Comprehensively Stress Testing the Economy in Closed Form

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Abstract

In response to the global financial crisis of 2008, the Federal Reserve decided to develop and implement stress tests to assess the soundness of the financial system. Each stress test involves crafting a potential real-world scenario and then quantifying the scenario’s effect on both financial actors in the economy and the financial system as a whole. There currently exist two weaknesses in the Federal Reserve’s stress testing approach. First, the number of stress tests faced by each financial institution is quite small, with many such stress test scenarios mimicking past historical events that are not necessarily reflective of future situations. Second, the Federal Reserve’s toolkit is not sufficiently macroprudential in nature, even though the financial crisis did cause many central banks to nominally transition from a microprudential regulatory approach to a macroprudential regulatory approach. In this work, we tackle these two issues. We show how to massively increase the number and types of possible stress tests without increasing the computational burden. To do this, we generate classes of stress tests with potentially very large cardinalities. For each class of stress tests, we then construct in closed form probability distributions that capture the range of possible balance sheet effects both for each individual financial institution and for the entire financial system. The approach that we take towards increasing the number of stress tests is fundamentally macroprudential. We moreover show how the topologies of the bipartite networks linking financial institutions to assets shape stress tests’ effects on the financial system.

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1 INTRODUCTION

The 2008 global financial crisis and the concomitant Great Recession consisted of an unprecedented series of events and adjustments to the macroeconomic and financial landscape. Within the domestic housing market, home prices fell approximately 30 percent from mid-2006 to mid-2009. In the United States, both investment bank Lehman Brothers and savings and loan institution Washington Mutual failed. Financial institutions Bear Stearns, Merrill Lynch, AIG, Freddie Mac, Fannie Mae, and Wachovia experienced some form of rescue or bailout. Globally, Northern Rock, HBOS, Royal Bank of Scotland, Bradford & Bingley, Fortis, Hypo Real Estate, and Alliance & Leicester likewise experienced some form of rescue, bailout, and/or nationalization. The unemployment rate in the United States increased from 5 percent in December 2007 to 10 percent in October 2009, real United States GDP contracted by 4.3 percent between the fourth quarter of 2007 and the second quarter of 2009, and the S&P 500 index fell 57 percent from its peak in October 2007 to its trough in March 2009. Net worth across United States households and nonprofit organizations fell from a peak of approximately 69 trillion dollars in 2007 to a trough of approximately 55 trillion dollars in 2009. To bring greater stability to the financial system, the United States federal government implemented the Trouble Asset Relief Program (TARP), in which it purchased troubled companies’ assets and equity. For TARP, Congress authorized the United States Treasury 475 billion dollars to make purchases; the Treasury principally used this money to stabilize banks, develop programs to increase credit availability, rescue the United States automobile industry, stabilize AIG, and buttress programs that prevent foreclosure. Globally, real GDP growth decreased from 5.6 percent in 2007 to 0.1 percent in 2008 and the financial crisis spurred a European sovereign debt crisis for the countries of Iceland, Portugal, Italy, Ireland, Greece, Spain, and Cyprus.

The global financial crisis led different governmental and supervisory authorities to massively reassess both their regulatory roles and existing financial regulation. The Basel Committee on Banking Supervision developed a set of recommendations for regulation, known as Basel III, in response to the global financial crisis. Basel III focused on strengthening regulation, supervision, and risk management of financial institutions to ensure financial stability. For example, Basel III sought to improve the quality of bank regulatory capital,

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increase the level of capital requirements, constrain excess leverage, and mitigate excess liquidity risk. The Basel Committee included representatives from central banks and regulatory authorities around the world, and in general, its members applied these standards in their own jurisdictions; indeed, the Federal Reserve announced that in December 2011 it would implement essentially all of the Basel III regulations. Within the United States, the Dodd-Frank Wall Street Reform and Consumer Protection Act, signed into federal law in July 2010, represented the federal government’s response to the global financial crisis. The Dodd-Frank Act made significant changes to financial regulation to improve financial stability and consumer protection. It promoted financial stability through creation of the Financial Stability Oversight Council and the Office of Financial Research, and it advocated for consumer protection in the financial industry through creation of the Consumer Financial Protection Bureau. The Dodd-Frank Act additionally introduced corporate governance reforms, executive compensation reforms, credit rating agency regulation, securitization retention requirements, procedures for regulatory enforcement, and regulation of over-the-counter derivatives, among other types of regulation.

Both Basel III and the Dodd-Frank Act led to major adjustments in financial regulation. They equipped regulators and supervisory institutions with new tools to assess the stability of individual financial institutions and the financial system as a whole, and they broadened these supervisory institutions’ mandates. For example, Basel III introduced a set of statistics for central banks to collect from the balance sheets of financial institutions. Basel III provided minimum and/or maximum allowable values for these statistics, which included capital ratios, and failure to meet these guidelines forced financial institutions to adjust their balance sheets. Given all of these new regulations and guidelines, we might wonder whether the resulting constrained financial system would now be able to survive and maintain normal operations after encountering the same economic conditions that had precipitated the 2008 global financial crisis. To answer this question, we would need to construct a scenario mimicking the start of the financial crisis and the Great Recession, and we would need to examine whether the financial system could withstand such stresses. This process of scenario design and stress testing is exactly what the Federal Reserve decided to do in response to the global financial crisis. A massive part of the Federal Reserve’s post-crisis regulatory toolkit involved designing stressful economic and financial scenarios and quantifying these scenarios’ effects on financial institutions’ capital holdings and balance sheets.

Following the financial crisis, in May 2009, the committee behind Basel III published guidelines for stress testing. Within the United States, the Dodd-Frank Act mandated that

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the Federal Reserve conduct annual supervisory stress tests for sufficiently large bank holding companies, which essentially includes large banks and other large financial institutions. The Federal Reserve decided to jointly implement these Dodd-Frank Act stress tests and a process of comprehensive capital analysis and review (CCAR); the former focused on studying financial institutions’ balance sheet items under stressed scenarios, while the latter focused on studying financial institutions’ capital adequacy under stressed scenarios, thus making the Federal Reserve’s annual supervisory stress tests and CCAR complementary. Not only is stress testing a core part of the supervisory toolkit, it is also a key tool for internal risk management by financial institutions. The Federal Reserve’s stress tests, in principal, can potentially capture broad risks, while individual financial institutions’ stress tests can capture the idiosyncratic risks that are germane for each institution.

Adhering to the requirements of the Dodd-Frank Act, the Federal Reserve undertakes the following stress testing process. It conducts annual supervisory stress tests for sufficiently large financial institutions. Each stress test involves crafting a different real-world scenario that perturbs certain economic quantities of interest. With the Federal Reserve required to carry out three stress tests, it therefore crafts three different scenarios: (1) a baseline scenario, (2) an adverse scenario, and (3) a severely adverse scenario. The Federal Reserve uses data provided by the financial institutions, and it examines how the three different stress scenarios impact each financial institutions’ balance sheets and capital holdings. There are many possible economic variables to perturb in constructing stress tests, so the Federal Reserve takes the following approach. It requires every sufficiently large financial institution to undertake a macroeconomic stress test. The baseline scenario, the adverse scenario, and the severely adverse scenario are all constructed from the perturbations of mainly macroeconomic variables, such as the unemployment rate or the growth rate of GDP. Financial institutions that have significant trading activity must also add a global market shock to their macroeconomic stress tests. The global market shock is an add-on component to the macroeconomic adverse scenario and the severely adverse macroeconomic scenario. For the global market shock, various market factors can adjust, such as equity prices, private equity values, and foreign exchange rates. Financial institutions with substantial trading or processing and custodian operations must additionally incorporate counterparty default into their stress tests. Counterparty default is an add-on component to each stress test. Each financial institution must estimate and report potential losses and effects on capital that would result if the institution’s largest counterparty unexpectedly defaulted. The Federal Reserve applies the same set of supervisory stress tests to each financial institution. In addition to these supervisory scenarios, every financial institution must carry out its own internal stress test. The results of the Federal Reserve’s supervisory stress tests and each financial institution’s
stress test are publicly disclosed.

The Federal Reserve’s stress testing approach has several limitations. The number of stress tests that get conducted annually is very small. Each year, the Federal Reserve only carries out three supervisory stress tests for every sufficiently large financial institution: a baseline scenario, an adverse scenario, and a severely adverse scenario. Each supervised financial institution must also execute one additional stress test. These four scenarios are not enough to ensure that the financial system has been comprehensively stress tested. Indeed, the Federal Reserve’s baseline scenario, adverse scenario, and severely adverse scenario sometimes involve perturbations of the same underlying economic variables. What then distinguishes these scenarios are the magnitudes by which these perturbed economic variables adjust; the severely adverse scenario, for example, might simply feature greater magnitudes of adjustment for the underlying economic variables than the adverse scenario. As a result, the supervisory scenarios do not necessarily capture different ways that a financial system can become stressed. Moreover, the stress test scenarios crafted by individual financial institutions are sometimes inspired by the Federal Reserve’s supervisory stress tests; consequently, these scenarios do not identify the idiosyncratic risks that can potentially destabilize individual financial institutions.

There exist additional weaknesses beyond the number of annual stress tests being very small. The supervisory stress test scenarios are often calibrated to past historical events. While that does serve as a reasonable benchmark, it is unlikely that history will exactly repeat itself. The economy may inevitably evolve along certain pathways that precipitate recessions and/or financial crises, but it is unlikely that these pathways will always be identical; the circumstances generating recessions and/or financial crises are not always the same. Financial institutions might be able to withstand stress test scenarios that mimic the onset of the 2008 global financial crisis, but that does not mean that the financial system is stable. The Federal Reserve may as a result be lulled into a false sense of financial system stability. Moreover, the Federal Reserve’s stress test scenarios have not substantially changed over the years. With financial institutions adjusting their operations and portfolios to satisfy the Federal Reserve’s stress tests, they may all become vulnerable to other realistic forms of risk. It is difficult to know what exactly should be the ideal set of stress test scenarios to ensure financial system stability. For instance, the Federal Reserve’s current approach to stress testing generally assumes that stress originates within the macroeconomy, hence its stress testing scenarios primarily being macroeconomic scenarios. It is, however, plausible that stresses within the financial system initiate financial crises, which would make market risk scenarios more relevant.

The current stress testing approach within the United States presents several legiti-
mate concerns. The present work devises a solution to address all of these concerns. We can tackle all of the stated weaknesses in the Federal Reserve’s current stress testing approach by massively increasing the number of distinct stress tests conducted annually. One might argue that drastically increasing the number of distinct stress test scenarios generates its own set of problems: for example, it can enormously increase the computational burden, and the process of generating additional scenarios can potentially be very haphazard. The present work addresses these critiques. It develops a systematic approach for scaling up the number of stress test scenarios, and it shows how to substantially increase the number of stress tests without increasing the computational burden. The procedure presented in this work for comprehensively stress testing the economy and the financial system can benefit the Federal Reserve, it can benefit individual financial institutions that carry out internal assessments of risk, and it can benefit central banks and supervisory institutions globally.

To illustrate the approach that the present work takes, let’s start off with an example. Imagine that there exists a stress test scenario in which the Indian rupee depreciates by 10 percent. We are interested in the effects of this depreciation on the balance sheets of individual financial institutions and the financial system as a whole. Given this particular stress test scenario, we can generate an entire class of stress tests. At the highest level, what we are interested in here is the effect of exchange rate risk on the financial system. We therefore develop a class of stress tests, and each stress test within this class is distinguished by the type of foreign currency facing a 10-percent depreciation; the first stress tests features a 10-percent depreciation of the Indian rupee, the second stress test features a 10-percent depreciation of the euro, the third stress test features a 10-percent depreciation of the Mexican peso, and so on. The number of stress tests within this class is then equal to the number of foreign currencies. Therefore, given our one initial stress test, we have generated an entire class of possible stress tests. For this class, we can construct in closed form a probability distribution that summarizes balance sheet effects for each individual financial institution, and we can construct a probability distribution that summarizes balance sheet effects for the entire financial system. Rather than having individual data points, we have entire probability distributions capturing exchange rate risk for the financial system.

Now, let’s instead imagine that there exists a stress test scenario in which the Indian rupee depreciates by 10 percent and the euro appreciates by 15 percent. Here, we have a different form of exchange rate risk. As in the previous example, we proceed to generate an entire class of stress test scenarios capturing this form of exchange rate risk. For every stress test within the class, we pick one currency to depreciate by 10 percent, and we pick a separate currency to appreciate by 15 percent. The class exhausts all possible combinations of appreciating and depreciating currencies, so that the total number of stress tests
within this class is combinatorial. Given this class of stress tests, there is a corresponding probability distribution summarizing balance sheet effects for each financial institution, and there is a corresponding probability distribution summarizing balance sheet effects for the overall financial system; we can solve for the major statistical features of these probability distributions in closed form.

The present work thus takes the following approach. It constructs different classes of stress tests. Each of these classes of stress tests is distinguished by its categories of risk. For example, one class of stress tests might feature certain levels of exchange rate risk and sectoral risk, while another class of stress tests might feature a particular magnitude increase in the probability of default for a type of debt. There are different ways that these types of risk can enter into the financial system; for example, exchange rate risk can separately manifest itself in assets denominated in different currencies. Each stress test scenario within a particular class therefore represents a different way that these types of risk manifest themselves. The set of stress test scenarios within a particular class is exhaustive; there are no additional ways that these types of stresses can be distributed within the financial system. For each class of stress tests, we construct a probability distribution that captures balance sheet effects for each individual financial institution, and we construct a probability distribution that captures balance sheet effects for the overall financial system. Through this approach, we massively increase the number of stress test scenarios in a systematic fashion without increasing the computational burden.

One last criticism of the Federal Reserve’s current stress testing approach is that it is not sufficiently macroprudential. Even though the global financial crisis exposed the weaknesses of regulatory approaches that are too microprudential, the Federal Reserve’s shift from a microprudential regulatory approach to one that is relatively more macroprudential has mostly been nominal. The present work takes steps towards making the Federal Reserve’s regulatory approach relatively more macroprudential. Macroprudential regulation is concerned with risks at the level of the financial system. The present work approaches the stress testing process by identifying categories of risk and then specifying the different ways that such risk can manifest itself within the financial system. This top-down perspective is fundamentally macroprudential. The present work moreover discerns how networks, a fundamentally macroprudential object, shape financial stability. One of the main networks studied in this work is a bipartite network that links financial institutions to assets. The present work shows, given a particular class of stress tests, how the topology of the bipartite network shapes stress tests’ effects on the financial system. More precisely, the topology of the bipartite network determines the shape of this corresponding probability distribution.
1.1 Relation to the Literature

The present work focuses on stress tests and how to massively improve this major part of the Federal Reserve’s supervisory toolkit. Hirtle and Lehnert (2015) provides background on stress testing in the United States and discusses the objectives of stress testing. Glasserman and Tangirala (2016), Demekas (2015), and Anderson (2016) also provide a history and overview of stress tests. Acharya et al. (2014) and Borio et al. (2014) discuss macroeconomic stress tests, in which macroeconomic factors adjust and cause the transmission of shocks to the financial system. A weakness of macroeconomic stress tests, as Bookstaber et al. (2014) discuss, is that they neglect scenarios in which shocks to the financial system themselves cause economic downturns. Consistent with this critique, the present work considers finance-specific stress tests scenarios that directly affect financial institutions’ balance sheets rather than scenarios primarily motivated by macroeconomic adjustments. Petrella and Resti (2013) and Scheurmann (2014) both weigh the costs and benefits of publicly disclosing the results of stress tests.

The literature has highlighted a couple of weaknesses regarding the Federal Reserve’s stress testing approach. First, as Glasserman and Tangirala (2016) mention, the Federal Reserve’s stress tests have not drastically changed over the years. Financial institutions have adjusted their balance sheets so that they can withstand the Federal Reserve’s stress tests, but in doing so, they may not be able to survive an actual stress scenario that differs from the ones implemented by the Federal Reserve. Second, the Federal Reserve carries out too few stress test scenarios; Bookstaber et al. (2014), Demekas (2015), and Glasserman and Tangirala (2016) all discuss the need to increase the number of stress test scenarios. Grundke (2011) addresses the issue of scenario selection by instead carrying out reverse stress tests; instead of deciding which scenarios are appropriate, the supervisory institution identifies a certain outcome or threshold of interest and then generates stress scenarios that would yield that particular outcome. Reverse stress tests have their own weaknesses. For reverse stress tests, the outcome or threshold must be very specific, and the number of stress test scenarios that can yield that particular outcome is potentially extremely large. It is often not feasible for the regulatory institution to entertain all of these possible stress scenarios. The present work instead focuses on massively increasing the number of stress test scenarios without increasing the computational burden. It generates classes of stress tests that each contain a very large number of individual stress tests. Associated with each class of stress tests is a corresponding probability distribution that summarizes the effects of those stress tests.

