Online Appendix

"Rearranging Attributes in Networked Economies"

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Appendix A: Additional Proofs

Proof of Theorem 3

Dropping the "N" subscripts, we first prove the following lemma:

Lemma 4

$$EZ^2 = \frac{1}{N-1}\sum_{i=1}^{N}\sum_{j=1}^{N}(w(i))^2(a(j))^2 + \frac{N^3}{N-1}\bar{w}^2\bar{a}^2 - \frac{N}{N-1}\bar{w}^2\sum_{i=1}^{N}(a(i))^2 - \frac{N}{N-1}\bar{a}^2\sum_{i=1}^{N}(w(i))^2.$$

Proof.

$$EZ^2 = E\left(\left[\sum_{i=1}^{N}w(i)\,a(X_i)\right]\left[\sum_{j=1}^{N}w(j)\,a(X_j)\right]\right)$$

$$= E\left[\sum_{i=1}^{N}(w(i))^2(a(X_i))^2\right] + E\left[\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N}w(i)\,w(j)\,a(X_i)\,a(X_j)\right]$$

$$= \frac{1}{N}\sum_{j=1}^{N}(a(j))^2\sum_{i=1}^{N}(w(i))^2 + \left[\frac{1}{N(N-1)}\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N}a(i)\,a(j)\right] \times \left[\sum_{i=1}^{N}\sum_{j=1}^{N}w(i)\,w(j) - \sum_{i=1}^{N}(w(i))^2\right]$$

$$= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}(w(i))^2(a(j))^2 + \frac{1}{N(N-1)}\left[\sum_{i=1}^{N}\sum_{j=1}^{N}a(i)\,a(j) - \sum_{i=1}^{N}(a(i))^2\right] \times \left[\sum_{i=1}^{N}w(i)\sum_{j=1}^{N}w(j) - \sum_{i=1}^{N}(w(i))^2\right]$$

and Lemma 4 follows. ■

We next prove the following lemma:

Lemma 5

$$
\begin{aligned}
EZ^3 = \frac{N}{(N-1)(N-2)} \sum_{i=1}^{N}\sum_{j=1}^{N} (a(i))^3 (w(j))^3 \\
+ \frac{3N(N+1)}{(N-1)(N-2)} \bar{a}\bar{w} \sum_{i=1}^{N}\sum_{j=1}^{N} (a(i))^2 (w(j))^2 \\
- \frac{3N}{(N-1)(N-2)} \bar{a} \sum_{i=1}^{N}\sum_{j=1}^{N} (a(i))^2 (w(j))^3 \\
- \frac{3N}{(N-1)(N-2)} \bar{w} \sum_{i=1}^{N}\sum_{j=1}^{N} (a(i))^3 (w(j))^2 \\
+ \frac{N^5}{(N-1)(N-2)} \bar{a}^3\bar{w}^3 + \frac{2N^2}{(N-1)(N-2)} \bar{a}^3 \sum_{i=1}^{N} (w(i))^3 \\
- \frac{3N^3}{(N-1)(N-2)} \bar{a}^3\bar{w} \sum_{i=1}^{N} (w(i))^2 \\
+ \frac{2N^2}{(N-1)(N-2)} \bar{w}^3 \sum_{i=1}^{N} (a(i))^3 - \frac{3N^3}{(N-1)(N-2)} \bar{a}\bar{w}^3 \sum_{i=1}^{N} (a(i))^2 .
\end{aligned}
$$

Proof.

$$
EZ^3 = E\left[\sum_{i=1}^{N} w(i)\, a(X_i)\right]^3 = E\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} w(i)\, w(j)\, w(k)\, a(X_i)\, a(X_j)\, a(X_k)\right]
$$

$$
\begin{aligned}
= E\Bigg[\sum_{i=1}^{N} (w(i)\, a(X_i))^3 + 3\sum_{i=1}^{N}\sum_{\substack{j=1 \\ i\neq j}}^{N} (w(i)\, a(X_i))^2\, w(j)\, a(X_j) \\
+ \sum_{i=1}^{N}\sum_{\substack{j=1 \\ i\neq j\neq k}}^{N}\sum_{k=1}^{N} w(i)\, a(X_i)\, w(j)\, a(X_j)\, w(k)\, a(X_k)\Bigg]
\end{aligned}
$$

$$= \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} (w(i))^3 (a(j))^3 + \frac{3}{N(N-1)} \left[\sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} (a(i))^2 a(j) \right]$$

$$\times \left[\sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} (w(i))^2 w(j) \right] + \frac{1}{N(N-1)(N-2)} \left[\sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j \neq k}}^{N} \sum_{k=1}^{N} a(i) a(j) a(k) \right]$$

$$\times \left[\sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j \neq k}}^{N} \sum_{k=1}^{N} w(i) w(j) w(k) \right]$$

