THE RADON TRANSFORM

JOSHUA BENJAMIN III

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jbenjamin@college.harvard.edu
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To My Haters
Chapter 1

Introduction

The reconstruction problem has long been of interest to mathematicians, and with the advancements in experimental science, applied fields also needed a solution to this issue. The reconstruction problem concerns distributions and profiles. Given the projected distribution (or profile), what can be known about the actual distribution? In 1917, Johann Radon of Austria presented a solution to the reconstruction problem (he was not attempting to solve this problem) with the Radon transform and its inversion formula. He developed this solution after building on the work of Hermann Minkowski and Paul Funk while maintaining private conversation with Gustav Herglotz. His work went largely unnoticed until 1972 when Allan McLeod Cormack and Arkady Vainshtein declared its importance to the field. Radon was concerned with $\mathbb{R}^2$ and $\mathbb{R}^3$, but his work was extended to $\mathbb{R}^n$ and some more general spaces. The Radon transform for $\mathbb{C}^n$ is known as the Penrose transform and it is related to integral geometry (the modern approach to integral geometry was largely inspired by integral transforms, specifically the Radon transform). In this paper, the author has decided to use functions of certain classes defined in class. Unless stated otherwise, all $f \in \mathcal{S}(\mathbb{R}^n)$ or $\mathcal{D}(\mathbb{R}^n)$. Respectively, Schwartz functions and functions of compact support. For reference on the generalizations of the transform and applications to integral geometry, see *The Radon Transform* by Sigurdur Helgason.
Chapter 2

The Radon Transform

The Radon Transform was originally used for determining a symmetric function on $S^2$ from its great circle integrals and a function of the plane $R^2$ from its line integrals. This was also extended by Radon to three dimensions. Radon found that

$$ f(x) = -\frac{1}{8\pi^2} \Delta x \left( \int_{S^2} J(\zeta, \langle \zeta, x \rangle) d\zeta \right) $$

(2.0.1)

Where $J(.)$ is the integral of $f$ over the hyperplane $\langle x, \zeta \rangle = p$ and $\zeta$ is a unit vector, and $\delta$ the laplacian operator. In the following sections are presented various forms of the Radon transform. Each one may be used to suit the purposes of the person computing and the information know regarding a specific problem.

2.1 Two Dimensions ($R^2$)

Since the Radon Transform is an operator like the Fourier transform we have two methods of writing. Starting with $R^2$

$$ \tilde{f} = \mathcal{R}f = \int_L f(x, y) ds $$

(2.1.1)

where $ds$ is an increment of length along a line $L$. Radon proved that if $f$ is continuous and has compact support, then $\mathcal{R}f$ is uniquely determined by integrating along all possible lines $L$. The transform over a line $L$ corresponds to a profile. This is called a sample of the Radon transform. Knowledge of the transform over all lines gives full knowledge of the transform. To add precision, let us consider the normal form a line and then shift the coordinates. With $p = x \cos \alpha + y \sin \alpha$ we can write

$$ \tilde{f}(p, \alpha) = \mathcal{R}f = \int_L f(x, y) ds $$

(2.1.2)
Now if we rotate by $\alpha$ and label the new axis $s$ are left with

\begin{align*}
x &= p \cos \alpha - s \sin \alpha \\
y &= p \sin \alpha + s \cos \alpha
\end{align*}

\[ \hat{f}(p, \alpha) = \mathcal{R} f = \int_{-\infty}^{\infty} f(p \cos \alpha - s \sin \alpha, p \sin \alpha + s \cos \alpha) \, ds \quad (2.1.3) \]

We can also use vector notation which will come in handy in higher dimensions. Let \( \mathbf{x} = (x, y) \) and take the unit vectors

\[ \zeta = (\cos \alpha, \sin \alpha) \]
\[ \zeta^\perp = (-\sin \alpha, \cos \alpha) \]

Since we can find a scalar $t$ such that $\mathbf{x} = p\zeta + t\zeta^\perp$, we now have

\[ \hat{f}(p, \zeta) = \mathcal{R} f = \int_{-\infty}^{\infty} f(p\zeta + t\zeta^\perp) \, dt \quad (2.1.4) \]