The present work interfaces with the literature on macroprudential regulation; it shows how to design stress tests so that they are more macroprudential in nature, rather than being strictly microprudential. Clement (2010) provides a history of the term “macro-
prudential,” and how it has been used over time. “Macroprudential” is traced back to 1979, in which it was mentioned at a meeting of the Cooke Committee, the predecessor of the Basel Committee on Banking Supervision. At the time, “macroprudential” meant “an enhanced focus on the financial system as a whole and its link to the macroeconomy.” Clark and Large (2011), Liebeg and Posch (2011), and Claessens (2015) all provide a modern overview of macroprudential regulation and its objectives. Macroprudential regulation is often concerned with risks at the level of the financial system; it is distinguished from microprudential regulation, which instead seeks to ensure the soundness of individual financial institutions one at a time. Hanson et al. (2011), Kashyap et al. (2011), and Borchgrevink et al. (2014) argue that macroprudential regulation arises out of a need to address market failures. Pecuniary externalities, such as fire sales of assets, interconnectedness externalities, and strategic complementarities all motivate macroprudential regulation because the standard microprudential toolkit does not address these market failures. Acharya (2009) discusses the role of capital requirements in a macroprudential framework. Borio (2003), Greenlaw et al. (2012), and Williams (2015) acknowledge that stress tests and current regulatory frameworks are still fairly microprudential in nature. Williams (2015) argues that microprudential regulations and supervision are unfortunately being used to attain macroprudential objectives due to the scarcity of explicitly macroprudential tools. The present work shows how to substantially enhance the Federal Reserve’s existing stress testing approach to make it much more macroprudential. The present work introduces classes of stress tests, which are distinguished by their types of risks. Each individual stress test within a particular class is then distinguished by the specific ways that these risks manifest themselves within the financial system, whether these risks appear in certain assets or financial institutions. Studying the different ways that risk can be distributed within the financial system, and quantifying the corresponding effects on the financial system is a fundamentally macroprudential perspective. Clement (2010) offers this perspective when discussing the history of the term “macroprudential.”

Financial networks and their topologies form an important part of the present work. The present work studies bipartite networks that link individual financial institutions to individual assets; edges are directed from the financial institutions to the assets in their portfolios, with the weight of each edge equal to the number of units of the asset held by the financial institution. Institutions’ overlapping portfolios here generate risk. As a result of the 2008 global financial crisis, there is quite a large literature on financial networks. Caccioli et al. (2014) and Levy-Carciente et al. (2015) study bipartite networks linking financial institutions to assets, and Marotta et al. (2015) studies bipartite credit networks linking banks to firms. Gualdi et al. (2016) examines portfolio overlap among financial
institutions as a channel for financial contagion. Now, a large part of the financial networks literature is focused on counterparty networks and financial contagion. Allen and Babus (2009), Gai and Kapadia (2010), Battiston et al. (2012), Elliott et al. (2014), and Acemoglu et al. (2015) all study how the actual structure of the counterparty network shapes systemic risk. Afonso et al. (2011) studies the impact of the global financial crisis on edge weights in a counterparty network of financial institutions. Cont et al. (2013) tries to determine the systemic importance of each financial institution in an interbank network. Zawadowski (2013) focuses on a counterparty network with bilateral over-the-counter contracts, and Markose et al. (2012) focuses on the network of credit default swaps within the United States at the time of the financial crisis. For Farboodi (2014), the structure of the financial network is determined endogenously as financial institutions make strategic borrowing and lending decisions. The present work studies how the topologies of financial networks shape stress tests' effects on the financial system. The present work carries out this analysis for entire classes of stress tests, not just individual stress tests.

In addition to interfacing with the overlapping literatures on stress testing, macroprudential regulation, and financial networks, the present work contributes to a new literature on networks and probability distributions in the economy. Both Schlossberger (2018b) and Schlossberger (2018a) provide a foundation for this literature. Schlossberger (2018b) develops a set of theoretical tools for mapping the topology of an economic network to a probability distribution of possible outcomes. Schlossberger (2018b) adapts these tools to study locally formed macroeconomic sentiment and how agents’ interaction structure shapes the capacity for there to exist non-fundamental swings in aggregate macroeconomic sentiment; Schlossberger (2018b) thereby enhances our understanding of animal spirits. Schlossberger (2018a) extends the set of theoretical tools from Schlossberger (2018b) so that they have broader applicability. As in Schlossberger (2018b), Schlossberger (2018a) also maps naturally occurring networks in the economy to different probability distributions of interest. Schlossberger (2018a), in particular, focuses on the effects that a given policy has on a population’s aggregate action when agents are networked and the actions that these agents take are interdependent. Schlossberger (2018a) therefore explicitly shows, for any given policy targeting a certain number of networked agents, how the topology of agents’ interaction network shapes the corresponding distribution of possible aggregate actions and the corresponding distribution of possible economic multipliers. The present work makes additional methodological and technical advances relative to Schlossberger (2018b) and Schlossberger (2018a); it further expands the set of tools for mapping networks to probability distributions. The present work studies how the topologies of bipartite networks linking financial institutions to assets shape stress tests’ effects on the financial system. Given a particular class of stress tests,
the present work shows how to map the topology of the bipartite network to a probability distribution capturing balance sheet effects.

1.2 Outline of Paper

We begin Section 2 by considering stress tests that directly shock the portfolios of financial institutions. While we are indeed interested in each stress test’s effects on the balance sheets of individual financial institutions and the financial system as a whole, we would like to drastically increase the number of such tests to more rigorously stress the financial system. In this section, we therefore show how to drastically increase the total number of stress tests without increasing the computational burden. We generate classes of stress tests, and for each class, we construct a probability distribution capturing possible balance sheet effects for each individual financial institution, and we construct a probability distribution capturing possible aggregate balance sheet effects for the entire financial system. In the United States, these shocks to financial institutions’ portfolios generally constitute the global market shock component of the Federal Reserve’s stress tests, and they can also constitute the macroeconomic component. We conclude in Section 3.

2 Classes of Stress Tests and Probability Distributions of Balance Sheet Effects

The Federal Reserve’s stress tests shock the portfolios of financial institutions through multiple conduits. Each stress test includes multiple categories of risk, including sovereign risk, exchange rate risk, and industry risk. We can take these categories of risk as key parts of economic and financial downturns, and we can examine all of the different ways that these categories of risk manifest themselves within the broader economy and financial system.

These different types of risk affect the values of assets; for instance, they change the prices of assets and they alter income streams. We are interested in stress tests’ effects on balance sheet items for individual financial institutions and the financial system as a whole, that is, the collection of financial institutions that comprise the financial system. In this section, we focus on stress tests’ effects on the market value of institutions’ net assets. Specifically, the market value of a financial institution’s net assets is equal to the market value of its assets minus the market value of its liabilities. Assets and liabilities for each financial institution are marked to market. A separate balance sheet item that we could instead look at is net income before taxes. Net income takes into account unrealized and realized gains and losses for securities, so net assets is a reasonable balance sheet item to
There are $M$ financial institutions indexed $1, \ldots, M$ and $N$ total securities indexed $1, \ldots, N$. Each security in a financial institution’s portfolio is either an asset or a liability. Define the $M \times N$ matrix $A$. The $ij^{th}$ element of $A$ is equal to the number of units of security $j$ held in the portfolio of financial institution $i$. Each row of $A$ represents a financial institution’s portfolio. The elements of $A$ can be positive, zero, or negative. $[A]_{ij} = 0$ means that institution $i$ does not have security $j$ in its portfolio, $[A]_{ij} > 0$ means that security $j$ is one of institution $i$’s assets, and $[A]_{ij} < 0$ means that security $j$ is one of institution $i$’s liabilities. The extent to which the rows of $A$ are similar determines the extent to which the portfolios of different financial institutions are overlapping. It determines the extent to which the financial system exhibits systematic risk. Define $p$ as the $N \times 1$ vector of securities prices. The market value of institution $i$’s net assets is $[A]_{i \cdot} p$, where $[A]_{i \cdot}$ is the $i^{th}$ row of $A$. The market value of net assets for the entire financial system is $1^T A p$.

In the background, we have this bipartite network linking financial institutions to securities. The weight of each directed edge is equal to the number of units of a particular security in the corresponding financial institution’s portfolio. We are interested in how the topology of this bipartite network shapes stress tests’ effects on individual financial institutions and the overall financial system.

We want to know how the stress test changes each individual institution’s net assets and the net assets for the financial system. In the Federal Reserve’s current stress testing approach, a stress test features shocks that cause certain securities’ prices to change; we then quantify the overall effect. In the present work, we would like to massively increase the number of annual stress tests being conducted without increasing the computational burden. Therefore, rather than identifying individual stress tests, we identify classes of stress tests. Each class of stress tests is distinguished by its categories of risk. Different classes of stress tests have different categories of risk. Categories of risk, for example, include a certain level of exchange rate risk, a certain level of sovereign debt risk, and a certain level of industry risk. There are different ways that each category of risk can manifest itself; for instance, there are many possible currencies in which exchange rate risk can appear. A given class of stress tests therefore contains many different individual stress tests. These individual stress tests account for all of the possible ways that the different categories of risk can manifest themselves.

For one stress test, we have one data point that captures the resulting level of net assets for an individual financial institution, and we have one data point that captures the resulting level of net assets for the whole financial system. In the present work, for one class of stress tests, we instead have an entire probability distribution that captures the possible
resulting levels of net assets for an individual financial institution, and we instead have an entire probability distribution that captures the possible resulting levels of net assets for the whole financial system. We move from data points to probability distributions.

In order to compute how a class of stress tests affects individual institutions’ net assets and aggregate net assets, we take the following approach. A class of stress tests features one or more categories of risk. We separately consider each category of risk. For each category of risk, we must select the relevant set of securities from the full set of securities and group together securities as needed. From the original matrix \( A \), we generate a new matrix \( \bar{A} \), and from the original price vector \( p \), we generate the new vector \( \bar{p} \). The next three examples illustrate how to generate \( \bar{A} \) and \( \bar{p} \) respectively from \( A \) and \( p \). For each individual financial institution, we then introduce a vector \( w_i \) that we use to compute the effects of a category of risk on the level of net assets for institution \( i \). For the entire financial system, we introduce the vector \( w_{agg} \) so that we can compute the effects of a category of risk on the level of aggregate net assets.

**Example 1** The category of risk is as follows: Forty percent of all AAA-rated mortgage-backed securities have experienced a decline in quality and therefore a specific reduction in price. Given this category of risk, there are many possible stress scenarios. Construct each stress scenario so that it features a different subset of AAA mortgage-backed securities experiencing a decline in quality. How do we construct \( \bar{A} \), \( \bar{p} \), \( w_i \) for all \( i \in \{1, \ldots, M\} \), and \( w_{agg} \)?

We start with the \( M \times N \) matrix \( A \), and we identify the indices \( j \in \{1, \ldots, N\} \) for AAA-rated mortgage-backed securities. There are \( L \) such securities. We then construct the \( M \times L \) matrix \( \bar{A} \) by extracting the relevant columns of AAA-rated mortgage-backed security portfolio holdings from the matrix \( A \). Meanwhile, we construct the \( L \times 1 \) price vector \( \bar{p} \), whose elements are the prices of AAA-rated mortgage-backed securities. Define \( \epsilon \) as the \( L \times 1 \) vector that captures potential changes to the prices of these mortgage-backed securities. The vector \( \epsilon \) represents a specific manifestation of the mortgage-backed security category of risk. It identifies the specific indices of those securities experiencing a price shock. Set \( w_i = [\bar{A}_{is}]^T \) and set \( w_{agg} = \bar{A}^T1 \). The value of the AAA-rated mortgage-backed security portfolio for financial institution \( i \) following the \( \epsilon \)-shock is: \( w_i^T(\bar{p} + \epsilon) \). The value of the AAA-rated mortgage-backed security portfolio for the entire financial system following the \( \epsilon \)-shock is: \( w_{agg}^T(\bar{p} + \epsilon) \).

**Example 2** The category of risk is as follows: Two industries receive negative shocks of particular magnitudes, while one separate industry receives a positive shock of a particular magnitude. Given this category of risk, stress scenarios are distinguished by the two types of
industries receiving negative shocks and the one type of industry receiving the positive shock. How do we construct $\bar{A}$, $\bar{p}$, $w_i$ for all $i \in \{1, \ldots, M\}$, and $w_{agg}$?

Suppose that there are $L$ total industries. We therefore construct an $M \times L$ matrix $\bar{A}$. Element $ij$ of matrix $\bar{A}$ is equal to the market value of net assets in industry $j$ for financial institution $i$. Specifically, define $K$ as the set of indices for securities in industry $j$. Then, $[\bar{A}]_{ij} = \sum_{k \in K} [A]_{ik} [p]_k$. $[\bar{A}]_{ij} = 0$ if institution $i$ does not hold securities from industry $j$. We introduce the $L \times 1$ vector $\epsilon$ that captures adjustments in value to securities across industries. It represents a specific way that industry risk can manifest itself, that is, it identifies the two industries receiving a negative shock and it identifies the one industry receiving a positive shock. For each institution $i \in \{1, \ldots, M\}$, we set $w_i = 1_{L \times 1}$ and $\bar{p} = [\bar{A}]^T i \cdot$. The value of net assets for financial institution $i$ following the industry shocks is: $w_i^T (\bar{p} + \epsilon)$. For the entire financial system, we set $w_{agg} = 1_{L \times 1}$ and $\bar{p} = [\bar{A}]^T 1$. The value of net assets for the entire financial system following the industry shocks is: $w_{agg}^T (\bar{p} + \epsilon)$.

**Example 3** The category of risk is as follows: One foreign currency massively depreciates relative to the U.S. dollar. Stress scenarios within this particular category of risk are distinguished by which currency is the one that is actually depreciating. How do we construct $\bar{A}$, $\bar{p}$, $w_i$ for all $i \in \{1, \ldots, M\}$, and $w_{agg}$?

Let’s assume that each financial institution prices its net assets in U.S. dollars. Each financial institution potentially holds securities that are originally denominated in foreign currencies. There are $L$ such foreign currencies. We therefore construct an $M \times L$ matrix $\bar{A}$. Element $ij$ of matrix $\bar{A}$ is equal to the dollar value of net assets for financial institution $i$ that are denominated in currency $j$. Specifically, define $K$ as the set of indices for securities that are denominated in currency $j$. Then, $[\bar{A}]_{ij} = \sum_{k \in K} [A]_{ik} [p]_k$. $[\bar{A}]_{ij} = 0$ if institution $i$ does not hold securities denominated in foreign currency $j$. We introduce the $L \times 1$ vector $\epsilon$ to capture exchange rate risk; it identifies the specific foreign currency facing depreciation and the size of the shock. For each financial institution $i \in \{1, \ldots, M\}$, we set $w_i = 1_{L \times 1}$ and $\bar{p} = [\bar{A}]^T i \cdot$. The value of net assets for financial institution $i$ following the exchange rate shock is: $w_i^T (\bar{p} + \epsilon)$. For the entire financial system, we set $w_{agg} = 1_{L \times 1}$ and $\bar{p} = [\bar{A}]^T 1$. The value of net assets for the entire financial system following the exchange rate shock is: $w_{agg}^T (\bar{p} + \epsilon)$.

Given a particular category of risk, as explored in the above three examples, we are interested in the distribution of possible effects on net assets for individual financial institutions and the overall financial system. Each category of risk affects securities prices differently. From our three examples, we see that the change in the market value of net assets for financial institution $i$ is $w_i^T \epsilon$, and the change in the market value of aggregate net assets
is $\mathbf{w}_{agg}^T \mathbf{e}$. For any given category of risk, there are many possible vectors $\mathbf{e}$; each individual vector $\mathbf{e}$ represents a different way that a category of risk manifests itself in securities prices. Also, note from the three examples that the dimensions of vectors $\mathbf{w}_i$ for all $i \in \{1, \ldots, M\}$, $\mathbf{w}_{agg}$, and $\mathbf{e}$ are all $L \times 1$. In certain settings, $L$ represents the total number of securities whose prices can potentially be affected by a particular category of risk. In other settings, to quantify how a certain category of risk affects the market value of net assets, we must group securities into different clusters. $L$ is then the total number of such clusters.

We are now going to explore four different environments. Each environment represents a different way by which risk can affect securities prices. Any given category of risk maps to one of these four environments.

