Econ 2110, fall 2016, Part IIIb
Statistical Decision Theory

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Roadmap

- IIIa
  - Basic definitions
  - Optimality criteria

- IIIb
  - Relationships between optimality criteria
  - Analogies to microeconomics
  - Two justifications of the Bayesian approach
  - Testing and the Neyman Pearson lemma
Part IIlb

Some relationships between these optimality criteria

Analogies to microeconomics

Two justifications of the Bayesian approach

Testing and the Neyman Pearson lemma
Some relationships between these optimality criteria

Proposition (Minimax decision functions)

If $\delta^*$ is admissible with constant risk, then it is a minimax decision function.

Proof:

- picture!

- Suppose that $\delta'$ had smaller worst-case risk than $\delta^*$

- Then

  $$R(\delta', \theta') \leq \sup_{\theta} R(\delta', \theta) < \sup_{\theta} R(\delta^*, \theta) = R(\delta^*, \theta'),$$

- used constant risk in the last equality

- This contradicts admissibility.
▶ despite this result, minimax decision functions are very hard to find
▶ Example:
  ▶ if $X \sim N(\mu, I)$, $\text{dim}(X) \geq 3$, then
  ▶ $X$ has constant risk (mean squared error) as estimator for $\mu$
  ▶ but: $X$ is not an admissible estimator for $\mu$
    therefore not minimax
Proposition (Bayes decisions are admissible)

Suppose:

- $\delta^*$ is the Bayes decision function
- $\pi(\theta) > 0$ for all $\theta$, $R(\delta^*, \pi) < \infty$
- $R(\delta^*, \theta)$ is continuous in $\theta$

Then $\delta^*$ is admissible.

(We will prove the reverse of this statement in the next section.)
Sketch of proof:

- picture!
- Suppose $\delta^*$ is not admissible
- $\Rightarrow$ dominated by some $\delta'$
  i.e. $R(\delta', \theta) \leq R(\delta^*, \theta)$ for all $\theta$ with strict inequality for some $\theta$
- Therefore
  \[
  R(\delta', \pi) = \int R(\delta', \theta) \pi(\theta) d\theta < \int R(\delta^*, \theta) \pi(\theta) d\theta = R(\delta^*, \pi)
  \]
- This contradicts $\delta^*$ being a Bayes decision function.
Proposition (Bayes risk and minimax risk)

The Bayes risk
\[ R(\pi) := \inf_\delta R(\delta, \pi) \]
is never larger than the minimax risk
\[ \bar{R} := \inf_\delta \sup_\theta R(\delta, \theta). \]

Proof:

\[
R(\pi) = \inf_\delta R(\delta, \pi) \\
\leq \sup_\pi \inf_\delta R(\delta, \pi) \\
\leq \inf_\delta \sup_\pi R(\delta, \pi) \\
= \inf_\delta \sup_\theta R(\delta, \theta) = \bar{R}.
\]

If there exists a prior \( \pi^* \) such that \( R(\pi) = \bar{R} \), it is called the least favorable distribution.
1) Welfare economics

<table>
<thead>
<tr>
<th>statistical decision theory</th>
<th>social welfare analysis</th>
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<tbody>
<tr>
<td>different parameter values $\theta$</td>
<td>different people $i$</td>
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<tr>
<td>risk $R(. , \theta)$</td>
<td>individuals’ utility $u_i(.)$</td>
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<td>dominance</td>
<td>Pareto dominance</td>
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<tr>
<td>admissibility</td>
<td>Pareto efficiency</td>
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<td>Bayes risk</td>
<td>social welfare function</td>
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<td>prior</td>
<td>welfare weights (distributional preferences)</td>
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<td>minimaxity</td>
<td>Rawlsian inequality aversion</td>
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2) choice under uncertainty / choice in strategic interactions

<table>
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<th>strategic interactions</th>
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<td>dominance of decision functions</td>
<td>dominance of strategies</td>
</tr>
<tr>
<td>Bayes risk</td>
<td>expected utility</td>
</tr>
<tr>
<td>Bayes optimality</td>
<td>expected utility maximization</td>
</tr>
<tr>
<td>minimaxity</td>
<td>(extreme) ambiguity aversion</td>
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Two justifications of the Bayesian approach

justification 1 – the complete class theorem

» last section: every Bayes decision function is admissible (under some conditions)

» the reverse also holds true (under some conditions): every admissible decision function is Bayes, or the limit of Bayes decision functions

» can interpret this as: all reasonable estimators are Bayes estimators

» will state a simple version of this result
Preliminaries

- set of risk functions that correspond to some $\delta$ is the risk set,
  \[ \mathcal{R} := \{ r(.) = R(., \delta) \text{ for some } \delta \} \]