To obtain Lemma 5, note the following decomposition:

$$\sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} (a(i))^2 a(j) = \sum_{i=1}^{N} \sum_{j=1}^{N} (a(i))^2 a(j) - \sum_{i=1}^{N} (a(i))^3,$$

and note the additional decomposition:

$$\sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j \neq k}}^{N} \sum_{k=1}^{N} a(i) a(j) a(k) = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} a(i) a(j) a(k) - \sum_{i=1}^{N} (a(i))^3$$

$$- 3 \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} (a(i))^2 a(j)$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} a(i) a(j) a(k) - \sum_{i=1}^{N} (a(i))^3 - 3 \left[\sum_{i=1}^{N} \sum_{j=1}^{N} (a(i))^2 a(j) - \sum_{i=1}^{N} (a(i))^3 \right].$$

■

Now,

$$E(Z - EZ)^3 = EZ^3 - 3(EZ)\left(EZ^2\right) + 2(EZ)^3.$$

Plugging in the expressions from Lemma 4, Lemma 5, and Theorem 1, we have:

$$E\left(Z-EZ\right)^3=\frac{N}{\left(N-1\right)\left(N-2\right)}NE\left[W-EW\right]^3NE\left[A-EA\right]^3.$$

Lastly, solving for Skew Z:

$$\text{Skew}\,Z=\frac{E\left[Z-EZ\right]^3}{\left(\text{Var}\,Z\right)^{3/2}}=\frac{\left(N-1\right)^{1/2}}{N-2}\,\text{Skew}\,W\ \ \text{Skew}\,A.\ \square$$

Proof of Theorem 4

Dropping the "N" subscripts, we first prove a couple of lemmata:

Lemma 6

$$\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j\neq k}}^{N}\sum_{k=1}^{N}\left(w\left(i\right)\right)^2w\left(j\right)w\left(k\right)=\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\left(w\left(i\right)\right)^2w\left(j\right)w\left(k\right)+2\sum_{i=1}^{N}\left(w\left(i\right)\right)^4$$
$$-2\sum_{i=1}^{N}\sum_{j=1}^{N}\left(w\left(i\right)\right)^3w\left(j\right)-\sum_{i=1}^{N}\sum_{j=1}^{N}\left(w\left(i\right)\right)^2\left(w\left(j\right)\right)^2.$$

Proof. We have

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\left(w\left(i\right)\right)^2w\left(j\right)w\left(k\right)=\sum_{i=1}^{N}\left(w\left(i\right)\right)^4+2\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N}\left(w\left(i\right)\right)^3w\left(j\right)$$
$$+\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N}\left(w\left(i\right)\right)^2\left(w\left(j\right)\right)^2+\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j\neq k}}^{N}\sum_{k=1}^{N}\left(w\left(i\right)\right)^2w\left(j\right)w\left(k\right).$$

With

$$\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N}\left(w\left(i\right)\right)^3w\left(j\right)=\sum_{i=1}^{N}\sum_{j=1}^{N}\left(w\left(i\right)\right)^3w\left(j\right)-\sum_{i=1}^{N}\left(w\left(i\right)\right)^4\ \text{ and}$$

$$\sum_{i=1}^{N}\sum_{\substack{j=1\\i\neq j}}^{N}(w(i))^2(w(j))^2=\sum_{i=1}^{N}\sum_{j=1}^{N}(w(i))^2(w(j))^2-\sum_{i=1}^{N}(w(i))^4,$$

Lemma 6 follows. ■

Lemma 7

$$\sum_{i=1}^{N}\sum_{\substack{j=1\\i\neq j\neq k\neq \ell}}^{N}\sum_{k=1}^{N}\sum_{\ell=1}^{N}w(i)\,w(j)\,w(k)\,w(\ell)=\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{\ell=1}^{N}w(i)\,w(j)\,w(k)\,w(\ell)$$
$$-6\sum_{i=1}^{N}(w(i))^4+8\sum_{i=1}^{N}\sum_{j=1}^{N}(w(i))^3\,w(j)+3\sum_{i=1}^{N}\sum_{j=1}^{N}(w(i))^2(w(j))^2$$
$$-6\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}(w(i))^2\,w(j)\,w(k).$$

Proof.

$$\sum_{i=1}^{N}\sum_{\substack{j=1\\i\neq j\neq k\neq \ell}}^{N}\sum_{k=1}^{N}\sum_{\ell=1}^{N}w(i)\,w(j)\,w(k)\,w(\ell)=\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{\ell=1}^{N}w(i)\,w(j)\,w(k)\,w(\ell)$$
$$-\sum_{i=1}^{N}(w(i))^4-4\sum_{i=1}^{N}\sum_{\substack{j=1\\i\neq j}}^{N}(w(i))^3\,w(j)-3\sum_{i=1}^{N}\sum_{\substack{j=1\\i\neq j}}^{N}(w(i))^2(w(j))^2$$
$$-6\sum_{i=1}^{N}\sum_{\substack{j=1\\i\neq j\neq k}}^{N}\sum_{k=1}^{N}(w(i))^2\,w(j)\,w(k).$$

