Optimal taxation and insurance using machine learning

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Abstract

How should one use (quasi-)experimental evidence when choosing policies such as tax rates, health insurance copay, unemployment benefit levels, class sizes in schools, etc.? This paper suggests an approach based on maximizing posterior expected social welfare, combining insights from (i) optimal policy theory as developed in the field of public finance, and (ii) machine learning using Gaussian process priors. We provide explicit formulas for posterior expected social welfare and optimal policies in a wide class of policy problems.

The proposed methods are applied to the choice of coinsurance rates in health insurance, using data from the RAND health insurance experiment. The key trade-off in this setting is between transfers towards the sick and insurance costs. The key empirical relationship the policy maker needs to learn about is the response of health care expenditures to coinsurance rates. Holding the economic model and distributive preferences constant, we obtain much smaller point estimates of the optimal coinsurance rate (18% vs. 50%) when applying our estimation method instead of the conventional "sufficient statistic" approach.

Keywords: optimal policy, Gaussian process priors, Bayesian nonparametrics, posterior expected welfare

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1 Introduction

How should empirical evidence be used to determine the optimal level of policy parameters such as tax rates, unemployment benefits, health insurance copay, or class sizes in school? A standard approach, labeled the "sufficient statistic approach" by Chetty (2009), uses the data to estimate a key behavioral elasticity, and then plugs this elasticity into formulas for optimal policy levels that are based on elasticities at the optimum. In this paper, an alternative approach is proposed and implemented in the context of choosing coinsurance rates for health insurance.[1]

Setup This paper takes the perspective of a policy maker who wants to maximize some notion of social welfare. We assume that the policy maker observes (quasi-)experimental data that allow her to learn about some behavioral relationship that is relevant for her decision. We assume further that the policy maker acts as a Bayesian decision maker. This assumption implies that she uses the available data to form a posterior expectation of social welfare given each possible policy choice, and that she chooses the policy that maximizes this posterior expectation.

The imposition of some additional structure allows us to derive explicit analytic solutions to the policy maker's problem. In Section 2 we assume that social welfare takes a form common to many problems in public finance, where the key trade-off is between a weighted sum of private utilities and public revenues. The empirical relationship that the policy maker needs to learn in these settings is the response of the tax base to tax rates, or of insurance claims to coinsurance rates. In Section 3 we consider Gaussian process priors for this behavioral relationship. The combination of the structure of the objective function and the structure of these priors implies that we can explicitly derive and characterize posterior expected social welfare. In contrast to the sufficient statistic method as discussed in Chetty (2009), our approach does not rely on extrapolation using log-linear functional form assumptions, and it takes uncertainty into account. The difference matters in practice, as we will see.

Contributions of this paper This paper contributes to the literature in several ways. First, for empirical researchers working on issues of public policy, this paper leverages the statistical insights of a well developed literature on machine learning using Gaussian process priors, spline regression, and reproducing kernel Hilbert spaces. This paper provides a simple framework to derive optimal policy choices given available data. The practical relevance of such a framework is demonstrated by our empirical application, where we find very different levels of optimal policy relative to those suggested by a conventional estimation approach (leaving the economic model and distributive preferences the same). Second, for statistical decision theorists, this paper suggests a class of objective functions ("loss functions") for statistical decision problems that have a substantive justification in economic theory, and which contrast with conventional loss functions such as quadratic error loss or mis-classification loss. Third, for practitioners of machine learning, this paper suggests a class of applications of machine learning methods where new predictive procedures might fruitfully be

[1] The coinsurance rate is the share of health care expenditures that the insured have to pay out of pocket.

leveraged for problems other than prediction.

Application In Section 4 the proposed approach is applied to the problem of setting coinsurance rates in health insurance. Lowering coinsurance leads to more redistribution from healthy contributors to those in need of health care. But it also increases insurance costs, both mechanically and through the behavioral response of possibly increased health care spending. We use data from the RAND health insurance experiment in order to estimate this behavioral response. We then use the estimated relationship to determine the optimal coinsurance rate. We find an optimal coinsurance rate of 18%. This contrasts markedly with the optimal coinsurance rate of 50% suggested by the conventional sufficient statistic approach under otherwise identical assumptions. Both of these numbers are based on the (arbitrary) normative assumption that the marginal value of a US$ for the sick is 1.5 times the marginal value of a US$ for the insurance provider.[2] For a range of alternative assumptions about this relative marginal value we find the same qualitative comparison.

