant to configuration:* $Y_{agg,q}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q, N, n, 0\right) = y_{agg}^{no}$*,* $M_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q, N, n, 0\right) = 0$*, and* $IRF_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q, N, n, 0\right) = 0$ *with probability 1, and* $Y_{agg,q}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q, N, n, 1\right) = y_{agg}^{no} + k_q n\epsilon$*,* $M_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q, N, n, 1\right) = k_q n$*, and* $IRF_q\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q, N, n, 1\right) = k_q n\epsilon$ *with probability 1.*

Proposition 20 nests the setting of Proposition 19 if we set $k_q = 1$. When $\bar{\mathbf{A}}$ is doubly stochastic and $\bar{\mathbf{A}}\circ\mathbf{O} = \bar{\mathbf{A}}$, that is, the environment only has coordination, $\mathbf{1}^T\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q = \mathbf{1}^T$ for all $q \in \mathbb{Z}_+$, which makes the aggregate action, economic multiplier, and impulse response invariant to configuration for all time periods $q \in \mathbb{Z}_+$. For a transfer or stimulus to generate a negative economic multiplier, we must deviate from those network topologies characterized in Proposition 20, for which the economic multiplier is invariant to configuration.

The next result allows us to rank networks so that the distributions of aggregate actions, dynamic multipliers, and impulse responses for the higher-ranked network first-order stochastically dominate the distributions of aggregate actions, dynamic multipliers, and impulse responses for the lower-ranked network:

Proposition 21 *If* $\left(\bar{\mathbf{A}}'\circ\mathbf{O}'\right)^q \geq \left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q\mathbf{P}$ *for some permutation matrix* $\mathbf{P}$*, then* $Y_{agg,q}\left(\left(\bar{\mathbf{A}}'\circ\mathbf{O}'\right)^q, N, n, 1\right) \succeq Y_{agg,q}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q, N, n, 1\right)$*,* $M_q\left(\left(\bar{\mathbf{A}}'\circ\mathbf{O}'\right)^q, N, n, 1\right) \succeq$

$M_q\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n, 1\right)$, *and* $IRF_q\left(\left(\bar{\mathbf{A}}' \circ \mathbf{O}'\right)^q, N, n, 1\right) \succeq IRF_q\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n, 1\right)$ *for all* $n \in \{1, \ldots, N-1\}$.

We can rank network topologies in settings with stimulus. Proposition 21 nests the case in which a linkage changes from negative to positive; in such an environment, $\bar{\mathbf{A}}' = \bar{\mathbf{A}}$, but $\mathbf{O}' > \mathbf{O}$. Once we move from anti-coordination to coordination for one pair of agents, the distribution of aggregate actions first-order stochastically dominates the original distribution. We are not able to rank network topologies in settings with transfers because the distribution of aggregate actions always has a mean equal to its no-intervention level, y_{agg}^{no}, for every feasible network structure.

Now that we have ranked networks, we proceed to determine the maximum and minimum possible dynamic multipliers among all networks. First, we must bound the allowable values for k_q:

Lemma 7 *For all* $q \geq 1$, $k_q \in [-1, 1]$.

$k_q = 1$ is attainable when $\bar{\mathbf{A}} \circ \mathbf{O} = \bar{\mathbf{A}}$, and $k_q = -1$ is attainable when $\bar{\mathbf{A}} \circ \mathbf{O} = -\bar{\mathbf{A}}$. We must also introduce two classes of graphs:

Definition 3 *Graph* $\mathcal{G}(\mathbf{Z})$ *is a* positive star graph *and a* negative star graph *when the weighted adjacency matrices are respectively:*

$$\mathbf{Z} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \mathbf{P} \quad \textit{and} \quad \mathbf{Z} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \mathbf{P}$$

for some $N \times N$ *permutation matrix* $\mathbf{P}$.

In the next proposition, we identify the extremal multipliers in settings with stimulus. These extremal values hold for all levels n, that is, for all possible levels of stimulus. We also identify the network topologies that generate these extremal multipliers, temporarily allowing for the existence of self-loops:

Proposition 22 *For every* $n \in \{1, \ldots, N-1\}$ *and all feasible matrices* $\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q$,

$$\max_{\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q} \left[\max \operatorname{supp} M_q\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n, 1\right)\right] = N \textit{ and}$$

$$\min_{\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q} \left[\min \operatorname{supp} M_q\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n, 1\right)\right] = -N.$$

These maximum and minimum values are attainable when graph $\mathcal{G}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)$ is respectively a positive star graph and a negative star graph.

Among all networks, in settings with stimulus, the maximum possible dynamic multiplier is N, and among all networks, in settings with stimulus, the minimum possible dynamic multiplier is $-N$. These maximum and minimum values are attainable for every level of stimulus, that is, for every integer $n \in \{1, \ldots, N-1\}$. If $\mathcal{G}\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)$ is a positive star graph, then the maximum possible dynamic multiplier is attainable for all periods $q \geq 1$ and for all levels $n \in \{1, \ldots, N-1\}$ of initial stimulus due to the idempotence of $\bar{\mathbf{A}} \circ \mathbf{O}$.

We conclude this section by noting that our environment with myopic coordination and anti-coordination is a dynamic one. Provided that $\bar{\mathbf{A}} \circ \mathbf{O}$ is semi-convergent, $\lim_{q\to\infty}\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q$ exists and we can characterize limiting distributions for the aggregate action, dynamic multiplier, and impulse response in settings with transfers and settings with stimulus as $q \to \infty$. From our random variables in Proposition 18, we see that these limiting distributions all depend on the following object: $\lim_{q\to\infty} \widehat{F}_{avg}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n\right)$. In settings with transfers, the limiting distributions depend on the quantity $\lim_{q\to\infty} k_q$ as well. To compute $\lim_{q\to\infty} \widehat{F}_{avg}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n\right)$, we must determine the limiting vector $\lim_{q\to\infty} \mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)$.

The remaining theoretical results allow us to solve for $\lim_{q\to\infty} \mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)$ after making specific assumptions. Let us first consider the case in which we only have coordination, so that $\bar{\mathbf{A}} \circ \mathbf{O} = \bar{\mathbf{A}}$:

Proposition 23 *If $\bar{\mathbf{A}}$ is primitive, then $\lim_{q\to\infty} \mathbf{d}_w^-\left(\bar{\mathbf{A}}^q\right) = \mathbf{w}_\infty\left(\bar{\mathbf{A}}\right)$, where the pair $\left(\mathbf{w}_\infty^T, 1\right)$ is the unique dominant left eigenpair of $\bar{\mathbf{A}}$, $\mathbf{w}_\infty^T \bar{\mathbf{A}} = \mathbf{w}_\infty^T$, and $\mathbf{w}_\infty^T \mathbf{1} = 1$.*

The limiting vector of agent weights is computed by solving for the left eigenvector of the matrix $\bar{\mathbf{A}}$ that pairs with the unit eigenvalue. Provided that $\bar{\mathbf{A}}$ is primitive, when $\bar{\mathbf{A}} \circ \mathbf{O} = \bar{\mathbf{A}}$, all agents converge to the same limiting action because $\lim_{q\to\infty} \bar{\mathbf{A}}^q = \mathbf{1}\left[\mathbf{w}_\infty\left(\bar{\mathbf{A}}\right)\right]^T$. From the row-stochasticity of $\bar{\mathbf{A}}$, we are also able to determine that $\lim_{q\to\infty} k_q = 1$.

Given some additional assumptions, in the coordinating environment we can compute the probability of a negative limiting dynamic multiplier in terms of the network's primitives. Let us assume that $\left[\bar{\mathbf{A}}\right]_{ij} > 0$ if and only if $\left[\bar{\mathbf{A}}\right]_{ji} > 0$ and agents assign an equal weight to all out-neighbors. Since all linkages in the network are accordingly reciprocal, we can compute a vector of degrees, $\mathbf{d}\left(\bar{\mathbf{A}}\right)$. The degree for agent i is equal to the number of non-zero elements in the ith row of $\bar{\mathbf{A}}$: $\left[\bar{\mathbf{A}}\right]_{i*}$. Given the vector of degrees, we define a random variable $D\left(\bar{\mathbf{A}}\right)$ whose realization is the degree for agent i: $\left[\mathbf{d}\left(\bar{\mathbf{A}}\right)\right]_i$.

Proposition 24 *Suppose that* $\bar{\mathbf{A}}$ *is primitive,* $\left[\bar{\mathbf{A}}\right]_{ij} > 0$ *if and only if* $\left[\bar{\mathbf{A}}\right]_{ji} > 0$*, and all non-zero elements within every row of* $\bar{\mathbf{A}}$ *have the same value. When* $n = 1$*,*
$\Pr\left[\lim_{q\to\infty} Y_{agg,q}\left(\bar{\mathbf{A}}^q, N, n, 0\right) < y_{agg}^{no}\right] = \Pr\left[\lim_{q\to\infty} M_q\left(\bar{\mathbf{A}}^q, N, n, 0\right) < 0\right] =$
$\Pr\left[D\left(\bar{\mathbf{A}}\right) < ED\left(\bar{\mathbf{A}}\right)\right]$.

When one agent receives a positive transfer at time period 0, the probability that the limiting dynamic multiplier ends up being negative is equal to the probability that the degree in the network is less than the expected degree.[4] Meanwhile, in a setting with stimulus, since agents are all coordinating, the probability of a negative limiting dynamic multiplier is always zero.

Lastly, we consider the case in which we have a mixture of coordinating and anti-coordinating behavior, so that $\bar{\mathbf{A}} \circ \mathbf{O} \neq \bar{\mathbf{A}}$. $\lim_{q\to\infty} \mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)$ exists when $\bar{\mathbf{A}} \circ \mathbf{O}$ is semi-convergent. Under certain assumptions, we can explicitly solve for $\lim_{q\to\infty} \mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)$. Let graph $\mathcal{G}\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)$ have edge weight $e_{i,j} = \left[\bar{\mathbf{A}} \circ \mathbf{O}\right]_{ij}$. We characterize $\lim_{q\to\infty} \mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)$ after providing definitions for structural balance and absolute row-stochasticity:

Definition 4 *Graph* $\mathcal{G}\left(\bar{\mathbf{A}} \circ \mathbf{O}\right) = \left(\mathcal{V}\left(\bar{\mathbf{A}} \circ \mathbf{O}\right), \mathcal{E}\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)\right)$ *is* structurally balanced *if there exists a partition of nodes* $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$*, with* $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$*, such that: (1) if nodes* $i, j \in \mathcal{V}_\ell$ *and* $(i, j, e_{i,j}) \in \mathcal{E}$ *for* $\ell \in \{1, 2\}$*, then* $\operatorname{sgn}(e_{i,j}) > 0$*, and (2) if* $i \in \mathcal{V}_\ell$ *and* $j \in \mathcal{V}_{-\ell}$ *with* $(i, j, e_{i,j}) \in \mathcal{E}$ *for* $\ell \in \{1, 2\}$*, then* $\operatorname{sgn}(e_{i,j}) < 0$.[5]

Definition 5 $\bar{\mathbf{A}} \circ \mathbf{O}$ *is* absolutely row-stochastic *if* $\left|\bar{\mathbf{A}} \circ \mathbf{O}\right|$*, the element-wise absolute value of matrix* $\bar{\mathbf{A}} \circ \mathbf{O}$*, is row-stochastic.*

We now solve for $\lim_{q\to\infty} \mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)$:

Proposition 25 *For* $\bar{\mathbf{A}} \circ \mathbf{O}$ *absolutely row-stochastic,* $\left|\bar{\mathbf{A}} \circ \mathbf{O}\right| = \bar{\mathbf{A}}$ *primitive, and* $\mathcal{G}\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)$ *structurally balanced,* $\lim_{q\to\infty} \mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right) = \lim_{q\to\infty} \frac{1}{N}\left[\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right]^T \mathbf{1}$ *exists, with*
$\lim_{q\to\infty} \left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q = \left(\mathbf{1}\left[\mathbf{w}_\infty\left(\bar{\mathbf{A}}\right)\right]^T\right) \circ \mathbf{O}$ *and* $\left[\mathbf{w}_\infty\left(\bar{\mathbf{A}}\right)\right]^T \bar{\mathbf{A}} = \left[\mathbf{w}_\infty\left(\bar{\mathbf{A}}\right)\right]^T$.

If $|\mathcal{V}_1| < |\mathcal{V}_2|$, then $\lim_{q\to\infty}\left[\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)\right]_i < 0$ if $i \in \mathcal{V}_1$ and $\lim_{q\to\infty}\left[\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)\right]_i > 0$ if $i \in \mathcal{V}_2$. Since $\lim_{q\to\infty} \mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right) \neq \lim_{q\to\infty} \frac{k_q}{N}\mathbf{1}$, in settings with transfers, we have a positive probability of a negative limiting dynamic multiplier for all levels of transfers, that is, for all $n \in \{1, \ldots, N-1\}$.

[4] We can construct propositions similar to Proposition 24 for networks $\mathcal{G}\left(\bar{\mathbf{A}}\right)$ that have other sets of features as well. Refer to Section 4 of Schlossberger (2018) for different closed-form formulations of $\mathbf{w}_\infty\left(\bar{\mathbf{A}}\right)$.

[5] Eger (2016b) refers to this property as +-opposition bipartiteness.

5 Networked Environments with Production

In this third and final setting with network-based interaction among agents, we consider a production network. We have a population of firms; every firm represents a different sector. Production by each firm potentially requires both labor and intermediate goods obtained from other firms. Linkages in the production network therefore capture the flow of intermediate goods between firms. Specifically, the directed edges of the network designate upstreamness in production; for each firm, these edges point towards the firm's suppliers.

Now, given the production network, we are interested in the nominal value of aggregate output, or GDP. In particular, we are interested in the distribution of possible levels of GDP that result when a random subset of sectors receives a positive demand shock. Sectors differ in their importance as suppliers to both other sectors and the final consumer, so depending on which group of sectors receives a positive demand shock, we can potentially have strong variation in GDP. We are also interested in the distribution of possible economic multipliers. We would like to know the change in GDP that results when a certain subset of sectors receives a positive demand shock of a particular magnitude. Depending on the structure of the production network, we can imagine that the multiplier on GDP varies with the group of sectors actually receiving the positive demand shock. For certain groups of sectors, the boost in GDP is larger than that for other groups.