### 2.1 First Risk Environment: Absolute Price Shocks, Same Across Securities Clusters

We now describe and analyze the first way that a category of risk can change the market value of net assets. We assume that $\ell \in \{1, \ldots, L\}$ clusters of securities experience a shock $\delta$ to their overall value. If each cluster only contains one security, then $\ell \in \{1, \ldots, L\}$ securities are individually experiencing a price shock $\delta$. For example, let’s suppose that the first $\ell$ clusters of securities are experiencing a shock $\delta$. Then, the change in net assets for institution $i$ is $\mathbf{w}_i^T \mathbf{e}$, and the aggregate change in net assets is $\mathbf{w}_{agg}^T \mathbf{e}$, where $[\mathbf{e}]_j = \delta$ for $j \in \{1, \ldots, L\}$ and $[\mathbf{e}]_j = 0$ for $j \in \{\ell + 1, \ldots, L\}$. $\delta < 0$ represents a negative shock to price or value. The vector $\mathbf{e}$ is capturing just one way that the $\delta$-shocks can manifest themselves. More precisely, $\mathbf{e} \equiv \mathbf{e} (L, \ell)$, which specifies that $\ell$ clusters are receiving a $\delta$-shock, while $L - \ell$ clusters are receiving a shock of zero. There is an entire set, $E (L, \ell)$, of vectors $\mathbf{e} (L, \ell)$ transmitting shocks to securities. $\mathbf{e} (L, \ell) \in E (L, \ell)$ and $\mathbf{e}' (L, \ell) \in E (L, \ell)$ are distinguished by the indices of the clusters that they target with a $\delta$-shock. They represent different configurations, relatable by permutation. Indeed, the cardinality of $E (L, \ell)$ is $L$. We assume that each configuration of shocks is equally likely.

$E (L, \ell)$ represents the set of all possible stress scenarios for a given category of risk. Associated with this category of risk is a probability distribution capturing possible changes in the market value of net assets for a specific financial institution. In addition, associated with this category of risk is a probability distribution capturing possible changes in the market value of net assets for the entire financial system. To construct these probability distributions and solve for their statistical features, we must introduce additional notation.

We define $\pi_i (\mathbf{w}_i, \mathbf{e}, L, \ell) = \mathbf{w}_i^T \mathbf{e} (L, \ell)$ as the change in net assets for institution $i$. We define $\pi_{agg} (\mathbf{w}_{agg}, \mathbf{e}, L, \ell) = \mathbf{w}_{agg}^T \mathbf{e} (L, \ell)$ as the change in net assets for the entire fi-
financial system. Random variable $\Pi_i (w_i, L, \ell)$ has realization $\pi_i (w_i, e_i, L, \ell)$, and random variable $\Pi_{agg} (w_{agg}, L, \ell)$ has realization $\pi_{agg} (w_{agg}, e_i, L, \ell)$. We assume that each stress scenario within a particular category of risk is equally likely. Corresponding to random variable $\Pi_i (w_i, L, \ell)$ is the CDF

$$G_{\Pi_i(w_i,L,\ell)} (t) = |E (L, \ell)| \sum_{(e_i, L, \ell) \in E(L,\ell)} 1_{\pi_i(w_i,e_i,L,\ell) \leq t}$$

with PMF $g_{\Pi_i(w_i,L,\ell)} (t)$, and corresponding to random variable $\Pi_{agg} (w_{agg}, L, \ell)$ is CDF

$$G_{\Pi_{agg}(w_{agg},L,\ell)} (t) = |E (L, \ell)| \sum_{(e_i, L, \ell) \in E(L,\ell)} 1_{\pi_{agg}(w_{agg},e_i,L,\ell) \leq t}$$

with PMF $g_{\Pi_{agg}(w_{agg},L,\ell)} (t)$. We define random variable $W_i$ with realization $[w_i]_j$, and we define random variable $W_{agg}$ with realization $[w_{agg}]_j$. We assume that each realization is equally likely. The elements of $w_{agg}$ and $w_i$ are not themselves random; we introduce random variables $W_{agg}$ and $W_i$ to make certain mathematical expressions more compact. We also set $1^T w_i = k_i$, and we set $1^T w_{agg} = k_{agg}$.

We can now solve for the features of $\Pi_i (w_i, L, \ell)$ and $\Pi_{agg} (w_{agg}, L, \ell)$. The next proposition characterizes their first moments:

**Proposition 1** The average change in net assets for financial institution $i$ is:

$$E \Pi_i (w_i, L, \ell) = \frac{k_i \ell}{L} \delta,$$

and the average change in net assets for the financial system is:

$$E \Pi_{agg} (w_{agg}, L, \ell) = \frac{k_{agg} \ell}{L} \delta.$$

The variances of the distributions capturing possible changes in net assets are as follows:

**Proposition 2** The change in net assets for financial institution $i$ has a variance of:

$$\text{Var} \Pi_i (w_i, L, \ell) = \delta^2 \ell \left(1 - \frac{\ell}{L}\right) \frac{L}{L-1} L \text{Var} W_i,$$

and the change in net assets for the entire financial system has a variance of:

$$\text{Var} \Pi_{agg} (w_{agg}, L, \ell) = \delta^2 \ell \left(1 - \frac{\ell}{L}\right) \frac{L}{L-1} L \text{Var} W_{agg}.$$
Specifically, \( \text{Var} W_i = \frac{1}{L} \sum_{j=1}^L \left( [w_i]_j - \frac{k_i}{L} \right)^2 \) and \( \text{Var} W_{agg} = \frac{1}{L} \sum_{j=1}^L \left( [w_{agg}]_j - \frac{k_{agg}}{L} \right)^2 \) are population variances.

We can additionally compute the lower and upper bounds on the supports of these distributions. The lower and upper bounds represent the range of possible changes in the market value of net assets for individual financial institutions and the overall financial system:

**Proposition 3** Construct the ordered multiset \( \{ \tilde{w}_j \}_{j=1}^L \) from the elements of \( w_i \) so that \( \tilde{w}_j \leq \tilde{w}_j' \) whenever \( j \leq j' \). When \( \delta < 0 \), the lower and upper bounds on the distribution of possible changes to net assets for institution \( i \) are:

\[
\min \text{ supp} \Pi_i (w_i, L, \ell) = \delta \sum_{j=1}^{L-\ell+1} \tilde{w}_j \quad \text{and} \quad \max \text{ supp} \Pi_i (w_i, L, \ell) = \delta \sum_{j=1}^\ell \tilde{w}_j.
\]

Now construct the ordered multiset \( \{ \tilde{x}_j \}_{j=1}^L \) from the elements of \( w_{agg} \) so that \( \tilde{x}_j \leq \tilde{x}_j' \) whenever \( j \leq j' \). When \( \delta < 0 \), the lower and upper bounds on the distribution of possible changes to net assets for the financial system are:

\[
\min \text{ supp} \Pi_{agg} (w_{agg}, L, \ell) = \delta \sum_{j=1}^{L-\ell+1} \tilde{x}_j \quad \text{and} \quad \max \text{ supp} \Pi_{agg} (w_{agg}, L, \ell) = \delta \sum_{j=1}^\ell \tilde{x}_j.
\]

We also want to construct asymptotic expansions that approximate the CDF for the distribution of changes in net assets for financial institution \( i \in \{1, \ldots, M\} \) and the overall financial system. In particular, we are interested in approximating \( G_{\Pi_i (w_i, L, \ell)} (t) \) for all \( i \in \{1, \ldots, M\} \) and \( G_{\Pi_{agg} (w_{agg}, L, \ell)} (t) \). We first introduce the function \( J (\tilde{w}, L, \ell, t) \):

\[
J (\tilde{w}, L, \ell, t) = \Phi (t) - H_2 (t) \phi (t) C_1 \sum_{j=1}^L \tilde{w}_j^3 - H_3 (t) \phi (t) \left[ C_2 \left( \sum_{j=1}^L \tilde{w}_j^3 - \frac{3}{L} \right) - \frac{1}{4L} \right] - H_5 (t) \phi (t) C_3 \left( \sum_{j=1}^L \tilde{w}_j^3 \right)^2,
\]

where \( C_1 = \frac{1-\frac{t^2}{6(\frac{t}{L})^2(1-\frac{t}{L})}}{6(\frac{t}{L})^2(1-\frac{t}{L})} \), \( C_2 = \frac{1-6(\frac{t}{L}) (1-\frac{t}{L})}{24(\frac{t}{L})^2 (1-\frac{t}{L})} \), \( C_3 = \frac{(1-\frac{t}{L})^2}{72(\frac{t}{L})^2 (1-\frac{t}{L})} \), \( \phi (t) = \Phi' (t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \), and \( H_j (t) \phi (t) = (-1)^j \frac{d^j}{dt^j} \phi (t) \). When we are interested in approximating \( G_{\Pi_i (w_i, L, \ell)} (t) \), we set \( \tilde{w}_j = \frac{[w_i]_j - EW_i}{\sqrt{L \text{Var} W_i}} \). When we are instead interested in approximating \( G_{\Pi_{agg} (w_{agg}, L, \ell)} (t) \), we set \( \tilde{w}_j = \frac{[w_{agg}]_j - EW_{agg}}{\sqrt{L \text{Var} W_{agg}}} \).
Proposition 4 Provided that condition (c) holds, for all \( i \in \{1, \ldots, M\} \),

\[
\left| \frac{G_{\Pi_i(w_i, L, \ell)}(t) - J(\hat{w}, L, \ell, t)}{(\text{Var} \Pi_i(w_i, L, \ell))^{1/2}} \right| < C_4 \times \sum_{j=1}^{L} |\hat{w}_j|^5
\]

with \( \hat{w}_j = \frac{[w_i]_j - EW_i}{\sqrt{L \text{Var} W_i}} \), and

\[
\left| \frac{G_{\Pi_{agg}(w_{agg}, L, \ell)}(t) - J(\hat{w}_{agg}, L, \ell, t)}{(\text{Var} \Pi_{agg}(w_{agg}, L, \ell))^{1/2}} \right| < C_4 \times \sum_{j=1}^{L} |\hat{w}_{agg,j}|^5
\]

with \( \hat{w}_{agg,j} = \frac{[w_{agg}]_j - EW_{agg}}{\sqrt{L \text{Var} W_{agg}}} \) for all \( t \), where \( C_4 \) is only a function of \( \ell \).

Condition (c) (Robinson (1978)) Given \( C' > 0 \), there exist \( \eta > 0 \), \( C > 0 \), and \( \kappa > 0 \) not depending on \( L \) such that, for any fixed \( t \), the number of indices \( j \), for which \( |\hat{w}_j x - t - 2\hat{r}\pi| > \eta \), for all \( x \in \left(C' [\max_i |\hat{w}_i|]^{-1}, C \left[ \sum_{i=1}^{L} |\hat{w}_i|^5 \right]^{-1} \right) \) and all \( \hat{r} = 0, \pm 1, \pm 2, \ldots \), is greater than \( \kappa L \), for all \( L \).

Condition (c) requires that the elements of \( \hat{w} \) not be clustered over too few values. Accordingly, condition (c) requires that the elements of \( w_i \) for each \( i \in \{1, \ldots, M\} \) and \( w_{agg} \) not be clustered over too few values. Given Proposition 4, we have that

\[
G_{\Pi_i(w_i, L, \ell)}(t) \approx J\left(\hat{w}, L, \ell, \frac{t - E\Pi_i(w_i, L, \ell)}{(\text{Var} \Pi_i(w_i, L, \ell))^{1/2}}\right) \quad \forall i \in \{1, \ldots, M\}
\]

with \( \hat{w}_j = \frac{[w_i]_j - EW_i}{\sqrt{L \text{Var} W_i}} \), and

\[
G_{\Pi_{agg}(w_{agg}, L, \ell)}(t) \approx J\left(\hat{w}_{agg}, L, \ell, \frac{t - E\Pi_{agg}(w_{agg}, L, \ell)}{(\text{Var} \Pi_{agg}(w_{agg}, L, \ell))^{1/2}}\right)
\]

with \( \hat{w}_{agg,j} = \frac{[w_{agg}]_j - EW_{agg}}{\sqrt{L \text{Var} W_{agg}}} \). For each individual financial institution, note that \( \sum_{j=1}^{L} \hat{w}_j^3 = L^{-1/2} \text{Skew} W_i \) and \( \sum_{j=1}^{L} \hat{w}_j^4 - \frac{3}{4} = L^{-1} \times (\text{Excess Kurtosis} W_i) \). For the overall financial system, note that \( \sum_{j=1}^{L} \hat{w}_{agg,j}^3 = L^{-1/2} \text{Skew} W_{agg} \) and \( \sum_{j=1}^{L} \hat{w}_{agg,j}^4 - \frac{3}{4} = L^{-1} \times (\text{Excess Kurtosis} W_{agg}) \). We can therefore approximate the CDFs of these distributions in terms of higher-order population moments.

When \( \ell = 1 \), we can solve for \( G_{\Pi_i(w_i, L, \ell)}(t) \), \( \forall i \in \{1, \ldots, M\} \), and \( G_{\Pi_{agg}(w_{agg}, L, \ell)}(t) \)
exactly. Observe that when $\ell = 1$, $\Pi_i(w_i, L, \ell) = \delta W_i$ and $\Pi_{agg}(w_{agg}, L, \ell) = \delta W_{agg}$, so

$$G_{\Pi_i(w_i, L, \ell)}(t) = \Pr[\Pi_i(w_i, L, \ell) \leq t] = \Pr[\delta W_i \leq t] = G_{W_i}\left(\frac{t}{\delta}\right) \forall i \in \{1, \ldots, M\},$$

and

$$G_{\Pi_{agg}(w_{agg}, L, \ell)}(t) = \Pr[\Pi_{agg}(w_{agg}, L, \ell) \leq t] = \Pr[\delta W_{agg} \leq t] = G_{W_{agg}}\left(\frac{t}{\delta}\right).$$

Example 4 Suppose that the category of risk is sovereign risk. Specifically, one country has a writedown of its sovereign debt, which causes the price of each sovereign bond for that country to decrease $y$ dollars. We are interested in the possible changes in net assets for each individual financial institution $i$, and we are interested in the possible changes in net assets for the overall financial system.

To compute the statistical features of $\Pi_i(w_i, L, \ell), \forall i \in \{1, \ldots, M\}$, and $\Pi_{agg}(w_{agg}, L, \ell)$, and to construct the probability distributions $G_{\Pi_i(w_i, L, \ell)}(t), \forall i \in \{1, \ldots, M\}$, and $G_{\Pi_{agg}(w_{agg}, L, \ell)}(t)$, we solve for all of the necessary variables. There are $L$ total countries that have issued sovereign bonds. These countries are indexed by $j \in \{1, \ldots, L\}$. $[w_i]_j$ is the number of sovereign bonds from country $j$ held by financial institution $i$. $[w_{agg}]_j$ is the number of sovereign bonds from country $j$ held by all $M$ financial institutions in the financial system. $[\bar{p}]_j$ is the dollar price of each sovereign bond from country $j$ prior to a potential writedown of debt. One country is writing down its debt, so $\ell = 1$. The price shock is $\delta = -y$. The vector $\epsilon(L, \ell)$ identifies the one country writing down its debt. $[\epsilon(L, \ell)]_j = -y$ if country $j$ is writing down its sovereign debt, and otherwise $[\epsilon(L, \ell)]_j = 0$. The set of all possible scenarios is $E(L, \ell)$, with $|E(L, \ell)| = \binom{L}{\ell} = L$. For each possible stress scenario, the change in net assets for institution $i$ is $\pi_i(w_i, \epsilon, L, \ell) = w_i^T \epsilon(L, \ell)$, and the change in net assets for the entire financial system is $\pi_{agg}(w_{agg}, \epsilon, L, \ell) = w_{agg}^T \epsilon(L, \ell)$. Since $\ell = 1$, $G_{\Pi_i(w_i, L, \ell)}(t) = G_{w_i}\left(-\frac{t}{y}\right)$ exactly, $\forall i \in \{1, \ldots, M\}$, and $G_{\Pi_{agg}(w_{agg}, L, \ell)}(t) = G_{W_{agg}}\left(-\frac{t}{y}\right)$ exactly.

## 2.2 Second Risk Environment: Percentage Price Shocks, Same Across Securities Clusters

In this second environment, a category of risk affects securities prices by triggering a percentage adjustment. We denote $\hat{\delta}$ as that percentage adjustment to securities prices; $\hat{\delta} < 0$ means that there is a negative shock to securities prices. We have $L$ total clusters of securities. If each cluster only contains one security, $\hat{\delta}$ represents the percentage adjustment to the prices of those securities. Meanwhile, if a cluster contains more than one security,
\(\delta\) represents the percentage adjustment to the market value of those securities. We simply adjust the price of each individual security in the cluster by the factor \(\delta\), and that is equivalent to adjusting the value of all securities in the cluster by the factor \(\delta\).