Technical perspective From a statistical perspective, the key features of our setting are as follows: (i) The decision maker's objective function is given by a known affine operator applied to an unknown causal relationship. The affine operator may involve operations such as integration, differentiation, multiplication by known functions, etc. (ii) (Quasi-)experimental variation of policy-parameters allows one to equate the relevant causal relationships to predictive relationships. (iii) A Gaussian process prior for these predictive relationships is used.

The combination of these features implies a tractable linear mapping with known weights from observed outcomes to the posterior expected social welfare function and its derivative, which in turn determine the expected welfare maximizing policy choice. This is sketched for the general case in Appendix B. Explicit weights for our application are derived in Appendix C. Given symmetry and unimodality of the posterior, the posterior expectation of the predictive relationship can also be written as a maximum a posteriori, or equivalently, as the solution to a penalized least squares regression. The penalty is equal to the reproducing kernel Hilbert space norm corresponding to the prior covariance kernel. Spline regression is a special case; cf. Wahba (1990) and van der Vaart and van Zanten (2008). Approximating the sample distribution of predictors by their population distribution yields an equivalent kernel representation for the weights mapping outcomes into welfare, as sketched in Appendix D; cf. Silverman (1984).

The social welfare function objective for a broad class of policy problems in public finance takes the form of an affine operator applied to behavioral relationships. Key reasons for this are the assumptions that (i) social welfare is a weighted sum of private utilities, (ii) individuals maximize their utility subject to constraints, and (iii) their choices generate no externalities. Under these assumptions the effect of behavioral responses to marginal policy changes on private welfare can be neglected, due to the envelope theorem reviewed in Appendix A, simplifying the form of social welfare.

[2]The choice of such welfare weights based on normative considerations is discussed in Saez and Stantcheva (2013).

Literature This paper draws on two distinct literatures, (i) optimal policy theory as discussed in the field of public finance, and (ii) statistical decision theory and machine learning using Gaussian process priors. Models of optimal policy in public finance have a long tradition going back at least to the discussion in Samuelson (1947) of social welfare functions, with classic contributions including Mirrlees (1971) and Baily (1978). The empirical implementation of such models using "sufficient statistics" is discussed in Chetty (2009) and Saez (2001). Gaussian process priors and nonparametric Bayesian function estimation are discussed extensively in Williams and Rasmussen (2006). Gaussian process priors are closely related to spline estimation and reproducing kernel Hilbert spaces, as discussed in Wahba (1990). When controlling for covariates we also make use of Dirichlet process priors, which are reviewed in Ghosh and Ramamoorthi (2003).

Road map The rest of this paper is structured as follows. Section 2 briefly reviews the theory of optimal insurance and optimal taxation, and reformulates the solution to these problems in a form amenable to our approach. Section 3 states our assumptions on the data generating process and the prior. We then derive simple closed form expressions for posterior expected social welfare and for the first order condition characterizing the optimal policy choice. Section 4 applies the proposed approach to data from the RAND health insurance experiment and provides estimates of the optimal coinsurance rate. Section 5 discusses a number of extensions of our framework, including conditional exogeneity, optimal experimental design for policy, and an alternative class of social welfare functions involving production. Section 6 concludes. The appendix discusses technical details, including the envelope theorem, a generalization of our setup involving affine operators, explicit weight functions for our application, and approximations using equivalent kernel weights.

2 Optimal insurance and optimal taxation

Many policy problems considered in the field of public finance share a similar structure. We first describe this structure in terms of the example of optimal health insurance, corresponding to the empirical application considered in Section 4 below. We then discuss how other policy problems, in particular optimal taxation, can be described in the same terms. A more detailed discussion of some of the ideas introduced in this section can be found in Chetty (2009).

The key takeaway of this section is equation (4). This equation is a reformulation of standard representations of social welfare. This representation is chosen such that it is amenable to our subsequent analysis using Gaussian process priors. Our approach is contrasted with more standard approaches using "sufficient statistic" formulas for optimal policy parameters in the context of the application in Section 4.

In the health insurance policy problem considered, the trade-off between two objectives (increasing insurance/redistribution versus lowering the cost to the provider) determines the optimal coinsurance rate. The key empirical ingredient informing the policy maker's choice is the behavioral response of health care usage to changes of the coinsurance rate.

Setup The insurance covers a population of insured individuals i. Let Y_i denote the health care expenditures of individual i, and let T_i denote the share of health care expenditures covered by the insurance, so that $1 - T_i$ is the coinsurance rate faced by individual i, and $Y_i \cdot (1 - T_i)$ are her out-of-pocket expenditures.