We proceed to describe this section's theoretical environment. There are N sectors with one good associated with each sector, so there are N total goods. Our representative consumer has Cobb-Douglas utility over the consumption bundle $\mathbf{c} = \begin{pmatrix} c_1 & \cdots & c_N \end{pmatrix}^T$:

$$U(\mathbf{c}) = \prod_{i=1}^{N} c_i^{\eta_i}.$$

We set $\sum_{i=1}^{N} \eta_i = 1$. Each good is produced by a competitive sector; the good is either consumed or used in the production of other goods. The production technology for the representative firm in each sector takes a Cobb-Douglas form:

$$x_i = A_i^{\alpha_i} \ell_i^{\alpha_i} \left(\prod_{j=1}^{N} x_{ji}^{[\mathbf{\Lambda}]_{ji}} \right)^{\beta_i}.$$

A_i is a sector-specific productivity parameter; productivity is labor-augmenting. ℓ_i denotes the amount of labor used in the production of good i. The representative consumer has no disutility from labor, so labor is inelastically supplied. x_{ji} denotes the quantity of the sector-j good required in the production of the sector-i good. Exponent $[\mathbf{\Lambda}]_{ji}$ captures how

intensely the sector-j good is used in the production of the sector-i good; $[\mathbf{\Lambda}]_{ji}$ represents the share of good j in total intermediate input use by sector i. We assume that the production technology for each sector is constant returns to scale, that is, $\alpha_i + \beta_i = 1 \ \forall i \in \{1, \ldots, N\}$. Parameters α_i and β_i can differ across sectors; sectors vary in the intensities with which they use labor and intermediate inputs. We also assume that $\mathbf{\Lambda}$ is column-stochastic; this ensures that the overall sectoral production function is constant returns to scale. The production network in this environment is $\mathbf{Z}' = \mathcal{G}(\mathbf{\Lambda})$.

In this economy, each firm $i \in \{1, \ldots, N\}$ maximizes its profit π_i: $\pi_i = p_i x_i - \sum_{j=1}^{N} p_j x_{ji} - w\ell_i$, where p_i is the price of good i and w is the wage rate. The representative consumer maximizes its utility subject to a resource constraint: $\sum_{i=1}^{N} p_i c_i = w \sum_{i=1}^{N} \ell_i + \sum_{i=1}^{N} \pi_i$. Labor market clearing requires that the supply of labor equals the total demand for labor: $1 = \sum_{i=1}^{N} \ell_i$, where we set the supply of labor equal to 1. Goods market clearing requires that, for each sector $i \in \{1, \ldots, N\}$, the supply of the good equals the demand for that good: $x_i = c_i + \sum_{j=1}^{N} x_{ij}$. We define the matrix $\mathbf{X}$ with elements x_{ij}, the quantity of the sector-i good used in the production of the good from sector j.

A competitive equilibrium in this economy can be characterized as follows:

Definition 6 *A* competitive equilibrium *is a collection of quantities,* $\mathbf{c}^*$, $\mathbf{x}^*$, $\mathbf{X}^*$, *and* $\boldsymbol{\ell}^*$, *and a collection of prices,* $\mathbf{p}^*$ *and* w^*, *such that:*

1. *The representative consumer maximizes utility subject to a budget constraint:*

$$\max_{c_1, \ldots, c_N} \prod_{i=1}^{N} c_i^{\eta_i} \quad s.t. \quad \sum_{i=1}^{N} p_i c_i = w \sum_{i=1}^{N} \ell_i + \sum_{i=1}^{N} \pi_i.$$

2. *Each firm* $i \in \{1, \ldots, N\}$ *maximizes profit given its production technology:*

$$\max_{x_{1i}, \ldots, x_{Ni}, \ell_i} p_i x_i - \sum_{j=1}^{N} p_j x_{ji} - w\ell_i \quad s.t. \quad x_i = A_i^{\alpha_i} \ell_i^{\alpha_i} \left(\prod_{j=1}^{N} x_{ji}^{[\mathbf{\Lambda}]_{ji}} \right)^{\beta_i} \ \forall i \in \{1, \ldots, N\}.$$

3. *The goods markets clear for all sectors* $i \in \{1, \ldots, N\}$ *and the labor market clears:*

$$x_i = c_i + \sum_{j=1}^{N} x_{ij} \quad \forall i \in \{1, \ldots, N\} \quad \textit{and} \quad \sum_{i=1}^{N} \ell_i = 1.$$

We then have the following result:

Proposition 26 *The economy admits a competitive equilibrium with quantities* $\mathbf{c}^*$, $\mathbf{x}^*$, $\mathbf{X}^*$, *and* $\boldsymbol{\ell}^*$, *and prices* $\mathbf{p}^*$ *and* w^*.

We define $y_i^* = p_i^* x_i^*$ as the nominal value of output in sector i and $\mathbf{y}^*$ as the vector of nominal values of output across all N sectors. We are interested in the nominal value of aggregate output, y_{agg}:

$$y_{agg} = \mathbf{1}^T \mathbf{y}^* = \sum_{i=1}^{N} p_i^* x_i^*.$$

In the next theoretical result, we compute $\mathbf{y}^*$ in closed form, from which we can then compute in closed form the nominal value of aggregate output, otherwise known as GDP: $y_{agg} = \mathbf{1}^T \mathbf{y}^*$.

Proposition 27 *The vector of equilibrium levels of nominal output for all N sectors is: $\mathbf{y}^* = (\mathbf{I} - \boldsymbol{\Lambda} \operatorname{diag}(\boldsymbol{\beta}))^{-1} \boldsymbol{\eta} w$. The nominal value of aggregate output is $y_{agg} = \mathbf{1}^T \mathbf{y}^*$.*

With $\boldsymbol{\Lambda}$ column-stochastic and $\beta_i \in (0,1)$ for every sector i, $\mathbf{I} - \boldsymbol{\Lambda} \operatorname{diag}(\boldsymbol{\beta})$ is invertible[6] and GDP is readily computable. We can re-write $\mathbf{y}^*$ as follows:

$$\mathbf{y}^* = \left(\sum_{j=0}^{\infty} (\boldsymbol{\Lambda} \operatorname{diag}(\boldsymbol{\beta}))^j \right) \boldsymbol{\eta} w = \left(\mathbf{I} + \boldsymbol{\Lambda} \operatorname{diag}(\boldsymbol{\beta}) + (\boldsymbol{\Lambda} \operatorname{diag}(\boldsymbol{\beta}))^2 + \cdots \right) \boldsymbol{\eta} w.^7$$

From this expansion, we see that aggregate expenditure in a particular sector depends on both demand from the representative consumer, the first term in the expansion, and industry demand from other sectors, the remaining terms in the expansion. Industry demand for a particular sector's good comes from both first-order and higher-order connections in the production network. Industry demand from first-order connections arises when there are sectors directly requiring that particular good as an intermediate input. Meanwhile, industry demand arises from higher-order connections when our good of interest indirectly appears in the output of another sector via a supply chain that is greater than length 1.

We would like to study what happens to GDP when a group of sectors receives a positive demand shock. To determine this effect, we first compute the baseline level of GDP in the absence of any transfers or stimulus, y_{agg}^{no}. We then compare it to the level of GDP following the implementation of a particular policy that adjusts the final demand of various sectors. The baseline level of GDP is as follows:

$$y_{agg}^{no} = \mathbf{1}^T \mathbf{y}^* = N \left[\mathbf{d}_w^- \left((\mathbf{I} - \boldsymbol{\Lambda} \operatorname{diag}(\boldsymbol{\beta}))^{-1} \right) \right]^T \boldsymbol{\eta} w,$$

where $\mathbf{d}_w^- \left((\mathbf{I} - \boldsymbol{\Lambda} \operatorname{diag}(\boldsymbol{\beta}))^{-1} \right) = \frac{1}{N} \left[(\mathbf{I} - \boldsymbol{\Lambda} \operatorname{diag}(\boldsymbol{\beta}))^{-1} \right]^T \mathbf{1}$ is the vector of average weighted in-degrees for the network $\mathcal{G}\left((\mathbf{I} - \boldsymbol{\Lambda} \operatorname{diag}(\boldsymbol{\beta}))^{-1} \right)$. Agents' weights sum to k:

[6] The invertibility of $\mathbf{I} - \boldsymbol{\Lambda} \operatorname{diag}(\boldsymbol{\beta})$ is proven in Lemma 9, which can be found in the Appendix.

[7] This expansion holds provided that 1 is smaller than the norm of the inverse of the largest eigenvalue of $\boldsymbol{\Lambda} \operatorname{diag}(\boldsymbol{\beta})$. As demonstrated by Lemma 9 in the Appendix, $r(\boldsymbol{\Lambda} \operatorname{diag}(\boldsymbol{\beta})) < 1$, so this condition holds.

$\mathbf{1}^T \mathbf{d}_w^- \left(\left(\mathbf{I} - \mathbf{\Lambda} \operatorname{diag}(\boldsymbol{\beta}) \right)^{-1} \right) = k$. In our environment, we have two relevant networks: (1) the production network, $\mathcal{G}(\mathbf{Z}') = \mathcal{G}(\mathbf{\Lambda})$, that captures the flow of intermediate goods, and (2) the network, $\mathcal{G}(\mathbf{Z}) = \mathcal{G}(\mathbf{I} - \mathbf{\Lambda} \operatorname{diag}(\boldsymbol{\beta}))^{-1}$, that determines each sector's effective weight in the production ecosystem. Sectors' weights, $\mathbf{d}_w^- \left(\left(\mathbf{I} - \mathbf{\Lambda} \operatorname{diag}(\boldsymbol{\beta}) \right)^{-1} \right)$, determine how much of an effect a demand shock has on overall GDP. The more influential the sector, the greater the effect on GDP.

We study two different settings in which a group of sectors receives a positive demand shock: (1) a setting with transfers and (2) a setting with stimulus. For the setting with transfers, a group of sectors receives a positive shock to final demand, while all other sectors receive a negative demand shock. For the setting with stimulus, a group of sectors receives a positive shock to final demand, while all other sectors receive no shock. We are interested in the level of GDP and the corresponding economic multiplier in these two settings. The economic multiplier captures the change in GDP when a group of sectors receives a positive shock to final demand that is of some particular magnitude.

To show how the nominal value of aggregate output changes in settings with transfers and stimulus, we first rewrite y_{agg}^{no}:

$$y_{agg}^{no} = N \left[\mathbf{d}_w^- \left(\left(\mathbf{I} - \mathbf{\Lambda} \operatorname{diag}(\boldsymbol{\beta}) \right)^{-1} \right) \right]^T \boldsymbol{\omega},$$

where $\boldsymbol{\omega} = \boldsymbol{\eta} w$. The vector $\boldsymbol{\omega}$ is a vector of sector-specific expenditures on final goods. These expenditures are made by the representative consumer: $p_i^* c_i^* = [\boldsymbol{\omega}]_i$. In a setting with transfers or externally funded stimulus, the vector of expenditures on final goods across the N sectors changes from $\boldsymbol{\omega}$ to $\boldsymbol{\omega} + \boldsymbol{\rho}$.

Specifically, in a setting with transfers, $[\boldsymbol{\rho}]_i = \epsilon$ if sector i is receiving a positive shock to final demand and $[\boldsymbol{\rho}]_i = -\frac{n\epsilon}{N-n}$ if sector i is not receiving a positive shock to final demand. In this setting with transfers, the total expenditure on final goods remains unchanged from its no-intervention level: $\mathbf{1}^T (\boldsymbol{\omega} + \boldsymbol{\rho}) = \mathbf{1}^T \boldsymbol{\omega}$. We now describe how the transfer of funds across sectors gets implemented. We have a government that is interested in making purchases. Let $p_i^* g_i^*$ be the amount of government expenditure in sector i, where g_i^* is the total number of units of good i purchased by the government. To finance its purchases, the government levies a lump-sum tax τ on the representative consumer. The amount of expenditure by the representative consumer in every sector i is then $p_i^* c_i^* = \eta_i (w - \tau)$. The amount of expenditure by the government is $p_i^* g_i^* = \eta_i \tau + \epsilon$ if sector i is receiving a positive transfer and $p_i^* g_i^* = \eta_i \tau - \frac{n\epsilon}{N-n}$ if sector i is not receiving a positive transfer. n sectors, in total, receive a positive shock to final demand. Note that the total amount of expenditure by the government is equal to its tax revenue: $\sum_{i=1}^N p_i^* g_i^* = \tau$. Also note that $p_i^* c_i^* + p_i^* g_i^* = [\boldsymbol{\omega}]_i + [\boldsymbol{\rho}]_i$

for every sector $i \in \{1, \ldots, N\}$.

In a setting with stimulus, $[\boldsymbol{\rho}]_i = \epsilon$ if sector i is receiving a positive shock to final demand and otherwise $[\boldsymbol{\rho}]_i = 0$. Now, the total expenditure on final goods increases by $n\epsilon$ units relative to the no-intervention level: $\mathbf{1}^T(\boldsymbol{\omega} + \boldsymbol{\rho}) = \mathbf{1}^T\boldsymbol{\omega} + n\epsilon$. The government receives these $n\epsilon$ units of wealth from an external source. The government sets $p_i^* g_i^* = \epsilon$ if it wishes to provide positive stimulus to sector i, and otherwise $p_i^* g_i^* = 0$. n sectors receive positive stimulus. Note that $\sum_{i=1}^N p_i^* g_i^* = n\epsilon$ and that $p_i^* c_i^* + p_i^* g_i^* = [\boldsymbol{\omega}]_i + [\boldsymbol{\rho}]_i$ for every sector $i \in \{1, \ldots, N\}$.

Depending on which sectors get a positive shock to final demand, we can have variation in the resulting level of GDP and the corresponding economic multiplier. Configuration vector $\mathbf{b}(N, n) \in \mathcal{B}(N, n)$ identifies which subset of $n \leq N$ sectors is receiving a positive demand shock. Element $b_i = 1$ if sector i receives a positive shock to final demand and otherwise $b_i = 0$.

We begin by characterizing GDP and the economic multiplier on GDP in a setting with transfers. Sectors $1, \ldots, n$ receive a positive shock to final demand and sectors $n+1, \ldots, N$ receive a negative shock to final demand so that there is zero net transfer across sectors. In this setting, $b_i = 1$ for $i \in \{1, \ldots, n\}$:

$$\begin{aligned} y_{agg}\left((\mathbf{I} - \boldsymbol{\Lambda}\,\mathrm{diag}(\boldsymbol{\beta}))^{-1}, \mathbf{b}, N, n, 0\right) = y_{agg}^{no} + N\epsilon\Big[\Big(\big[\mathbf{d}_w^-\left((\mathbf{I} - \boldsymbol{\Lambda}\,\mathrm{diag}(\boldsymbol{\beta}))^{-1}\right)\big]_1 + \cdots \\ + \big[\mathbf{d}_w^-\left((\mathbf{I} - \boldsymbol{\Lambda}\,\mathrm{diag}(\boldsymbol{\beta}))^{-1}\right)\big]_n\Big) - \frac{n}{N-n}\Big(\big[\mathbf{d}_w^-\left((\mathbf{I} - \boldsymbol{\Lambda}\,\mathrm{diag}(\boldsymbol{\beta}))^{-1}\right)\big]_{n+1} + \cdots \\ + \big[\mathbf{d}_w^-\left((\mathbf{I} - \boldsymbol{\Lambda}\,\mathrm{diag}(\boldsymbol{\beta}))^{-1}\right)\big]_N\Big)\Big] \end{aligned}$$

with $m\left((\mathbf{I} - \boldsymbol{\Lambda}\,\mathrm{diag}(\boldsymbol{\beta}))^{-1}, \mathbf{b}, N, n, 0\right) = \frac{dy_{agg}\left((\mathbf{I} - \boldsymbol{\Lambda}\,\mathrm{diag}(\boldsymbol{\beta}))^{-1}, \mathbf{b}, N, n, 0\right)}{d\epsilon}$. We continue by characterizing GDP and the economic multiplier on GDP in a setting with stimulus. Now, sectors $1, \ldots, n$ receive a positive shock to final demand while sectors $n+1, \ldots, N$ receive zero shock to final demand. Setting $b_i = 1$ for $i \in \{1, \ldots, n\}$, we have

$$\begin{aligned} y_{agg}\left((\mathbf{I} - \boldsymbol{\Lambda}\,\mathrm{diag}(\boldsymbol{\beta}))^{-1}, \mathbf{b}, N, n, 1\right) = y_{agg}^{no} + N\epsilon\Big(\big[\mathbf{d}_w^-\left((\mathbf{I} - \boldsymbol{\Lambda}\,\mathrm{diag}(\boldsymbol{\beta}))^{-1}\right)\big]_1 + \cdots \\ + \big[\mathbf{d}_w^-\left((\mathbf{I} - \boldsymbol{\Lambda}\,\mathrm{diag}(\boldsymbol{\beta}))^{-1}\right)\big]_n\Big) \end{aligned}$$

with $m\left((\mathbf{I} - \boldsymbol{\Lambda}\,\mathrm{diag}(\boldsymbol{\beta}))^{-1}, \mathbf{b}, N, n, 1\right) = \frac{dy_{agg}\left((\mathbf{I} - \boldsymbol{\Lambda}\,\mathrm{diag}(\boldsymbol{\beta}))^{-1}, \mathbf{b}, N, n, 1\right)}{d\epsilon}$.