- will assume convexity of $\mathcal{R}$
  – no big restriction, since we can always randomly “mix” decision functions

- a class of decision functions $\delta$ is a complete class if it contains every admissible decision function $\delta^*$
Theorem (Complete class theorem)

Suppose

- the set $\Theta$ of possible values for $\theta$ is compact
- the risk set $R$ is convex
- all decision functions have continuous risk

Then the Bayes decision functions constitute a complete class: For every admissible decision function $\delta^*$, there exists a prior distribution $\pi$ such that $\delta^*$ is a Bayes decision function for $\pi$. 
Two justifications of the Bayesian approach

Figure: Complete class theorem
Intuition for the complete class theorem

- any choice of decision procedure has to trade off risk across $\theta$
- slope of feasible risk set
  - = relative “marginal cost” of decreasing risk at different $\theta$
- pick a risk function on the admissible frontier
- can rationalize it with a prior
  - = “marginal benefit” of decreasing risk at different $\theta$
- for example, minimax decision rule:
  - rationalizable by least favorable prior
  - slope of feasible set at constant risk admissible point
- analogy to social welfare: any policy choice or allocation corresponds to distributional preferences / welfare weights
Proof of complete class theorem:

- application of the separating hyperplane theorem, to the space of functions of $\theta$, with the inner product
  \[
  \langle f, g \rangle = \int f(\theta)g(\theta)d\theta.
  \]

- for intuition: focus on binary $\theta$, $\theta \in \{0, 1\}$, and $\langle f, g \rangle = \sum_\theta f(\theta)g(\theta)$

- Let $\delta^*$ be admissible. Then $R(., \delta^*)$ belongs to the lower boundary of $\mathcal{R}$.

- convexity of $\mathcal{R}$, separating hyperplane theorem
  separating $\mathcal{R}$ from risk functions dominating $\delta^*$
there exists a function $\tilde{\pi}$ (with finite integral) such that for all $\delta$

$$\langle R(., \delta^*), \tilde{\pi} \rangle \leq \langle R(., \delta), \tilde{\pi} \rangle.$$ 

by construction $\tilde{\pi} \geq 0$

thus $\pi := \tilde{\pi} / \int \tilde{\pi}$ defines a prior distribution.

$\delta^*$ minimizes

$$\langle R(., \delta^*), \pi \rangle = R(\delta^*, \pi)$$

among the set of feasible decision functions

and is therefore the optimal Bayesian decision function for the prior $\pi$. 
justification 2 – subjective probability theory

- going back to Savage (1954) and Anscombe and Aumann (1963).
- and maybe in Econ 2010.
Suppose a decision maker ranks risk functions $R(., \delta)$ by a preference relationship $\succeq$

properties $\succeq$ might have:

1. **completeness**: any pair of risk functions can be ranked
2. **monotonicity**: if the risk function $R$ is (weakly) lower than $R'$ for all $\theta$, than $R$ is (weakly) preferred
3. **independence**:

   $$R^1 \succeq R^2 \iff \alpha R^1 + (1 - \alpha) R^3 \succeq \alpha R^2 + (1 - \alpha) R^3$$

   for all $R^1, R^2, R^3$ and $\alpha \in [0, 1]$

Important: this independence has nothing to do with statistical independence
Theorem

If ⪰ is complete, monotonic, and satisfies independence, then there exists a prior $\pi$ such that

$$ R(., \delta^1) \succeq R(., \delta^2) \iff R(\pi, \delta^1) \leq R(\pi, \delta^2). $$

Intuition of proof:

- Independence and completeness imply linear, parallel indifference sets
- Monotonicity makes sure prior is non-negative
Sketch of proof:
Using independence repeatedly, we can show that for all $R^1, R^2, R^3 \in \mathbb{R}^X$, and all $\alpha > 0$,

1. $R^1 \succeq R^2$ iff $\alpha R^1 \succeq \alpha R^2$,
2. $R^1 \succeq R^2$ iff $R^1 + R^3 \succeq R^2 + R^3$,
3. $\{ R : R \succeq R^1 \} = \{ R : R \succeq 0 \} + R^1$,
4. $\{ R : R \succeq 0 \}$ is a convex cone.
5. $\{ R : R \succeq 0 \}$ is a half space.