Individuals might adjust their health care expenditures depending on the coinsurance rate they face. We can capture this response by considering the structural function

$$Y_i = g(T_i, \epsilon_i). \tag{1}$$

In this equation, ϵ_i captures unobserved heterogeneity which is assumed to be invariant under counterfactual policies.[3] Corresponding to this structural function we can consider the average structural function

$$m(t) = E[g(t, \epsilon_i)]. \tag{2}$$

In this equation, the expectation averages over the distribution of unobserved heterogeneity ϵ_i across the population of insured individuals. The function $m(t)$ describes the average level of health care expenditures if all individuals were to face the policy level t. We assume that this function is differentiable.

Policy objective Given this setting, we can now describe how a marginal change of the policy t, when applied to all of the insured, would affect the policy maker's objectives. A marginal change dt of t affects insurance expenditures in two ways, mechanically, and through individuals' behavioral response. The insurance provider's expenditures per person are given by $t \cdot m(t)$. The mechanical effect of the change of t on the provider's expenditures, holding constant individuals' health care expenditures, is given by $m(t)dt$. This mechanical effect can be calculated by accounting, given the expenditures $m(t)$. It does not require estimation of a causal effect. The behavioral effect on expenditures is given by $t \cdot m'(t)dt$. This behavioral effect poses the key empirical challenge. To calculate it we need to know the causal effect $m'(t)$ of a change in t on expenditures $m(t)$.

The effect of the marginal change of t on the welfare of the insured is a subtler matter. There is again a mechanical monetary effect proportional to $m(t)dt$, since the sick have to pay less for their health care when t is increased. This effect can again be calculated by accounting. But what about the effect of behavioral responses on private welfare? As it turns out these don't affect private welfare under standard utilitarian assumptions for a very general class of models; this includes models that allow for multiple behavioral margins, dynamic choices, discrete choices, etc. This follows from the so-called envelope theorem. Appendix A provides a brief discussion of this point; see also Milgrom and Segal (2002) and Chetty (2009).

To trade off between her two conflicting objectives, the policy maker has to decide on the marginal value $\lambda > 1$ of an additional dollar transferred to the sick relative to the cost of an additional dollar of expenditures. The parameter λ reflects both social preferences for redistribution to the sick, cf. Saez and Stantcheva (2013), as well as private risk aversion to unforeseen health shocks; we will assume λ known for

[3] This structural function could equivalently be written in terms of potential outcomes $Y_i^t = g(t, \epsilon_i)$, so that $Y_i = Y_i^{T_i}$.

simplicity of exposition.[4] Adding up the effects of a policy change on the welfare of the insured (weighted by λ) and on provider revenues, we get the marginal effect of a change in t on social welfare,

$$u'(t) = (\lambda - 1) \cdot m(t) - t \cdot m'(t) = \lambda m(t) - \tfrac{\partial}{\partial t}(t \cdot m(t)). \tag{3}$$

Integrating and imposing the normalization $u(0) = 0$ yields social welfare,

$$u(t) = \lambda \int_0^t m(x)dx - t \cdot m(t). \tag{4}$$

This objective function is a variant of the classic "Harberger triangle," where the latter corresponds to the special case where $\lambda = 1$. The first-order condition for the optimal coinsurance rate t^* when $m(\cdot)$ is known is given by $u'(t^*) = (\lambda - 1) \cdot m(t^*) - t^* \cdot m'(t^*) = 0$.

Formally equivalent policy problems There are many problems of optimal policy choice in public finance which share a similar structure. One example is optimal unemployment insurance, as discussed by Baily (1978) and subsequent papers. Chetty (2006), building on insights of Feldstein (1999), has argued that a very general class of models of unemployment insurance lead to the same formulas characterizing optimal benefits, which are in fact equivalent to the one derived above. In the context of unemployment insurance, t would be interpreted as the level of unemployment benefits, and Y as the share of days spent unemployed in a given time period by a given individual. λ is the relative value of additional income for the unemployed, and $m(t)$ is the unemployment rate given policy level t.

Optimal taxation problems such as optimal income taxation can also be reformulated in this way. An example is the choice of the tax rate for the top tax bracket, as in Saez (2001). In this setting, t is the top tax rate, and Y is the taxable income declared by an individual. λ is the marginal value assigned to additional income for rich people relative to additional government revenues, and $m(t)$ is the size of the tax base in the top bracket.