The setup in this environment is as follows: Given \(L\) total clusters of securities, \(\ell \in \{1, \ldots, L\}\) clusters experience a \(\delta\)-shock, which leads to a percentage adjustment to the market values of securities in those clusters. Let’s suppose that the first \(\ell\) clusters are experiencing the \(\delta\)-shock, while clusters \(\ell + 1, \ldots, L\) are not experiencing any shock. The change in the market value of net assets for financial institution \(i\) is \(w_i^T \epsilon(L, \ell)\), for all \(i \in \{1, \ldots, M\}\), where \([\epsilon(L, \ell)]_j = \delta\bar{p}_j\) for \(j \in \{1, \ldots, \ell\}\), and \([\epsilon(L, \ell)]_j = 0\) for \(j \in \{\ell + 1, \ldots, L\}\). The change in the market value of net assets for the entire financial system is \(w_{agg}^T \epsilon(L, \ell)\), where \([\epsilon(L, \ell)]_j = \delta\bar{p}_j\) for \(j \in \{1, \ldots, \ell\}\), and \([\epsilon(L, \ell)]_j = 0\) for \(j \in \{\ell + 1, \ldots, L\}\). \(\epsilon(L, \ell) \in E(L, \ell)\) represents one possible stress scenario given that we have this one category of risk. The number of ways that this category of risk can manifest itself within the financial system is combinatorial: \(|E(L, \ell)| = \binom{L}{\ell}\).

We use the same set of notation as in the previous environment. We define \(\pi_i (w_i, \epsilon, L, \ell) = w_i^T \epsilon(L, \ell)\) as the change in net assets for institution \(i \in \{1, \ldots, M\}\) given stress scenario \(\epsilon(L, \ell) \in E(L, \ell)\), and we define \(\pi_{agg} (w_{agg}, \epsilon, L, \ell) = w_{agg}^T \epsilon(L, \ell)\) as the change in net assets for the entire financial system given stress scenario \(\epsilon(L, \ell) \in E(L, \ell)\). In this environment, vector \(\epsilon(L, \ell)\) takes the following form: \([\epsilon(L, \ell)]_j = \delta\bar{p}_j\) if cluster \(j\) is being stressed, while \([\epsilon(L, \ell)]_j = 0\) if cluster \(j\) is not being stressed. A total of \(\ell\) clusters are being stressed. We are interested in characterizing the statistical properties of random variables \(\Pi_i (w_i, L, \ell), \forall i \in \{1, \ldots, M\}\), and \(\Pi_{agg} (w_{agg}, L, \ell)\) as well as their corresponding CDFs. Given the category of risk, there is a probability distribution capturing possible changes in net assets for each individual financial institution, and there is a probability distribution capturing possible changes in net assets for the entire financial system.

To construct these probability distributions and solve for their statistical features, we need to rewrite certain expressions of interest. Define the \(L \times 1\) vector \(b(L, \ell)\), with \([b(L, \ell)]_j = 1\) if \([\epsilon(L, \ell)]_j = \delta\bar{p}_j\) and \([b(L, \ell)]_j = 0\) if \([\epsilon(L, \ell)]_j = 0\). The vector \(b(L, \ell)\) identifies the indices of those clusters being stressed. Additionally, define the vectors \(v_i\), \(\forall i \in \{1, \ldots, M\}\), and \(v_{agg}\):

\[
v_i = \begin{pmatrix} w_{i1} \bar{p}_{i1} & \cdots & w_{i\ell} \bar{p}_{i\ell} \end{pmatrix}^T, \quad \text{and} \quad v_{agg} = \begin{pmatrix} w_{agg1} \bar{p}_{i1} & \cdots & w_{agg\ell} \bar{p}_{i\ell} \end{pmatrix}^T.
\]

We then establish the following lemma:
Lemma 1 Given stress scenario $\epsilon (L, \ell)$, the change in the market value of net assets for financial institution $i$, $\forall i \in \{1, \ldots, M\}$, is $w_i^T \epsilon (L, \ell) = \left( \hat{\delta} [\bar{p}]_1 \right) v_i^T b (L, \ell)$, and the change in the market value of net assets for the entire financial system is $w_{agg}^T \epsilon (L, \ell) = \left( \hat{\delta} [\bar{p}]_1 \right) v_{agg}^T b (L, \ell)$.

We set $k_i = 1^T v_i$, $\forall i \in \{1, \ldots, M\}$, and $k_{agg} = 1^T v_{agg}$. In addition, we define random variable $V_i$ with realization $[v_i]_j$, and we define random variable $V_{agg}$ with realization $[v_{agg}]_j$. Each realization is equally likely. We can now solve, in closed form, for the statistical features of $\Pi_i (w_i, L, \ell)$ and $\Pi_{agg} (w_{agg}, L, \ell)$. We have the following results:

Proposition 5 The average change in net assets for financial institution $i$ is:

$$E \Pi_i (w_i, L, \ell) = \frac{k_i \ell}{L} \hat{\delta} [\bar{p}]_1,$$

and the average change in net assets for the financial system is:

$$E \Pi_{agg} (w_{agg}, L, \ell) = \frac{k_{agg} \ell}{L} \hat{\delta} [\bar{p}]_1,$$

Proposition 6 The change in net assets for financial institution $i$ has a variance of:

$$Var \Pi_i (w_i, L, \ell) = \left( \hat{\delta} [\bar{p}]_1 \right)^2 \frac{\ell}{L} \left( 1 - \frac{\ell}{L} \right) \frac{L}{L-1} L Var V_i,$$

and the change in net assets for the entire financial system has a variance of:

$$Var \Pi_{agg} (w_{agg}, L, \ell) = \left( \hat{\delta} [\bar{p}]_1 \right)^2 \frac{\ell}{L} \left( 1 - \frac{\ell}{L} \right) \frac{L}{L-1} L Var V_{agg}.$$

Specifically, $Var V_i = \frac{1}{L} \sum_{j=1}^{L} \left( [v_i]_j - k_i \right)^2$ and $Var V_{agg} = \frac{1}{L} \sum_{j=1}^{L} \left( [v_{agg}]_j - k_{agg} \right)^2$.

Proposition 7 Construct the ordered multiset $\{\tilde{v}_j\}^{L}_{j=1}$ from the elements of $v_i$ so that $\tilde{v}_j \leq \tilde{v}_{j'}$ whenever $j \leq j'$. When $\hat{\delta} < 0$, the lower and upper bounds on the distribution of possible changes to net assets for institution $i$ are:

$$\min supp \Pi_i (w_i, L, \ell) = \hat{\delta} [\bar{p}]_1 \sum_{j=L-\ell+1}^{L} \tilde{v}_j \quad \text{and} \quad \max supp \Pi_i (w_i, L, \ell) = \hat{\delta} [\bar{p}]_1 \sum_{j=1}^{\ell} \tilde{v}_j.$$

Now construct the ordered multiset $\{\tilde{x}_j\}^{L}_{j=1}$ from the elements of $v_{agg}$ so that $\tilde{x}_j \leq \tilde{x}_{j'}$ whenever $j \leq j'$. When $\hat{\delta} < 0$, the lower and upper bounds on the distribution of possible changes
to net assets for the overall financial system are:

$$\min \text{ supp } \Pi_{agg}(w_{agg}, L, \ell) = \tilde{\delta}[\bar{p}]_1 \sum_{j=L-\ell+1}^{L} \tilde{x}_j \quad \text{and} \quad \max \text{ supp } \Pi_{agg}(w_{agg}, L, \ell) = \tilde{\delta}[\bar{p}]_1 \sum_{j=1}^{\ell} \tilde{x}_j.$$ 

**Proposition 8** Provided that condition (c) holds, for all $i \in \{1, \ldots, M\}$,

$$\left| \frac{G_{\Pi_i}(w_i, L, \ell) - E\Pi_i(w_i, L, \ell)}{\text{Var } \Pi_i(w_i, L, \ell)^{1/2}} (t) - J(\hat{w}, L, \ell, t) \right| < C_4 \times \sum_{j=1}^{L} |\hat{w}_j|^5$$

with $\hat{w}_j = \frac{[v_i]_i - EV_i}{\sqrt{L \text{Var } V_i}}$, and

$$\left| \frac{G_{\Pi_{agg}(w_{agg}, L, \ell) - E\Pi_{agg}(w_{agg}, L, \ell)}}{\text{Var } \Pi_{agg}(w_{agg}, L, \ell)^{1/2}} (t) - J(\hat{w}, L, \ell, t) \right| < C_4 \times \sum_{j=1}^{L} |\hat{w}_j|^5$$

with $\hat{w}_j = \frac{[v_{agg}]_i - EV_{agg}}{\sqrt{L \text{Var } V_{agg}}}$ for all $t$, where $C_4$ is only a function of $\frac{\ell}{L}$.

Condition (c) requires that the elements of $v_i$, for all $i \in \{1, \ldots, M\}$, and the elements of $v_{agg}$ not be clustered over too few values. Given Proposition 8, we have that

$$G_{\Pi_i(w_i, L, \ell)}(t) \approx J\left(\hat{w}, L, \ell, \frac{t - E\Pi_i(w_i, L, \ell)}{\text{Var } \Pi_i(w_i, L, \ell)^{1/2}}\right), \quad \forall i \in \{1, \ldots, M\},$$

with $\hat{w}_j = \frac{[v_i]_i - EV_i}{\sqrt{L \text{Var } V_i}}$, and

$$G_{\Pi_{agg}(w_{agg}, L, \ell)}(t) \approx J\left(\hat{w}, L, \ell, \frac{t - E\Pi_{agg}(w_{agg}, L, \ell)}{\text{Var } \Pi_{agg}(w_{agg}, L, \ell)^{1/2}}\right),$$

with $\hat{w}_j = \frac{[v_{agg}]_i - EV_{agg}}{\sqrt{L \text{Var } V_{agg}}}$. For each individual financial institution, note that $\sum_{j=1}^{L} \hat{w}_j^3 = L^{-1/2} \text{Skew } V_i$ and $\sum_{j=1}^{L} \hat{w}_j^4 - \frac{3}{L} = L^{-1} \times (\text{Excess Kurtosis } V_i)$. For the overall financial system, note that $\sum_{j=1}^{L} \hat{w}_j^3 = L^{-1/2} \text{Skew } V_{agg}$ and $\sum_{j=1}^{L} \hat{w}_j^4 - \frac{3}{L} = L^{-1} \times (\text{Excess Kurtosis } V_{agg})$. We can therefore approximate the CDFs of these distributions in terms of higher-order population moments.

When $\ell = 1$, we can solve for $G_{\Pi_i(w_i, L, \ell)}(t), \forall i \in \{1, \ldots, M\}$, and $G_{\Pi_{agg}(w_{agg}, L, \ell)}(t)$ exactly. When $\ell = 1$, $\Pi_i(w_i, L, \ell) = \tilde{\delta}[\bar{p}]_1 V_i$ and $\Pi_{agg}(w_{agg}, L, \ell) = \tilde{\delta}[\bar{p}]_1 V_{agg}$, so $G_{\Pi_i(w_i, L, \ell)}(t) = G_{V_i}\left(\frac{t}{\delta[\bar{p}]_1}\right)$ and $G_{\Pi_{agg}(w_{agg}, L, \ell)}(t) = G_{V_{agg}}\left(\frac{t}{\delta[\bar{p}]_1}\right)$. 

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Example 5 Suppose that the category of risk is exchange rate risk. Specifically, one foreign currency depreciates by \( \hat{y} \) percent relative to the U.S. dollar. Given that the market value of net assets for each financial institution is priced in U.S. dollars, we are interested in the possible changes in net assets for each individual financial institution \( i, i \in \{1, \ldots, M\} \), and we are interested in the possible changes in net assets for the overall financial system.

To compute the statistical features of \( \Pi_i (w_i, L, \ell), \forall i \in \{1, \ldots, M\} \), and \( \Pi_{agg} (w_{agg}, L, \ell) \), and to construct the probability distributions \( G_{\Pi_i(w_i, L, \ell)} (t), \forall i \in \{1, \ldots, M\} \), and \( G_{\Pi_{agg}(w_{agg}, L, \ell)} (t) \), we solve for all of the necessary variables. There are \( L \) total foreign currencies, and each financial institution potentially holds securities denominated in foreign currencies. Each foreign currency is indexed by \( j \in \{1, \ldots, L\} \). Both \( w_i = 1_{L \times 1} \) and \( w_{agg} = 1_{L \times 1} \). Rather than there being just one vector \( \bar{p} \), this example requires us to have an \( L \times 1 \) vector \( \bar{p}_i \) for each financial institution \( i \in \{1, \ldots, M\} \) and an \( L \times 1 \) vector \( \bar{p}_{agg} \) for the overall financial system. \( [\bar{p}_i]_j \) is equal to the dollar value of all securities denominated in foreign currency \( j \) for financial institution \( i \) before any currency depreciation. Meanwhile, \( [\bar{p}_{agg}]_j \) is equal to the dollar value of all securities denominated in foreign currency \( j \) for all financial institutions in the system before any currency depreciation. We then construct \( v_i, \forall i \in \{1, \ldots, M\} \), from the vectors \( w_i \) and \( \bar{p}_i \), and we construct \( v_{agg} \) from the vectors \( w_{agg} \) and \( \bar{p}_{agg} \). One foreign currency depreciates, so \( \ell = 1 \). Given this depreciation of a foreign currency, \( \hat{\delta} = -\hat{y} \). To see why \( \hat{\delta} = -\hat{y} \), follow the derivation in the footnote.\(^7\) The vector \( \epsilon (L, \ell) \) identifies which foreign currency is depreciating. For financial institution \( i \), \( \epsilon (L, \ell) = \hat{\delta} \bar{p}_i \circ b (L, \ell) \), and for the entire financial system, \( \epsilon (L, \ell) = \hat{\delta} \bar{p}_{agg} \circ b (L, \ell) \), with \( 1^T b (L, \ell) = \ell = 1 \). With \( L \) foreign currencies, there are \( L \) possible stress scenarios for this category of risk; each scenario features a different depreciating foreign currency. For each scenario, the change in net assets for institution \( i \) is \( \pi_i (w_i, \epsilon, L, \ell) = \left( \hat{\delta} [\bar{p}_i]_1 \right) v_i^T b (L, \ell) \), and the change in net assets for the entire financial system is \( \pi_{agg} (w_{agg}, \epsilon, L, \ell) = \left( \hat{\delta} [\bar{p}_{agg}]_1 \right) v_{agg}^T b (L, \ell) \). Since \( \ell = 1 \),

\[ \frac{\xi_{S/F}^{(1)} - \xi_{S/F}^{(0)}}{\xi_{S/F}^{(0)}} = -\hat{y}, \]

so that \( \xi_{S/F}^{(1)} = \xi_{S/F}^{(0)} \times (1 - \hat{y}) \), we then have

\[ P_{S}^{(1)} = P_{F}^{(0)} \times P_{S/F}^{(1)} = P_{F}^{(0)} \times P_{S/F}^{(0)} \times (1 - \hat{y}) = P_{S}^{(0)} \times (1 - \hat{y}) = P_{S}^{(0)} + \hat{\delta} P_{S}^{(0)}, \]

setting \( \hat{\delta} = -\hat{y} \).
\[ G_{\Pi_i(w_i, L, \ell)}(t) = G_{V_i} \left( \frac{t}{\delta[p_i]} \right) \] exactly, \( \forall i \in \{1, \ldots, M\} \), and \( G_{\Pi_{agg}(w_{agg}, L, \ell)}(t) = G_{V_{agg}} \left( \frac{t}{\delta[p_{agg}]} \right) \) exactly.

Example 6 Suppose that the category of risk is credit risk. Specifically, forty percent of all AAA-rated mortgage-backed securities have been downgraded to a CCC rating. As a result, the price of each affected mortgage-backed security has declined 80 percent from its original level. We are interested in the possible changes in net assets for each individual financial institution \( i, \forall i \in \{1, \ldots, M\} \), and we are interested in the possible changes in net assets for the overall financial system.