Given Lemma 6, we have:

$$\sum_{\substack{i=1}}^{N}\sum_{\substack{j=1 \\ i\neq j\neq k\neq \ell}}^{N}\sum_{k=1}^{N}\sum_{\ell=1}^{N} w(i)\,w(j)\,w(k)\,w(\ell) = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{\ell=1}^{N} w(i)\,w(j)\,w(k)\,w(\ell)$$
$$-\sum_{i=1}^{N}(w(i))^4 - 4\left[\sum_{i=1}^{N}\sum_{j=1}^{N}(w(i))^3\,w(j) - \sum_{i=1}^{N}(w(i))^4\right]$$
$$-3\left[\sum_{i=1}^{N}\sum_{j=1}^{N}(w(i))^2\,(w(j))^2 - \sum_{i=1}^{N}(w(i))^4\right]$$
$$-6\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}(w(i))^2\,w(j)\,w(k) + 2\sum_{i=1}^{N}(w(i))^4\right.$$
$$\left. -2\sum_{i=1}^{N}\sum_{j=1}^{N}(w(i))^3\,w(j) - \sum_{i=1}^{N}\sum_{j=1}^{N}(w(i))^2\,(w(j))^2\right]$$

and Lemma 7 follows. ■

Lemma 8

$$EZ^4 = \frac{N(N+1)}{(N-1)(N-2)(N-3)}\sum_{i=1}^{N}\sum_{j=1}^{N}(w(i))^4\,(a(j))^4$$
$$+4N\bar{w}\bar{a}\left[\frac{N^2+N+4}{(N-1)(N-2)(N-3)}\right]\sum_{i=1}^{N}\sum_{j=1}^{N}(w(i))^3\,(a(j))^3$$
$$-\frac{4\bar{a}N(N+1)}{(N-1)(N-2)(N-3)}\sum_{i=1}^{N}\sum_{j=1}^{N}(w(i))^4\,(a(j))^3$$
$$-\frac{4\bar{w}N(N+1)}{(N-1)(N-2)(N-3)}\sum_{i=1}^{N}\sum_{j=1}^{N}(w(i))^3\,(a(j))^4$$
$$+\frac{3(N^2-3N+3)}{N(N-1)(N-2)(N-3)}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{\ell=1}^{N}(w(i))^2\,(w(j))^2\,(a(k))^2\,(a(\ell))^2$$

$$- \frac{3}{(N-2)(N-3)} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} (w(i))^2 (w(j))^2 (a(k))^4$$

$$- \frac{3}{(N-2)(N-3)} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} (w(i))^4 (a(j))^2 (a(k))^2$$

$$+ \frac{6N^3\bar{w}^2\bar{a}^2 (N+3)}{(N-1)(N-2)(N-3)} \sum_{i=1}^{N} \sum_{j=1}^{N} (w(i))^2 (a(j))^2$$

$$+ \frac{12N^2\bar{a}^2}{(N-1)(N-2)(N-3)} \sum_{i=1}^{N} \sum_{j=1}^{N} (w(i))^4 (a(j))^2$$

$$- \frac{12N^2\bar{w}\bar{a}^2 (N+1)}{(N-1)(N-2)(N-3)} \sum_{i=1}^{N} \sum_{j=1}^{N} (w(i))^3 (a(j))^2$$

$$- \frac{6N^2\bar{a}^2}{(N-1)(N-2)(N-3)} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} (w(i))^2 (w(j))^2 (a(k))^2$$

$$+ \frac{12N^2\bar{w}^2}{(N-1)(N-2)(N-3)} \sum_{i=1}^{N} \sum_{j=1}^{N} (w(i))^2 (a(j))^4$$

$$- \frac{12N^2\bar{w}^2\bar{a} (N+1)}{(N-1)(N-2)(N-3)} \sum_{i=1}^{N} \sum_{j=1}^{N} (w(i))^2 (a(j))^3$$

$$+ \frac{12\bar{a}}{(N-2)(N-3)} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} (w(i))^2 (w(j))^2 (a(k))^3$$

$$- \frac{6N^2\bar{w}^2}{(N-1)(N-2)(N-3)} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} (w(i))^2 (a(j))^2 (a(k))^2$$

$$+ \frac{12\bar{w}}{(N-2)(N-3)} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} (w(i))^3 (a(j))^2 (a(k))^2 + \frac{N^7\bar{w}^4\bar{a}^4}{(N-1)(N-2)(N-3)}$$

$$- \frac{6N^3\bar{a}^4}{(N-1)(N-2)(N-3)} \sum_{i=1}^{N} (w(i))^4 + \frac{8N^4\bar{w}\bar{a}^4}{(N-1)(N-2)(N-3)} \sum_{i=1}^{N} (w(i))^3$$