A representation of the policy objective in the form of equation (4) is more generally possible in settings which satisfy the following assumptions: The policy maker's objective is to maximize a weighted sum of private utilities. Individuals are maximizing utility. Policy choices (such as tax rates or replacement rates) affect private choices. The government is subject to a budget constraint, or equivalently has alternative expenditures and revenues which pin down the marginal value of government revenues. If there are no externalities, these assumptions imply that the behavioral effects of policy choices on private welfare are zero at the margin, due to envelope conditions. This implies that welfare under a given policy choice only depends on some key behavioral relationship, for instance the tax base as a function of tax rates.

[4] In settings where λ is considered a parameter to be estimated, if it is estimated using data which are independent from those considered below, then λ can be replaced by a posterior expectation $\widehat{\lambda}$ throughout. Our results continue to apply verbatim for such settings.

3 Experimental variation, Gaussian process prior, and posterior

Our discussion so far described social welfare $u(\cdot)$ and the optimal policy t^* in terms of the true average response function $m(\cdot)$ under counterfactual policies t. The function $m(\cdot)$ is not known to the policy maker in general, however, so she has to use empirical evidence to form beliefs about this function. As a baseline case, we discuss a randomized experiment.

Sampling and experimental variation Assume that we observe n i.i.d. draws of (Y_i, T_i) from the population of interest. Assume further that T_i was randomly assigned in an experiment, so that T_i is statistically independent of the unobserved heterogeneity ϵ_i. These assumptions imply

$$E[Y_i|T_i = t] = E[g(t, \epsilon_i)|T_i = t] = E[g(t, \epsilon_i)] = m(t). \tag{5}$$

Assume next that Y_i is normally distributed given T_i, with constant variance

$$Y_i|T_i = t \sim N(m(t), \sigma^2). \tag{6}$$

In Section 5 below we discuss extensions, including the case of conditional exogeneity of treatment T_i given observables W_i, and the case of non-normal residuals $Y_i - m(T_i)$.

Prior The key empirical relationship that the policy maker of Section 2 has to learn is the average structural function $m(\cdot)$. This function describes average health care expenditures given the coinsurance rate. We assume that the policy maker has a prior for $m(\cdot)$ which takes the form

$$m(\cdot) \sim GP(\mu(\cdot), C(\cdot, \cdot)), \tag{7}$$

where $GP(\mu(\cdot), C(\cdot, \cdot)$ denotes the law of a Gaussian process which is such that $E[m(t)] = \mu(t)$ and $\mathrm{Cov}(m(t), m(t')) = C(t, t')$, and where both the mean function $\mu(\cdot)$ and the covariance kernel $C(\cdot, \cdot)$ are assumed to be differentiable. We impose further that the policy maker's prior is such that the function $m(\cdot)$ is independent of the probability distribution P_T of T. Such priors are discussed in detail in Williams and Rasmussen (2006).

Posterior expectation of the average response function m Recall that we assume the availability of a random sample Y_i, T_i, $i = 1, \ldots, n$, satisfying equation (6). What is the posterior expectation $\widehat{m}(t)$ of $m(t)$ given such data? Denote $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ and $\boldsymbol{T} = (T_1, \ldots, T_n)$, and let

$$\begin{aligned}
\mu_i &= E[m(T_i)|\boldsymbol{T}] = \mu(T_i), \\
C_{i,j} &= \mathrm{Cov}(m(T_i), m(T_j)|\boldsymbol{T}) = C(T_i, T_j), \text{ and} \\
C_i(t) &= \mathrm{Cov}(m(t), m(T_i)|\boldsymbol{T}) = C(t, T_i).
\end{aligned} \tag{8}$$

Let furthermore $\boldsymbol{\mu}$, $\boldsymbol{C}(t)$, and $\boldsymbol{C}$ denote the vectors and matrix collecting these terms for $i, j = 1, \ldots, n$. Since our setting implies joint normality of $\boldsymbol{Y}$ and $m(t)$ conditional on $\boldsymbol{T}$, the posterior expectation of $m(t)$ takes the form of a posterior best linear predictor:

$$\begin{aligned}\widehat{m}(t) = E[m(t)|\boldsymbol{Y}, \boldsymbol{T}] &= E[m(t)|\boldsymbol{T}] + \mathrm{Cov}(m(t), \boldsymbol{Y}|\boldsymbol{T}) \cdot \mathrm{Var}(\boldsymbol{Y}|\boldsymbol{T})^{-1} \cdot (\boldsymbol{Y} - E[\boldsymbol{Y}|\boldsymbol{T}]) \\ &= \mu(t) + \boldsymbol{C}(t) \cdot \left[\boldsymbol{C} + \sigma^2 \boldsymbol{I}\right]^{-1} \cdot (\boldsymbol{Y} - \boldsymbol{\mu}). \end{aligned} \tag{9}$$