To compute the statistical features of \( \Pi_i(w_i, L, \ell), \forall i \in \{1, \ldots, M\} \), and \( \Pi_{agg}(w_{agg}, L, \ell) \), and to construct the probability distributions \( G_{\Pi_i(w_i, L, \ell)}(t), \forall i \in \{1, \ldots, M\} \), and \( G_{\Pi_{agg}(w_{agg}, L, \ell)}(t) \), we solve for all of the necessary variables. There are \( L \) total initially AAA-rated mortgage-backed securities. We index each of these securities by \( j \in \{1, \ldots, L\} \). \([w_i]_j\) is equal to the number of units of mortgage-backed security \( j \) held by financial institution \( i \), which may likely equal 1. \([w_{agg}]_j\) is equal to the number of units of mortgage-backed security \( j \) held by the entire financial system. \([p]_j\) is equal to the original value of security \( j \). We construct \( v_i \) from \( w_i \) and \( \bar{p} \), and we construct \( v_{agg} \) from \( w_{agg} \) and \( \bar{p} \). We set \( \ell = 0.4L \), and we set \( \delta = -0.8 \). The vector \( e(L, \ell) = \delta[p] \circ b(L, \ell) \) identifies which group of securities has been downgraded, with a consequent reduction in price; \( 1^Tb(L, \ell) = 0.4L \). There are many possible vectors \( e(L, \ell) \); each vector \( e(L, \ell) \) captures a different group of securities being downgraded. \( |E(L, \ell)| = \binom{L}{\ell} = \binom{L}{0.4L} \). For any given group of stressed securities, the change in net assets for institution \( i \) is \( \pi_i(w_i, e, L, \ell) = \left( \delta[p]_1 \right)^T v_i^T b(L, \ell) \), and the change in net assets for the entire financial system is \( \pi_{agg}(w_{agg}, e, L, \ell) = \left( \delta[p]_1 \right)^T v_{agg}^T b(L, \ell) \). We can construct both \( G_{\Pi_i(w_i, L, \ell)}(t), \forall i \in \{1, \ldots, M\} \), and \( G_{\Pi_{agg}(w_{agg}, L, \ell)}(t) \) via asymptotic expansion, following Proposition 8.

### 2.3 Third Risk Environment: Absolute Price Shocks, Different Across Securities Clusters

In the third environment, the category of risk affects securities prices in a manner different from the previous two environments. In the previous two environments, every stressed cluster of securities had the same price adjustment. For the first environment, every stressed cluster had the same magnitude of adjustment to its price or value, and for the second environment, every stressed cluster had the same percentage adjustment to its price or value. In the present environment, different stressed clusters can potentially have different magnitudes of adjustment to their prices or values. With \( L \) total clusters of securities, we
introduce the $L \times 1$ vector $\delta$. This vector specifies price adjustments to various clusters. Specifically, one cluster’s price or value adjusts by $[\delta]_1$, a separate cluster’s price or value adjusts by $[\delta]_2$, and so on. As an example, let’s suppose that the value of the securities in cluster $\ell$ adjusts by $[\delta]_{\ell}$, for all $\ell \in \{1, \ldots, L\}$. Then, setting $\epsilon = \delta$, the change in the market value of net assets for financial institution $i$ is $w_i^T \epsilon(\delta)$, and the change in the market value of net assets for the entire financial system is $w_{agg}^T \epsilon(\delta)$.

There are many possible ways that this category of risk, encapsulated by $\delta$, can manifest itself in securities prices. Setting $\epsilon = \delta$ represents one possible stress scenario. We can generate additional stress scenarios by rearranging the elements of $\delta$. For example, a new stress scenario $\epsilon'$ is a permutation of the elements of $\delta$. In particular, there exists a permutation matrix $P$ such that $\epsilon' = P \delta$. $E(\delta)$ is the set of all possible stress scenarios given $\delta$. The total number of possible scenarios, $|E(\delta)|$, is $L!$. Depending on the properties of $\delta$, we might have $P \delta = P' \delta$ for $P \neq P'$, which means that the multiset $E(\delta)$ contains identical stress scenarios. This multiplicity is fine when we compute the statistical features of our probability distributions. These probability distributions will correctly capture how a particular category of risk, $\delta$, affects net assets for individual financial institutions and the overall financial system.

As in the previous environments, we define $\pi_i(w_i, \epsilon, \delta) = w_i^T \epsilon(\delta)$ as the change in net assets for institution $i \in \{1, \ldots, M\}$ given stress scenario $\epsilon = P \delta$, and we define $\pi_{agg}(w_{agg}, \epsilon, \delta) = w_{agg}^T \epsilon(\delta)$ as the change in net assets for the entire financial system given stress scenario $\epsilon = P \delta$. Note that the arguments of $\pi_i(\cdot)$ and $\pi_{agg}(\cdot)$ have changed from the previous two environments due to the different nature of the stress scenarios. We are interested in characterizing the statistical properties of random variables $\Pi_i(w_i, \delta)$, $\forall i \in \{1, \ldots, M\}$, and $\Pi_{agg}(w_{agg}, \delta)$, which respectively represent the possible changes in net assets for financial institution $i$ and the overall financial system given risk category $\delta$.

We also define random variable $W_i$ with realization $[w_i]_j$, and we define random variable $W_{agg}$ with realization $[w_{agg}]_j$. Furthermore, $1^T w_i = k_i$ and $1^T w_{agg} = k_{agg}$.

**Proposition 9** The average change in net assets for financial institution $i$ given $\delta$ is:

$$E\Pi_i(w_i, \delta) = \left(\frac{1^T \delta}{L}\right) k_i,$$

and the average change in net assets for the entire financial system given $\delta$ is:

$$E\Pi_{agg}(w_{agg}, \delta) = \left(\frac{1^T \delta}{L}\right) k_{agg}.$$
changes in net assets for financial institution given the category of risk $\delta$. We would also like to compute the variance of the distribution capturing possible changes in net assets for the financial system given the category of risk $\delta$. To solve in closed form for these second moments we need to introduce some additional notation. $\Pi_i (w_i, \delta) = w_i^T \Delta$ and $\Pi_{agg} (w_{agg}, \delta) = w_{agg}^T \Delta$, where $\Delta$ is an $L \times 1$ random vector whose elements are the random variables $\Delta_j$, $j \in \{1, \ldots, L\}$. The random variables $\Delta_1, \ldots, \Delta_L$ are identically distributed but not independent. Each random variable $\Delta_j$, $j \in \{1, \ldots, L\}$, has the underlying CDF $G_{\Delta_j} (t) = \frac{1}{L} \sum_{m=1}^{L} \mathbb{1}_{[\delta]_m \leq t}$; it’s the empirical probability distribution formed from the elements of $\delta$. Each random variable $\Delta_j$ essentially draws values from the elements of $\delta$. These draws are done without replacement; this is what makes the random variables $\Delta_1, \ldots, \Delta_L$ not independent. When drawing a value for $\Delta_1$, there are $L$ scalars to choose from, while there is only one scalar to choose from when drawing for $\Delta_L$. Note that $E \Delta_j = \frac{1}{L} \delta$ and $\text{Var} \Delta_j = \frac{1}{L} \sum_{m=1}^{L} ([\delta]_m - E \Delta_j)^2$.

**Proposition 10** The change in net assets for financial institution $i$ has a variance of:

\[
\text{Var} \Pi_i (w_i, \delta) = (\text{Var} \Delta_j) \sum_{m=1}^{L} ([w_i]_m)^2 + (E [\Delta_j \Delta_r] - (E \Delta_j) (E \Delta_r)) \left[ (L - 1) \sum_{m=1}^{L} ([w_i]_m)^2 - L^2 \text{Var} W_i \right],
\]

and the change in net assets for the entire financial system has a variance of:

\[
\text{Var} \Pi_{agg} (w_{agg}, \delta) = (\text{Var} \Delta_j) \sum_{m=1}^{L} ([w_{agg}]_m)^2 + (E [\Delta_j \Delta_r] - (E \Delta_j) (E \Delta_r)) \left[ (L - 1) \sum_{m=1}^{L} ([w_{agg}]_m)^2 - L^2 \text{Var} W_{agg} \right].
\]

We need to compute $E [\Delta_j \Delta_r]$, keeping in mind that sampling is done without replacement. When the support of $\Delta_j$ is small, it is straightforward to compute $E [\Delta_j \Delta_r]$ by hand.

We proceed to compute the lower and upper bounds on the supports of $\Pi_i (w_i, \delta)$ and $\Pi_{agg} (w_{agg}, \delta)$:

**Proposition 11** Construct the ordered multiset $\{\tilde{w}_j\}_{j=1}^{L}$ from the elements of $w_i$ so that
\[ \hat{w}_j \leq \hat{w}_{j'} \text{ whenever } j \leq j'. \] Also construct the ordered multiset \( \{ \tilde{\delta}_j \}_{j=1}^L \) from the elements of \( \delta \) so that \( \tilde{\delta}_j \leq \tilde{\delta}_{j'} \text{ whenever } j \leq j'. \) Then,

\[
\min \sup \Pi_i (w_i, \delta) = \tilde{w}_1 \tilde{\delta}_L + \tilde{w}_2 \tilde{\delta}_{L-1} + \cdots + \tilde{w}_L \tilde{\delta}_1 \text{ and}
\]

\[
\max \sup \Pi_i (w_i, \delta) = \tilde{w}_1 \tilde{\delta}_1 + \tilde{w}_2 \tilde{\delta}_2 + \cdots + \tilde{w}_L \tilde{\delta}_L.
\]

Now construct the ordered multiset \( \{ \tilde{x}_j \}_{j=1}^L \) from the elements of \( w_{agg} \) so that \( \tilde{x}_j \leq \tilde{x}_{j'} \text{ whenever } j \leq j'. \) Then,

\[
\min \sup \Pi_{agg} (w_{agg}, \delta) = \tilde{x}_1 \tilde{\delta}_L + \tilde{x}_2 \tilde{\delta}_{L-1} + \cdots + \tilde{x}_L \tilde{\delta}_1 \text{ and}
\]

\[
\max \sup \Pi_{agg} (w_{agg}, \delta) = \tilde{x}_1 \tilde{\delta}_1 + \tilde{x}_2 \tilde{\delta}_2 + \cdots + \tilde{x}_L \tilde{\delta}_L.
\]

While we cannot compute \( G_{\Pi_i(w_i, \delta)}(t) \) and \( G_{\Pi_{agg}(w_{agg}, \delta)}(t) \) by hand, we can simulate them on the computer.

**Example 7** Suppose that the category of risk is solvency risk. Specifically, 100 public companies have filed for Chapter 11 bankruptcy. Each of these public companies undergoes corporate debt restructuring. In particular, each public company renegotiates its debt obligations so that there is a dollar amount of debt forgiven. We are interested in the possible changes in net assets for each individual financial institution \( i, \forall i \in \{1, \ldots, M\} \), and we are interested in the possible changes in net assets for the overall financial system.

To compute the statistical features of \( \Pi_i (w_i, \delta), \forall i \in \{1, \ldots, M\} \), and \( \Pi_{agg} (w_{agg}, \delta) \), we solve for all of the necessary variables. There are \( L \) total public companies with issued corporate debt. Each public company is indexed by \( j \in \{1, \ldots, L\} \). \([w_i]_j \) is equal to the number of bonds from public company \( j \) held by financial institution \( i \). \([w_{agg}]_j \) is equal to the total number of bonds from public company \( j \) held by all institutions in the financial system. \([\tilde{p}]_j \) is the price of a corporate bond issued by public company \( j \). We now introduce the \( L \times 1 \) vector \( \delta \). \([\delta]_j, j \in \{1, \ldots, 100\} \), equals the change in the price of a company’s bonds for one of the 100 public companies that filed for Chapter 11 bankruptcy. Therefore \([\delta]_j < 0 \) for \( j \in \{1, \ldots, 100\} \). There is no change in the price of bonds for the other public companies, so \([\delta]_j = 0 \) for \( j \in \{101, \ldots, L\} \). The \( L \times 1 \) vector \( \epsilon(\delta) = P\delta \), for permutation matrix \( P \), identifies the indices of those public companies experiencing a write-down in corporate debt. Different vectors \( \epsilon(\delta), \epsilon'(\delta) \) identify different groups of 100 companies facing some

level of debt relief. The set of all possible scenarios, \( E(\delta) \), has a very large cardinality. If each restructured company has a different debt write-down amount, \(|E(\delta)| = L!\) with there being \( \frac{L!}{(L-100)!} \) unique stress scenarios. Given \( \epsilon(\delta) \), the change in net assets for institution \( i \) is \( \pi_i(\mathbf{w}_i, \epsilon, \delta) = \mathbf{w}_i^T \epsilon(\delta) \), and the change in net assets for the entire financial system is \( \pi_{agg}(\mathbf{w}_{agg}, \epsilon, \delta) = \mathbf{w}_{agg}^T \epsilon(\delta) \).

### 2.4 Fourth Risk Environment: Percentage Price Shocks, Different Across Securities Clusters

In this final environment, we have \( L \) clusters of securities. Each category of risk affects securities prices in the following manner. Every cluster of securities faces a percentage adjustment to its market value. Unlike the second environment, different stressed clusters of securities can face different percentage adjustments to their market values. We introduce the \( \ell \times 1 \) vector \( \hat{\delta} \), which captures how a particular category of risk can impact securities prices. Specifically, one cluster faces a percentage adjustment \( [\hat{\delta}]_1 \), another cluster faces a percentage adjustment \( [\hat{\delta}]_2 \), and so on. The \( \ell \times 1 \) vector \( \epsilon \) represents the specific way that a category of risk can manifest itself. For example, when \( \epsilon = \hat{\delta} \), cluster \( \ell \) faces a percentage price adjustment of \( [\hat{\delta}]_\ell \), for all \( \ell \in \{1, \ldots, L\} \). There are many possible ways that the category of risk can manifest itself in securities prices. Each stress scenario \( \epsilon \) is a permutation of \( \hat{\delta} \), that is \( \epsilon = \mathbf{P}\hat{\delta} \) for some permutation matrix \( \mathbf{P} \). The set of all possible scenarios is \( E(\hat{\delta}) \), with \( |E(\hat{\delta})| = L! \) potentially including duplicates.

For a given configuration of shocks \( \epsilon(\hat{\delta}) \), the change in the market value of net assets for financial institution \( i \) is \( \mathbf{w}_i^T(\epsilon(\hat{\delta}) \circ \mathbf{p}) \), while the change in the market value of net assets for the overall financial system is \( \mathbf{w}_{agg}^T(\epsilon(\hat{\delta}) \circ \mathbf{p}) \). The operator \( \circ \) denotes the element-wise Hadamard product. We define \( \pi_i(\mathbf{w}_i, \epsilon, \hat{\delta}) = \mathbf{w}_i^T(\epsilon(\hat{\delta}) \circ \mathbf{p}) \) as the change in net assets for institution \( i \in \{1, \ldots, M\} \) given stress scenario \( \epsilon(\hat{\delta}) = \mathbf{P}\hat{\delta} \), and we define \( \pi_{agg}(\mathbf{w}_{agg}, \epsilon, \hat{\delta}) = \mathbf{w}_{agg}^T(\epsilon(\hat{\delta}) \circ \mathbf{p}) \) as the change in net assets for all financial institutions given stress scenario \( \epsilon(\hat{\delta}) = \mathbf{P}\hat{\delta} \). We are interested in characterizing the statistical properties of random variables \( \Pi_i(\mathbf{w}_i, \hat{\delta}), \forall i \in \{1, \ldots, M\} \), and \( \Pi_{agg}(\mathbf{w}_{agg}, \hat{\delta}) \), which respectively represent the possible changes in net assets for financial institution \( i \) and the overall financial system given risk category \( \hat{\delta} \).

To solve for these statistical features, we first need to introduce the additional vari-
ables $v_i$, $\forall i \in \{1, \ldots, M\}$, and $v_{agg}$, which we define as follows:

$$v_i = \left(\frac{[w_i]}{p_1}, \ldots, \frac{[w_i]}{p_L}\right)^T, \quad \text{and}$$

$$v_{agg} = \left(\frac{[w_{agg}]}{p_1}, \ldots, \frac{[w_{agg}]}{p_L}\right)^T.$$

We then have the following lemma:

**Lemma 2** Given stress scenario $\epsilon(\hat{\delta})$, the change in the market value of net assets for financial institution $i$, $\forall i \in \{1, \ldots, M\}$, is:

$$w_i^T \left(\epsilon(\hat{\delta}) \circ \bar{p}\right) = [\bar{p}]_1 v_i^T \epsilon(\hat{\delta}) ,$$

and the change in the market value of net assets for the entire financial system is:

$$w_{agg}^T \left(\epsilon(\hat{\delta}) \circ \bar{p}\right) = [\bar{p}]_1 v_{agg}^T \epsilon(\hat{\delta}) .$$

We let $1^T v_i = k_i, \forall i \in \{1, \ldots, M\}$, and we let $1^T v_{agg} = k_{agg}$. We also define random variable $V_i$ with realization $[v_i]_j$, and we define random variable $V_{agg}$ with realization $[v_{agg}].$ Each realization is equally likely, and that allows us to define population moments for $V_i$ and $V_{agg}$.