$$+ \frac{3N^3\bar{a}^4}{(N-1)(N-2)(N-3)} \sum_{i=1}^{N} \sum_{j=1}^{N} (w(i))^2 (w(j))^2$$

$$- \frac{6N^5\bar{w}^2\bar{a}^4}{(N-1)(N-2)(N-3)} \sum_{i=1}^{N} (w(i))^2$$

$$- \frac{6N^3\bar{w}^4}{(N-1)(N-2)(N-3)} \sum_{i=1}^{N} (a(i))^4 + \frac{8N^4\bar{w}^4\bar{a}}{(N-1)(N-2)(N-3)} \sum_{i=1}^{N} (a(i))^3$$
$$+ \frac{3N^3\bar{w}^4}{(N-1)(N-2)(N-3)} \sum_{i=1}^{N} \sum_{j=1}^{N} (a(i))^2 (a(j))^2$$
$$- \frac{6N^5\bar{w}^4\bar{a}^2}{(N-1)(N-2)(N-3)} \sum_{i=1}^{N} (a(i))^2 .$$

Proof. Solving for EZ^4:

$$\begin{aligned} EZ^4 &= E\left[\sum_{i=1}^{N} w(i)\, a(X_i)\right]^4 \\ &= E\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{\ell=1}^{N} w(i)\, w(j)\, w(k)\, w(\ell)\, a(X_i)\, a(X_j)\, a(X_k)\, a(X_\ell)\right]. \end{aligned}$$

We have the following decomposition:

$$\begin{aligned} \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{\ell=1}^{N} y(i)\, y(j)\, y(k)\, y(\ell) &= \sum_{i=1}^{N} (y(i))^4 + 4\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} (y(i))^3\, y(j) \\ &+ 3\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} (y(i))^2 (y(j))^2 + 6\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}{}_{i\neq j\neq k} (y(i))^2\, y(j)\, y(k) \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{\ell=1}^{N}{}_{i\neq j\neq k\neq \ell}\, y(i)\, y(j)\, y(k)\, y(\ell) . \end{aligned}$$

Then:

$$EZ^4 = E\left[\sum_{i=1}^{N}\left(w\left(i\right)a\left(X_i\right)\right)^4 + 4\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N}\left(w\left(i\right)a\left(X_i\right)\right)^3\left(w\left(j\right)a\left(X_j\right)\right)\right.$$
$$+3\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N}\left(w\left(i\right)a\left(X_i\right)\right)^2\left(w\left(j\right)a\left(X_j\right)\right)^2$$
$$+6\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j\neq k}}^{N}\sum_{k=1}^{N}\left(w\left(i\right)a\left(X_i\right)\right)^2\left(w\left(j\right)a\left(X_j\right)\right)\left(w\left(k\right)a\left(X_k\right)\right)$$
$$\left.+\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j\neq k\neq \ell}}^{N}\sum_{k=1}^{N}\sum_{\ell=1}^{N}\left(w\left(i\right)a\left(X_i\right)\right)\left(w\left(j\right)a\left(X_j\right)\right)\left(w\left(k\right)a\left(X_k\right)\right)\left(w\left(\ell\right)a\left(X_\ell\right)\right)\right]$$

Given Lemmata 6 and 7, we have:

$$EZ^4 = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(w\left(i\right)\right)^4\left(a\left(j\right)\right)^4 + \frac{4}{N(N-1)}\left[\sum_{k=1}^{N}\sum_{\ell=1}^{N}\left(a\left(k\right)\right)^3 a\left(\ell\right) - \sum_{k=1}^{N}\left(a\left(k\right)\right)^4\right]$$
$$\times\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\left(w\left(i\right)\right)^3 w\left(j\right) - \sum_{i=1}^{N}\left(w\left(i\right)\right)^4\right]$$
$$+\frac{3}{N(N-1)}\left[\sum_{k=1}^{N}\sum_{\ell=1}^{N}\left(a\left(k\right)\right)^2\left(a\left(\ell\right)\right)^2 - \sum_{k=1}^{N}\left(a\left(k\right)\right)^4\right]$$
$$\times\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\left(w\left(i\right)\right)^2\left(w\left(j\right)\right)^2 - \sum_{i=1}^{N}\left(w\left(i\right)\right)^4\right]$$
$$+\frac{6}{N(N-1)(N-2)}\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\left(a\left(i\right)\right)^2 a\left(j\right)a\left(k\right) + 2\sum_{i=1}^{N}\left(a\left(i\right)\right)^4\right.$$
$$\left.-2\sum_{i=1}^{N}\sum_{j=1}^{N}\left(a\left(i\right)\right)^3 a\left(j\right) - \sum_{i=1}^{N}\sum_{j=1}^{N}\left(a\left(i\right)\right)^2\left(a\left(j\right)\right)^2\right]$$

