Introduction to Convex Optimization, Game Theory and Variational Inequalities

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G. Scutari, D. Palomar, F. Facchinei, and J. Pang, "Convex Optimization, Game Theory, and Variational Inequality Theory," IEEE Signal Processing Magazine, vol. 27, no. 3, pp. 35–49, May 2010.

Introduction

- Optimization Problems:
  - Linear programming:

$$\begin{aligned} \max_x \; & c^T x \\ \text{s.t. } & Ax \leq b \\ & x \geq 0 \end{aligned}$$

  - LASSO problem:

$$\min_x \; \|y - Ax\|^2 + \lambda |x|$$

  - Support Vector Machines

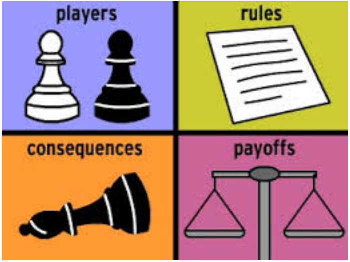
$$\begin{aligned} \min_{w,b} \; & \|w\|_2 \\ \text{s.t. } & y_i \left(w^T x_i - b\right) \geq 1. \end{aligned}$$

  - k-means Clustering:

$$\arg\min_S \sum_{i=1}^{k} \sum_{x \in S_i} \|x - \mu_i\|^2$$

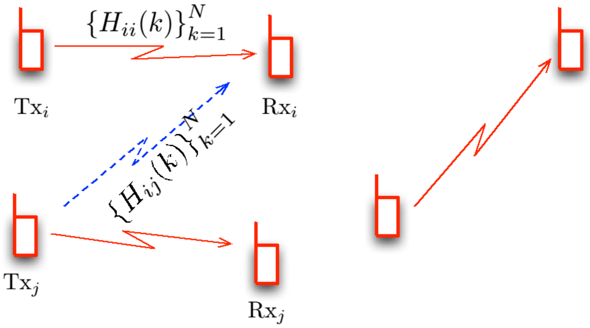
Introduction

- Game Theory:
  - Rough definition: Coupled optimization problems
  - Players interaction: Distributed modelling
  - Purpose?: solution concept.
  - Examples: resource sharing of wireless networks, p2p networks, smart grids



Game Theory examples (I)

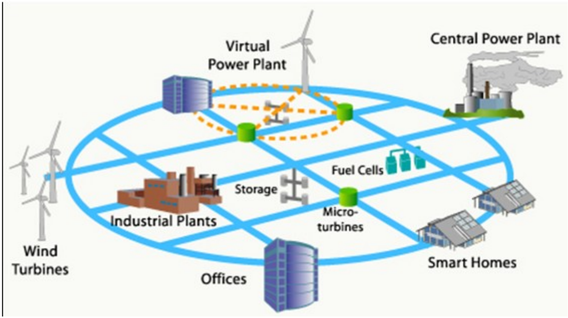
- Consider a peer-to-peer (ad-hoc) wireless network with Q users:



Ad-hoc Network (=Interference channel)

Game Theory examples (II)

- Consider a Demand-side-management perspective in a smart grid



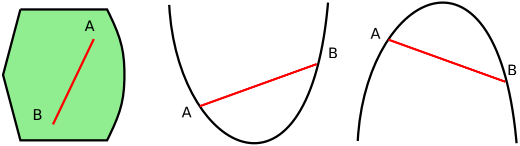
Distributed generation, consumtion, storage (=big data)

Convex Optimization Theory

Convex Problem

$$\min \; f(x)$$
$$\text{s.t. } x \in \mathcal{K}$$

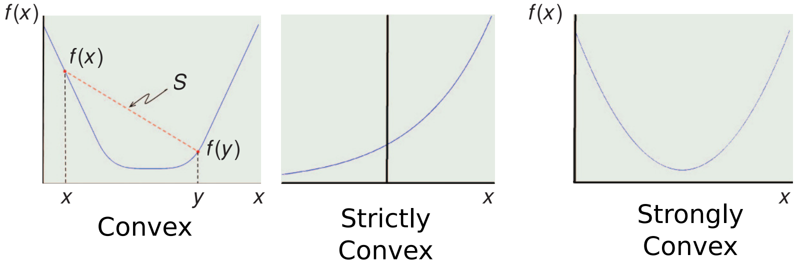
- $\mathcal{K}$ is closed and convex.
- $f(x)$ is convex.
- Convex set: $\alpha x + (1-\alpha)y \in \mathcal{K}$, for all $x, y \in \mathcal{K}$ and $\alpha \in [0,1]$.
  - Unit ball: $\mathcal{K} = \{x \in \mathbb{R}^n | \, \|x\| \leq 1\}$.
  - Positive quadrant (cone): $\mathcal{K} = \{x \in \mathbb{R}^n | x_i \geq 0\}$.