We are interested in all possible levels of GDP and all possible multipliers given that n sectors receive a positive shock to final demand. We therefore introduce random variables that allow us to characterize the distribution of possible levels of GDP as well as the distribution of possible economic multipliers given n:

Proposition 28 *In a setting with transfers, the GDP and the corresponding economic multiplier are:*

$$Y_{agg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n,0\right)=y_{agg}^{no}+\frac{N^2\epsilon}{N-n}\left[\widehat{F}_{avg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n\right)-\frac{kn}{N}\right]\ \textit{and}$$
$$M\left(\left(\mathbf{I}-\mathbf{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n,0\right)=\frac{N^2}{N-n}\left[\widehat{F}_{avg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n\right)-\frac{kn}{N}\right].$$

In a setting with stimulus, the GDP and the corresponding economic multiplier are:

$$Y_{agg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n,1\right)=y_{agg}^{no}+N\epsilon\widehat{F}_{avg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n\right)\ \textit{and}$$
$$M\left(\left(\mathbf{I}-\mathbf{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n,1\right)=N\widehat{F}_{avg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1}\right).$$

The expressions for $Y_{agg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n,0\right)$, $M\left(\left(\mathbf{I}-\mathbf{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n,0\right)$, $Y_{agg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n,1\right)$, and $M\left(\left(\mathbf{I}-\mathbf{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n,1\right)$ in Proposition 28 directly map to Equations 2-5 in Section 2, where we set $\mathbf{Z}=\left(\mathbf{I}-\mathbf{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1}$ and $\gamma_1=1$.

All of the theoretical results from Section 2 therefore map to the present environment. Given that we are in a setting with transfers or a setting with stimulus and n sectors have received a positive shock to final demand, we can directly compute in closed-form the resulting mean level of GDP and the mean economic multiplier. We can compute the variance of these distributions as well. Moreover, for a particular production network $\mathcal{G}\left(\mathbf{\Lambda}\right)$, we can determine the lowest possible level of GDP, the highest possible level of GDP, the lowest possible economic multiplier, and the highest possible economic multiplier given that n sectors are receiving a positive shock to final demand. For all feasible production networks $\mathcal{G}\left(\mathbf{\Lambda}\right)$ with N sectors, for any given N, and n sectors receiving a positive shock to final demand, we can approximate the cumulative distribution functions for GDP and the corresponding economic multiplier. Quite importantly, we can analytically determine the probability that a particular policy measure, whether it be a transfer of funds across sectors or externally funded stimulus, leads to a reduction in GDP below its no-intervention level and a negative economic multiplier. GDP dips below its no-intervention level when the economic multiplier is negative.

In settings with stimulus, it turns out that the probability of a negative multiplier is always zero. GDP following an intervention that targets n sectors for stimulus is at least as large as GDP in the absence of any intervention. This property holds for all feasible levels of stimulus, that is, for all $n\in\{1,\ldots,N-1\}$:

Proposition 29 *For every $n \in \{1, \ldots, N-1\}$,*

$$\Pr\left[Y_{agg}\left((\mathbf{I} - \mathbf{\Lambda}\,\mathrm{diag}\,(\boldsymbol{\beta}))^{-1}, N, n, 1\right) \geq y_{agg}^{no}\right] = 1 \text{ and}$$
$$\Pr\left[M\left((\mathbf{I} - \mathbf{\Lambda}\,\mathrm{diag}\,(\boldsymbol{\beta}))^{-1}, N, n, 1\right) \geq 0\right] = 1.$$

Stimulus always weakly increases GDP. Given ϵ units of stimulus for n sectors, the multiplier is always non-negative.

The results that we have provided thus far hold for any feasible network structure $\mathcal{G}(\mathbf{\Lambda})$. We now study what happens to the distribution of GDP and the distribution of corresponding economic multipliers when the network is trivial. Specifically, we consider an environment in which there is no network-based interaction. To be consistent with the other sections, we declare an absence of network-based interaction when $\mathbf{\Lambda} = \mathbf{I}$, the identity matrix:

Proposition 30 *In the absence of any network-based interaction,*
$Y_{agg}\left((\mathbf{I} - \mathbf{\Lambda}\,\mathrm{diag}\,(\boldsymbol{\beta}))^{-1}, N, n, 0\right)$, $M\left((\mathbf{I} - \mathbf{\Lambda}\,\mathrm{diag}\,(\boldsymbol{\beta}))^{-1}, N, n, 0\right)$,
$Y_{agg}\left((\mathbf{I} - \mathbf{\Lambda}\,\mathrm{diag}\,(\boldsymbol{\beta}))^{-1}, N, n, 1\right)$, *and* $M\left((\mathbf{I} - \mathbf{\Lambda}\,\mathrm{diag}\,(\boldsymbol{\beta}))^{-1}, N, n, 1\right)$ *have the same functional form as in Proposition 28, with*

$$\left[\mathbf{d}_w^{-}\left((\mathbf{I} - \mathbf{\Lambda}\,\mathrm{diag}\,(\boldsymbol{\beta}))^{-1}\right)\right]^T = \frac{1}{N}\begin{pmatrix} \frac{1}{1-\beta_1} & \cdots & \frac{1}{1-\beta_N} \end{pmatrix}.$$

Provided that we do not have $\beta_1 = \cdots = \beta_N$, the distributions of GDP and economic multipliers are non-degenerate. Both GDP and the corresponding economic multiplier can vary with the particular configuration of transfers or stimulus even when there is no underlying network structure. This feature of non-degeneracy distinguishes the present environment from those of the other two sections. In the other two sections, once we removed the network structure, the distribution of the aggregate action and the distribution of the corresponding economic multiplier both became degenerate. In those two environments, by removing the agent interaction structure, all agents became identical; the aggregate action and the economic multiplier were the same regardless of which subset of agents received a positive transfer or positive stimulus. In the present environment with production, even though we are eliminating any heterogeneity that arises from the topology of the production network, there is still heterogeneity across sectors; each sector differs in the intensity with which it uses intermediate inputs, which makes the level of GDP vary and the value of the economic multiplier vary with the particular configuration of transfers or stimulus.

Once $\beta_1 = \cdots = \beta_N \equiv \beta$, both GDP and the corresponding economic multiplier are invariant to configuration for all levels, n, of transfers or stimulus:

Proposition 31 *Both GDP and the corresponding economic multiplier are invariant to configuration if and only if* $\beta_1 = \cdots = \beta_N \equiv \beta$*:*

$$Y_{agg}\left((\mathbf{I} - \mathbf{\Lambda}\,\mathrm{diag}\,(\boldsymbol{\beta}))^{-1}, N, n, 0\right) = y_{agg}^{no} \quad \textit{and} \quad M\left((\mathbf{I} - \mathbf{\Lambda}\,\mathrm{diag}\,(\boldsymbol{\beta}))^{-1}, N, n, 0\right) = 0$$

with probability 1, and

$$Y_{agg}\left((\mathbf{I} - \mathbf{\Lambda}\,\mathrm{diag}\,(\boldsymbol{\beta}))^{-1}, N, n, 1\right) = y_{agg}^{no} + \frac{n\epsilon}{1-\beta} \quad \textit{and} \quad M\left((\mathbf{I} - \mathbf{\Lambda}\,\mathrm{diag}\,(\boldsymbol{\beta}))^{-1}, N, n, 1\right) = \frac{n}{1-\beta}$$

with probability 1.

When every sector has the same labor intensity, that is, $\beta_1 = \cdots = \beta_N$, both GDP and the corresponding multiplier are invariant to configuration for any production network $\mathcal{G}(\mathbf{\Lambda})$, provided that $\mathbf{\Lambda}$ is column-stochastic. In such an environment, the distribution of GDP and the distribution of possible economic multipliers are both degenerate. In settings with transfers, GDP equals its no-intervention level regardless of which group of sectors receives a positive shock and regardless of how many sectors receive this positive shock. This result holds for all feasible networks $\mathcal{G}(\mathbf{\Lambda})$. In settings with stimulus, there is an increase in GDP relative to its no-intervention level, but this level of GDP remains the same regardless of which group of sectors receives positive stimulus.

6 Conclusion

This paper studies economic systems with N networked agents, $n \leq N$ of whom each initially receive a positive shock that is either financed by internal transfers or external stimulus. This work examines the resulting probability distributions of possible aggregate actions and economic multipliers given n. Agents' actions are interconnected, with such interdependency captured by the topology of an interaction network. As the particular configuration of positive shocks to agents varies, holding n fixed, agents' actions adjust so that the aggregate action varies as well. In general, the distribution of possible aggregate actions and the distribution of possible economic multipliers are non-degenerate. We explore these distributions of aggregate actions and corresponding economic multipliers in three different networked environments: (1) those featuring strategic complements and substitutes, (2) those featuring coordination and anti-coordination, and (3) those featuring production.

Despite strong differences across these three environments, the core mathematics is the same. For each environment, given agents' decision-making behavior and the underlying network structure, we construct a network-derived vector of agent weights, the vector of

average weighted in-degrees for graph $\mathcal{G}(\mathbf{Z})$. It is from this vector of agent weights that we can characterize in closed form all essential features of the distribution of aggregate actions and the corresponding distribution of economic multipliers. For all feasible population sizes N, number of agents receiving a positive shock, n, and network topologies, we can compute the mean aggregate action and the mean economic multiplier. We can also compute in closed form the variance for these two distributions, the bounds on the support of these distributions, the corresponding limiting distributions as $N \to \infty$, and approximations to the CDFs of these two distributions for finite N. We study the aggregate action and the economic multiplier when there is no underlying network structure and when the network topology is such that these two quantities are invariant to configuration. In addition, for certain environments, we can rank networks so that the distributions of possible aggregate actions and economic multipliers corresponding to a higher-ranked network first-order stochastically dominate the distributions of possible aggregate actions and economic multipliers for a lower-ranked network. As a result, the higher-ranked the network, the more effective the policy.

Quite importantly, we develop and use a set of tools that allows us to analytically compute the probability of a negative economic multiplier given agents' decision-making behavior and a particular aggregate level, $n\epsilon$, of transfers or stimulus. The set of network-derived agent weights ultimately shapes the probability that a particular environment generates negative economic multipliers. In settings with transfers, practically every network structure generates a negative multiplier with a positive probability. Quantifying the probability of a negative economic multiplier or an aggregate action below its no-intervention level is important because it captures the extent to which a policy is ineffective. Policy-making entities craft policy measures to achieve particular objectives, such as jump-starting an economy during recession. If the naturally occurring agent interaction structure is such that there is a non-negligible positive probability of a negative economic multiplier, then the policy-making entity may rethink its policy prescription. When the policy is financed by internal transfers, there is no outright cost to the policy-making entity, except that it is transferring funds away from individuals. However, when the policy is financed by external stimulus, perhaps by issuance of debt, the policy-making entity ultimately must provide repayment, and if the economic multiplier following stimulus is negative, it becomes all the more difficult to repay borrowed funds. If the policy-making entity does not have a good grasp of the topology of agents' interaction structure, then enacting a policy can be quite risky, as there are entire non-degenerate distributions of possible resulting aggregate actions and economic multipliers.

In essence, this work extends and applies a set of tools that allows us to construct in closed form policy-induced distributions of possible aggregate actions and economic multi-

pliers in environments with complex, network-based agent interactions. The tools and the methodology implemented in this work are general. Hopefully they can be applied to a broad range of diverse settings and provide substantive insights.