The statistical features of $\Pi_i \left(w_i, \hat{\delta}\right)$ and $\Pi_{agg} \left(w_{agg}, \hat{\delta}\right)$ are as follows:

**Proposition 12** The average change in net assets for financial institution $i$ given $\hat{\delta}$ is:

$$E\Pi_i \left(w_i, \hat{\delta}\right) = [\bar{p}]_1 \left(\frac{1^T \hat{\delta}}{L}\right) k_i,$$

and the average change in net assets for the financial system given $\hat{\delta}$ is:

$$E\Pi_{agg} \left(w_{agg}, \hat{\delta}\right) = [\bar{p}]_1 \left(\frac{1^T \hat{\delta}}{L}\right) k_{agg}.$$
the underlying CDF
\[ G_{\Delta_j}(t) = \frac{1}{L} \sum_{m=1}^{L} \mathbb{1}_{[\delta]_m \leq t}, \]

it’s the empirical probability distribution formed from the elements of \( \hat{\delta} \). Each random variable \( \hat{\Delta}_j \) essentially draws values from the elements of \( \hat{\delta} \). These draws are done without replacement; this is what makes the random variables \( \hat{\Delta}_1, \ldots, \hat{\Delta}_L \) not independent. If we draw random variables \( \hat{\Delta}_j \) sequentially, there are \( L \) scalars to choose from for \( \hat{\Delta}_1 \), while there is only one scalar to choose from for \( \hat{\Delta}_L \). The first two population moments for \( \hat{\Delta}_j \) are
\[
E\{\hat{\Delta}_j\} = \frac{1}{L} \sum_{m=1}^{L} (\hat{\delta}_m - E\{\hat{\Delta}_j\})^2.
\]

**Proposition 13** Given \( \hat{\delta} \), the change in net assets for financial institution \( i \) has a variance of:
\[
\text{Var} \Pi_i(w_i, \hat{\delta}) = (\bar{p}_1)^2 \times \left( \text{Var} \hat{\Delta}_j \sum_{m=1}^{L} (v_i)_m^2 \right. \\
+ \left. \left( E\{\hat{\Delta}_j\hat{\Delta}_r\} - (E\hat{\Delta}_j)(E\hat{\Delta}_r) \right) \left( (L - 1) \sum_{m=1}^{L} (v_i)_m^2 - L^2 \text{Var} V_i \right) \right),
\]
and the change in net assets for the entire financial system has a variance of:
\[
\text{Var} \Pi_{agg}(w_{agg}, \hat{\delta}) = (\bar{p}_1)^2 \times \left( \text{Var} \hat{\Delta}_j \sum_{m=1}^{L} (v_{agg})_m^2 \right. \\
+ \left. \left( E\{\hat{\Delta}_j\hat{\Delta}_r\} - (E\hat{\Delta}_j)(E\hat{\Delta}_r) \right) \left( (L - 1) \sum_{m=1}^{L} (v_{agg})_m^2 - L^2 \text{Var} V_{agg} \right) \right),
\]

We must compute \( E\{\hat{\Delta}_j\hat{\Delta}_r\} \), keeping in mind that sampling is done without replacement. It is straightforward to compute \( E\{\hat{\Delta}_j\hat{\Delta}_r\} \), especially when \( \hat{\Delta}_j \) has a small support.

We next compute the lower and upper bounds on the supports of \( \Pi_i(w_i, \hat{\delta}) \), and \( \Pi_{agg}(w_{agg}, \hat{\delta}) \). These lower and upper bounds capture the range of possible adjustments to the market value of net income given \( \hat{\delta} \) for each individual financial institution and the overall financial system.

**Proposition 14** Construct the ordered multiset \( \{\tilde{v}_j\}_{j=1}^{L} \) from the elements of \( v_i \) so that \( \tilde{v}_j \leq \tilde{v}_{j'} \) whenever \( j \leq j' \). Also construct the ordered multiset \( \{\tilde{\delta}_j\}_{j=1}^{L} \) from the elements of
\( \hat{\delta} \) so that \( \hat{\delta}_j \leq \hat{\delta}_{j'} \) whenever \( j \leq j' \). Then

\[
\min \sup \Pi_i \left( w_i, \hat{\delta} \right) = [\bar{p}]_1 \times \left( \bar{v}_1 \hat{\delta}_L + \bar{v}_2 \hat{\delta}_{L-1} + \cdots + \bar{v}_L \hat{\delta}_1 \right) \quad \text{and} \\
\max \sup \Pi_i \left( w_i, \hat{\delta} \right) = [\bar{p}]_1 \times \left( \bar{\tilde{v}}_1 \hat{\delta}_L + \bar{\tilde{v}}_2 \hat{\delta}_{L-1} + \cdots + \bar{\tilde{v}}_L \hat{\delta}_1 \right).
\]

Now construct the ordered multiset \( \{ \bar{x}_j \}_{j=1}^L \) from the elements of \( v_{agg} \) so that \( \bar{x}_j \leq \bar{x}_{j'} \) whenever \( j \leq j' \). Then,

\[
\min \sup \Pi_{agg} \left( w_{agg}, \hat{\delta} \right) = [\bar{p}]_1 \times \left( \bar{x}_1 \hat{\delta}_L + \bar{x}_2 \hat{\delta}_{L-1} + \cdots + \bar{x}_L \hat{\delta}_1 \right) \quad \text{and} \\
\max \sup \Pi_{agg} \left( w_{agg}, \hat{\delta} \right) = [\bar{p}]_1 \times \left( \bar{\tilde{x}}_1 \hat{\delta}_L + \bar{\tilde{x}}_2 \hat{\delta}_{L-1} + \cdots + \bar{\tilde{x}}_L \hat{\delta}_1 \right).
\]

While we cannot compute \( G_{\Pi_i \left( w_i, \hat{\delta} \right)} \) and \( G_{\Pi_{agg} \left( w_{agg}, \hat{\delta} \right)} \) by hand, we can simulate them by computer.

**Example 8** Suppose that the category of risk is industry risk. Specifically, the prices of securities in one industry decline by 20 percent, the prices of securities in another industry decline by 10 percent, and the prices of securities in a third industry increase by 12 percent. We are interested in the possible changes in net assets for each individual financial institution \( i, \forall i \in \{1, \ldots, M\} \), and we are interested in the possible changes in net assets for the overall financial system.

To compute the statistical features of \( \Pi_i \left( w_i, \hat{\delta} \right), \forall i \in \{1, \ldots, M\} \), and \( \Pi_{agg} \left( w_{agg}, \hat{\delta} \right) \), we solve for all of the necessary variables. There are \( L \) total industries. Each industry is indexed by \( j \in \{1, \ldots, L\} \). We set \( w_i = 1_{L \times 1} \), \( \forall i \in \{1, \ldots, M\} \), and \( w_{agg} = 1_{L \times 1} \). Rather than there being just one vector \( \bar{p} \), this example requires us to have an \( L \times 1 \) vector \( \bar{p}_i \) for each financial institution \( i \in \{1, \ldots, M\} \) and an \( L \times 1 \) vector \( \bar{p}_{agg} \) for the overall financial system. \( [\bar{p}_i]_j \) is equal to the original total market value for all securities in industry \( j \) held by financial institution \( i \). Meanwhile, \( [\bar{p}_{agg}]_j \) is equal to the original total market value for all securities in industry \( j \) held by all financial institutions in the financial system. We then construct \( v_i, \forall i \in \{1, \ldots, M\} \), from the vectors \( w_i \) and \( \bar{p}_i \), and we construct \( v_{agg} \) from the vectors \( w_{agg} \) and \( \bar{p}_{agg} \). The \( L \times 1 \) vector \( \hat{\delta} \) captures percentage changes to the values of securities in each sector. For this category of risk, we set \( \hat{\delta}_1 = -0.20 \), we set \( \hat{\delta}_2 = -0.10 \), we set \( \hat{\delta}_3 = 0.12 \), and we set \( \hat{\delta}_j = 0 \) for \( j \in \{4, \ldots, L\} \). Given \( \hat{\delta} \), we then construct \( \epsilon \left( \hat{\delta} \right) \).

In particular, \( \epsilon \left( \hat{\delta} \right) = P \hat{\delta} \) for permutation matrix \( P \). \( \left[ \epsilon \left( \hat{\delta} \right) \right]_j \) represents the percentage change in the value of securities from industry \( j \). There are many possible vectors \( \epsilon \left( \hat{\delta} \right) \).
in the set $E(\hat{\delta})$: $|E(\hat{\delta})| = L!$, with there being $\frac{L!}{(L-3)!} = L \times (L-1) \times (L-2)$ unique stress scenarios. For a scenario $\epsilon(\hat{\delta})$, the change in net assets for financial institution $i$ is $\pi_i(w_i, \epsilon, \hat{\delta}) = [\bar{p}_i]_1 v_i^T \epsilon(\hat{\delta})$, and the change in net assets for the overall financial system is $\pi_{agg}(w_{agg}, \epsilon, \hat{\delta}) = [\bar{p}_{agg}]_1 v_{agg}^T \epsilon(\hat{\delta})$.

### 2.5 Combining Categories of Risk to Generate Entire Classes of Stress Tests

Thus far, we have been studying four different environments that specify how categories of risk affect securities prices. Each environment represents a different way that a category of risk can affect net assets for individual financial institutions and the overall financial system. Categories of risk in the first environment generate an absolute adjustment to the values of a certain number of securities clusters, with stressed clusters facing the same magnitude of adjustment. Categories of risk in the second environment generate a percentage adjustment to the values of a certain number of securities clusters, with stressed clusters facing the same percentage adjustment. Categories of risk in the third environment generate potentially different levels of adjustment to the values of securities clusters, and categories of risk in the fourth environment generate potentially different percentage adjustments to the values of securities clusters. Any category of risk fits into one of these four environments.

We can map the category of risk to a probability distribution capturing possible balance sheet effects for each individual financial institution, $\forall i \in \{1, \ldots, M\}$, and we can map the category of risk to a probability distribution capturing possible balance sheet effects for the overall financial system. When the individual category of risk fits into the first or second environments, we can either construct the corresponding probability distributions in closed form, or we can construct asymptotic expansions that strongly approximate the CDFs of these probability distributions. When the individual category of risk fits into the third or fourth environments, we can solve in closed form for the major statistical features of the corresponding probability distributions. For each category of risk, we have many possible stress scenarios, and we are able to condense the effects of these stress scenarios into probability distributions.

Now that we have thoroughly studied individual categories of risk, we would like to construct classes of stress tests. Each class of stress tests features $Q$ categories of risk. We therefore index each category of risk by $q \in \{1, \ldots, Q\}$. We define random variable $\Pi_q^i(\cdot)$ as the change in net assets for financial institution $i$ given category of risk $q$, $\forall i \in \{1, \ldots, M\}$, and we define random variable $\Pi_{agg}^q(\cdot)$ as the change in net assets for the overall financial
system given category of risk $q$. Corresponding to these random variables are the CDFs $G_{\Pi_i^q(t)}$ and $G_{\Pi_{agg}^q(t)}$ and the PMFs $g_{\Pi_i^q(\cdot)}(t)$ and $g_{\Pi_{agg}^q(\cdot)}(t)$.

For a given class of stress tests, we introduce random variables $\Pi_i^{\text{class}}$, $\forall i \in \{1, \ldots, M\}$, and $\Pi_{agg}^{\text{class}}$, which respectively capture the entire change in net assets for financial institution $i$ and the entire change in net assets for the overall financial system. We then have the following relationships:

$$\Pi_i^{\text{class}} = \Pi_i^1(\cdot) + \Pi_i^2(\cdot) + \cdots + \Pi_i^Q(\cdot)$$
with $g_{\Pi_i^{\text{class}}}(t) = \left(g_{\Pi_i^1(\cdot)} * g_{\Pi_i^2(\cdot)} * \cdots * g_{\Pi_i^Q(\cdot)}\right)(t)$, and

$$\Pi_{agg}^{\text{class}} = \Pi_{agg}^1(\cdot) + \Pi_{agg}^2(\cdot) + \cdots + \Pi_{agg}^Q(\cdot)$$
with $g_{\Pi_{agg}^{\text{class}}}(t) = \left(g_{\Pi_{agg}^1(\cdot)} * g_{\Pi_{agg}^2(\cdot)} * \cdots * g_{\Pi_{agg}^Q(\cdot)}\right)(t)$,

$\forall i \{1, \ldots, M\}$, where the $*$ operator denotes convolution.\(^9\) Let $|E^{\text{class}}|$ be the unique number of stress tests in the entire class given that category of risk $q$ has $|E^q|$ unique stress scenarios. Then,

$$|E^{\text{class}}| = |E^1| \times |E^2| \times \cdots \times |E^Q|,$$

which can be extremely large.

We can solve for the major statistical features of $\Pi_i^{\text{class}}$, $\forall i \in \{1, \ldots, M\}$, and $\Pi_{agg}^{\text{class}}$ in closed form. For $i \in \{1, \ldots, M\}$,

$$E\Pi_i^{\text{class}} = \sum_{q=1}^{Q} E\Pi_i^q(\cdot), \quad \text{Var}\Pi_i^{\text{class}} = \sum_{q=1}^{Q} \text{Var}\Pi_i^q(\cdot),$$

$$\min \text{ supp } \Pi_i^{\text{class}} = \sum_{q=1}^{Q} \min \text{ supp } \Pi_i^q(\cdot), \quad \text{and} \quad \max \text{ supp } \Pi_i^{\text{class}} = \sum_{q=1}^{Q} \max \text{ supp } \Pi_i^q(\cdot).$$

Meanwhile, for $\Pi_{agg}^{\text{class}}$,

$$E\Pi_{agg}^{\text{class}} = \sum_{q=1}^{Q} E\Pi_{agg}^q(\cdot), \quad \text{Var}\Pi_{agg}^{\text{class}} = \sum_{q=1}^{Q} \text{Var}\Pi_{agg}^q(\cdot),$$

$$\min \text{ supp } \Pi_{agg}^{\text{class}} = \sum_{q=1}^{Q} \min \text{ supp } \Pi_{agg}^q(\cdot), \quad \text{and} \quad \max \text{ supp } \Pi_{agg}^{\text{class}} = \sum_{q=1}^{Q} \max \text{ supp } \Pi_{agg}^q(\cdot).$$

When category of risk $q$ fits into the first or second environment, we can approximate $g_{\Pi_i^q(\cdot)}(t)$

---

\(^9\)If more than one category of risk fits into the third environment and/or the fourth environment, we assume that each category of risk causes a percentage change to the original value of the securities cluster. We do not consider the case in which percentage changes to the value of a securities cluster are applied sequentially.
and $g_{\Pi_{agg}(\cdot)}(t)$ using the relevant asymptotic expansion. We first establish the following approximations, assuming that the category of risk fits into the first environment:

$$G_{\Pi_{i}(\cdot)}(t) \approx J \left( \hat{w}, L, \ell, \frac{t - E\Pi_{i}^{q}(\cdot)}{(\text{Var } \Pi_{i}^{q}(\cdot))^{1/2}} \right), \forall i \in \{1, \ldots, M\},$$

with $\hat{w}_i = \frac{[w_i]_j - EW_i}{\sqrt{L \text{Var } W_i}}$, and

$$G_{\Pi_{agg}(\cdot)}(t) \approx J \left( \hat{w}_{agg}, L, \ell, \frac{t - E\Pi_{agg}^{q}(\cdot)}{(\text{Var } \Pi_{agg}^{q}(\cdot))^{1/2}} \right),$$

with $\hat{w}_{agg} = \frac{[w_{agg}]_j - EW_{agg}}{\sqrt{L \text{Var } W_{agg}}}$. Depending on the particular category of risk, we need to determine the relevant support for the probability mass functions. If we suppose that the supports of $g_{\Pi_{i}(\cdot)}(t), \forall i \in \{1, \ldots, M\}$, and $g_{\Pi_{agg}(\cdot)}(t)$ take integer values, that is, $t \in \mathbb{Z}$, then

$$g_{\Pi_{i}(\cdot)}(t) \approx \lim_{\kappa \uparrow 0.5} J \left( \hat{w}, L, \ell, \frac{(t + \kappa) - E\Pi_{i}^{q}(\cdot)}{(\text{Var } \Pi_{i}^{q}(\cdot))^{1/2}} \right) - J \left( \hat{w}, L, \ell, \frac{(t - 0.5) - E\Pi_{i}^{q}(\cdot)}{(\text{Var } \Pi_{i}^{q}(\cdot))^{1/2}} \right), \forall i \in \{1, \ldots, M\},$$

with $\hat{w}_j = \frac{[w_i]_j - EW_i}{\sqrt{L \text{Var } W_i}}$, and

$$g_{\Pi_{agg}(\cdot)}(t) \approx \lim_{\kappa \uparrow 0.5} J \left( \hat{w}_{agg}, L, \ell, \frac{(t + \kappa) - E\Pi_{agg}^{q}(\cdot)}{(\text{Var } \Pi_{agg}^{q}(\cdot))^{1/2}} \right) - J \left( \hat{w}_{agg}, L, \ell, \frac{(t - 0.5) - E\Pi_{agg}^{q}(\cdot)}{(\text{Var } \Pi_{agg}^{q}(\cdot))^{1/2}} \right),$$

with $\hat{w}_{agg} = \frac{[w_{agg}]_j - EW_{agg}}{\sqrt{L \text{Var } W_{agg}}}$. We consider three different examples that characterize possible changes in net assets for individual financial institutions and the overall financial system given a particular stress test class:

**Example 9** Our class of stress tests is formed from the following category of risk: credit risk. Forty percent of all AAA-rated mortgage-backed securities have been downgraded to a CCC rating. As a result, the price of each affected mortgage-backed security has declined 80 percent from its original level. In addition, the euro has depreciated by fifteen percent relative to the U.S. dollar. We are interested in the possible changes in net assets for each individual financial institution $i, \forall i \in \{1, \ldots, M\}$, and we are interested in the possible changes in net assets for the overall financial system.