Posterior expectation of social welfare u and its derivative u' What ultimately matters from the policy maker's perspective is not the response function $m(\cdot)$ itself, but how expected social welfare $\widehat{u}(t)$ depends on her policy choice t. Recall from equation (4) that $u(t) = \lambda \int_0^t m(x)dx - t \cdot m(t)$. The function $u(\cdot)$ is thus a linear transformation of $m(\cdot)$. This implies that it has a Gaussian process prior distribution, like $m(\cdot)$ itself, where

$$\nu(t) = E[u(t)] = \lambda \int_0^t \mu(x)dx - t \cdot \mu(t), \qquad \text{and} \tag{10}$$

$$D(t, t') = \mathrm{Cov}(u(t), m(t'))) = \lambda \cdot \int_0^t C(x, t')dx - t \cdot C(t, t'). \tag{11}$$

The posterior expectation of $u(t)$ then equals

$$\begin{aligned}\widehat{u}(t) = E[u(t)|\boldsymbol{Y}, \boldsymbol{T}] &= E[u(t)|\boldsymbol{T}] + \mathrm{Cov}(u(t), \boldsymbol{Y}|\boldsymbol{T}) \cdot \mathrm{Var}(\boldsymbol{Y}|\boldsymbol{T})^{-1} \cdot (\boldsymbol{Y} - E[\boldsymbol{Y}|\boldsymbol{T}]) \\ &= \nu(t) + \boldsymbol{D}(t) \cdot \left[\boldsymbol{C} + \sigma^2 \boldsymbol{I}\right]^{-1} \cdot (\boldsymbol{Y} - \boldsymbol{\mu}). \end{aligned} \tag{12}$$

It is in this formula that the pieces of our optimal policy setup and of the Gaussian process prior setup start to come together.

The optimal policy choice given the data We assume that the policy maker aims to maximize expected social welfare.[5] The optimal t, maximizing posterior expected social welfare given the experimental observations $\boldsymbol{Y}, \boldsymbol{T}$, satisfies

$$\widehat{t}^* = \widehat{t}^*(\boldsymbol{Y}, \boldsymbol{T}) \in \underset{t}{\operatorname{argmax}}\ \widehat{u}(t). \tag{13}$$

The first order condition for this optimization problem is given by

$$\tfrac{\partial}{\partial t} \widehat{u}(\widehat{t}^*) = E[u'(\widehat{t}^*)|\boldsymbol{Y}, \boldsymbol{T}] = \nu'(\widehat{t}^*) + \boldsymbol{B}(\widehat{t}^*) \cdot \left[\boldsymbol{C} + \sigma^2 \boldsymbol{I}\right]^{-1} \cdot (\boldsymbol{Y} - \boldsymbol{\mu}) = 0. \tag{14}$$

where

$$B(t, t') = \mathrm{Cov}\left(\tfrac{\partial}{\partial t} u(t), m(t')\right) = \tfrac{\partial}{\partial t} D(t, t') = (\lambda - 1) \cdot C(t, t') - t \cdot \tfrac{\partial}{\partial t} C(t, t'). \tag{15}$$

[5] Note that the maximizer of expected welfare, chosen by a Bayesian decision maker, is in general different from the expectation of the maximizer of welfare.

Numerically, the maximizer of $\widehat{u}$ might be found using a grid search algorithm or the Newton-Raphson algorithm. Explicit expressions for $D(\cdot,\cdot)$ and $B(\cdot,\cdot)$, for a specific choice of $C(\cdot,\cdot)$, are derived in Appendix C.