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Appendix: Proofs

Proof of Lemma 1

First method of proof: $E\widehat{F}_{avg}(\mathbf{Z},N,n) = E\left(\left[\mathbf{d}_w^-(\mathbf{Z})\right]^T \mathbf{B}(N,n)\right)$, where $\mathbf{B}(N,n)$ is a random vector whose elements are $B_i \sim \text{Bern}\left(\frac{n}{N}\right), i \in \{1,\ldots,N\}$ and $\left[\mathbf{d}_w^-(\mathbf{Z})\right]^T \mathbf{1} = k$. Therefore,

$$E\widehat{F}_{avg}(\mathbf{Z},N,n) = E\left(\left[\mathbf{d}_w^-(\mathbf{Z})\right]_1 B_1 + \left[\mathbf{d}_w^-(\mathbf{Z})\right]_2 B_2 + \ldots + \left[\mathbf{d}_w^-(\mathbf{Z})\right]_N B_N\right) = \sum_{i=1}^{N}\left[\mathbf{d}_w^-(\mathbf{Z})\right]_i EB_i = \frac{kn}{N}. \ \square$$

Second method of proof:

$$\begin{aligned} E\widehat{F}_{avg}(\mathbf{Z},N,n) &= \frac{1}{|\mathcal{B}(N,n)|}\sum_{\mathbf{b}(N,n)\in\mathcal{B}(N,n)}\left[\mathbf{d}_w^-(\mathbf{Z})\right]^T \mathbf{b}(N,n) \\ &= \frac{1}{|\mathcal{B}(N,n)|}\sum_{i=1}^{N}|\{\mathbf{b}\in\mathcal{B}(N,n): b_i = 1\}|\left[\mathbf{d}_w^-(\mathbf{Z})\right]_i \\ &= \frac{|\{\mathbf{b}\in\mathcal{B}(N,n): b_i = 1\}|}{|\mathcal{B}(N,n)|}\sum_{i=1}^{N}\left[\mathbf{d}_w^-(\mathbf{Z})\right]_i = \left(\frac{\binom{N}{n}\times\frac{n}{N}}{\binom{N}{n}}\right)\times k = \frac{kn}{N}. \end{aligned}$$

By symmetry, $|\{\mathbf{b}\in\mathcal{B}(N,n): b_i = 1\}| = |\{\mathbf{b}\in\mathcal{B}(N,n): b_j = 1\}|$, $\forall i,j\in\{1,\ldots,N\}$. $\square$

Proof of Lemma 2

The proof of this Lemma immediately follows from the proof of Theorem 9 in Schlossberger (2018). In Schlossberger (2018), $\text{Var}\, D_w^-(\mathbf{Z}) = \frac{1}{N}\sum_{i=1}^{N}\left([\mathbf{d}_w^-(\mathbf{Z})]_i - \frac{1}{N}\right)^2$ because $\mathbf{1}^T\mathbf{d}_w^-(\mathbf{Z}) = 1$, while in the present work, $\text{Var}\, D_w^-(\mathbf{Z}) = \frac{1}{N}\sum_{i=1}^{N}\left([\mathbf{d}_w^-(\mathbf{Z})]_i - \frac{k}{N}\right)^2$ because $\mathbf{1}^T\mathbf{d}_w^-(\mathbf{Z}) = k$. $\square$

Proof of Lemma 3

The proof of this Lemma immediately follows from the proof of Theorem 10 in Schlossberger (2018). $\square$

Proof of Lemma 4

The scalar $\widehat{f}_{avg}(\mathbf{Z},\mathbf{b},N,n) = \left[\mathbf{d}_w^-(\mathbf{Z})\right]^T \mathbf{b}(N,n)$ is invariant to configuration if and only if $\left[\mathbf{d}_w^-(\mathbf{Z})\right]^T \mathbf{b}(N,n) = \left[\mathbf{d}_w^-(\mathbf{Z})\right]^T \mathbf{b}'(N,n)$ for all $\mathbf{b}(N,n), \mathbf{b}'(N,n) \in \mathcal{B}(N,n)$, with this relation holding for each integer $n \in [0,N]$. Let $n = 1$, and define $\mathbf{e}_i$ to be the ith unit vector whose i^{th} element equals 1 and all other elements equal zero. Then $\left[\mathbf{d}_w^-(\mathbf{Z})\right]^T \mathbf{b}(N,1) = \left[\mathbf{d}_w^-(\mathbf{Z})\right]^T \mathbf{b}'(N,1)$ if and only if $\left[\mathbf{d}_w^-(\mathbf{Z})\right]^T \mathbf{e}_i = \left[\mathbf{d}_w^-(\mathbf{Z})\right]^T \mathbf{e}_j$, that is, if and only if $\left[\mathbf{d}_w^-(\mathbf{Z})\right]_i = \left[\mathbf{d}_w^-(\mathbf{Z})\right]_j$ for all $i,j \in \{1,\ldots,N\}$. Since $\mathbf{1}^T\mathbf{d}_w^-(\mathbf{Z}) = k$, $\left[\mathbf{d}_w^-(\mathbf{Z})\right]_i = \frac{k}{N}$ for all $i \in \{1,\ldots,N\}$. Given that $\mathbf{d}_w^-(\mathbf{Z}) =$

$\frac{k}{N}\mathbf{1}$ when $\widehat{f}_{avg}(\mathbf{Z},\mathbf{b},N,n)$ is invariant to configuration, $\widehat{f}_{avg}(\mathbf{Z},\mathbf{b},N,n) = [\mathbf{d}_w^-(\mathbf{Z})]^T\mathbf{b}(N,n) = \frac{k}{N}\mathbf{1}^T\mathbf{b}(N,n) = \frac{kn}{N}$. $\square$

Proof of Lemma 5

The proof of this Lemma immediately follows from the proof of Theorem 13 in Schlossberger (2018). $\square$

Proof of Lemma 6

The proof of Lemma 6 makes use of the following result from Erdös and Rényi (1959), with notation modified for the present work:

Lemma 8 (Erdös and Rényi (1959), Theorem 1) *Consider the infinite triangular matrix of real elements*

$$\begin{array}{cccccc}
w_{11}^{'} & & & & & \\
w_{21}^{'} & w_{22}^{'} & & & & \\
\cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
w_{N1}^{'} & w_{N2}^{'} & \cdot & \cdot & \cdot & w_{NN}^{'} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \quad \cdot
\end{array}$$

with $\mathbf{w}_N^{'}$ *denoting the* N^{th} *row of the matrix and* $\sum_{j=1}^{N} w_{Nj}^{'} = 0$. *For any real value* t, *determine* $T\left(\mathbf{w}_N^{'},N,n,t\right)$, *that is, the total number of sums*

$$a\left(\mathbf{w}_N^{'},N,n\right) = w_{Ni_1}^{'} + w_{Ni_2}^{'} + \cdots + w_{Ni_n}^{'}, \qquad 1 \le i_1 < i_2 < \cdots < i_n \le N,$$

whose value does not exceed $t\sigma\left(\mathbf{w}_N^{'},N,n\right) \equiv t\sqrt{\frac{n}{N}\left(1-\frac{n}{N}\right)\sum_{j=1}^{N} w_{Nj}^{'2}}$. *Let CDF* $G_{\frac{A\left(\mathbf{w}_N^{'},N,n\right)}{\sigma\left(\mathbf{w}_N^{'},N,n\right)}}(t) = \frac{T\left(\mathbf{w}_N^{'},N,n,t\right)}{\binom{N}{n}}$. *With*

$$\kappa\left(\mathbf{w}_N^{'},N,n,\epsilon'\right) \equiv \frac{1}{\sum_{j=1}^{N} w_{Nj}^{'2}} \sum_{\substack{j\in\{1,\ldots,N\}\ s.t. \\ \left|w_{Nj}^{'}\right| > \epsilon'\sigma\left(\mathbf{w}_N^{'},N,n\right)}} w_{Nj}^{'2}$$

if $\lim_{N\to\infty}\kappa\left(\mathbf{w}_N^{'},N,n,\epsilon'\right) = 0$ *for any* $\epsilon' > 0$, *then* $\lim_{N\to\infty} G_{\frac{A}{\sigma}}(t) = \Phi(t)$ *for any real* t, *where* $\Phi(\cdot)$ *denotes the standard normal CDF.*

For a given population size N, set $\mathbf{w}_N^{'} = \mathbf{d}_{N,w}^-(\mathbf{Z}) - \frac{k}{N}$, where $\mathbf{d}_{N,w}^-(\mathbf{Z})$ is the vector of average weighted in-degrees ($\mathbf{d}_w^-(\mathbf{Z})$) discussed in the text, and subscript N is added to make the

population size explicit. Then $\sum_{j=1}^{N} w_{Nj}^{'} = \sum_{j=1}^{N} \left(\left[\mathbf{d}_{N,w}^{-} (\mathbf{Z}) \right]_j - \frac{k}{N} \right) = 0$. Scalar quantity

$$\begin{aligned}
\widehat{f}_{avg} (\mathbf{Z}, \mathbf{b}, N, n) - \frac{kn}{N} &= \left[\mathbf{d}_{N,w}^{-} (\mathbf{Z}) \right]^T \mathbf{b} (N, n) - \frac{kn}{N} \\
&= \left(\left[\mathbf{d}_{N,w}^{-} (\mathbf{Z}) \right]_{i_1} - \frac{k}{N} \right) + \left(\left[\mathbf{d}_{N,w}^{-} (\mathbf{Z}) \right]_{i_2} - \frac{k}{N} \right) + \cdots + \left(\left[\mathbf{d}_{N,w}^{-} (\mathbf{Z}) \right]_{i_n} - \frac{k}{N} \right) \\
&= w_{Ni_1}^{'} + w_{Ni_2}^{'} + \cdots + w_{Ni_n}^{'},
\end{aligned}$$

where $1 \leq i_1 < i_2 < \cdots < i_n \leq N$, given a configuration $\mathbf{b} (N, n) \in \mathcal{B} (N, n)$. Thus, by Lemma 8,

$$\begin{aligned}
G_{\frac{A\left(\mathbf{w}_N^{'}, N, n\right)}{\sigma\left(\mathbf{w}_N^{'}, N, n\right)}} (t) = \frac{T\left(\mathbf{w}_N^{'}, N, n, t\right)}{\binom{N}{n}} &= \frac{1}{\binom{N}{n}} \sum_{\forall 1 \leq i_1 < i_2 < \cdots < i_n \leq N} \mathbb{1}_{w_{Ni_1}^{'} + w_{Ni_2}^{'} + \cdots + w_{Ni_n}^{'} \leq t\sigma\left(\mathbf{w}_N^{'}, N, n\right)} \\
&= \frac{1}{\binom{N}{n}} \sum_{\mathbf{b}(N,n) \in \mathcal{B}(N,n)} \mathbb{1}_{\widehat{f}_{avg}(\mathbf{Z}, \mathbf{b}, N, n) - \frac{kn}{N} \leq t\sigma\left(\mathbf{w}_N^{'}, N, n\right)} = G_{\frac{\widehat{F}_{avg}(\mathbf{Z}, N, n) - \frac{kn}{N}}{\sigma\left(\mathbf{w}_N^{'}, N, n\right)}} (t),
\end{aligned}$$

so $\lim_{N \to \infty} G_{\frac{\widehat{F}_{avg}(\mathbf{Z}, N, n) - \frac{kn}{N}}{\sigma\left(\mathbf{w}_N^{'}, N, n\right)}} (t) = \Phi (t)$, where $\sigma \left(\mathbf{w}_N^{'}, N, n \right) = \sqrt{\frac{n}{N} \left(1 - \frac{n}{N}\right) \sum_{j=1}^{N} \left(\left[\mathbf{d}_{N,w}^{-} (\mathbf{Z}) \right]_j - \frac{k}{N} \right)^2}$.
□

Proof of Proposition 1

This Proposition follows immediately from Equations 2-5 and Lemma 1. □

Proof of Proposition 2

This Proposition follows immediately from Equations 2-5 and Lemma 2. □

Proof of Proposition 3

This Proposition follows immediately from Equations 2-5 and Lemma 3. □

Proof of Proposition 4

Given Equations 2-5, $y_{agg} (\mathbf{Z}, \mathbf{b}, N, n, 0)$, $m (\mathbf{Z}, \mathbf{b}, N, n, 0)$, $y_{agg} (\mathbf{Z}, \mathbf{b}, N, n, 1)$, and $m (\mathbf{Z}, \mathbf{b}, N, n, 1)$ are all invariant to configuration if and only if $\widehat{f}_{avg} (\mathbf{Z}, \mathbf{b}, N, n)$ is invariant to configuration. This Proposition then follows from Lemma 4. □

Proof of Proposition 5

Given Equation 2 for $Y_{agg}(\mathbf{Z}, N, n, 0)$:

$$\begin{aligned}
G_{\frac{Y_{agg}(\mathbf{Z},N,n,0)-EY_{agg}(\mathbf{Z},N,n,0)}{(\operatorname{Var} Y_{agg}(\mathbf{Z},N,n,0))^{1/2}}}(t) &= \Pr\left[\frac{Y_{agg}(\mathbf{Z}, N, n, 0) - EY_{agg}(\mathbf{Z}, N, n, 0)}{(\operatorname{Var} Y_{agg}(\mathbf{Z}, N, n, 0))^{1/2}} \leq t\right] \\
&= \Pr\left[\frac{y_{agg}^{no} + \gamma_1 \frac{N^2\epsilon}{N-n}\left(\widehat{F}_{avg}(\mathbf{Z}, N, n) - \frac{kn}{N}\right) - y_{agg}^{no}}{\gamma_1 \frac{N^2\epsilon}{N-n}\left(\operatorname{Var} \widehat{F}_{avg}(\mathbf{Z}, N, n)\right)^{1/2}} \leq t\right] \\
&= G_{\frac{\widehat{F}_{avg}(\mathbf{Z},N,n)-E\widehat{F}_{avg}(\mathbf{Z},N,n)}{(\operatorname{Var} \widehat{F}_{avg}(\mathbf{Z},N,n))^{1/2}}}(t).
\end{aligned}$$

Given Equation 3 for $M(\mathbf{Z}, N, n, 0)$:

$$\begin{aligned}
G_{\frac{M(\mathbf{Z},N,n,0)-EM(\mathbf{Z},N,n,0)}{(\operatorname{Var} M(\mathbf{Z},N,n,0))^{1/2}}}(t) &= \Pr\left[\frac{M(\mathbf{Z}, N, n, 0) - EM(\mathbf{Z}, N, n, 0)}{(\operatorname{Var} M(\mathbf{Z}, N, n, 0))^{1/2}} \leq t\right] \\
&= \Pr\left[\frac{\gamma_1 \frac{N^2}{N-n}\left(\widehat{F}_{avg}(\mathbf{Z}, N, n) - \frac{kn}{N}\right)}{\gamma_1 \frac{N^2}{N-n}\left(\operatorname{Var} \widehat{F}_{avg}(\mathbf{Z}, N, n)\right)^{1/2}} \leq t\right] \\
&= G_{\frac{\widehat{F}_{avg}(\mathbf{Z},N,n)-E\widehat{F}_{avg}(\mathbf{Z},N,n)}{(\operatorname{Var} \widehat{F}_{avg}(\mathbf{Z},N,n))^{1/2}}}(t).
\end{aligned}$$

Given Equation 4 for $Y_{agg}(\mathbf{Z}, N, n, 1)$:

$$\begin{aligned}
G_{\frac{Y_{agg}(\mathbf{Z},N,n,1)-EY_{agg}(\mathbf{Z},N,n,1)}{(\operatorname{Var} Y_{agg}(\mathbf{Z},N,n,1))^{1/2}}}(t) &= \Pr\left[\frac{Y_{agg}(\mathbf{Z}, N, n, 1) - EY_{agg}(\mathbf{Z}, N, n, 1)}{(\operatorname{Var} Y_{agg}(\mathbf{Z}, N, n, 1))^{1/2}} \leq t\right] \\
&= \Pr\left[\frac{y_{agg}^{no} + \gamma_1 N\epsilon \widehat{F}_{avg}(\mathbf{Z}, N, n) - y_{agg}^{no} - \gamma_1 kn\epsilon}{\gamma_1 N\epsilon\left(\operatorname{Var} \widehat{F}_{avg}(\mathbf{Z}, N, n)\right)^{1/2}} \leq t\right] \\
&= G_{\frac{\widehat{F}_{avg}(\mathbf{Z},N,n)-E\widehat{F}_{avg}(\mathbf{Z},N,n)}{(\operatorname{Var} \widehat{F}_{avg}(\mathbf{Z},N,n))^{1/2}}}(t).
\end{aligned}$$