$$\times \left[\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} (w(i))^2 w(j) w(k) + 2 \sum_{i=1}^{N} (w(i))^4 - 2 \sum_{i=1}^{N} (w(i))^3 w(j) \right.$$
$$\left. - \sum_{i=1}^{N} \sum_{j=1}^{N} (w(i))^2 (w(j))^2 \right]$$
$$+ \frac{1}{N(N-1)(N-2)(N-3)} \left[\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{\ell=1}^{N} a(i) a(j) a(k) a(\ell) \right.$$
$$-6 \sum_{i=1}^{N} (a(i))^4 + 8 \sum_{i=1}^{N} \sum_{j=1}^{N} (a(i))^3 a(j) + 3 \sum_{i=1}^{N} \sum_{j=1}^{N} (a(i))^2 (a(j))^2$$
$$\left. -6 \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} (a(i))^2 a(j) a(k) \right]$$
$$\times \left[\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{\ell=1}^{N} w(i) w(j) w(k) w(\ell) - 6 \sum_{i=1}^{N} (w(i))^4 + 8 \sum_{i=1}^{N} \sum_{j=1}^{N} (w(i))^3 w(j) \right.$$
$$\left. +3 \sum_{i=1}^{N} \sum_{j=1}^{N} (w(i))^2 (w(j))^2 - 6 \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} (w(i))^2 w(j) w(k) \right]$$

Lemma 8 follows. ■

Now,

$$E(Z - EZ)^4 = EZ^4 - 4\left(EZ^3\right)(EZ) + 6\left(EZ^2\right)(EZ)^2 - 3(EZ)^4 .$$

Plugging in the expressions from Theorem 1 and Lemmata 4, 5, and 8, we obtain:

$$E(Z-EZ)^4 = \frac{3}{(N+1)(N-1)}\left[\sum_{i=1}^{N}(w(i)-\bar{w})^2\right]^2\left[\sum_{i=1}^{N}(a(i)-\bar{a})^2\right]^2$$
$$+\frac{1}{(N+1)N(N-1)(N-2)(N-3)}\left(N(N+1)\sum_{i=1}^{N}(w(i)-\bar{w})^4\right.$$
$$\left.-3(N-1)\left[\sum_{i=1}^{N}(w(i)-\bar{w})^2\right]^2\right)\left(N(N+1)\sum_{i=1}^{N}(a(i)-\bar{a})^4\right.$$
$$\left.-3(N-1)\left[\sum_{i=1}^{N}(a(i)-\bar{a})^2\right]^2\right).$$

Given the expression for $\operatorname{Var} Z$ in Theorem 2, we solve for $\operatorname{Kurt} Z$:

$$\begin{aligned}\operatorname{Kurt} Z &= \frac{E(Z-EZ)^4}{(\operatorname{Var} Z)^2}\\ &= \frac{3(N-1)}{N+1}+\frac{(N+1)(N-1)}{N(N-2)(N-3)}\left[\operatorname{Kurt} W-\frac{3(N-1)}{(N+1)}\right]\left[\operatorname{Kurt} A-\frac{3(N-1)}{N+1}\right].\end{aligned}$$

□

Proof of Theorem 9

This theorem follows immediately from the work of Calvó-Armengol, Patacchini, and Zenou (2009). These authors, in their Appendix A, build on the work of Ballester, Calvó-Armengol, and Zenou (2006). They consider a network game with the following payoffs (line A3 of their paper with notation modified for the present work):

$$u_i(\mathbf{e}_N,\mathbf{G}) = \alpha(i)e_N(i)-\frac{1}{2}(\delta-\gamma)(e_N(i))^2-\gamma\sum_{k=1}^{N}e_N(i)e_N(k)+\lambda\sum_{k=1}^{N}[\mathbf{G}]_{ik}\,e_N(i)e_N(k).$$

The model of peer influence in the present text is the same as the one above, but with the following parameter substitutions: $\alpha(i) = \mu n_i + a_N(\chi_{Ni})$, $\delta = 1$, $\gamma = 0$, and $\lambda = \phi$. Given these parameter substitutions, we re-write Theorem 1 Part (b) of Calvó-Armengol, Patacchini, and Zenou (2009) as the following lemma:

Lemma 9 (Calvó-Armengol, Patacchini, and Zenou (2009), Theorem 1(b)) *If $\phi\rho(\mathbf{G}) < 1$, then this game has a unique Nash equilibrium in pure strategies $\mathbf{e}_N^*$, which is interior and given by*

$$\mathbf{e}_N^* = (\mathbf{I} - \phi\mathbf{G})^{-1} \boldsymbol{\alpha},$$

where $\boldsymbol{\alpha} = \mu\mathbf{G}\mathbf{1} + \boldsymbol{\alpha}_N(\boldsymbol{\chi}_N)$.