Here we have $Q = 1$. Let scalar $\gamma_i$ be the change in the market value of net assets for financial institution $i, \forall i \in \{1, \ldots, M\}$, when the euro depreciates fifteen percent relative to the U.S. dollar. Let scalar $\gamma_{agg}$ be the change in the market value of net assets for the entire financial system when the euro depreciates fifteen percent relative to the U.S. dollar. Random variable $\Pi_{i}^{q}(\cdot)$ captures possible changes in net assets for financial institution $i$ arising from
the downgrade of mortgage-backed securities, and random variable \( \Pi_{agg}^1 (\cdot) \) captures possible changes in net assets for the overall financial system arising from this form of credit risk. We then have:

\[
\begin{align*}
\Pi_i^{class} &= \Pi_i^1 (\cdot) + \gamma_i, \forall i \in \{1, \ldots, M\}, \text{ and} \\
\Pi_{class}^{class} &= \Pi_{agg}^1 (\cdot) + \gamma_{agg}.
\end{align*}
\]

Random variables \( \Pi_i^1 (\cdot) \) and \( \Pi_{agg}^1 (\cdot) \) are constructed by following Example 6. We can construct \( G_{\Pi_i^1} (\cdot) (t) \) and \( G_{\Pi_{agg}^1} (\cdot) (t) \) via asymptotic expansion. With \( L \) total initial AAA-rated mortgage-backed securities, the number of stress test scenarios in this class is \( |E^{class}| = \left( \frac{L}{0.4L} \right) \). Corresponding to all of these stress tests are probability distributions that capture possible changes in net assets for each individual financial institution and the overall financial system.

**Example 10** *Our class of stress tests is formed from two categories of risk:*

(1) *Credit risk.* Forty percent of all AAA-rated mortgage-backed securities have been downgraded to a CCC rating. As a result, the price of each affected mortgage-backed security has declined 80 percent from its original level.

(2) *Exchange rate risk.* One foreign currency has depreciated by 15 percent relative to the U.S. dollar.

We are interested in the possible changes in net assets for each individual financial institution \( i, \forall i \in \{1, \ldots, M\} \), and we are interested in the possible changes in net assets for the overall financial system.

Here we have \( Q = 2 \). Random variable \( \Pi_i^1 (\cdot) \) captures possible changes in net assets for financial institution \( i \) arising from the downgrade of mortgage-backed securities, and random variable \( \Pi_{agg}^1 (\cdot) \) captures possible changes in net assets for the overall financial system arising from this form of credit risk. We can construct \( \Pi_i^1 (\cdot) \) and \( \Pi_{agg}^1 (\cdot) \) by following Example 6. We can also strongly approximate \( G_{\Pi_i^1} (\cdot) (t) \) and \( G_{\Pi_{agg}^1} (\cdot) (t) \) via asymptotic expansion. Now, random variable \( \Pi_i^2 (\cdot) \) captures possible changes in net assets for financial institution \( i \) arising from foreign currency depreciation, and random variable \( \Pi_{agg}^2 (\cdot) \) captures possible changes in net assets for the overall financial system arising from foreign currency depreciation. We can construct \( \Pi_i^2 (\cdot) \) and \( \Pi_{agg}^2 (\cdot) \) by following Example 5 and as discussed
in that example, we can solve for \( G_{\Pi^2_i}(t) \) and \( G_{\Pi^2_{agg}}(t) \), and their corresponding PMFs, exactly. We then have:

\[
\Pi^\text{class}_i = \Pi^1_i(\cdot) + \Pi^2_i(\cdot), \forall i \in \{1, \ldots, M\}, \quad \text{and}
\]
\[
\Pi^\text{class}_{agg} = \Pi^1_{agg}(\cdot) + \Pi^2_{agg}(\cdot), \quad \text{with}
\]
\[
g_{\Pi^\text{class}_i}(t) = \left( g_{\Pi^1_i}(\cdot) \ast g_{\Pi^2_i}(\cdot) \right)(t), \forall i \in \{1, \ldots, M\}, \quad \text{and}
\]
\[
g_{\Pi^\text{class}_{agg}}(t) = \left( g_{\Pi^1_{agg}}(\cdot) \ast g_{\Pi^2_{agg}}(\cdot) \right)(t).
\]

We can generate \( g_{\Pi^1_i}(t) \) and \( g_{\Pi^1_{agg}}(t) \) from the asymptotic expansions that strongly approximate the CDFs \( G_{\Pi^1_i}(t) \) and \( G_{\Pi^1_{agg}}(t) \). With \( L_1 \) total initial AAA-rated mortgage-backed securities and \( L_2 \) total foreign currencies, the number of distinct stress test scenarios in this class is \( |E^{\text{class}}| = (0.4L_1)^M \times L_2 \).

**Example 11** Our class of stress tests is formed from three categories of risk:

1. **Credit risk.** Forty percent of all AAA-rated mortgage-backed securities have been downgraded to a CCC rating. As a result, the price of each affected mortgage-backed security has declined 80 percent from its original level.
2. **Exchange rate risk.** One foreign currency has depreciated by 15 percent relative to the U.S. dollar.
3. **Sovereign risk.** One country has a writedown of its sovereign debt, which causes the price of each sovereign bond for that country to decrease 10 U.S. dollars.

We are interested in the possible changes in net assets for each individual financial institution \( i \), \( \forall i \in \{1, \ldots, M\} \), and we are interested in the possible changes in net assets for the overall financial system.

Here we have \( Q = 3 \). We build on Example 10. Random variable \( \Pi^3_{i}(\cdot) \) captures possible changes in net assets for financial institution \( i \) arising from a writedown of sovereign debt, and \( \Pi^3_{agg}(\cdot) \) captures possible changes in net assets for the overall financial system arising from this form of sovereign risk. We can construct \( \Pi^3_{i}(\cdot) \) and \( \Pi^3_{agg}(\cdot) \) by following Example 4, and as discussed in that example, we can solve for \( G_{\Pi^3_i}(t) \) and \( G_{\Pi^3_{agg}}(t) \), and their corresponding PMFs, exactly. We then have

\[
\Pi^\text{class}_i = \Pi^1_i(\cdot) + \Pi^2_i(\cdot) + \Pi^3_i(\cdot), \forall i \in \{1, \ldots, M\}, \quad \text{and}
\]
\[
\Pi^\text{class}_{agg} = \Pi^1_{agg}(\cdot) + \Pi^2_{agg}(\cdot) + \Pi^3_{agg}(\cdot), \quad \text{with}
\]
\[ g_{\Pi_i^{\text{class}}} (t) = \left( g_{\Pi_1^{\text{class}}} (\cdot) * g_{\Pi_2^{\text{class}}} (\cdot) * g_{\Pi_3^{\text{class}}} (\cdot) \right) (t), \forall i \in \{1, \ldots, M\}, \quad \text{and} \]

\[ g_{\Pi_i^{\text{agg}}} (t) = \left( g_{\Pi_1^{\text{agg}}} (\cdot) * g_{\Pi_2^{\text{agg}}} (\cdot) * g_{\Pi_3^{\text{agg}}} (\cdot) \right) (t). \]

With \( L_1 \) total initial AAA-rated mortgage-backed securities, \( L_2 \) total foreign currencies, and \( L_3 \) total countries that have issued sovereign bonds, the total number of stress test scenarios in this class is

\[ |E^{\text{class}}| = \binom{L_1}{0.4 L_1} \times L_2 \times L_3. \]

### 3 Conclusion

The global financial crisis of 2008 forced a regulatory paradigm shift for central banks and other supervisory institutions around the world. As a result of the global financial crisis, supervisory institutions expanded their mandates. They rethought their specific functions, and they developed new approaches to regulation. In particular, central banks such as the Federal Reserve began to develop and implement stress tests as part of a new financial stability mandate, and they nominally shifted their regulatory approaches from ones that were purely microprudential to ones that were both macroprudential and microprudential.

Within the United States, even though stress tests were used in the financial industry prior to the global financial crisis, annual supervisory stress tests only became a mandatory part of the Federal Reserve’s regulatory toolkit with the passage of the 2010 Dodd-Frank Act. Globally, it was the post-crisis Basel III capital framework that articulated principles of stress testing for regulatory institutions; these global institutions could then use the Basel III framework as a constructive set of guidelines for developing their own stress tests.

The Federal Reserve’s set of stress tests, while designed to assess the stability of individual financial institutions and the financial system as a whole, can be made much more comprehensive and impactful. We want the Federal Reserve’s stress testing tools to be maximally informative, but in their current form, they only provide a limited view of the financial landscape and the broader macroeconomy. The number of stress tests that the Federal Reserve executes annually is very small, and the stress test scenarios are often calibrated to past historical events. As a result, current stress tests do not adequately assess the financial system’s overall health and ability to maintain operations and obligations in the presence of a broad range of possible negative shocks. The financial system can potentially be ill-equipped to handle certain realistic stressed scenarios, but this weakness will never be uncovered while employing the Federal Reserve’s current stress testing approach. The present work shows how to massively increase the total number of stress test scenarios without increasing the computational burden; this work therefore substantially strengthens the Federal Reserve’s stress testing process. Rather than collecting a small number of data
points summarizing financial institutions’ balance sheet effects for each individual stress test, the Federal Reserve can instead construct entire probability distributions that capture financial institutions’ balance sheet effects for a class of stress tests. Associated with each class of stress tests is a corresponding probability distribution for each individual financial institution and a probability distribution for the financial system as a whole.

The approach to stress tests in this work differs from the Federal Reserve’s existing approach. The Federal Reserve currently develops a very small number of stress tests that generally mimic past historical events. The present work, meanwhile, articulates a different framework for the Federal Reserve; according to the present work, the Federal Reserve would first identify different classes of stress tests, and then within each class, the Federal Reserve would generate an exhaustive list of constituent stress tests. The Federal Reserve would form each class of stress tests by identifying certain categories of risk. Different stress tests within the same class have the same categories of risk, but the manner by which such risks manifest themselves within the financial system would differ. The approach of this work for generating large classes of stress tests is top-down and macroprudential in spirit; we are assuming that the financial system inherently has certain types of overall stressors, but we are agnostic to how these stressors ultimately appear in the system. We consider all of the multitudinous ways by which these stressors can potentially manifest themselves.

Going forward, it is important that we craft a methodology for how to construct classes of stress tests. We would want to know which categories of risk are the most relevant and therefore appropriate for more thorough examination. We would also want to know how many different classes of stress tests are sufficient to enable thorough analysis of the financial system. Having the correct set of tools is crucial for the Federal Reserve and other central banks. These tools, when properly designed and employed, provide regulatory institutions with the ability to pinpoint potential sources of weakness within the financial system, evaluate the financial system’s overall health, and more generally survey the financial landscape. The financial system is inherently global, and shocks within the financial system transmit to the broader macroeconomy, so it is extremely important that we develop and employ the right set of regulatory tools.
REFERENCES


CLARK, A. AND A. LARGE (2011): Macroprudential policy: Addressing the things we don’t know, Group of Thirty.


APPENDIX: PROOFS

Proof of Proposition 1

\[ E\Pi_i (w_i, L, \ell) = E \left[ w_i^T \delta B (L, \ell) \right] \text{, where } B (L, \ell) \text{ is a random vector whose elements are } B_j \sim \text{Bern} \left( \frac{\ell}{L} \right), j \in \{1, \ldots, L\} \text{ and } 1^T w_i = k_i. \text{ Therefore,} \]

\[ E\Pi_i (w_i, L, \ell) = \delta E \left( [w_i]_1 B_1 + \cdots + [w_i]_L B_L \right) = \delta \sum_{j=1}^{L} [w_i]_j EB_j = \delta k_i \frac{\ell}{L}. \]

\[ E\Pi_{agg} (w_{agg}, L, \ell) = E \left[ w_{agg}^T \delta B (L, \ell) \right] \text{, where } 1^T w_{agg} = k_{agg}. \text{ Therefore,} \]

\[ E\Pi_{agg} (w_{agg}, L, \ell) = \delta E \left( [w_{agg}]_1 B_1 + \cdots + [w_{agg}]_L B_L \right) = \delta \sum_{j=1}^{L} [w_{agg}]_j EB_j = \delta k_{agg} \frac{\ell}{L}. \]

□

Proof of Proposition 2

\[ \text{Var} \Pi_i (w_i, L, \ell) = \text{Var} \left( [w_i]_1 B_1 + \cdots + [w_i]_L B_L \right) = \delta^2 \frac{\ell}{L} \left( 1 - \frac{\ell}{L} \right) \frac{L}{L-1} L \text{Var} W_i \]

from Theorem 9 in Schlossberger (2018b) and Lemma 2 in Schlossberger (2018a). Similarly, with

\[ \text{Var} \Pi_{agg} (w_{agg}, L, \ell) = \text{Var} \left( [w_{agg}]_1 B_1 + \cdots + [w_{agg}]_L B_L \right) = \delta^2 \frac{L}{L-1} L \text{Var} W_{agg}. \]

we then have \[ \text{Var} \Pi_{agg} (w_{agg}, L, \ell) = \delta^2 \frac{L}{L-1} L \text{Var} W_{agg}. \] □

Proof of Proposition 3

\[ \pi_i (w_i, \epsilon, L, \ell) = w_i^T \epsilon (L, \ell) = \delta \sum_{j \in \{1, \ldots, L\} \text{ s.t. } \epsilon_j = \delta} [w_i]_j \text{ and} \]

\[ \pi_{agg} (w_{agg}, \epsilon, L, \ell) = w_{agg}^T \epsilon (L, \ell) = \delta \sum_{j \in \{1, \ldots, L\} \text{ s.t. } \epsilon_j = \delta} [w_{agg}]_j. \]

With \( \delta < 0 \), the Proposition then follows. □
Proof of Proposition 4

\( \Pi_i (w_i, L, \ell) = \delta X (w_i, L, \ell) \), where random variable \( X (w_i, L, \ell) \) has realizations \( x (w_i, b, L, \ell) = w_i^T b (L, \ell) \), the elements of \( b (L, \ell) \) are either 0 or 1, and \( 1^T b (L, \ell) = \ell \). From Theorem 13 in Schlossberger (2018b), provided that condition (c) holds, and after re-labelling some variables, we have that

\[ \left| G_{X(w_i,L,\ell)-E X(w_i,L,\ell)} (t) - J (\hat{w}, L, \ell, t) \right| < C_4 \times \sum_{j=1}^{L} |\hat{w}_j|^5 \]

with \( \hat{w}_j = \frac{[w_i]_j - EW_i}{\sqrt{\text{Var} W_i}} \). Now, \( G_{\Pi_i(w_i,L,\ell)-E \Pi_i(w_i,L,\ell)} (t) = G_{X(w_i,L,\ell)-E X(w_i,L,\ell)} (t) \), so the Proposition holds for the individual financial institutions \( i \in \{1, \ldots, M\} \). Similarly, \( \Pi_{agg} (w_{agg}, L, \ell) = \delta X (w_{agg}, L, \ell) \), where random variable \( X (w_{agg}, L, \ell) \) has realizations \( x (w_{agg}, b, L, \ell) = w_{agg}^T b (L, \ell) \), the elements of \( b (L, \ell) \) are either 0 or 1, and \( 1^T b (L, \ell) = \ell \). Given Theorem 13 in Schlossberger (2018b), Proposition 4 therefore also holds when approximating the CDF for the entire financial system. \( \square \)