Given Equation 5 for $M(\mathbf{Z}, N, n, 1)$:

$$\begin{aligned}
G_{\frac{M(\mathbf{Z},N,n,1)-EM(\mathbf{Z},N,n,1)}{(\operatorname{Var} M(\mathbf{Z},N,n,1))^{1/2}}}(t) &= \Pr\left[\frac{M(\mathbf{Z}, N, n, 1) - EM(\mathbf{Z}, N, n, 1)}{(\operatorname{Var} M(\mathbf{Z}, N, n, 1))^{1/2}} \leq t\right] \\
&= \Pr\left[\frac{\gamma_1 N \widehat{F}_{avg}(\mathbf{Z}, N, n) - \gamma_1 kn}{\gamma_1 N\left(\operatorname{Var} \widehat{F}_{avg}(\mathbf{Z}, N, n)\right)^{1/2}} \leq t\right] \\
&= G_{\frac{\widehat{F}_{avg}(\mathbf{Z},N,n)-E\widehat{F}_{avg}(\mathbf{Z},N,n)}{(\operatorname{Var} \widehat{F}_{avg}(\mathbf{Z},N,n))^{1/2}}}(t).
\end{aligned}$$

The Proposition then follows from Lemma 5. $\square$

Proof of Proposition 6

From the proof of Proposition 5, we have that

$$G_{\frac{Y_{agg}(\mathbf{Z},N,n,\ell)-EY_{agg}(\mathbf{Z},N,n,\ell)}{\left(\operatorname{Var} Y_{agg}(\mathbf{Z},N,n,\ell)\right)^{1/2}}}(t) = G_{\frac{\widehat{F}_{avg}(\mathbf{Z},N,n)-E\widehat{F}_{avg}(\mathbf{Z},N,n)}{\left(\operatorname{Var} \widehat{F}_{avg}(\mathbf{Z},N,n)\right)^{1/2}}}(t) \text{ and}$$

$$G_{\frac{M(\mathbf{Z},N,n,\ell)-EM(\mathbf{Z},N,n,\ell)}{\left(\operatorname{Var} M(\mathbf{Z},N,n,\ell)\right)^{1/2}}}(t) = G_{\frac{\widehat{F}_{avg}(\mathbf{Z},N,n)-E\widehat{F}_{avg}(\mathbf{Z},N,n)}{\left(\operatorname{Var} \widehat{F}_{avg}(\mathbf{Z},N,n)\right)^{1/2}}}(t)$$

for $\ell \in \{0,1\}$. Therefore, by Lemma 6, if $\lim_{N\to\infty} \kappa_N(\epsilon') = 0$ for any $\epsilon' > 0$, then

$$\lim_{N\to\infty} G_{\frac{Y_{agg}(\mathbf{Z},N,n,\ell)-EY_{agg}(\mathbf{Z},N,n,\ell)}{\left(\operatorname{Var} Y_{agg}(\mathbf{Z},N,n,\ell)\right)^{1/2}}}(t) = \Phi(t) \text{ and}$$

$$\lim_{N\to\infty} G_{\frac{M(\mathbf{Z},N,n,\ell)-EM(\mathbf{Z},N,n,\ell)}{\left(\operatorname{Var} M(\mathbf{Z},N,n,\ell)\right)^{1/2}}}(t) = \Phi(t)$$

for $\ell \in \{0,1\}$ and for all real t. $\square$

Proof of Proposition 7

The statements that $\Pr\left[Y_{agg}(\mathbf{Z},N,n,0) < y_{agg}^{no}\right] = \Pr\left[M(\mathbf{Z},N,n,0) < 0\right] = \Pr\left[\widehat{F}_{avg}(\mathbf{Z},N,n) < \frac{kn}{N}\right]$ and $\Pr\left[Y_{agg}(\mathbf{Z},N,n,1) < y_{agg}^{no}\right] = \Pr\left[M(\mathbf{Z},N,n,1) < 0\right] = \Pr\left[\widehat{F}_{avg}(\mathbf{Z},N,n) < 0\right]$ follow from Equations 2-5. From Lemma 5, provided that condition (c) holds, $G_{\frac{\widehat{F}_{avg}(\mathbf{Z},N,n)-E\widehat{F}_{avg}(\mathbf{Z},N,n)}{\left(\operatorname{Var} \widehat{F}_{avg}(\mathbf{Z},N,n)\right)^{1/2}}}(t) \approx J(\mathbf{Z},N,n,t)$ and therefore $G_{\widehat{F}_{avg}(\mathbf{Z},N,n)}(t) \approx J\left(\mathbf{Z},N,n,\frac{t-E\widehat{F}_{avg}(\mathbf{Z},N,n)}{\left(\operatorname{Var} \widehat{F}_{avg}(\mathbf{Z},N,n)\right)^{1/2}}\right)$. $\square$

Proof of Proposition 8

For every $n \in \{1,\ldots,N-1\}$, we show that there is a positive probability of both a negative multiplier and an aggregate action below the no-intervention level, y_{agg}^{no}, provided that $\mathbf{d}_w^-(\mathbf{Z}) \neq \frac{k}{N}\mathbf{1}$. Now, $\Pr\left[M(\mathbf{Z},N,n,0) < 0\right] > 0$ if and only if there exists a configuration $\mathbf{b}(N,n) \in \mathcal{B}(N,n)$ for which $\widehat{f}_{avg}(\mathbf{Z},\mathbf{b},N,n) < \frac{kn}{N}$; the same condition is required for $\Pr\left[Y_{agg}(\mathbf{Z},N,n,0) < y_{agg}^{ns}\right] > 0$. We therefore prove that, for every integer $n \in \{1,\ldots,N-1\}$, $\min \operatorname{supp} \widehat{F}_{avg}(\mathbf{Z},N,n) < \frac{kn}{N}$.

Order the elements of $\mathbf{d}_w^-(\mathbf{Z})$ to form the vector $\widetilde{\mathbf{w}}$, with $\widetilde{w}_i \leq \widetilde{w}_{i'}$ if and only if $i \leq i'$. $\mathbf{1}^T\mathbf{d}_w^-(\mathbf{Z}) = k$, so $\mathbf{1}^T\widetilde{\mathbf{w}} = k$ as well. We want to show that $\min \operatorname{supp} \widehat{F}_{avg}(\mathbf{Z},N,n) = \sum_{i=1}^n \widetilde{w}_i < \frac{kn}{N}$ for every integer $n \in \{1,\ldots,N-1\}$. The proof follows by induction. We first show that $\widetilde{w}_1 < \frac{k}{N}$ and $\widetilde{w}_1 + \widetilde{w}_2 < \frac{2k}{N}$; then, given that $\widetilde{w}_1 + \cdots + \widetilde{w}_{n-1} < \frac{k(n-1)}{N}$, we prove that $\widetilde{w}_1 + \cdots + \widetilde{w}_n < \frac{kn}{N}$ for a general $n \in \{3,\ldots,N-1\}$.

Showing that $\widetilde{w}_1 < \frac{k}{N}$: Suppose that $\widetilde{w}_1 = \frac{k}{N}$. With $\widetilde{w}_i \geq \widetilde{w}_1\ \forall i \in \{2,\ldots,N\}$, $\sum_{i=1}^N \widetilde{w}_i = k$ if and only if $\widetilde{w}_1 = \cdots = \widetilde{w}_N = \frac{k}{N}$, which we have ruled out by assumption. Suppose that $\widetilde{w}_1 > \frac{k}{N}$.

With $\widetilde{w}_i \geq \widetilde{w}_1 \ \forall i \in \{2, \ldots, N\}$, $\sum_{i=1}^{N} \widetilde{w}_i > k$. Since we must have $\sum_{i=1}^{N} \widetilde{w}_i = k$, it follows that $\widetilde{w}_1 < \frac{k}{N}$.

Showing that $\widetilde{w}_1 + \widetilde{w}_2 < \frac{2k}{N}$: Since $\widetilde{w}_1 < \frac{k}{N}$, set $\widetilde{w}_1 = \frac{k}{N} - \kappa$ for positive κ. We want $\widetilde{w}_2 < \frac{k}{N} + \kappa$ so that $\widetilde{w}_1 + \widetilde{w}_2 < \frac{2k}{N}$. Suppose that $\widetilde{w}_2 \geq \frac{k}{N} + \kappa$. Then $\widetilde{w}_i \geq \frac{k}{N} + \kappa \ \forall i \in \{3, \ldots, N\}$, and $\sum_{i=1}^{N} \widetilde{w}_i = \widetilde{w}_1 + \widetilde{w}_2 + \sum_{i=3}^{N} \widetilde{w}_i \geq \frac{2k}{N} + (N-2)\left(\frac{k}{N} + \kappa\right) = k + (N-2)\kappa > k$. Since we must have $\sum_{i=1}^{N} \widetilde{w}_i = k$, it follows that $\widetilde{w}_1 + \widetilde{w}_2 < \frac{2k}{N}$.

Showing that $\widetilde{w}_1 + \cdots + \widetilde{w}_n < \frac{kn}{N}$ for a general integer $n \in \{3, \ldots, N-1\}$: Suppose that $\widetilde{w}_1 + \cdots + \widetilde{w}_{n-1} < \frac{k(n-1)}{N}$. Set $\widetilde{w}_1 + \cdots + \widetilde{w}_{n-1} = \frac{k(n-1)}{N} - \kappa'$ for positive κ'. We want $\widetilde{w}_n < \frac{k}{N} + \kappa'$. Suppose that $\widetilde{w}_n \geq \frac{k}{N} + \kappa'$. Then $\widetilde{w}_i \geq \frac{k}{N} + \kappa'$ for $i \in \{n+1, \ldots, N\}$ and $\sum_{i=1}^{N} \widetilde{w}_i = \sum_{i=1}^{n-1} \widetilde{w}_i + \sum_{i=n}^{N} \widetilde{w}_i \geq \frac{k(n-1)}{N} - \kappa' + (N-n+1)\left(\frac{k}{N} + \kappa'\right) = k + (N-n)\kappa' > k$. Therefore, $\widetilde{w}_n < \frac{k}{N} + \kappa'$ and $\widetilde{w}_1 + \cdots + \widetilde{w}_n < \frac{kn}{N}$.

Therefore, for every $n \in \{1, \ldots, N-1\}$, provided that $\mathbf{d}_w^-(\mathbf{Z}) \neq \frac{k}{N}\mathbf{1}$, $\min \operatorname{supp} \widehat{F}_{avg}(\mathbf{Z}, N, n) < \frac{kn}{N}$ and we thus have $\Pr[M(\mathbf{Z}, N, n, 0) < 0] = \Pr\left[Y_{agg}(\mathbf{Z}, N, n, 0) < y_{agg}^{ns}\right] > 0$. $\square$

Proof of Proposition 9

The proof of Proposition 9 follows from the proof of Theorem 1 in Ballester, Calvó-Armengol, and Zenou (2006) coupled with Remarks 2 and 3. $\square$

Proof of Proposition 10

In a setting with transfers,

$$\begin{aligned} Y_{agg}\left(-\mathbf{\Sigma}^{-1}, N, n, 0\right) &= y_{agg}^{no} + \psi N \epsilon \left[\widehat{F}_{avg}\left(-\mathbf{\Sigma}^{-1}, N, n\right) - \frac{n}{N-n}\left(k - \widehat{F}_{avg}\left(-\mathbf{\Sigma}^{-1}, N, n\right)\right)\right] \\ &= y_{agg}^{no} + \frac{\psi N^2 \epsilon}{N-n}\left[\widehat{F}_{avg}\left(-\mathbf{\Sigma}^{-1}, N, n\right) - \frac{kn}{N}\right]. \end{aligned}$$

In a setting with stimulus,

$$Y_{agg}\left(-\mathbf{\Sigma}^{-1}, N, n, 1\right) = y_{agg}^{no} + \psi N \epsilon \widehat{F}_{avg}\left(-\mathbf{\Sigma}^{-1}, N, n\right).$$

$M\left(-\mathbf{\Sigma}^{-1}, N, n, 0\right) = \frac{dY_{agg}\left(-\mathbf{\Sigma}^{-1}, N, n, 0\right)}{d\epsilon}$ and $M\left(-\mathbf{\Sigma}^{-1}, N, n, 1\right) = \frac{dY_{agg}\left(-\mathbf{\Sigma}^{-1}, N, n, 1\right)}{d\epsilon}$. $\square$

Proof of Proposition 11

We first define an M-matrix:

Definition 7 *Matrix* $\mathbf{Z}$ *is an* M-matrix *if it has the form* $\mathbf{Z} = s\mathbf{I} - \mathbf{Q}$, *with* $s > 0$, $\mathbf{Q} \geq \mathbf{0}$, *and* $s \geq r(\mathbf{Q})$. *Matrix* $\mathbf{Z}$ *is a* non-singular M-matrix *if* $s > r(\mathbf{Q})$.

In environments without strategic substitutes, matrix $-\mathbf{\Sigma}$ is an M-matrix: $s = -\sigma$ and $\mathbf{Q} = \mathbf{\Sigma} - \sigma\mathbf{I}$. Provided that $-\sigma > r(\mathbf{\Sigma} - \sigma\mathbf{I})$, matrix $-\mathbf{\Sigma}$ is a non-singular M-matrix. Then $(-\mathbf{\Sigma})^{-1} \geq \mathbf{0}$. With $\mathbf{d}_w^-\left(-\mathbf{\Sigma}^{-1}\right) = \frac{1}{N}\left(-\mathbf{\Sigma}^{-1}\right)^T\mathbf{1}$, it follows that $\mathbf{d}_w^-\left(-\mathbf{\Sigma}^{-1}\right) \geq \mathbf{0}$. Given the expressions for $Y_{agg}\left(-\mathbf{\Sigma}^{-1}, N, n, 1\right)$ and $M\left(-\mathbf{\Sigma}^{-1}, N, n, 1\right)$ in Proposition 10, $Y_{agg}\left(-\mathbf{\Sigma}^{-1}, N, n, 1\right) \geq y_{agg}^{no}$ with probability 1, and $M\left(-\mathbf{\Sigma}^{-1}, N, n, 1\right) \geq 0$ with probability 1. □

Proof of Proposition 12

In the absence of any network-based interaction, $\mathbf{\Sigma} = \sigma\mathbf{I}$, which makes $-\mathbf{\Sigma}^{-1} = -\frac{1}{\sigma}\mathbf{I}$, $\mathbf{d}_w^-\left(-\mathbf{\Sigma}^{-1}\right) = -\frac{1}{N\sigma}\mathbf{1}$, and $k = -\frac{1}{\sigma}$. As a result, $\widehat{F}_{avg}\left(-\mathbf{\Sigma}^{-1}, N, n\right) = -\frac{n}{N\sigma} > 0$ with probability 1. Given the expressions for aggregate output and the corresponding economic multiplier in Proposition 10, we obtain our result. □