The last claim requires completeness. It immediately implies the existence of $\pi$. Monotonicity implies that $\pi$ is not negative.
Remark

- personally, I’m more convinced by the complete class theorem than by normative subjective utility theory
- admissibility seems a very sensible requirement
- whereas “independence” of the preference relationship seems more up for debate
Testing and the Neyman Pearson lemma

- testing as a decision problem
- goal: decide whether $H_0 : \theta \in \Theta_0$ is true
- decision $a \in \{0, 1\}$ (true / not true)
- statistical test is a decision function $\varphi : X \mapsto \{0, 1\}$
- $\varphi = 1$ corresponds to rejecting the null hypothesis
- more generally: randomized tests $\varphi : X \mapsto [0, 1]$
- reject $H_0$ with probability $\varphi(X)$
  (for technical reasons only, as we will see)
### Two types of classification error

<table>
<thead>
<tr>
<th>decision $a$</th>
<th>truth</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta \in \Theta_0$</td>
<td>$\theta \notin \Theta_0$</td>
</tr>
<tr>
<td>0</td>
<td>$\smiley$</td>
</tr>
<tr>
<td>1</td>
<td>Type I error</td>
</tr>
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</table>
The power function

- suppose \( X \sim f_\theta(x) \)
- \( f \): probability mass function or probability density function
- probability of rejecting \( H_0 \) given \( \theta \): power function

\[
\beta(\theta) = E_\theta[\varphi(X)] = \int \varphi(x)f_\theta(x)dx.
\]
suppose that \( \theta \) has only two points of support, \( \theta_0 \) and \( \theta_1 \)

then

1. \( P(\text{Type I error}) = \beta(\theta_0) \).
2. \( P(\text{Type II error}) = 1 - \beta(\theta_1) \).

\( \beta(\theta_0) \) is called "level" or "significance" of the test, often denoted \( \alpha \).

\( \beta(\theta_1) \) is called the "power" of a test, and is often denoted \( \beta \).

would like to have a small \( \alpha \) and a large \( \beta \)
Figure: testing as a decision problem
Suppose we want $\varphi^*$ that solves

$$\max_{\varphi} \beta(\theta_1) \quad \text{s.t. } \beta(\theta_0) = \alpha$$

for a prespecified level $\alpha$.

**Lemma (Neyman-Pearson)**

The solution to this problem is given by

$$\varphi^*(x) = \begin{cases} 
1 & \text{for } f_1(x) > \lambda f_0(x) \\
\kappa & \text{for } f_1(x) = \lambda f_0(x) \\
0 & \text{for } f_1(x) < \lambda f_0(x)
\end{cases}$$

where $\lambda$ and $\kappa$ are chosen such that $\int \varphi^*(x)f_0(x)dx = \alpha$. 
Practice problem
Try to prove this!

Hint:
our problem is to solve

$$\max_\varphi \int \varphi(x)f_1(x)dx$$

subject to

$$\int \varphi(x)f_0(x)dx = \alpha$$

and

$$\varphi(x) \in [0, 1].$$
Recall the proposed solution,

\[
\varphi^*(x) = \begin{cases} 
1 & \text{for } f_1(x) > \lambda f_0(x) \\
\kappa & \text{for } f_1(x) = \lambda f_0(x) \\
0 & \text{for } f_1(x) < \lambda f_0(x)
\end{cases}
\]

**Proof:**

- let \( \varphi(x) \) be any other test of level \( \alpha \)
  
i.e. \( \int \varphi(x)f_0(x)dx = \alpha \).
- need to show that
  \( \int \varphi^*(x)f_1(x)dx \geq \int \varphi(x)f_1(x)dx \).
- Note that
  \[
  \int (\varphi^*(x) - \varphi(x))(f_1(x) - \lambda f_0(x))dx \geq 0
  \]
  since \( \varphi^*(x) = 1 \geq \varphi(x) \) for all \( x \) such that \( f_1(x) - \lambda f_0(x) > 0 \) and \( \varphi^*(x) = 0 \leq \varphi(x) \) for all \( x \) such that \( f_1(x) - \lambda f_0(x) < 0 \).
Therefore, using \( \alpha = \int \varphi(x)f_0(x)dx = \int \varphi^*(x)f_0(x)dx \),

\[
\int (\varphi^*(x) - \varphi(x))(f_1(x) - \lambda f_0(x))dx \\
= \int (\varphi^*(x) - \varphi(x))f_1(x)dx \\
= \int \varphi^*(x)f_1(x)dx - \int \varphi(x)f_1(x)dx \geq 0
\]

as required.

proof in the discrete case: identical with all summations replaced by integrals.
Practice problem

- you observe $X \sim N(\mu, 1)$
- you know that either $\mu = 0$ or $\mu = 1$
- construct the test of largest power for $H_0 : \mu = 0$ and any level $\alpha$