The posterior variance of m, u and u' In order to choose the optimal policy $\widehat{t^*}$ we only need to know the posterior expectation $\widehat{u}(t)$ of $u(t)$. In order to perform Bayesian inference, however, we might also be interested in the posterior variance of m, u and u'. Given joint normality of $\boldsymbol{Y}$ and $m(t)$ given $\boldsymbol{T}$, the posterior variance of $m(t)$ is given by the difference between the prior variance of $m(t)$, and the prior variance of the estimator $\widehat{m}(t)$,

$$\mathrm{Var}(m(t)|\boldsymbol{Y},\boldsymbol{T}) = \mathrm{Var}(m(t)|\boldsymbol{T}) - \mathrm{Var}(\widehat{m}(t)|\boldsymbol{T}). \tag{16}$$

Similarly, $\mathrm{Var}(u(t)|\boldsymbol{Y},\boldsymbol{T}) = \mathrm{Var}(u(t)|\boldsymbol{T}) - \mathrm{Var}(\widehat{u}(t)|\boldsymbol{T})$ and $\mathrm{Var}(u'(t)|\boldsymbol{Y},\boldsymbol{T}) = \mathrm{Var}(u'(t)|\boldsymbol{T}) - \mathrm{Var}(\widehat{u}'(t)|\boldsymbol{T})$. The prior variance of $m(t)$ is given by $\mathrm{Var}(m(t)) = C(t,t)$ by assumption, while

$$\begin{aligned}\mathrm{Var}(u(t)|\boldsymbol{T}) &= \lambda^2 \cdot \int_0^t \int_0^t C(x,x')dx'dx - 2\lambda t \cdot \int_0^t C(x,t)dx + t^2 \cdot C(t,t),\\ \mathrm{Var}(u'(t)|\boldsymbol{T}) &= (\lambda-1)^2 \cdot C(t,t) - 2(\lambda-1)\cdot \tfrac{\partial}{\partial t'} C(t,t')|_{t'=t} + t^2 \cdot \tfrac{\partial^2}{\partial t' \partial t} \cdot C(t,t')|_{t'=t}.\end{aligned} \tag{17}$$

The prior variances of the estimators (posterior expectations) equal

$$\begin{aligned}\mathrm{Var}(\widehat{m}(t)|\boldsymbol{T}) &= C(t)\cdot\left[C+\sigma^2\boldsymbol{I}\right]^{-1}\cdot C(t)',\\ \mathrm{Var}(\widehat{u}(t)|\boldsymbol{T}) &= D(t)\cdot\left[C+\sigma^2\boldsymbol{I}\right]^{-1}\cdot D(t)', \text{ and}\\ \mathrm{Var}(\widehat{u}'(t)|\boldsymbol{T}) &= B(t)\cdot\left[C+\sigma^2\boldsymbol{I}\right]^{-1}\cdot B(t)'.\end{aligned} \tag{18}$$

Choice of covariance kernel To fully specify the prior for $m(\cdot)$, we need to describe its prior moments, that is the mean function $\mu(\cdot)$ and the covariance kernel $C(\cdot,\cdot)$. Following common practice in the machine learning literature (cf. Williams and Rasmussen, 2006) we take $\mu = 0$ and consider covariance kernels of the form

$$C(t_1,t_2) = v_0 + v_1 \cdot t_1 t_2 + \exp\left(-|t_1-t_2|^2/(2l)\right). \tag{19}$$

The first two terms correspond to the covariance kernel of a linear trend $\beta_0 + \beta_1 t$ where β_0 and β_1 are uncorrelated and have variance v_0 and v_1. If v_0 and v_1 are chosen to be large, this prior (i) allows for arbitrary functional forms of the relationship between t and Y, (ii) is relatively uninformative about the intercept and slope of the relationship between t and Y, while (iii) providing shrinkage towards smooth functions.

Covariates and conditional independence Thus far we have assumed that T_i varies randomly (independently of ϵ_i) in our data. In practice, we can often more plausibly justify conditional independence given additional observed covariates W_i. If independence holds only conditionally, i.e., $T_i \perp \epsilon_i | W_i$, we can consider a Gaussian process prior for $k(t,w) = E[Y|T=t, W=t]$. Such a conditional approach is also warranted in experimental settings, such as the one considered in section 4, when we wish to adjust for random imbalances of covariates. The details of the conditional approach are spelled out in Section 5 below.

4 Application - The RAND health insurance experiment

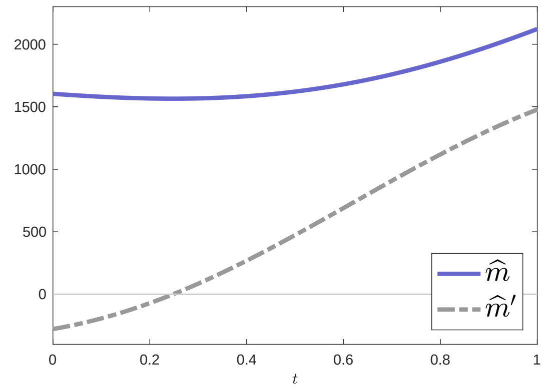
We now turn to our empirical application, using the data of the RAND health insurance experiment to estimate the behavioral response function $m(\cdot)$ as well as the social welfare function $u(\cdot)$, which in turn is used to determine the optimal coinsurance rate $\widehat{t^*}$.