Proof of Proposition 13

When $\mathbf{1}^T\mathbf{\Sigma} = \delta\mathbf{1}^T$, $\mathbf{1}^T\mathbf{\Sigma}\mathbf{\Sigma}^{-1} = \delta\mathbf{1}^T\mathbf{\Sigma}^{-1}$, so $\mathbf{1}^T\mathbf{\Sigma}^{-1} = \frac{1}{\delta}\mathbf{1}^T$, and the sum of each column of $\mathbf{\Sigma}^{-1}$ is $\frac{1}{\delta}$. We define $\mathbf{d}_w^-\left(-\mathbf{\Sigma}^{-1}\right)$ equal to $-\frac{1}{N}\left(\mathbf{\Sigma}^{-1}\right)^T\mathbf{1}$. Each element of $\mathbf{d}_w^-\left(-\mathbf{\Sigma}^{-1}\right)$ has the same value, and since we set $\mathbf{1}^T\mathbf{d}_w^-\left(-\mathbf{\Sigma}^{-1}\right) = k$, $\mathbf{d}_w^-\left(-\mathbf{\Sigma}^{-1}\right) = \frac{k}{N}\mathbf{1}$ with $k = -\frac{1}{\delta}$. Then, $\widehat{F}_{avg}\left(-\mathbf{\Sigma}^{-1}, N, n\right) = \frac{kn}{N}$ with probability 1, and given the expressions for aggregate output and the corresponding economic multiplier in Proposition 10, our result follows. □

Proof of Proposition 14

For all $n \in \{1, \ldots, N-1\}$, $Y_{agg}\left(-\left(\mathbf{\Sigma}'\right)^{-1}, N, n, 0\right) \succeq Y_{agg}\left(-\mathbf{\Sigma}^{-1}, N, n, 0\right)$ if $y_{agg}\left(-\left(\mathbf{\Sigma}'\right)^{-1}, \mathbf{b}, N, n, 0\right) \geq y_{agg}\left(-\mathbf{\Sigma}^{-1}, \mathbf{b}, N, n, 0\right)$ for every configuration $\mathbf{b} \in \mathcal{B}(N, n)$. Similarly, for all $n \in \{1, \ldots, N-1\}$, $Y_{agg}\left(-\left(\mathbf{\Sigma}'\right)^{-1}, N, n, 1\right) \succeq Y_{agg}\left(-\mathbf{\Sigma}^{-1}, N, n, 1\right)$ if $y_{agg}\left(-\left(\mathbf{\Sigma}'\right)^{-1}, \mathbf{b}, N, n, 1\right) \geq y_{agg}\left(-\mathbf{\Sigma}^{-1}, \mathbf{b}, N, n, 1\right)$ for every configuration $\mathbf{b} \in \mathcal{B}(N, n)$. Each configuration $\mathbf{b}(N, n)$ of stimulus represents an adjustment to the wealth vector: $\boldsymbol{\omega} \to \boldsymbol{\omega} + \boldsymbol{\rho}$. We therefore wish to show that, for any given vector $\boldsymbol{\omega}' = \boldsymbol{\omega} + \boldsymbol{\rho}$, the aggregate action in the $\mathbf{\Sigma}'$ environment exceeds the aggregate action in the $\mathbf{\Sigma}$ environment. Define $\mathbf{y}^*\left(-\left(\mathbf{\Sigma}'\right)^{-1}, \mathbf{b}, N, n\right)$ as the equilibrium vector of agent actions in the $\mathbf{\Sigma}'$ environment with wealth vector $\boldsymbol{\omega}'$ and corresponding aggregate action $y_{agg}\left(-\left(\mathbf{\Sigma}'\right)^{-1}, \mathbf{b}, N, n\right) = \mathbf{1}^T\mathbf{y}^*\left(-\left(\mathbf{\Sigma}'\right)^{-1}, \mathbf{b}, N, n\right)$. Similarly define $\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1}, \mathbf{b}, N, n\right)$.

Set $\mathbf{\Sigma}' = \mathbf{\Sigma} + \mathbf{D}$. $[\mathbf{D}]_{ij} \geq 0$ for all pairs (i, j) with at least one strict inequality $[\mathbf{D}]_{ij} > 0$. With $\beta > \lambda r(\mathbf{G})$ and $\beta' > \lambda' r(\mathbf{G}')$, $\mathbf{y}^*\left(-\left(\mathbf{\Sigma}'\right)^{-1}, \mathbf{b}, N, n\right)$, $\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1}, \mathbf{b}, N, n\right) > \mathbf{0}$. We also have that $-\mathbf{\Sigma}\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1}, \mathbf{b}, N, n\right) = \psi\boldsymbol{\omega}'$ and $-\mathbf{\Sigma}'\mathbf{y}^*\left(-\left(\mathbf{\Sigma}'\right)^{-1}, \mathbf{b}, N, n\right) = \psi\boldsymbol{\omega}'$. Decompose $\mathbf{\Sigma}$ as $\mathbf{\Sigma} = -\beta\mathbf{I} - \gamma\mathbf{U} + \lambda\mathbf{G}$, and decompose $\mathbf{\Sigma}'$ as $\mathbf{\Sigma}' = \beta'\mathbf{I} - \gamma'\mathbf{U} + \lambda'\mathbf{G}'$, where $\mathbf{U} = \mathbf{1}\mathbf{1}^T$.

For any vector $\boldsymbol{\omega}'$, $-\left(\mathbf{\Sigma}' - \mathbf{D}\right)\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1}, \mathbf{b}, N, n\right) = \psi\boldsymbol{\omega}$, and therefore

$$\left(\beta'\mathbf{I} - \lambda'\mathbf{G}'\right)\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1}, \mathbf{b}, N, n\right) = \psi\boldsymbol{\omega}' - \gamma'\mathbf{U}\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1}, \mathbf{b}, N, n\right) - \mathbf{D}\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1}, \mathbf{b}, N, n\right). \quad (6)$$

We also have $-\mathbf{\Sigma}'\mathbf{y}^*\left(-\left(\mathbf{\Sigma}'\right)^{-1},\mathbf{b},N,n\right)=\psi\boldsymbol{\omega}'$, and therefore

$$\left(\beta'\mathbf{I}-\lambda'\mathbf{G}'\right)\mathbf{y}^*\left(-\left(\mathbf{\Sigma}'\right)^{-1},\mathbf{b},N,n\right)=\psi\boldsymbol{\omega}'-\gamma'\mathbf{U}\mathbf{y}^*\left(-\left(\mathbf{\Sigma}'\right)^{-1},\mathbf{b},N,n\right). \tag{7}$$

Setting $\gamma'=0$ and subtracting Equation 6 from Equation 7 gives

$$\left(\beta'\mathbf{I}-\lambda'\mathbf{G}'\right)\left[\mathbf{y}^*\left(-\left(\mathbf{\Sigma}'\right)^{-1},\mathbf{b},N,n\right)-\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1},\mathbf{b},N,n\right)\right]=\mathbf{D}\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1},\mathbf{b},N,n\right).$$

$\left(\beta'\mathbf{I}-\lambda'\mathbf{G}'\right)=\lambda'\left(\frac{\beta'}{\lambda'}\mathbf{I}-\mathbf{G}'\right)$, with $\lambda'>0$. $\left(\frac{\beta'}{\lambda'}\mathbf{I}-\mathbf{G}'\right)$ is an M-matrix that is non-singular when $\frac{\beta'}{\lambda'}>\rho\left(\mathbf{G}'\right)$, which immediately follows from the initial assumptions. Therefore, $\left(\frac{\beta'}{\lambda'}\mathbf{I}-\mathbf{G}'\right)$ is inverse-positive, and

$$\mathbf{y}^*\left(-\left(\mathbf{\Sigma}'\right)^{-1},\mathbf{b},N,n\right)-\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1},\mathbf{b},N,n\right)=\left(\lambda'\right)^{-1}\left(\frac{\beta'}{\lambda'}\mathbf{I}-\mathbf{G}'\right)^{-1}\mathbf{D}\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1},\mathbf{b},N,n\right)>\mathbf{0},$$

so for any vector of wealth $\boldsymbol{\omega}'$, $\mathbf{y}^*\left(-\left(\mathbf{\Sigma}'\right)^{-1},\mathbf{b},N,n\right)>\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1},\mathbf{b},N,n\right)$. $\square$

Proof of Proposition 15

Set $\mathbf{\Sigma}'=\mathbf{\Sigma}+\mathbf{D}$, with $[\mathbf{D}]_{ij}\geq 0$ for all pairs (i,j) and at least one strict inequality $[\mathbf{D}]_{ij}>0$. Also set $\boldsymbol{\omega}=\omega\mathbf{1}$. We have $-\mathbf{\Sigma}\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1}\right)=\psi\omega\mathbf{1}$ and $-\mathbf{\Sigma}'\mathbf{y}^*\left(-\left(\mathbf{\Sigma}'\right)^{-1}\right)=\psi\omega\mathbf{1}$ with $\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1}\right),\mathbf{y}^*\left(-\left(\mathbf{\Sigma}'\right)^{-1}\right)>\mathbf{0}$ since $\beta>\lambda r\left(\mathbf{G}\right)$ and $\beta'>\lambda' r\left(\mathbf{G}'\right)$. We decompose $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$ as follows: $\mathbf{\Sigma}=-\beta\mathbf{I}-\gamma\mathbf{U}+\lambda\mathbf{G}$ and $\mathbf{\Sigma}'=-\beta'\mathbf{I}-\gamma'\mathbf{U}+\lambda'\mathbf{G}'$, with $\mathbf{U}=\mathbf{1}\mathbf{1}^T$.

Then, $-\left(\mathbf{\Sigma}'-\mathbf{D}\right)\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1}\right)=\psi\omega\mathbf{1}$, and therefore

$$\left(\beta'\mathbf{I}-\lambda'\mathbf{G}'\right)\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1}\right)=\psi\boldsymbol{\omega}-\gamma'\mathbf{U}\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1}\right)-\mathbf{D}\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1}\right). \tag{8}$$

Meanwhile, $-\mathbf{\Sigma}'\mathbf{y}^*\left(-\left(\mathbf{\Sigma}'\right)^{-1}\right)=\psi\omega\mathbf{1}$, and therefore,

$$\left(\beta'\mathbf{I}-\lambda'\mathbf{G}'\right)\mathbf{y}^*\left(-\left(\mathbf{\Sigma}'\right)^{-1}\right)=\psi\omega\mathbf{1}-\gamma'\mathbf{U}\mathbf{y}^*\left(-\left(\mathbf{\Sigma}'\right)^{-1}\right). \tag{9}$$

Setting $\gamma'=0$ and subtracting Equation 8 from Equation 9 gives

$$\left(\beta'\mathbf{I}-\lambda'\mathbf{G}'\right)\left(\mathbf{y}^*\left(-\left(\mathbf{\Sigma}'\right)^{-1}\right)-\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1}\right)\right)=\mathbf{D}\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1}\right).$$

It follows that

$$\mathbf{y}^*\left(-\left(\mathbf{\Sigma}'\right)^{-1}\right)-\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1}\right)=\left(\lambda'\right)^{-1}\left(\frac{\beta'}{\lambda'}\mathbf{I}-\mathbf{G}'\right)^{-1}\mathbf{D}\mathbf{y}^*\left(-\mathbf{\Sigma}^{-1}\right)>\mathbf{0}$$

since $\lambda'>0$ and $\left(\frac{\beta'}{\lambda'}\mathbf{I}-\mathbf{G}'\right)$ is an M-matrix that is inverse-positive because $\frac{\beta'}{\lambda'}>r\left(\mathbf{G}'\right)$ by assump-

tion. Thus, $\mathbf{y}^* \left(-\left(\boldsymbol{\Sigma}'\right)^{-1}\right) > \mathbf{y}^* \left(-\boldsymbol{\Sigma}^{-1}\right)$.

Now, $-\boldsymbol{\Sigma}\mathbf{y}^* \left(-\boldsymbol{\Sigma}^{-1}\right) = \psi\omega\mathbf{1}$ and $-\boldsymbol{\Sigma}'\mathbf{y}^* \left(-\left(\boldsymbol{\Sigma}'\right)^{-1}\right) = \psi\omega\mathbf{1}$. Since $\beta > \lambda r\left(\mathbf{G}\right)$ and $\beta' > \lambda' r\left(\mathbf{G}\right)$, $\mathbf{y}^* \left(-\boldsymbol{\Sigma}^{-1}\right) = \psi\omega\left(-\boldsymbol{\Sigma}^{-1}\right)\mathbf{1}$ and $\mathbf{y}^* \left(-\left(\boldsymbol{\Sigma}'\right)^{-1}\right) = \psi\omega\left(-\left(\boldsymbol{\Sigma}'\right)^{-1}\right)\mathbf{1}$. $\boldsymbol{\Sigma}, \boldsymbol{\Sigma}'$ are symmetric, so $\mathbf{y}^* \left(-\boldsymbol{\Sigma}^{-1}\right) = \psi\omega\left(-\boldsymbol{\Sigma}^{-1}\right)^T\mathbf{1} = \psi N\omega\mathbf{d}_w^- \left(-\boldsymbol{\Sigma}^{-1}\right)$ and $\mathbf{y}^* \left(-\left(\boldsymbol{\Sigma}'\right)^{-1}\right) = \psi\omega\left(-\left(\boldsymbol{\Sigma}'\right)^{-1}\right)^T\mathbf{1} = \psi N\omega\mathbf{d}_w^- \left(-\left(\boldsymbol{\Sigma}'\right)^{-1}\right)$. Since $\mathbf{y}^* \left(-\left(\boldsymbol{\Sigma}'\right)^{-1}\right) > \mathbf{y}^* \left(-\boldsymbol{\Sigma}^{-1}\right)$, it immediately follows that $\mathbf{d}_w^- \left(-\left(\boldsymbol{\Sigma}'\right)^{-1}\right) > \mathbf{d}_w^- \left(-\boldsymbol{\Sigma}^{-1}\right)$. Then $\widehat{F}_{avg}\left(-\left(\boldsymbol{\Sigma}'\right)^{-1}, N, n\right) \succeq \widehat{F}_{avg}\left(-\boldsymbol{\Sigma}^{-1}, N, n\right)$ for all $n \in \{1, \ldots, N\}$ and we obtain the result. $\square$

Proof of Proposition 16

Agent i's optimization problem is:

$$\max_{y_{i,q}} u_{i,q} = \max_{y_{i,q}} \; -\sum_{j=1}^{N} \left[\bar{\mathbf{A}}\right]_{ij} \left(y_{i,q} - \left[\mathbf{T}\right]_{ij}\left(y_{j,q-1}\right)\right)^2.$$

From the first-order condition, $\sum_{j=1}^{N} \left[\bar{\mathbf{A}}\right]_{ij} y_{i,q}^* = \sum_{j=1}^{N} \left[\bar{\mathbf{A}}\right]_{ij} \left[\mathbf{T}\right]_{ij}\left(y_{j,q-1}\right)$. By the row-stochasticity of $\bar{\mathbf{A}}$, $y_{i,q}^* = \sum_{j=1}^{N} \left[\bar{\mathbf{A}}\right]_{ij} \left[\mathbf{T}\right]_{ij}\left(y_{j,q-1}\right)$, and so $y_{i,q}^* = \left[\bar{\mathbf{A}} \circ \mathbf{T}\right]_{i*} \mathbf{y}_{q-1}$. It then follows that $\mathbf{y}_q^* = \left(\bar{\mathbf{A}} \circ \mathbf{T}\right)^q \mathbf{y}_0$. $\square$