Convex Problem

1. Finding if a problem is convex: inspection, operations that preserve convexity, definition
2. Properties of the problem:
   1. Convexity $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(x)$
   2. Strict convexity $f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha)f(x)$
   3. Strong convexity $f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha)f(x) - \frac{c}{2}\|x-y\|^2$

strongly convex $\Rightarrow$ strictly convex $\Rightarrow$ convex



Characterization of Solutions

- Minimum: A feasible point $x^* \in \mathcal{K}$ is said to be optimal if

$$f(x^*) \leq f(x) \qquad \forall x \in \mathcal{K}.$$

- Minimum principle:

$$\boxed{(y - x^*)\,\nabla f(x^*) \geq 0 \qquad \forall y \in \mathcal{K}}$$

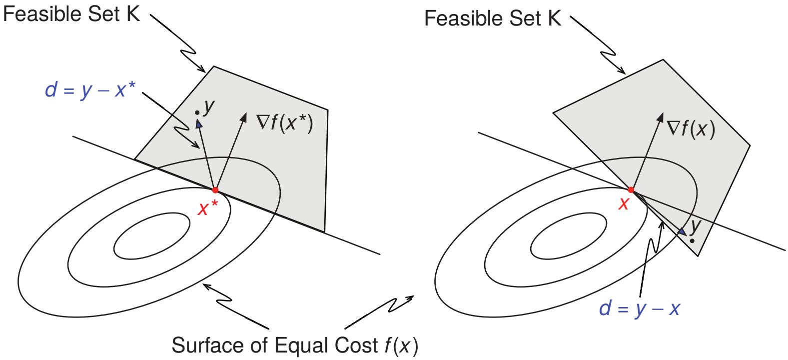
- Unconstrained optimization $\Leftrightarrow \nabla f(x^*) = 0$.

Existance and uniqueness

Convex $\Rightarrow$ Multiple Solutions (convex set)
Strictly Convex $\Rightarrow$ 1 solution (at most)
Strongly Convex $\Rightarrow$ Unique solution

Graphical Interpretation

$$(y - x^*)\,\nabla f(x^*) \geq 0 \qquad \forall y \in \mathcal{K}$$

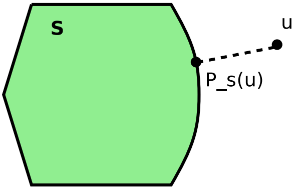


Example I: Projection

- Euclidean Projection

$$\min_x \|x - u\|_2^2$$
$$\text{s.t. } x \in \mathcal{K}$$

$$\Pi_{\mathcal{K}}(u) \equiv \hat{x} = \arg\min_{x \in \mathcal{K}} \|x - u\|_2^2$$



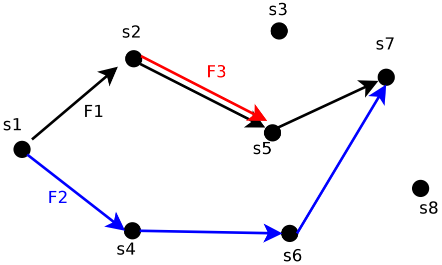
- Gradient Projection Algorithm

$$\min_x f(x)$$
$$\text{s.t.} x \in \mathcal{K}$$

$$x^{k+1} = \Pi_{\mathcal{K}}\left(x^k + \alpha \nabla_x f\left(x^k\right)\right)$$

Example II: Network Flow Control

$$\max_{x_i} \sum_{b \in B} U_i(x_i)$$
$$\text{s.t. } A^T x \leq c$$
$$x_i \geq 0$$