Background and data The following discussion is based on the review of the RAND experiment provided by Aron-Dine et al. (2013). The RAND experiment, which took place between 1974 and 1981, provided health insurance to more than 5,800 individuals from about 2,000 households in six different locations across the United States. Families participating in the experiment were assigned to plans with one of six coinsurance rates. Four of the six plans simply set different overall coinsurance rates of 95, 50, 25, or 0 percent (free care). The other two plans were somewhat more complicated, with higher coinsurance rates for dental and outpatient mental health services, or for outpatient services in general. For the sake of simplicity of our discussion, data from the last two plans are neglected; the analysis focuses on the first four plans. The probability of assignment to each of these was .32 for the free care plan, .11 for the 25% coinsurance plan, .07 for the 50% coinsurance plan, and .19 for the 95% coinsurance plan.

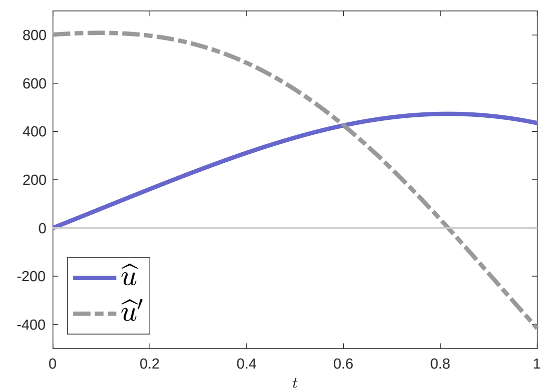
Families were additionally randomly assigned, within each of the six plans, to different out-of-pocket maximums, referred to as the "Maximum Dollar Expenditure." The possible Maximum Dollar Expenditure limits were 5, 10, or 15 percent of family income, up to a maximum of $750 or $1,000 (roughly $3,000 or $4,000 in 2011 dollars). We pool data across Maximum Dollar Expenditure amounts, and only consider the effect of coinsurance rates on expenditures.

Replication of results from Aron-Dine et al. (2013) As a first step, we replicate some of the results of Aron-Dine et al. (2013). We estimate predicted expenditures, using specifications corresponding to those used by Aron-Dine et al. (2013) for rows 2 and 3 in each of the panels of their Table 3. The chosen regression specification controls for month $\times$ site fixed effects and year fixed effects; this is necessary, since treatment was only conditionally random. The chosen specification additionally corrects for under-reporting of spending, by proportionally scaling up spending for outpatient services based on estimated rates of under-reporting. As discussed in Aron-Dine et al. (2013), this adjustment has only a minor impact on results. Our Table 1 reports predicted values for the share of families with any spending and for the average amount of spending within each of the treatment categories. Column 3 and 4 of this table control additionally for a rich set of predetermined covariates to correct for imbalance in the assignment. This correction again has only a minor effect. As can be seen from this table, spending is essentially unaffected by the coinsurance rate in the range from 95% coinsurance to 25% coinsurance. Only when approaching the free-care treatment does there appear to be an effect of the coinsurance rate on spending.

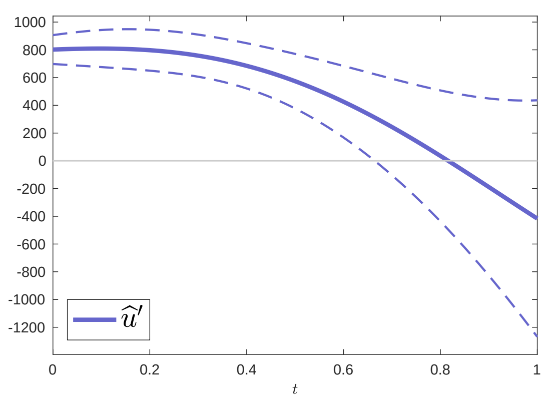
Figure 1: Health expenditures m, social welfare u and its derivative, and credible sets for u'. Estimates based on the RAND health insurance experiment data.