Note that the equation of a line can be written as

\[ p = \zeta \cdot \mathbf{x} = x \cos \alpha + y \sin \alpha \]

This allows the transform to be expressed as an integral over $\mathbb{R}^2$ where the Dirac-delta function selects the line $p = \zeta \cdot \mathbf{x}$ from $\mathbb{R}^2$

\[ \hat{f}(p, \zeta) = \mathcal{R} f = \int_{\mathbb{R}^2} f(\mathbf{x}) \delta(p - \zeta \cdot \mathbf{x}) \, dxdy \quad (2.1.5) \]

Which when $d\mathbf{x} = dxdy$ we have

\[ \hat{f}(p, \zeta) = \mathcal{R} f = \int f(\mathbf{x}) \delta(p - \zeta \cdot \mathbf{x}) \, d\mathbf{x} \quad (2.1.6) \]

### 2.2 Higher Dimensions (\( \mathbb{R}^n \))

The Dirac delta function version of the transform is due to the work of Gel’fand, Graev, and Vilenkin in their 1966 paper. In $\mathbb{R}^n$ we will use $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $d\mathbf{x} = dx_1dx_2...dx_n$ so we still have

\[ \hat{f}(p, \zeta) = \mathcal{R} f = \int f(\mathbf{x}) \delta(p - \zeta \cdot \mathbf{x}) \, d\mathbf{x} \quad (2.2.1) \]

This shows that we integrate over each hyperplane in the given space. This can be expressed as

\[ \hat{f}(\zeta) = \int_{\mathbb{R}^n} f(\mathbf{x}) \, dm(\mathbf{x}) \quad (2.2.2) \]

Where $dm$ is the Euclidean measure on the hyperplane $\zeta$
2.3 Dual Transform

The dual transform is defined as follows

\[ \mathcal{R}^* \varphi = \tilde{\varphi}(x) = \int_{x \in \xi} \varphi(\xi) d\mu(\xi) \] (2.3.1)

Where \( d\mu \) is the measure of the compact set \( \{ \xi \in P^n : x \in \xi \} \). The measure of the entire set is 1 making it a probability measure.

2.4 Gaussian distribution

Example 2.4.1. Let \( f(x, y) = e^{-x^2-y^2} \). The transform is

\[ \tilde{f} = \mathcal{R} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} \delta(p - \zeta_1 x - \zeta_2 y) dxdy \]

Take \( \zeta = (\zeta_1, \zeta_2) \) and with an orthogonal linear transformation

\[ \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \zeta_1 & \zeta_2 \\ -\zeta_2 & \zeta_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]

We have

\[ \tilde{f}(p, \zeta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-u^2-v^2} \delta(p - u) dudv \]
\[ = e^{-p^2} \int_{-\infty}^{\infty} e^{-v^2} dv \]
\[ = \sqrt{\pi} e^{-p^2} \]

We see that with for every \( n \in \mathbb{N} \) dimensions we will integrate \( n - 1 \) Gaussian distributions. So for arbitrary \( n \)

\[ \mathcal{R}\{e^{-x_1^2-x_2^2-\ldots-x_n^2} \} = (\sqrt{\pi})^{n-1} e^{-p^2} \]
Chapter 3

Properties

Most of the properties of the transform will follow directly from
\[ Rf = \int f(x)\delta(p - \zeta \cdot x)dx \]

For more on the properties delineated here, Gel'fand, Graev, and Vilenkin is a reference that provides a discussion.

3.1 Homogeneity and scaling

Note that
\[ \tilde{f}(tp, t\zeta) = Rf = \int f(x)\delta(tp - t\zeta \cdot x)dx \]
\[ = |t|^{-1} \int f(x)\delta(p - \zeta \cdot x)dx \]
so long as \( t \neq 0 \). So the Radon transform produces an even function of degree \(-1\). We know that it is even because of evaluating the following relation when \( t = -1 \).
\[ \tilde{f}(tp, t\zeta) = |t|^{-1} \tilde{f}(p, \zeta) \quad (3.1.1) \]