Proof of Proposition 17

Given that $\mathbf{y}_q^* = y\mathbf{1}$ for $q < 0$, the proof follows by induction. For $q = 0$, $\mathbf{y}_q^* = y\mathbf{1} + \left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^0 \boldsymbol{\rho} = y\mathbf{1} + \boldsymbol{\rho}$. For $q = 1$, given Proposition 16, $\mathbf{y}_1^* = \left(\bar{\mathbf{A}} \circ \mathbf{O}\right)\mathbf{y}_0$, so

$$\begin{aligned}
y_{i,1}^* &= \sum_{\substack{j \in \{1,\ldots,N\} \\ \text{s.t. } [\mathbf{T}]_{ij} = \mathcal{F}}} \left[\bar{\mathbf{A}}\right]_{ij} y_{j,0} + \sum_{\substack{j \in \{1,\ldots,N\} \\ \text{s.t. } [\mathbf{T}]_{ij} = \mathcal{D}}} \left[\bar{\mathbf{A}}\right]_{ij} \mathcal{D}\left(y_{j,0}\right) \\
&= \sum_{\substack{j \in \{1,\ldots,N\} \\ \text{s.t. } [\mathbf{T}]_{ij} = \mathcal{F}}} \left[\bar{\mathbf{A}}\right]_{ij} \left(y + \left[\boldsymbol{\rho}\right]_j\right) + \sum_{\substack{j \in \{1,\ldots,N\} \\ \text{s.t. } [\mathbf{T}]_{ij} = \mathcal{D}}} \left[\bar{\mathbf{A}}\right]_{ij} \left(2y - \left(y + \left[\boldsymbol{\rho}\right]_j\right)\right) \\
&= y + \sum_{\substack{j \in \{1,\ldots,N\} \\ \text{s.t. } [\mathbf{T}]_{ij} = \mathcal{F}}} \left[\bar{\mathbf{A}}\right]_{ij} \left[\boldsymbol{\rho}\right]_j - \sum_{\substack{j \in \{1,\ldots,N\} \\ \text{s.t. } [\mathbf{T}]_{ij} = \mathcal{D}}} \left[\bar{\mathbf{A}}\right]_{ij} \left[\boldsymbol{\rho}\right]_j \\
&= y + \left[\bar{\mathbf{A}} \circ \mathbf{O}\right]_{i*} \boldsymbol{\rho},
\end{aligned}$$

where $\left[\mathbf{O}\right]_{ij} = 1$ if $\left[\mathbf{T}\right]_{ij} = \mathcal{F}$ and $\left[\mathbf{O}\right]_{ij} = -1$ if $\left[\mathbf{T}\right]_{ij} = \mathcal{D}$. It then follows that $\mathbf{y}_1^* = y\mathbf{1} + \left(\bar{\mathbf{A}} \circ \mathbf{O}\right)\boldsymbol{\rho}$.

We now assume that $\mathbf{y}_{q-1}^* = y\mathbf{1} + \left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^{q-1}\boldsymbol{\rho}$, and we demonstrate that $\mathbf{y}_q^* = y\mathbf{1} +$

$\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q \boldsymbol{\rho}$:

$$
\begin{aligned}
y^*_{i,q} &= \left[\bar{\mathbf{A}} \circ \mathbf{T}\right]_{i*} \mathbf{y}_{q-1} \\
&= \sum_{\substack{j\in\{1,\ldots,N\} \\ \text{s.t. } [\mathbf{T}]_{ij}=\mathcal{F}}} \left[\bar{\mathbf{A}}\right]_{ij} y_{j,q-1} + \sum_{\substack{j\in\{1,\ldots,N\} \\ \text{s.t. } [\mathbf{T}]_{ij}=\mathcal{D}}} \left[\bar{\mathbf{A}}\right]_{ij} \mathcal{D}\left(y_{j,q-1}\right) \\
&= \sum_{\substack{j\in\{1,\ldots,N\} \\ \text{s.t. } [\mathbf{T}]_{ij}=\mathcal{F}}} \left[\bar{\mathbf{A}}\right]_{ij} \left(y + \left[\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^{q-1}\right]_{j*} \boldsymbol{\rho}\right) + \sum_{\substack{j\in\{1,\ldots,N\} \\ \text{s.t. } [\mathbf{T}]_{ij}=\mathcal{D}}} \left[\bar{\mathbf{A}}\right]_{ij} \left(2y - \left(y + \left[\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^{q-1}\right]_{j*} \boldsymbol{\rho}\right)\right) \\
&= y + \sum_{\substack{j\in\{1,\ldots,N\} \\ \text{s.t. } [\mathbf{T}]_{ij}=\mathcal{F}}} \left[\bar{\mathbf{A}}\right]_{ij} \left[\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^{q-1}\right]_{j*} \boldsymbol{\rho} - \sum_{\substack{j\in\{1,\ldots,N\} \\ \text{s.t. } [\mathbf{T}]_{ij}=\mathcal{D}}} \left[\bar{\mathbf{A}}\right]_{ij} \left[\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^{q-1}\right]_{j*} \boldsymbol{\rho} \\
&= y + \left[\bar{\mathbf{A}} \circ \mathbf{O}\right]_{j*} \left(\left[\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^{q-1}\right]_{j*} \boldsymbol{\rho}\right) \\
&= y + \left[\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^{q}\right]_{j*} \boldsymbol{\rho},
\end{aligned}
$$

so $\mathbf{y}^*_q = y\mathbf{1} + \left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q \boldsymbol{\rho}$. $\square$

Proof of Proposition 18

The period-q aggregate action is: $y_{agg,q}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, \mathbf{b}, N, n\right) = y^{no}_{agg} + N\left[\mathbf{d}^-_w\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)\right]^T \boldsymbol{\rho}$. In a setting with positive transfers to n agents and negative transfers to $N-n$ agents, $[\boldsymbol{\rho}]_i = \epsilon$ if agent i is receiving a positive transfer, and $[\boldsymbol{\rho}]_i = -\frac{n\epsilon}{N-n}$ if agent i is receiving a negative transfer. Therefore,

$$
\begin{aligned}
Y_{agg,q}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n, 0\right) &= y^{no}_{agg} + N\epsilon \widehat{F}_{avg}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n\right) - N\frac{n\epsilon}{N-n}\left(k_q - \widehat{F}_{avg}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n\right)\right) \\
&= y^{no}_{agg} + \frac{N^2\epsilon}{N-n}\left(\widehat{F}_{avg}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n\right) - \frac{k_q n}{N}\right)
\end{aligned}
$$

With $M_q\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n, 0\right) = \frac{dY_{agg,q}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n, 0\right)}{d\epsilon}$ and $IRF_q\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n, 0\right) = Y_{agg,q}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n, 0\right) - y^{no}_{agg}$, the first part of the Proposition follows.

In a setting with stimulus, $[\boldsymbol{\rho}]_i = \epsilon$ if agent i is receiving stimulus, and $[\boldsymbol{\rho}]_i = 0$ if agent i is not receiving stimulus. Therefore,

$$
Y_{agg,q}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n, 1\right) = y^{no}_{agg} + N\epsilon \widehat{F}_{avg}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n\right).
$$

With $M_q\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n, 1\right) = \frac{dY_{agg,q}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n, 1\right)}{d\epsilon}$ and $IRF_q\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n, 1\right) = Y_{agg,q}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n, 1\right) - y^{no}_{agg}$, the second part of the Proposition follows. $\square$

Proof of Proposition 19

Set $\bar{\mathbf{A}} \circ \mathbf{O} = \mathbf{I}$. As a result, $\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right) = \frac{1}{N}\mathbf{1}$ and $k_q = 1$ for all $q \in \mathbb{Z}_+$. Then, $\widehat{F}_{avg}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n\right) = \frac{n}{N}$ with probability 1 for all $q \in \mathbb{Z}_+$. The result follows from Proposition 18. $\square$

Proof of Proposition 20

When $\mathbf{1}^T\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q = k_q\mathbf{1}^T$, $\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right) = \frac{k_q}{N}\mathbf{1}$ and $\widehat{F}_{avg}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n\right) = \frac{k_q n}{N}$ with probability 1. Proposition 20 then follows from Proposition 18. $\square$

Proof of Proposition 21

If $\left(\bar{\mathbf{A}}' \circ \mathbf{O}'\right)^q \geq \left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\mathbf{P}$ for some permutation matrix $\mathbf{P}$, then $\left[\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}}' \circ \mathbf{O}'\right)^q\right)\right]^T \geq \left[\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)\right]^T\mathbf{P}$ and $\widehat{F}_{avg}\left(\left(\bar{\mathbf{A}}' \circ \mathbf{O}'\right)^q, N, n\right) \succeq \widehat{F}_{avg}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n\right)$. Proposition 21 then follows from Proposition 18. $\square$

Proof of Lemma 7

By definition, $\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right) = \frac{1}{N}\left[\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right]^T\mathbf{1}$ and $k_q = \mathbf{1}^T\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right)$, so $Nk_q = \mathbf{1}^T\left[\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right]^T\mathbf{1} = \sum_{i=1}^N\sum_{j=1}^N\left[\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right]_{ij}$. $k_q \in [-1, 1]$ for all $q \geq 1$ if and only if $\sum_{i=1}^N\sum_{j=1}^N\left[\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right]_{ij} \in [-N, N]$ for all $q \geq 1$. When $q = 1$, $-\bar{\mathbf{A}} \leq \bar{\mathbf{A}} \circ \mathbf{O} \leq \bar{\mathbf{A}}$ element-wise, so $-N = \mathbf{1}^T\left(-\bar{\mathbf{A}}\right)\mathbf{1} \leq \mathbf{1}^T\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)\mathbf{1} \leq \mathbf{1}^T\bar{\mathbf{A}}\mathbf{1} = N$, and $\sum_{i=1}^N\sum_{j=1}^N\left[\bar{\mathbf{A}} \circ \mathbf{O}\right]_{ij} \in [-N, N]$.

For $q > 1$, $\left[\bar{\mathbf{A}}^q\right]_{ij}$, $\left[\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right]_{ij}$, and $-\left[\bar{\mathbf{A}}^q\right]_{ij}$ are each the sum of a collection of terms, with each term a product of elements $\left[\bar{\mathbf{A}}\right]_{\ell\ell'}$. The collection of terms, ignoring sign, is the same for $\left[\bar{\mathbf{A}}^q\right]_{ij}$, $\left[\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right]_{ij}$, and $-\left[\bar{\mathbf{A}}^q\right]_{ij}$. For $\left[\bar{\mathbf{A}}^q\right]_{ij}$, all of the terms have a positive sign; for $-\left[\bar{\mathbf{A}}^q\right]_{ij}$, all of the terms have a negative sign; and for $\left[\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right]_{ij}$, when $\bar{\mathbf{A}} \circ \mathbf{O} \neq \bar{\mathbf{A}}$ and $\bar{\mathbf{A}} \circ \mathbf{O} \neq -\bar{\mathbf{A}}$, the terms have a mixture of positive and negative signs. Therefore, $-\bar{\mathbf{A}}^q \leq \left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q \leq \bar{\mathbf{A}}^q$ element-wise. $\bar{\mathbf{A}}$ is row-stochastic, with row stochasticity preserved under matrix multiplication, so $\mathbf{1}^T\bar{\mathbf{A}}^q\mathbf{1} = N$. For $q > 1$, $-N = \mathbf{1}^T\left(-\bar{\mathbf{A}}^q\right)\mathbf{1} \leq \mathbf{1}^T\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\mathbf{1} \leq \mathbf{1}^T\bar{\mathbf{A}}^q\mathbf{1} = N$, and $\sum_{i=1}^N\sum_{j=1}^N\left[\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right]_{ij} \in [-N, N]$. $\square$

Proof of Proposition 22

From Proposition 18, $M_q\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n, 1\right) = N\widehat{F}_{avg}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n\right)$. With $k_q \in [-1, 1]$, for every $n \in \{1, \ldots, N-1\}$,

$$\max_{\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q}\left[\max \operatorname{supp} \widehat{F}_{avg}\left(\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q, N, n\right)\right] = 1,$$

which occurs when $\left[\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q\right)\right]^T = \begin{pmatrix}0 & 0 & \dots & 0 & 1\end{pmatrix}\mathbf{P}$ for some permutation matrix $\mathbf{P}$. Meanwhile,

$$\min_{\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q}\left[\min\operatorname{supp}\widehat{F}_{avg}\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q, N, n\right)\right] = -1,$$

which occurs when $\left[\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q\right)\right]^T = \begin{pmatrix}0 & 0 & \dots & 0 & -1\end{pmatrix}\mathbf{P}$ for some permutation matrix $\mathbf{P}$. Consistent with $\left[\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q\right)\right]^T = \begin{pmatrix}0 & 0 & \dots & 0 & 1\end{pmatrix}\mathbf{P}$ is a positive star graph, and consistent with $\left[\mathbf{d}_w^-\left(\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q\right)\right]^T = \begin{pmatrix}0 & 0 & \dots & 0 & -1\end{pmatrix}\mathbf{P}$ is a negative star graph. □

Proof of Proposition 23

The proof of Proposition 23 follows immediately from Theorem 1 in Schlossberger (2018). Provided that $\bar{\mathbf{A}}$ is primitive, $\lim_{q\to\infty}\left[\bar{\mathbf{A}}^q\right]_{ij} = \left[\mathbf{w}_\infty^T\right]_j$ for all $i \in \{1,\dots,N\}$, and therefore $\lim_{q\to\infty}\mathbf{d}_w^-\left(\bar{\mathbf{A}}^q\right) = \mathbf{w}_\infty\left(\bar{\mathbf{A}}\right)$. □

Proof of Proposition 24

If $\bar{\mathbf{A}}$ is primitive, $\left[\bar{\mathbf{A}}\right]_{ij} > 0$ if and only if $\left[\bar{\mathbf{A}}\right]_{ji} > 0$, and all non-zero elements within every row of $\bar{\mathbf{A}}$ have the same value, then by Theorem 2 in Schlossberger (2018), $\mathbf{w}_\infty\left(\bar{\mathbf{A}}\right) = \frac{\mathbf{d}}{\mathbf{1}^T\mathbf{d}}$. Since $\lim_{q\to\infty}\bar{\mathbf{A}}^q = \mathbf{1}\left[\mathbf{w}_\infty\left(\bar{\mathbf{A}}\right)\right]^T$, $\lim_{q\to\infty}\mathbf{d}_w^-\left(\bar{\mathbf{A}}^q\right) = \frac{\mathbf{d}}{\mathbf{1}^T\mathbf{d}}$. Setting $\bar{\mathbf{A}}\circ\mathbf{O} = \bar{\mathbf{A}}$,

$$\lim_{q\to\infty} Y_{agg,q}\left(\bar{\mathbf{A}}^q, N, n, 0\right) = y_{agg}^{no} + \frac{N^2\epsilon}{N-n}\lim_{q\to\infty}\left(\widehat{F}_{avg}\left(\bar{\mathbf{A}}^q, N, n\right) - \frac{k_q n}{N}\right).$$