Karush-Kuhn-Tucker (KKT)

$$\min_{x} \; f(x)$$
$$\text{s.t. } g(x) \leq 0$$

- Let's define the Lagrangian:

$$L\left(x, \lambda\right) = f\left(x\right) + \mu^T g_l\left(x\right)$$

- Optimality criteria: KKT conditions

$$\nabla_x f(x) + \mu^T \nabla_x g\left(x\right) = 0$$
$$0 \leq \mu \perp -g(x) \geq 0$$

- Dual problem (assumption: strong duality holds)

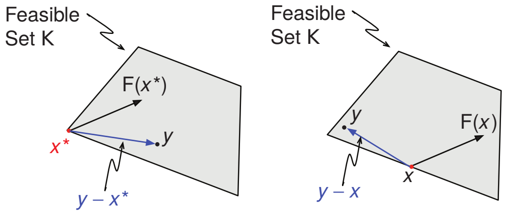
$$q(\mu) = \min_{x} \; f\left(x\right) + \mu^T g_l\left(x\right)$$
$$\max_{\mu \geq 0} \; q(\mu)$$

Variational Inequalities

Variational Inequality Problem

- Given a closed and convex set $\mathcal{K} \in \mathbb{R}^n$,
- a continuous mapping $\mathbf{F} : \mathcal{K} \to \mathbb{R}^n$,
  then, the $VI(\mathcal{K}, \mathbf{F})$ problem is to find a vector such

$$(\mathbf{y} - \mathbf{x}^\star)\,\mathbf{F}(\mathbf{x}^\star) \geq 0, \quad \forall \mathbf{y} \in \mathcal{K}$$



The importance of VI: that they provide a theory in which to test existance/uniqueness of solutions, and algorithms to find those solutions!!

Variational Inequality Examples

- Optimization problem: $\min_{x \in \mathcal{K}} f(x)$

$$\mathcal{K} = \{x \in \mathcal{K}\}$$
$$F = \nabla_x f(x)$$
$$VI(\mathcal{K}, F)$$

- System of equations: find an $x^* \in \mathbb{R}^n$ such that $F(x^*) = 0$

$$\mathcal{K} = \mathbb{R}^n$$
$$VI(\mathbb{R}^n, F)$$

- Nonlinear complementarity problem: $0 \leq \mu^* \perp F(\mu^*) \geq 0$

$$\mathcal{K} = \{\mu \geq 0\}$$
$$VI(\mathbb{R}^n_+, F)$$

- Non-cooperative Games
- $VI\left(\mathcal{K}, \mathbf{F}\right)$ represents a wider range of problems than classical optimization whenever $\mathbf{F} \neq \nabla f$.

Existence of the solution

Given the VI($\mathcal{K}, F$), suppose that

1. The set $\mathcal{K} \subseteq \mathbb{R}^n$ is compact and convex, and
2. The function $F : \mathcal{K} \to \mathbb{R}^n$ be continuous.

Then, the $VI\,(\mathcal{K}, \mathbf{F})$ has a nonempty and compact solution set.

- The requirement on $\mathcal{K}$ being compact can be very restrictive. Other results exists.

Monotonicity properties of functions:

1. Monotone:

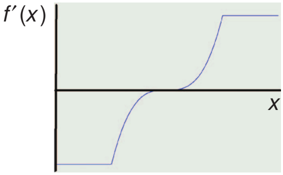
$$F(x) - F(y)^T(x - y) \geq 0 \qquad \forall x, y \in \mathcal{K}$$

2. Strictly monotone:

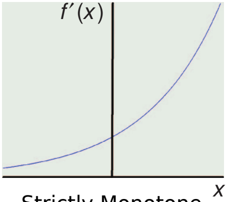
$$F(x) - F(y)^T(x - y) > 0 \qquad \forall x, y \in \mathcal{K} \text{ and } x \neq y$$

3. Strongly monotone:

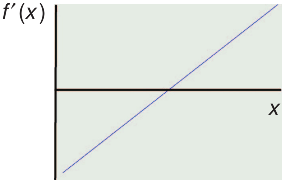
$$F(x) - F(y)^T(x - y) > c\,\|x - y\|^2 \qquad \forall x, y \in \mathcal{K}$$