3.2 Linearity

The Radon transform is in fact a linear transformation, which can be seen by the following calculations. Take \( f, g \) to be functions and \( c_1, c_2 \) to be constants.
\[ R\{c_1f + c_2g\} = \int [c_1f(x) + c_2g(x)]\delta(p - \zeta \cdot x)dx \]
\[ = c_1\tilde{f} + c_2\tilde{g} \]
\[ R\{c_1f + c_2g\} = c_1Rf + c_2Rg \quad (3.2.1) \]
3.3 Transform of a linear transformation

Define two vectors $x, y$ and matrices $A^{-1} = B$ so that $x = A^{-1}y = By$. Note that,

$$
\mathcal{R}\{f(Ax)\} = \int f(Ax)\delta(p - \zeta \cdot x)dx
$$

$$
= |\det(B)| \int f(y)\delta(p - \zeta \cdot By)dy
$$

$$
= |\det(B)| \int f(y)\delta(p - B^T \zeta \cdot y)dy
$$

$$
\mathcal{R}\{f(Ax)\} = |\det(B)| \tilde{f}(p, B^T \zeta) \quad (3.3.1)
$$

The two special cases are when $|\det(B)| = 1$ and when $A = \lambda I$ (is a multiple of the identity. For each of those cases we have,

$$
\mathcal{R}\{f(Ax)\} = \tilde{f}(p, A\zeta) \quad (3.3.2)
$$

$$
\mathcal{R}\{f(\lambda x)\} = \frac{1}{\lambda^n} \tilde{f}(p, \frac{\zeta}{\lambda}) \quad (3.3.3)
$$

3.4 Shifting

Shifting is a direct result

$$
\mathcal{R}\{f(x - a)\} = \int f(x - a)\delta(p - \zeta \cdot x)dx
$$

$$
= \int f(x - a)\delta(p - \zeta \cdot a - \zeta \cdot y)dy
$$

$$
\mathcal{R}\{f(x - a)\} = \tilde{f}(p - \zeta \cdot a, \zeta) \quad (3.4.1)
$$
Chapter 4

Inversion

The inversion formula presented by Radon (given in chapter 1) hides the more general form. The author has decided to give the formulas for the even and odd dimension, and the unification formula as presented by Helgason. For proof of the existence of the formula, consult Deans or Helgason. The inversion formulas for the odd and even dimension Radon transform respectively are,

4.1 Inversion formulas

\[
f(x) = C_n \Delta_x^{(n-1)/2} \int_{|\zeta|=1} \tilde{f}(\zeta \cdot x, \zeta) d\zeta \quad (4.1.1)
\]

\[
f(x) = \frac{C_n}{i\pi} \int_{|\zeta|=1} d\zeta \int_{-\infty}^{\infty} \frac{\partial^2 \tilde{f}(p, \zeta)}{\partial p^2} \frac{p - \zeta \cdot x}{p^2} \quad (4.1.2)
\]

Where

\[
C_n = \frac{(-1)^{(n-1)/2}}{2(2\pi)^{(n-1)}} = \frac{1}{2(2\pi i)^{n-1}}
\]

Now define the adjoint operator

\[
\mathcal{R}^* \psi = \int_{|\zeta|=1} \psi(\zeta \cdot x, \zeta) d\zeta
\]

Note that on a Hilbert space, the dual is the adjoint. Now define the second order differential operator (from Helgason as \(\square\), but the author decides to denote with \(L\)),

\[
L \psi(p, \zeta) = \frac{\partial^2 \psi(p, \zeta)}{\partial p^2}
\]

From these, we have the intertwining property of the Radon transform and its dual/adjoint.

\[
\mathcal{R} \Delta f = L \mathcal{R} f \quad (4.1.3)
\]
\[ \mathcal{R}^* L \varphi = \Delta \mathcal{R}^* \varphi \quad (4.1.4) \]
\[ \mathcal{R}^\dagger L \psi = \Delta \mathcal{R}^\dagger \psi \quad (4.1.5) \]

### 4.2 Unification

Define a function

\[ \Lambda g(t) = \begin{cases} 
\Lambda_0 g = C_n \left( \frac{\partial}{\partial p} \right)^{n-1} g(p) \bigg|_{p=t} & (\text{n odd}) \\
\Lambda_1 g = \frac{C_n}{t} \left[ \mathcal{H} \left\{ \left( \frac{\partial}{\partial p} \right)^{n-1} g(p) \right\} \right] (t) & (\text{n even}) 
\end{cases} \quad (4.2.1) \]