$\lim_{q\to\infty} k_q = \lim_{q\to\infty}\left[\mathbf{d}_w^-\left(\bar{\mathbf{A}}^q\right)\right]^T\mathbf{1} = 1$. When $n = 1$, $\lim_{q\to\infty}\widehat{F}_{avg}\left(\bar{\mathbf{A}}^q, N, n\right) = \frac{D\left(\bar{\mathbf{A}}\right)}{\mathbf{1}^T\mathbf{d}}$, so

$$\begin{aligned}\Pr\left[\lim_{q\to\infty} Y_{agg,q}\left(\bar{\mathbf{A}}^q, N, n, 0\right) < y_{agg}^{no}\right] &= \Pr\left[\lim_{q\to\infty}\widehat{F}_{avg}\left(\bar{\mathbf{A}}^q, N, n\right) < \frac{k_q n}{N}\right] \\ &= \Pr\left[\frac{D\left(\bar{\mathbf{A}}\right)}{\mathbf{1}^T\mathbf{d}} < \frac{1}{N}\right] = \Pr\left[D\left(\bar{\mathbf{A}}\right) < \frac{\mathbf{1}^T\mathbf{d}}{N}\right],\end{aligned}$$

and the Proposition follows. □

Proof of Proposition 25

We wish to compute $\lim_{q\to\infty}\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)^q$. Graph $\mathcal{G}\left(\bar{\mathbf{A}}\circ\mathbf{O}\right) = \left(\mathcal{V}\left(\bar{\mathbf{A}}\circ\mathbf{O}\right), \mathcal{E}\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)\right)$, $\left|\mathcal{V}\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)\right| = N$, is structurally balanced, so we can partition $\mathcal{V}\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)$ into two disjoint subsets: $\mathcal{V} = \mathcal{V}_1\cup\mathcal{V}_2$. Moreover, since $\mathcal{G}\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)$ is structurally balanced and $\left|\bar{\mathbf{A}}\circ\mathbf{O}\right| = \bar{\mathbf{A}}$, there exists an $N\times N$ diagonal matrix $\mathbf{\Omega}$ for which $\bar{\mathbf{A}} = \mathbf{\Omega}\left(\bar{\mathbf{A}}\circ\mathbf{O}\right)\mathbf{\Omega}$. To make this equality hold, choose an $\ell\in\{1,2\}$ and set $\left[\mathbf{\Omega}\right]_{ii} = 1$ for $\forall i\in\mathcal{V}_\ell$ and $\left[\mathbf{\Omega}\right]_{ii} = -1$ for $\forall i\in\mathcal{V}_{-\ell}$. Matrix $\mathbf{\Omega}$ equals its inverse,

$\mathbf{\Omega}^{-1}$, so $\bar{\mathbf{A}}^q = \left(\mathbf{\Omega}\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)\mathbf{\Omega}\right)^q = \mathbf{\Omega}\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q \mathbf{\Omega}$ and $\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q = \mathbf{\Omega}\bar{\mathbf{A}}^q\mathbf{\Omega}$. Therefore,

$$\lim_{q\to\infty}\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q = \lim_{q\to\infty}\mathbf{\Omega}\bar{\mathbf{A}}^q\mathbf{\Omega} = \mathbf{\Omega}\left(\lim_{q\to\infty}\bar{\mathbf{A}}^q\right)\mathbf{\Omega} = \mathbf{\Omega}\mathbf{1}\left[\mathbf{w}_\infty\left(\bar{\mathbf{A}}\right)\right]^T\mathbf{\Omega},$$

where $\left[\mathbf{w}_\infty\left(\bar{\mathbf{A}}\right)\right]^T\bar{\mathbf{A}} = \left[\mathbf{w}_\infty\left(\bar{\mathbf{A}}\right)\right]^T$. Specifically, $\left[\lim_{q\to\infty}\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right]_{ij} = \left[\mathbf{w}_\infty\left(\bar{\mathbf{A}}\right)\right]_j$ if $i, j \in \mathcal{V}_\ell$ and $\left[\lim_{q\to\infty}\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q\right]_{ij} = -\left[\mathbf{w}_\infty\left(\bar{\mathbf{A}}\right)\right]_j$ if $i \in \mathcal{V}_\ell$ and $j \in \mathcal{V}_{-\ell}$ for $\ell \in \{1, 2\}$. Equivalently, $\lim_{q\to\infty}\left(\bar{\mathbf{A}} \circ \mathbf{O}\right)^q = \left(\mathbf{1}\left[\mathbf{w}_\infty\left(\bar{\mathbf{A}}\right)\right]^T\right) \circ \mathbf{O}$, where $[\mathbf{O}]_{ij} = 1$ if $i, j \in \mathcal{V}_\ell$ and $[\mathbf{O}]_{ij} = -1$ if $i \in \mathcal{V}_\ell$ and $j \in \mathcal{V}_{-\ell}$ for $\ell \in \{1, 2\}$. $\square$

Proof of Proposition 26

Solving the representative consumer's problem:

$$\max_{c_1,\ldots,c_N}\prod_{i=1}^N c_i^{\eta_i} \quad \text{s.t.} \quad \sum_{i=1}^N p_i c_i = w\sum_{i=1}^N \ell_i + \sum_{i=1}^N \pi_i.$$

Setting $\sum_{i=1}^N \ell_i = 1$, the first-order condition is:

$$p_i c_i = \eta_i\left(w + \sum_{i=1}^N \pi_i\right). \tag{10}$$

Solving the firm's optimization problem:

$$\max_{x_{1i},\ldots,x_{Ni},\ell_i} p_i x_i - \sum_{j=1}^N p_j x_{ji} - w\ell_i \quad \text{s.t.} \quad x_i = A_i^{\alpha_i}\ell_i^{\alpha_i}\left(\prod_{j=1}^N x_{ji}^{[\mathbf{\Lambda}]_{ji}}\right)^{\beta_i}.$$

Rewriting, we have:

$$\max_{x_{1i},\ldots,x_{Ni},\ell_i} p_i A_i^{\alpha_i}\ell_i^{\alpha_i}\left(\prod_{j=1}^N x_{ji}^{[\mathbf{\Lambda}]_{ji}}\right)^{\beta_i} - \sum_{j=1}^N p_j x_{ji} - w\ell_i.$$

The first-order conditions are:

$$w = \frac{\alpha_i p_i x_i}{\ell_i} \tag{11}$$

and

$$p_j = \frac{[\mathbf{\Lambda}]_{ji}\,\beta_i p_i x_i}{x_{ji}}. \tag{12}$$

The profit of each firm is:

$$\pi_i = p_i x_i - \sum_{j=1}^N p_j x_{ji} - w\ell_i.$$

From Equations 11 and 12,

$$\pi_i = p_i x_i - \sum_{j=1}^{N} [\mathbf{\Lambda}]_{ji} \beta_i p_i x_i - \alpha_i p_i x_i.$$

Matrix $\mathbf{\Lambda}$ is column-stochastic, so

$$\pi_i = (1 - \alpha_i - \beta_i) p_i x_i = 0. \tag{13}$$

From Equations 10 and 13,

$$c_i = \frac{\eta_i w}{p_i}, \tag{14}$$

and from Equation 12,

$$x_{ij} = \frac{[\mathbf{\Lambda}]_{ij} \beta_j p_j x_j}{p_i}. \tag{15}$$

Substituting the expressions for c_i and x_{ij} in Equations 14 and 15 into the goods market clearing condition, $x_i = c_i + \sum_{j=1}^{N} x_{ij}$, we have:

$$p_i x_i = \eta_i w + \sum_{j=1}^{N} [\mathbf{\Lambda}]_{ij} \beta_j p_j x_j.$$

Define $y_j = p_j x_j$. Then $\mathbf{y} = \boldsymbol{\eta} w + \mathbf{\Lambda} \operatorname{diag}(\boldsymbol{\beta}) \mathbf{y}$.

Lemma 9 $\mathbf{I} - \mathbf{\Lambda} \operatorname{diag}(\boldsymbol{\beta})$ *is invertible.*

Proof. $\mathbf{I} - \mathbf{\Lambda} \operatorname{diag}(\boldsymbol{\beta})$ is an M-matrix. It is therefore invertible if $1 > r(\mathbf{\Lambda} \operatorname{diag}(\boldsymbol{\beta}))$. We know that any matrix-induced norm satisfies the inequality $\|\mathbf{Z}\| > |\mu|$ for any matrix $\mathbf{Z}$ with eigenvalue μ. We take the infinity norm of matrix $(\mathbf{\Lambda} \operatorname{diag}(\boldsymbol{\beta}))^T$:

$$\left\| (\mathbf{\Lambda} \operatorname{diag}(\boldsymbol{\beta}))^T \right\|_\infty = \max_{i \in \{1,\ldots,N\}} \sum_{j=1}^{N} \left[(\mathbf{\Lambda} \operatorname{diag}(\boldsymbol{\beta}))^T \right]_{ij} = \max_{i \in \{1,\ldots,N\}} \beta_i$$

when $\mathbf{\Lambda}$ is column-stochastic. Since $\max_{i \in \{1,\ldots,N\}} \beta_i \in (0,1)$, $r(\mathbf{\Lambda} \operatorname{diag}(\boldsymbol{\beta})) < 1$, and $\mathbf{I} - \mathbf{\Lambda} \operatorname{diag}(\boldsymbol{\beta})$ is invertible. ■

Following Lemma 9, we obtain $\mathbf{y}^* = (\mathbf{I} - \mathbf{\Lambda} \operatorname{diag}(\boldsymbol{\beta}))^{-1} \boldsymbol{\eta} w$. □

Proof of Proposition 27

The proof of Proposition 27 immediately follows from the proof of Proposition 26. □

Proof of Proposition 28

In a setting with transfers,

$$Y_{agg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1},N,n,0\right)=y_{agg}^{no}+N\epsilon\left[\widehat{F}_{avg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1},N,n\right)\right.$$
$$\left.-\frac{n}{N-n}\left(k-\widehat{F}_{avg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1},N,n\right)\right)\right],$$

so $Y_{agg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1},N,n,0\right)=y_{agg}^{no}+\frac{N^2\epsilon}{N-n}\left[\widehat{F}_{avg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1},N,n\right)-\frac{kn}{N}\right]$ with $M\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1},N,n,0\right)=\frac{dY_{agg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1},N,n,0\right)}{d\epsilon}$.

In a setting with stimulus,

$$Y_{agg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1},N,n,1\right)=y_{agg}^{no}+N\epsilon\widehat{F}_{avg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1},N,n\right)$$

and $M\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1},N,n,1\right)=\frac{dY_{agg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1},N,n,1\right)}{d\epsilon}$. □

Proof of Proposition 29

From the expressions for $Y_{agg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1},N,n,1\right)$ and $M\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1},N,n,1\right)$ we see that $\Pr\left[Y_{agg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1},N,n,1\right)\geq y_{agg}^{no}\right]=\Pr\left[M\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1},N,n,1\right)\geq 0\right]=1$ for every $n\in\{1,\ldots,N-1\}$ if and only if $\Pr\left[\widehat{F}_{avg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1},N,n\right)\geq 0\right]=1$. Now, $\Pr\left[\widehat{F}_{avg}\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1},N,n\right)\geq 0\right]=1$ for every $n\in\{1,\ldots,N-1\}$ if and only if $\mathbf{d}_w^-\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1}\right)\geq\mathbf{0}$. Since $\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})$ is an M-matrix with $1>r\left(\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)$, $\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})$ is inverse-positive which makes $\mathbf{d}_w^-\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1}\right)\geq\mathbf{0}$. □

Proof of Proposition 30

In general, the vector of average weighted in-degrees is

$$\left[\mathbf{d}_w^-\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1}\right)\right]^T=\frac{1}{N}\mathbf{1}^T\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1}.$$

Setting $\mathbf{\Lambda}=\mathbf{I}$, $\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})=\operatorname{diag}(\mathbf{1}-\boldsymbol{\beta})$ and $\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1}=\operatorname{diag}\begin{pmatrix}\frac{1}{1-\beta_1} & \cdots & \frac{1}{1-\beta_N}\end{pmatrix}$; the expression for $\mathbf{d}_w^-\left(\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1}\right)$ then follows. □

Proof of Proposition 31

We first demonstrate the following: If $\beta_1=\cdots=\beta_N\equiv\beta$, then both GDP and the corresponding economic multiplier are invariant to configuration. Suppose that $\beta_1=\cdots=\beta_N\equiv\beta$. Then $\operatorname{diag}(\boldsymbol{\beta})=\beta\mathbf{I}$. $\mathbf{1}^T\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)=\mathbf{1}^T\left(\mathbf{I}-\beta\mathbf{\Lambda}\right)=(1-\beta)\,\mathbf{1}^T$, so $\mathbf{1}^T\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)\left(\mathbf{I}-\mathbf{\Lambda}\operatorname{diag}(\boldsymbol{\beta})\right)^{-1}=$

$(1-\beta)\,\mathbf{1}^T\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1}$ and $\frac{1}{1-\beta}\mathbf{1}^T=\mathbf{1}^T\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1}$. Therefore, $\left[\mathbf{d}_w^-\left(\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1}\right)\right]^T=\frac{1}{N}\frac{1}{1-\beta}\mathbf{1}^T$ and $k=\frac{1}{1-\beta}$. $\widehat{F}_{avg}\left(\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n\right)=\frac{n}{N}\frac{1}{1-\beta}$ with probability 1, and we obtain the expressions for $Y_{agg}\left(\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n,0\right)$, $M\left(\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n,0\right)$, $Y_{agg}\left(\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n,1\right)$, and $M\left(\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n,1\right)$.

We next demonstrate the following: If both GDP and the corresponding economic multiplier are invariant to configuration, then $\beta_1=\cdots=\beta_N\equiv\beta$. Suppose that both GDP and the corresponding economic multiplier are invariant to configuration. If $Y_{agg}\left(\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n,0\right)$, $M\left(\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n,0\right)$, $Y_{agg}\left(\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n,1\right)$, and $M\left(\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n,1\right)$ are invariant to configuration, then $\widehat{F}_{avg}\left(\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1},N,n\right)$ is invariant to configuration. We must then have $\mathbf{d}_w^-\left(\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1}\right)=\frac{k}{N}\mathbf{1}$, or equivalently, $\mathbf{1}^T\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1}=k\mathbf{1}^T$. It follows that $\mathbf{1}^T\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)^{-1}\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)=k\mathbf{1}^T\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)$, so $\mathbf{1}^T\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)=\frac{1}{k}\mathbf{1}^T$. In general, $\mathbf{1}^T\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)=\begin{pmatrix}1-\beta_1 & \cdots & 1-\beta_N\end{pmatrix}$ by the column-stochasticity of $\boldsymbol{\Lambda}$. For $\mathbf{1}^T\left(\mathbf{I}-\boldsymbol{\Lambda}\,\mathrm{diag}\left(\boldsymbol{\beta}\right)\right)=\frac{1}{k}\mathbf{1}^T$, we must have $\beta_1=\cdots=\beta_N\equiv\beta$, and we then have $N\left(1-\beta\right)=Nk$, or equivalently, $k=\frac{1}{1-\beta}$. $\square$