We derive this Nash equilibrium through utility maximization:

$$\max_{e_N(i)} u_i(\mathbf{e}_N, \mathbf{G}) = \max_{e_N(i)} [\mu n_i + a_N(\chi_{Ni})] e_N(i) - \frac{1}{2}(e_N(i))^2 + \phi \sum_{k=1}^{N} [\mathbf{G}]_{ik} e_N(i) e_N(k).$$

The first-order condition is:

$$e_N^*(i) = \mu n_i + a_N(\chi_{Ni}) + \phi \sum_{k=1}^{N} [\mathbf{G}]_{ik} e_N^*(k).$$

At the population level,

$$\mathbf{e}_N^* = \mu\mathbf{G}\mathbf{1} + \mathbf{a}_N(\boldsymbol{\chi}_N) + \phi\mathbf{G}\mathbf{e}_N^*,$$

which yields

$$\mathbf{e}_N^* = (\mathbf{I} - \phi\mathbf{G})^{-1} (\mu\mathbf{G}\mathbf{1} + \mathbf{a}_N(\boldsymbol{\chi}_N)).$$

Theorem 9 then immediately follows from Lemma 9. □

Proof of Theorem 10

The optimization problem for consumer j is:

$$\max_{x_{Nj}, y_{Nj}} u_j\left(x_{Nj}, \Phi_j\left(y_{Nj}, \{y_{Nk}\}_{k\in\mathcal{N}(j)}\right)\right) \text{ s.t. } x_{Nj} + p(1+T) y_{Nj} = a_{Nj}.$$

With

$$u_j\left(x_{Nj}, \Phi_j\left(y_{Nj}, \{y_{Nk}\}_{k\in\mathcal{N}(j)}\right)\right) = x_{Nj}^{\sigma}\left(\Phi_j\left(y_{Nj}, \{y_{Nk}\}_{k\in\mathcal{N}(j)}\right)\right)^{1-\sigma}$$

and

$$\Phi_j\left(y_{Nj}, \{y_{Nk}\}_{k\in\mathcal{N}(j)}\right) = y_{Nj} + \alpha n_j\left[y_{Nj} - \frac{1}{n_j}\sum_{k\in\mathcal{N}(j)} y_{Nk}\right],$$

the optimization problem becomes:

$$\max_{x_{Nj},y_{Nj}} x_{Nj}^{\sigma}\left(y_{Nj} + \alpha n_j\left[y_{Nj} - \frac{1}{n_j}\sum_{k\in\mathcal{N}(j)} y_{Nk}\right]\right)^{1-\sigma} \quad \text{s.t. } x_{Nj} + p\left(1+T\right)y_{Nj} = a_{Nj}.$$

Dropping the asterisks in x_{Nj}^* and y_{Nj}^*, the three first-order conditions are:

$$\sigma x_{Nj}^{\sigma-1}\left(y_{Nj} + \alpha n_j\left[y_{Nj} - \frac{1}{n_j}\sum_{k\in\mathcal{N}(j)} y_{Nk}\right]\right)^{1-\sigma} = \lambda \tag{1}$$

$$x_{Nj}^{\sigma}\left(1-\sigma\right)\left(y_{Nj} + \alpha n_j\left[y_{Nj} - \frac{1}{n_j}\sum_{k\in\mathcal{N}(j)} y_{Nk}\right]\right)^{-\sigma}\left(1+\alpha n_j\right) = \lambda p\left(1+T\right) \tag{2}$$

$$x_{Nj} + p\left(1+T\right)y_{Nj} = a_{Nj} \tag{3}$$

Now,

$$\begin{aligned}\sigma x_{Nj}^{\sigma-1}&\left(y_{Nj} + \alpha n_j\left[y_{Nj} - \frac{1}{n_j}\sum_{k\in\mathcal{N}(j)} y_{Nk}\right]\right)^{1-\sigma} \\ &= \frac{1}{p\left(1+T\right)}\left(1-\sigma\right)x_{Nj}^{\sigma}\left(y_{Nj} + \alpha n_j\left[y_{Nj} - \frac{1}{n_j}\sum_{k\in\mathcal{N}(j)} y_{Nk}\right]\right)^{-\sigma}\left(1+\alpha n_j\right)\end{aligned}$$

$$\sigma x_{Nj}^{\sigma-1}\left(y_{Nj}\left(1+\alpha n_j\right)-\alpha\sum_{k\in\mathcal{N}(j)}y_{Nk}\right)^{1-\sigma}$$

$$=\frac{1-\sigma}{p\left(1+T\right)}x_{Nj}^{\sigma}\left(y_{Nj}\left(1+\alpha n_j\right)-\alpha\sum_{k\in\mathcal{N}(j)}y_{Nk}\right)^{-\sigma}\left(1+\alpha n_j\right)$$