Where

\[ g_H(t) = \mathcal{H} g = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(p)}{p - t} dp \]

is the Hilbert transform (from Bracewell) This gives us

\[ f = \mathcal{R}^\dagger \Lambda \mathcal{R} f \quad (4.2.2) \]

Where the adjoint can be replaced by the dual. Note the identity

\[ \mathcal{R}^\dagger \Lambda \mathcal{R} = I \]
Chapter 5

Relation to the Fourier Transform

The following chapter presents a connection between the Fourier transform (denoted $\mathcal{F}$) and the Radon transform $\mathcal{R}$.

5.1 Fourier and Radon

The n-dimensional Fourier transform ($\mathcal{F}_n$) can be written as follows:

$$\hat{f}(\eta) = \int_{-\infty}^{\infty} dt \int f(x) e^{-i2\pi t \delta(t - \eta \cdot x)} dx$$

Where $t \in \mathbb{R}$ and $\eta$ is a vector in Fourier frequency space. Substitute $\eta = s\zeta$ and $t = sp$ where $s \in \mathbb{R}$ and $\zeta$ is a unit vector as before. Now note

$$\hat{f}(s\zeta) = |s| \int_{-\infty}^{\infty} dp \int f(x) e^{-i2\pi sp \delta(sp - s\zeta \cdot x)} dx$$

$$= \int_{-\infty}^{\infty} e^{-i2\pi sp} dp \int f(x) \delta(p - \zeta \cdot x) dx$$

$$\hat{f}(s\zeta) = \int_{-\infty}^{\infty} \hat{f}(p, \zeta) e^{-i2\pi sp} dp$$

(5.1.1)

The right-hand side gives the Fourier transform in 1 dimension on the radial direction of the Radon transform denoted by $\mathcal{F}_1$. 
5.2 Another Inversion

Observe that the preceding equation sets up the relation

\[ f \xrightarrow{R} \hat{f} \xrightarrow{F_n} F_1 \xrightarrow{F_1} \hat{f} \]

Stated in terms of operators:

\[ Rf = F_1^{-1} F_n f \]  \hspace{1cm} (5.2.1)

Which yields inversion formula

\[ f = F_1^{-1} F_n Rf \]  \hspace{1cm} (5.2.2)
Chapter 6

Application

There are too many applications to list in such a short exposition, but for a more comprehensive treatment of applications of the Radon transform, consult Stanley Deans book, *The Radon Transform and Some of Its Applications*. The author has decided to give the example of tomography. Tomography is a non-invasive imaging technique allowing for the visualization of the internal structures of an object without the superposition of over and under-lying structures that often are a part of traditional projection images. For example, in a conventional chest radiograph, the heart, lungs, and ribs are all superimposed on the same film, whereas a computed tomography (CT) slice captures each organ in its actual three-dimensional position. Tomography has found widespread application in many scientific fields, including physics, chemistry, astronomy, geophysics, and medicine. While X-ray CT may be the most familiar application of tomography, tomography can be performed using other imaging modalities, including ultrasound, magnetic resonance, nuclear-medicine, and microwave techniques. Each tomographic modality measures a different physical quantity:

CT: The number of x-ray photons transmitted through the patient along individual projection lines.
Nuclear medicine: The number of photons emitted from the patient along individual projection lines.
Ultrasound diffraction tomography: The amplitude and phase of scattered waves along a particular line connecting the source and detector.

The task in all cases is to estimate from these measurements the distribution of a particular physical quantity in the object. The quantities that can be reconstructed are:

CT: The distribution of linear attenuation coefficient in the slice being imaged.
Nuclear medicine: The distribution of the radiotracer administered to the patient in the slice being imaged.
Ultrasound diffraction tomography: the distribution of refractive index in the slice being imaged.

The measurements made in each modality can be converted into samples of the Radon transform of the distribution that needs to be reconstructed. For example, in CT, divid-
ing the measured photon counts by the incident photon counts and taking the negative logarithm yields samples of the Radon transform of the linear attenuation map. The Radon transform and its inverse provide the mathematical basis for reconstructing tomographic images from measured projection or scattering data.
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