Proof of Lemma 1

With \( \circ \) denoting the Hadamard product,

\[ w_i^T \epsilon (L, \ell) = ([w_i]_1 [w_i]_2 \cdots [w_i]_L) \widehat{\delta} \begin{pmatrix} [\bar{p}]_1 \\ [\bar{p}]_2 \\ \vdots \\ [\bar{p}]_L \end{pmatrix} \circ \begin{pmatrix} [b (L, \ell)]_1 \\ [b (L, \ell)]_2 \\ \vdots \\ [b (L, \ell)]_L \end{pmatrix} \]

\[ = \left( [w_i]_1 [w_i]_2 [\bar{p}]_2 \cdots [w_i]_L [\bar{p}]_L \right) \begin{pmatrix} \widehat{\delta} [\bar{p}]_1 \\ \widehat{\delta} [\bar{p}]_2 \\ \vdots \\ \widehat{\delta} [\bar{p}]_L \end{pmatrix} \circ \begin{pmatrix} [b (L, \ell)]_1 \\ [b (L, \ell)]_2 \\ \vdots \\ [b (L, \ell)]_L \end{pmatrix} \]

\[ = \left( \widehat{\delta} [\bar{p}]_1 \right) v_{agg}^T (1_{L \times 1} \circ b (L, \ell)) \]

\[ = \widehat{\delta} [\bar{p}]_1 v_{agg}^T b (L, \ell). \]

To show that \( w_{agg}^T \epsilon (L, \ell) = \widehat{\delta} [\bar{p}]_1 v_{agg}^T b (L, \ell) \), simply replace \( w_i \) with \( w_{agg} \) in the above derivation. \( \square \)
Proof of Proposition 5

\[ E\Pi_i(w_i, L, \ell) = E[\hat{\delta}[\bar{p}]_1 v_i^T B(L, \ell)], \]
where \( B(L, \ell) \) is a random vector whose elements are \( B_j \sim \text{Bern}\left(\ell \frac{1}{L}\right), j \in \{1, \ldots, L\} \), and \( 1^T v_i = k_i \). Therefore,

\[ E\Pi_i(w_i, L, \ell) = \hat{\delta}[\bar{p}]_1 \sum_{j=1}^L [v_i]_j E B_j = \hat{\delta}[\bar{p}]_1 k_i \frac{\ell}{L}. \]

Meanwhile, \( E\Pi_{agg}(w_{agg}, L, \ell) = E[\hat{\delta}[\bar{p}]_1 v_{agg}^T B(L, \ell)] \), where \( 1^T v_{agg} = k_{agg} \). Then,

\[ E\Pi_{agg}(w_{agg}, L, \ell) = \hat{\delta}[\bar{p}]_1 \sum_{j=1}^L [v_{agg}]_j E B_j = \hat{\delta}[\bar{p}]_1 k_{agg} \frac{\ell}{L}. \]

Proof of Proposition 6

This proposition immediately follows from Proposition 2. Simply substitute \( w_i \) for \( v_i \), \( w_{agg} \) for \( v_{agg} \), and \( \delta \) for \( \hat{\delta}[\bar{p}]_1 \). \( \square \)

Proof of Proposition 7

\[ \pi_i(w_i, \epsilon, L, \ell) = w_i^T \epsilon = \hat{\delta}[\bar{p}]_1 v_i^T b(L, \ell) = \hat{\delta}[\bar{p}]_1 \sum_{j \in \{1, \ldots, L\}} \text{s.t. } [b]_j = 1 [v_i]_j, \]
and

\[ \pi_{agg}(w_{agg}, \epsilon, L, \ell) = w_{agg}^T \epsilon = \hat{\delta}[\bar{p}]_1 v_{agg}^T b(L, \ell) = \hat{\delta}[\bar{p}]_1 \sum_{j \in \{1, \ldots, L\}} \text{s.t. } [b]_j = 1 [v_{agg}]_j. \]

With \( \hat{\delta} < 0 \), the Proposition then follows. \( \square \)

Proof of Proposition 8

This proposition follows from Proposition 4. Set \( \Pi_i(w_i, L, \ell) = \hat{\delta}[\bar{p}]_1 X(v_i, L, \ell) \), where random variable \( X(v_i, L, \ell) \) has realizations \( x(v_i, b, L, \ell) = v_i^T b(L, \ell) \). Similarly, set \( \Pi_{agg}(w_{agg}, L, \ell) = \hat{\delta}[\bar{p}]_1 X(v_{agg}, L, \ell) \), where random variable \( X(v_{agg}, L, \ell) \) has realizations \( x(v_{agg}, b, L, \ell) = v_{agg}^T b(L, \ell) \). \( \square \)
Proof of Proposition \[9\]

First method of proof: Define \(\{\sigma_j(\delta)\}_{j=1}^{L!}\) as the family of all possible permutations of the elements in \(\delta\). Then,

\[
E\Pi_i(w_i,\delta) = \frac{1}{L!} \sum_{j=1}^{L!} w_i^T \sigma_j(\delta)
\]

\[
= \frac{1}{L!} \sum_{j=1}^{L!} \sum_{m=1}^L [w_i]_m [\sigma_j(\delta)]_m
\]

\[
= \sum_{m=1}^L [w_i]_m \left[ \frac{1}{L!} \sum_{j=1}^{L!} [\sigma_j(\delta)]_m \right]
\]

\[
= \sum_{m=1}^L [w_i]_m \left[ \frac{1}{L!} \sum_{n=1}^{L} (L-1)! [\delta]_n \right]
\]

\[
= \sum_{m=1}^L [w_i]_m \left[ \frac{1}{L} \sum_{n=1}^{L} [\delta]_n \right]
\]

\[
= \left( \frac{1^T \delta}{L} \right) k_i.
\]

Substituting \(w_i\) for \(w_{agg}\), \(E\Pi_{agg}(w_{agg},\delta) = \left( \frac{1^T \delta}{L} \right) k_{agg}\). \(\square\)

Second method of proof: \(E\Pi_i(w_i,\delta) = E[ w_i^T \Delta] \), where \(\Delta\) is an \(L \times 1\) random vector whose elements are \(\Delta_j\), \(j \in \{1, \ldots, L\}\). The random variables \(\Delta_1, \ldots, \Delta_L\) are identically distributed, with the corresponding CDF \(G_{\Delta_j}(t) = \frac{1}{L} \sum_{m=1}^L \mathbf{1}_{[\delta]_m \leq t}\). Then,

\[
E\Pi_i(w_i,\delta) = E( [w_i]_1 \Delta_1 + \cdots + [w_i]_L \Delta_L)
\]

\[
= \sum_{j=1}^L [w_i]_j E\Delta_j
\]

\[
= \left( \frac{1^T \delta}{L} \right) \sum_{j=1}^L [w_i]_j
\]

\[
= \left( \frac{1^T \delta}{L} \right) k_i.
\]

Substituting \(w_i\) for \(w_{agg}\), \(E\Pi_{agg}(w_{agg},\delta) = \left( \frac{1^T \delta}{L} \right) k_{agg}\). \(\square\)
Proof of Proposition 10

\[ \text{Var } \Pi_{agg}(w_{agg}, \delta) = \text{Var } (w_{agg}^T \Delta) \]

\[ = \text{Var } ([w_{agg}]_1 \Delta_1 + \cdots + [w_{agg}]_L \Delta_L) \]

\[ = \sum_{j=1}^{L} \text{Var } ([w_{agg}]_j \Delta_j) + \sum_{j=1}^{L} \sum_{m=1}^{L} \text{Cov } ([w_{agg}]_j \Delta_j, [w_{agg}]_m \Delta_m) \]

\[ = \sum_{j=1}^{L} (|w_{agg}|_j)^2 \text{Var } \Delta_j + \sum_{j=1}^{L} \sum_{m=1}^{L} |w_{agg}|_j |w_{agg}|_m \text{Cov } (\Delta_j, \Delta_m) \]

\[ = (\text{Var } \Delta_j) \sum_{j=1}^{L} (|w_{agg}|_j)^2 + (E [\Delta_j \Delta_m] - (E \Delta_j) (E \Delta_m)) \sum_{j=1}^{L} \sum_{m=1}^{L} |w_{agg}|_j |w_{agg}|_m. \]

We would like to find a simple expression to replace \( \sum_{j=1}^{L} \sum_{m=1}^{L} |w_{agg}|_j |w_{agg}|_m \). Note the following:

\[ \text{Var } W_{agg} = \frac{1}{L} \sum_{j=1}^{L} \left( |w_{agg}|_j - \frac{\sum_{m=1}^{L} |w_{agg}|_m}{L} \right)^2 \]

\[ = \frac{1}{L} \left( \sum_{j=1}^{L} (|w_{agg}|_j)^2 - \frac{1}{L} \left( \sum_{j=1}^{L} |w_{agg}|_j \right)^2 \right) \]

\[ = \frac{1}{L} \left( \sum_{j=1}^{L} (|w_{agg}|_j)^2 - \frac{1}{L} \left( \sum_{j=1}^{L} |w_{agg}|_j \right)^2 + \sum_{j=1}^{L} \sum_{m=1}^{L} |w_{agg}|_j |w_{agg}|_m \right) \]

\[ = \frac{1}{L} \left( \sum_{j=1}^{L} (|w_{agg}|_j)^2 - \frac{1}{L-1} \sum_{j=1}^{L} \sum_{m=1, j \neq m}^{L} |w_{agg}|_j |w_{agg}|_m \right). \]

Therefore,

\[ \sum_{j=1}^{L} \sum_{m=1, j \neq m}^{L} |w_{agg}|_j |w_{agg}|_m = (L-1) \sum_{j=1}^{L} (|w_{agg}|_j)^2 - L^2 \text{Var } W_{agg}, \]

and

\[ \text{Var } \Pi_{agg}(w_{agg}, \delta) = (\text{Var } \Delta_j) \sum_{j=1}^{L} (|w_{agg}|_j)^2 \]

\[ + (E [\Delta_j \Delta_m] - (E \Delta_j) (E \Delta_m)) \left[ (L-1) \sum_{j=1}^{L} (|w_{agg}|_j)^2 - L^2 \text{Var } W_{agg} \right]. \]

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To compute $\text{Var} \Pi_i (w_i, \delta)$ for $i \in \{1, \ldots, M\}$, replace $w_{agg}$ with $w_i$, and replace $W_{agg}$ with $W_i$. □

**Proof of Proposition 11**

$$\pi_i (w_i, e, \delta) = w_i^T e (\delta)$$
and
$$\pi_{agg} (w_{agg}, e, \delta) = w_{agg}^T e (\delta).$$
The elements in the multisets \{\tilde{w}_j\}_{j=1}^{L}, \{\tilde{x}_j\}_{j=1}^{L}, and \{\tilde{\delta}_j\}_{j=1}^{L}
are real-valued and weakly increasing with index. By the rearrangement inequality,

$$w_L \delta_1 + w_{L-1} \delta_2 + \cdots + w_2 \delta_2 + \cdots + w_1 \delta_2 \leq \tilde{w}_1 \sigma(1) \tilde{\delta}_1 + \tilde{w}_2 \sigma(2) \tilde{\delta}_2 + \cdots + \tilde{w}_L \sigma(L) \tilde{\delta}_L \leq \tilde{w}_1 \tilde{\delta}_1 + \tilde{w}_2 \tilde{\delta}_2 + \cdots + \tilde{w}_L \tilde{\delta}_L$$

and

$$\tilde{x}_L \delta_1 + \tilde{x}_{L-1} \delta_2 + \cdots + \tilde{x}_1 \delta_2 \leq \tilde{x}_1 \sigma(1) \tilde{\delta}_1 + \tilde{x}_2 \sigma(2) \tilde{\delta}_2 + \cdots + \tilde{x}_L \sigma(L) \tilde{\delta}_L \leq \tilde{x}_1 \tilde{\delta}_1 + \tilde{x}_2 \tilde{\delta}_2 + \cdots + \tilde{x}_L \tilde{\delta}_L$$

for all possible permutations $\tilde{w}_\sigma(1), \tilde{w}_\sigma(2), \ldots, \tilde{w}_\sigma(L)$ of elements $\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_L$, and for all possible permutations $\tilde{x}_\sigma(1), \tilde{x}_\sigma(2), \ldots, \tilde{x}_\sigma(L)$ of elements $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_L$. □

**Proof of Lemma 2**

$$w_i^T (e (\tilde{\delta}) \circ \bar{p}) = \begin{bmatrix} w_i \end{bmatrix} \begin{bmatrix} \begin{bmatrix} e (\tilde{\delta}) \end{bmatrix}_1 \\ \begin{bmatrix} e (\tilde{\delta}) \end{bmatrix}_2 \\ \vdots \\ \begin{bmatrix} e (\tilde{\delta}) \end{bmatrix}_L \end{bmatrix} \begin{bmatrix} \bar{p}_1 \\ \bar{p}_2 \\ \vdots \\ \bar{p}_L \end{bmatrix}$$

$$= \begin{bmatrix} w_i \end{bmatrix} \begin{bmatrix} \begin{bmatrix} e (\tilde{\delta}) \end{bmatrix}_1 \\ \begin{bmatrix} e (\tilde{\delta}) \end{bmatrix}_2 \\ \vdots \\ \begin{bmatrix} e (\tilde{\delta}) \end{bmatrix}_L \end{bmatrix} \begin{bmatrix} \bar{p}_1 \\ \bar{p}_2 \\ \vdots \\ \bar{p}_L \end{bmatrix}$$

$$= [\bar{p}]_1 v_i^T e (\tilde{\delta}).$$

To show that $w_{agg}^T (e (\tilde{\delta}) \circ \bar{p}) = [\bar{p}]_1 v_{agg}^T e (\tilde{\delta})$, simply replace $w_i$ with $w_{agg}$ in the above derivation. □

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**Proof of Proposition 12**

Following the first method of proof in Proposition 9, define \( \{ \sigma_j(\hat{\delta}) \}_{j=1}^{L!} \) as the family of all possible permutations of the elements in \( \hat{\delta} \). Then,

\[
E \Pi_i \left( w_i, \delta \right) = \frac{1}{L!} \sum_{j=1}^{L!} [\bar{p}]_1 \nu_i^T \sigma_j(\hat{\delta})
\]

\[
= [\bar{p}]_1 \frac{1}{L!} \sum_{j=1}^{L!} \nu_i^T \sigma_j(\hat{\delta})
\]

\[
= [\bar{p}]_1 \left( \frac{1^T \hat{\delta}}{L} \right) k_i.
\]

Substituting \( w_i \) for \( w_{agg} \), \( E \Pi_{agg} \left( w_{agg}, \hat{\delta} \right) = [\bar{p}]_1 \left( \frac{1^T \hat{\delta}}{L} \right) k_{agg}. \]

**Proof of Proposition 13**

\[
\text{Var} \Pi_i \left( w_i, \delta \right) = \text{Var} \left( [\bar{p}]_1 \nu_i^T \hat{\Delta} \right) = ([\bar{p}]_1)^2 \text{Var} \left( \nu_i^T \hat{\Delta} \right)
\]

and

\[
\text{Var} \Pi_{agg} \left( w_{agg}, \hat{\delta} \right) = \text{Var} \left( [\bar{p}]_1 \nu_{agg}^T \hat{\Delta} \right) = ([\bar{p}]_1)^2 \text{Var} \left( \nu_{agg}^T \hat{\Delta} \right).
\]

To compute \( \text{Var} \left( \nu_i^T \hat{\Delta} \right) \), we substitute \( w_i \) with \( \nu_i \) and \( \Delta \) with \( \hat{\Delta} \) in Proposition 10. To compute \( \text{Var} \left( \nu_{agg}^T \hat{\Delta} \right) \), we substitute \( w_{agg} \) with \( \nu_{agg} \) and \( \Delta \) with \( \hat{\Delta} \) in Proposition 10. This Proposition then follows.

**Proof of Proposition 14**

For all \( i \in \{1, \ldots, M\} \), \( \pi_i \left( w_i, \epsilon, \hat{\delta} \right) = [\bar{p}]_1 \nu_i^T \epsilon \left( \hat{\delta} \right) \) and \( \pi_{agg} \left( w_{agg}, \epsilon, \hat{\delta} \right) = [\bar{p}]_1 \nu_{agg}^T \epsilon \left( \hat{\delta} \right) \).

The elements in the multisets \( \{ \bar{v}_j \}_{j=1}^{L} \), \( \{ \bar{x}_j \}_{j=1}^{L} \), and \( \{ \bar{\delta}_j \}_{j=1}^{L} \) are real-valued and weakly increasing with index. This Proposition then follows from the rearrangement inequality, as detailed in Proposition 11. □