$$\sigma\left(y_{Nj}\left(1+\alpha n_j\right)-\alpha\sum_{k\in\mathcal{N}(j)}y_{Nk}\right)=\frac{1-\sigma}{p\left(1+T\right)}\left(1+\alpha n_j\right)x_{Nj}$$

$$x_{Nj}=\frac{\sigma p\left(1+T\right)}{1-\sigma}\frac{y_{Nj}\left(1+\alpha n_j\right)-\alpha\sum_{k\in\mathcal{N}(j)}y_{Nk}}{1+\alpha n_j}$$

$$x_{Nj}=\frac{\sigma p\left(1+T\right)}{1-\sigma}\left(y_{Nj}-\frac{\alpha}{1+\alpha n_j}\sum_{k\in\mathcal{N}(j)}y_{Nk}\right)\tag{4}$$

Plugging (4) into the budget constraint and solving for y_{Nj}^*:

$$x_{Nj}+p\left(1+T\right)y_{Nj}=a_{Nj}$$

$$\frac{\sigma p\left(1+T\right)}{1-\sigma}\left(y_{Nj}-\frac{\alpha}{1+\alpha n_j}\sum_{k\in\mathcal{N}(j)}y_{Nk}\right)+p\left(1+T\right)y_{Nj}=a_{Nj}$$

$$\frac{\sigma p\left(1+T\right)}{1-\sigma}y_{Nj}+p\left(1+T\right)y_{Nj}-\frac{\sigma p\left(1+T\right)}{1-\sigma}\frac{\alpha}{1+\alpha n_j}\sum_{k\in\mathcal{N}(j)}y_{Nk}=a_{Nj}$$

$$\frac{p\left(1+T\right)}{1-\sigma}y_{Nj}-\frac{\sigma p\left(1+T\right)}{1-\sigma}\frac{\alpha}{1+\alpha n_j}\sum_{k\in\mathcal{N}(j)}y_{Nk}=a_{Nj}$$

$$\frac{p\left(1+T\right)}{1-\sigma}y_{Nj}=a_{Nj}+\frac{\sigma p\left(1+T\right)}{1-\sigma}\frac{\alpha}{1+\alpha n_j}\sum_{k\in\mathcal{N}(j)}y_{Nk}$$

$$y_{Nj}^*=\frac{1-\sigma}{p\left(1+T\right)}\left[a_{Nj}+\frac{\sigma}{1-\sigma}\frac{\alpha}{1+\alpha n_j}p\left(1+T\right)\sum_{k\in\mathcal{N}(j)}y_{Nk}^*\right]\tag{5}$$

Plugging (5) into the budget constraint and solving for x_{Nj}^*:

$$x_{Nj} + p\left(1+T\right) y_{Nj} = a_{Nj}$$

$$x_{Nj} + p\left(1+T\right) \frac{1-\sigma}{p\left(1+T\right)} \left[a_{Nj} + \frac{\sigma}{1-\sigma} \frac{\alpha}{1+\alpha n_j} p\left(1+T\right) \sum_{k \in \mathcal{N}(j)} y_{Nk} \right] = a_{Nj}$$

$$x_{Nj} + \left(1-\sigma\right) \left[a_{Nj} + \frac{\sigma}{1-\sigma} \frac{\alpha}{1+\alpha n_j} p\left(1+T\right) \sum_{k \in \mathcal{N}(j)} y_{Nk} \right] = a_{Nj}$$

$$x_{Nj} = a_{Nj} - \left(1-\sigma\right) \left[a_{Nj} + \frac{\sigma}{1-\sigma} \frac{\alpha}{1+\alpha n_j} p\left(1+T\right) \sum_{k \in \mathcal{N}(j)} y_{Nk} \right]$$

$$x_{Nj}^* = \sigma \left[a_{Nj} - \frac{\alpha}{1+\alpha n_j} p\left(1+T\right) \sum_{k \in \mathcal{N}(j)} y_{Nk}^* \right] \tag{6}$$

Solving for $\mathbf{y}_N^*$:

$$y_{Nj}^* = \frac{1-\sigma}{p\left(1+T\right)} \left(a_{Nj} + \frac{\sigma}{1-\sigma} \frac{\alpha}{1+\alpha n_j} p\left(1+T\right) \sum_{k \in \mathcal{N}(j)} y_{Nk}^* \right)$$

$$\mathbf{y}_N^* = \frac{1-\sigma}{p\left(1+T\right)} \mathbf{a}_N + \alpha\sigma \mathbf{P}\mathbf{G}^N \mathbf{P}^T \mathbf{y}_N^*$$

$$\left(\mathbf{I} - \alpha\sigma \mathbf{P}\mathbf{G}^N \mathbf{P}^T\right) \mathbf{y}_N^* = \frac{1-\sigma}{p\left(1+T\right)} \mathbf{a}_N$$

$$\mathbf{P}\left(\mathbf{I} - \alpha\sigma \mathbf{G}^N\right) \mathbf{P}^T \mathbf{y}_N^* = \frac{1-\sigma}{p\left(1+T\right)} \mathbf{a}_N$$

$$\mathbf{y}_N^* = \frac{1-\sigma}{p\left(1+T\right)} \mathbf{P}\left(\mathbf{I} - \alpha\sigma \mathbf{G}^N\right)^{-1} \mathbf{P}^T \mathbf{a}_N \tag{7}$$