Monotone



Strictly Monotone



Strongly Monotone

Existence and uniqueness of the solutions

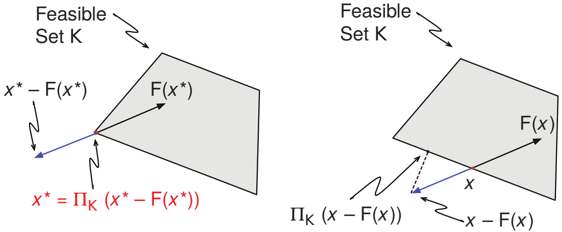
1. If F is monotone $\Longrightarrow$ the solution set of the VI$(\mathcal{K}, F)$ is closed and convex
2. If F is strictly monotone $\Longrightarrow$ the VI admits at most one solution
3. If F is strongly monotone $\Longrightarrow$ the VI admits a unique solution

- If the $VI(\mathcal{K}, F)$ corresponds to a optimization problem $\min_{x \in \mathcal{K}} f(x)$, then

$$
\begin{aligned}
i) f \text{ convex} &\quad\Longleftrightarrow \nabla f \text{ monotone} \\
ii) f \text{ strictly convex} &\quad\Longleftrightarrow \nabla f \text{ strictly monotone} \\
iii) f \text{ strongly convex} &\quad\Longleftrightarrow \nabla f \text{ strongly monotone}
\end{aligned}
$$

Characterization of the solution

$$\boxed{x^* \text{ is a solution of the } VI(\mathcal{K}, F) \iff x^* = \Pi_{\mathcal{K}}\left(x^* - F(x^*)\right)}$$



- The fixed-point equation invites for an iterative algorithm

$$x^{k+1} = \Pi_{\mathcal{K}}\left(x^k - \alpha F(x^k)\right)$$

- Convergence is globally guaranteed under monotonity requirements.
- There are also necessary KKT conditions for solutions (as in the convex problem)

Game Theory

Non-cooperative Game Theory

- Resolution of problems with interacting decision-makers (called players).

$$\mathcal{G} =< \prod_i \mathcal{K}_i, \mathbf{f} >$$

- Noncooperative: selfish players try to optimize their own objective function.t

$$\begin{matrix} \min f_i\left(x_i, \mathbf{x}_{-i}\right) \\ \text{s.t.} x_i \in \mathcal{K}_i \end{matrix} i = 1, ..., Q$$

where $\mathbf{x}_{-i} = [x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_Q]^T$.

- **Nash Equilibrium** (NE): a point $x^* \in \mathcal{K}$ is NE, iff

$$f_i\left(x_i^\star, \mathbf{x}_{-i}^\star\right) \leq f_i\left(y_i, \mathbf{x}_{-i}^\star\right), \quad \forall y_i \in \mathcal{K}_i, \quad \forall i$$

where $\mathcal{K} = \prod_i \mathcal{K}_i$.

Types of Nash Equilibrium Problems

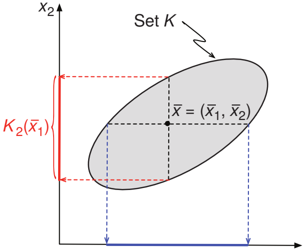
- NE Problems (NEP)

$$\begin{array}{ll} \min_{x_i} f_i\left(x_i, \mathbf{x}_{-i}\right) & \\ & i = 1, ..., Q \\ \text{s.t.} x_i \in \mathcal{K}_i & \end{array}$$

- Generalized NEP (GNEP)

$$\begin{array}{ll} \min_{x_i} f_i\left(x_i, \mathbf{x}_{-i}\right) & \\ & i = 1, ..., Q \\ \text{s.t.} x_i \in \mathcal{K}_i\left(\mathbf{x}_{-i}\right) & \end{array}$$

- GNEP with shared constratins: $\mathcal{K}_i\left(\mathbf{x}_{-i}\right) = \left\{x_i : \mathbf{g}\left(x_i, \mathbf{x}_{-i}\right) \leq 0\right\}$



VI Reformulation of the NEP

- $\mathcal{K} = \prod_i \mathcal{K}_i$ and $\mathbf{f} = (f_i(\mathbf{x}))_{i=1}^{Q}$

Equivalence with VI

Given the game $\mathcal{G} =< \mathcal{K}, \mathbf{f} >$,

1. the strategy ste $\mathcal{K}_i$ are closed and convex;
2. the payoff functions $f_i(x_i, \mathbf{x}_{-i})$ are continuously differentiable in $\mathbf{x}$ and convex in x_i for every fixed $\mathbf{x}_{-i}$.