Solving for z_N^*:

$$z_N^* = Tp\mathbf{1}^T \mathbf{y}_N^*$$

$$z_N^* = Tp\mathbf{1}^T \left[\frac{1-\sigma}{p\left(1+T\right)} \mathbf{P}\left(\mathbf{I} - \alpha\sigma \mathbf{G}^N\right)^{-1} \mathbf{P}^T \mathbf{a}_N \right]$$

$$z_N^* = \frac{T(1-\sigma)}{1+T} \mathbf{1}^T \left(\mathbf{I} - \alpha\sigma\mathbf{G}^N\right)^{-1} \mathbf{P}^T \mathbf{a}_N \tag{8}$$

The matrix, $\mathbf{I} - \alpha\sigma\mathbf{G}^N$, is invertible if $\alpha\sigma$ is smaller than the inverse of the modulus of the largest eigenvalue of $\mathbf{G}^N$. By the Perron-Frobenius theorem, the largest eigenvalue of $\mathbf{G}^N$ is less than the maximum sum across all rows of $\mathbf{G}^N$. The sum of row j for matrix $\mathbf{G}^N$ is: $\frac{[\mathbf{G}]_{j*}\mathbf{1}}{1+\alpha[\mathbf{G}]_{j*}\mathbf{1}}$, which monotonically increases with $[\mathbf{G}]_{j*}\mathbf{1}$ and tends to $\frac{1}{\alpha}$ as $[\mathbf{G}]_{j*}\mathbf{1} \to \infty$. The modulus of the largest eigenvalue of $\mathbf{G}^N$ is then less than $\frac{1}{\alpha}$. With $\sigma < 1$ by assumption, the condition for invertibility of $\mathbf{I} - \alpha\sigma\mathbf{G}^N$ holds. □

Proof of Theorem 11

The proof of Theorem 11 is identical to the proof of Theorem 9, except that we make the following parameter substitutions: $\alpha(j) = \pi + \xi - p_N(j) f$, $\delta = 1$, $\gamma = 0$, and $\lambda = \phi$, and the relevant matrix is $\mathbf{PGP}^T$, not $\mathbf{G}$. We derive the Nash equilibrium through utility maximization.

Individual j chooses effort $e_N(j)$ to maximize the following utility function:

$$\max_{e_N(j)} (\pi + \xi - p_N(j) f) e_N(j) - \frac{1}{2}(e_N(j))^2 + \phi \sum_{k=1}^{N} \left[\mathbf{PGP}^T\right]_{jk} e_N(j) e_N(k).$$

The first-order condition for individual j is:

$$e_N^*(j) = (\pi + \xi - p_N(j) f) + \phi \sum_{k=1}^{N} \left[\mathbf{PGP}^T\right]_{jk} e_N^*(k).$$

At the population level:

$$\mathbf{e}_N^* = (\pi + \xi)\mathbf{1} - \mathbf{p}_N f + \phi\mathbf{PGP}^T\mathbf{e}_N^*.$$

Solving for $\mathbf{e}_N^*$:

$$\left(\mathbf{I} - \phi\mathbf{PGP}^T\right)\mathbf{e}_N^* = (\pi + \xi)\mathbf{1} - \mathbf{p}_N f$$

$$\mathbf{P}(\mathbf{I} - \phi\mathbf{G})\mathbf{P}^T\mathbf{e}_N^* = (\pi + \xi)\mathbf{1} - \mathbf{p}_N f$$

$$\mathbf{e}_N^* = \mathbf{P}\left(\mathbf{I} - \phi\mathbf{G}\right)^{-1}\mathbf{P}^T\left[\left(\pi + \xi\right)\mathbf{1} - \mathbf{p}_N f\right].$$

Finally, solving for e_{agg}^*:

$$e_{agg}^* = \mathbf{1}^T\mathbf{e}_N^* = \mathbf{1}^T\left(\mathbf{I} - \phi\mathbf{G}\right)^{-1}\mathbf{P}^T\left[\left(\pi + \xi\right)\mathbf{1} - \mathbf{p}_N f\right].$$

□

Appendix B: References

Ballester, C., A. Calvó-Armengol, and Y. Zenou (2006): "Who's who in networks. Wanted: The key player," *Econometrica*, 74, 1403-1417.

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