Then the game $\mathcal{G}$ is equivalent to the $VI(\mathcal{K}, \mathbf{F})$, where $\mathbf{F}(x) = (\nabla_{x_i} f_i(\mathbf{x}))_{i=1}^{Q}$.

Characterization of NE

- The minimum principle (NE): For every $i \in \{1, \ldots, Q\}$,

$$\boxed{(y_i - x_i^*)^T \nabla_{x_i} f_i(x_i, x_{-i}^*) \geq 0, \qquad \forall y_i \in \mathcal{K}_i}$$

- The NE necessary condition can be equivalently expressed with the solution of VI.
- If we can express a game with VI, we can use existence and uniqueness results of VI to infer NE solutions.
- Moreover, we have a choice of algorithms to find the solution.

Best Response Algorithm

- Let $\mathcal{B}_i(x_{-i})$ be the set of optimal solutions of the ith optimization problem

$$\min_{x_i} f_i\left(x_i, \mathbf{x}_{-i}\right)$$
$$\text{s.t.} x_i \in \mathcal{K}_i$$

  and set $\mathcal{B}(\mathbf{x}) = \mathcal{B}_1(x_{-1}) \times \mathcal{B}_2(x_{-2}) \times \cdots \times \mathcal{B}_Q(x_{-Q})$

- A point is a NE iff

$$\mathbf{x}^* \in \mathcal{B}(\mathbf{x}^*)$$

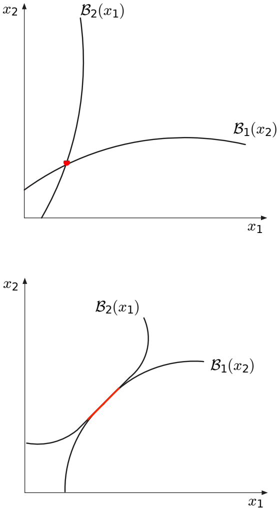
  which is another fixed-point equation.

- An iterative algorithm of the form

$$x_i^{k+1} = \mathcal{B}(x_{-i}^k)$$

  with $x_{-i}^k = \left(x_1^{k+1}, x_2^{k+1}, \ldots, x_{i-1}^{k+1}, x_{i+1}^k, \ldots, x_Q^k\right)$ (Gauss-Seidel), converges if the VI associated to the NEP is strongly monotone.

Best Response Algorithm



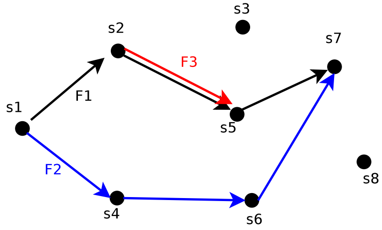
Example: Network Flow Problem

Q users, shared constraints GNEP

$$\max_{x_i} U_i(x_i)$$
$$\text{s.t. } A^T\mathbf{x} \le c$$

Lagrangian and KKT:

$$L_i\left(x_i, \mathbf{x}_{-i}, \lambda_i\right) = U_i\left(x_i\right) + \lambda^T\left(A^T\mathbf{x} - c\right)$$
$$\nabla_{x_i} L_i(x_i, x_{-i}, \lambda) = 0 \qquad \forall i$$
$$0 \le \lambda^\star \perp -\left(A^T\mathbf{x}^\star - c\right) \ge 0$$



Variational Inequality $VI(\mathcal{K}, \mathbf{F})$:

$$\begin{aligned} \mathbf{F}(\mathbf{x}) &= \left(\nabla_{x_i} U_i(x_i)\right)_{i=1}^{Q} \\ \mathcal{K}_i(\mathbf{x}_{-i}) &: \left\{x_i \ge 0 | g(x_i, \mathbf{x}_{-i}) = A^T\mathbf{x} - c \le 0\right\} \end{aligned}$$

Thank you!!

Any questions?