Notes: This graph shows our estimate of m, which describes average health care expenditures as a function of the subsidy rate t, and our estimate of m'.



Notes: This graph shows our estimate of u, which describes social welfare as a function of t, and our estimate of u'.



Notes: This graph shows point-wise 95% credible sets for u'.

Table 1: Predicted average expenditures for different coinsurance rates

	(1) Share with any	(2) Spending in $	(3) Share with any	(4) Spending in $
Free Care ($t = 1$)	0.931	2166.1	0.932	2173.9
	(0.006)	(78.76)	(0.006)	(72.06)
25% Coinsurance ($t = .75$)	0.853	1535.9	0.852	1580.1
	(0.013)	(130.5)	(0.012)	(115.2)
50% Coinsurance ($t = .5$)	0.832	1590.7	0.826	1634.1
	(0.018)	(273.7)	(0.016)	(279.6)
95% Coinsurance ($t = .05$)	0.808	1691.6	0.810	1639.2
	(0.011)	(95.40)	(0.009)	(88.48)
N	14777	14777	14777	14777

Notes: This table shows OLS estimates of average health care expenditures for the different treatment arms. Columns (1) and (2) control for month × site fixed effects and year fixed effects, columns (3) and (4) control additionally for a large set of further pre-determined covariates. All regressions are pooled across Maximum Dollar Expenditure values.

Estimation of $m(\cdot)$ We next apply the method proposed in Section 3 to these data. Consider first estimation of m, the response function which gives expected spending as a function of the subsidy rate t. The subsidy rate t equals 1 minus the coinsurance rate. We use a Gaussian process prior with squared exponential covariance kernel plus an "uninformative" (dispersed) linear component.[6] We use the same controls as in column 4 of table 1, so that our estimate $\widehat{m}$ is effectively a smooth interpolation of the estimates in this column. The first panel of Figure 1 shows our estimate $\widehat{m}$, as well as the estimated slope of m, $\widehat{m}'$. As to be expected based on the predicted values of table 1, $\widehat{m}$ is flat over most of it's support and curves upward toward the right, as t approaches 1, corresponding to the free care plan.

Estimation of $u(\cdot)$ and of t^* We next calculate the posterior expected social welfare $\widehat{u}$, as in equation (12), and its derivative $\widehat{u}'$. We assume that the preference for redistribution to the sick is given by $\lambda = 1.5$. This is a key parameter reflecting a normative choice by the policy maker; alternative values for λ are considered below. The specific parameter is chosen for illustration only, and our findings should be interpreted in this light. The second panel of Figure 1 plots our estimate $\widehat{u}$ of social welfare, and its derivative $\widehat{u}'$. The optimal policy choice $\widehat{t^*}$ solves the first order condition $\widehat{u}'(\widehat{t^*}) = 0$. We find an optimal policy choice of $\widehat{t^*} = 0.82$, corresponding to a coinsurance rate of 18%. As the objective function is fairly flat around this point, the free care plan performs almost as well in terms of expected social welfare.

[6]Specifically, we use the type of prior discussed in Section 5 below, with covariance kernel C^k of the form

$$\frac{1}{\sigma^2} C^k((t_1, w_1), (t_2, w_2)) = v_0 + v_1 \cdot t_1 t_2 + \exp\left(-\frac{1}{2}\left(\|t_1 - t_2\|^2 + \|w_1 - w_2\|^2\right)\right), \tag{20}$$

where we choose $v_0 = 100$ and $v_1 = 50$. $\|t_1 - t_2\|$ is the absolute difference in coinsurance, and $\|w_1 - w_2\|$ is the Euclidean norm of the difference in covariates. Covariates are scaled such that (i) year fixed effects and month × location fixed effects have a distance of 2 when they are unequal, and (ii) all other covariates have a distance of .2 when they are one standard deviation apart. For the distribution of covariates P_W we consider an "uninformative" Dirichlet prior with $\alpha = 0$, which implies that $\widehat{P_W}$ is equal to the empirical distribution of W.

Credible sets The last panel in Figure 1 plots a point-wise 95% credible set for $u'(t)$ for each t, centered at $\widehat{u}'(t)$ and with length equal to 1.96 times the posterior standard deviation of $u'(t)$. A frequentist confidence band, not shown here, looks almost identical. The intersection of this credible band with the horizontal axis yields a corresponding credible set for the optimal policy choice, which in this case ranges from a subsidy rate t of 68% to a subsidy rate of 100%, that is free care.

Comparison to the conventional "sufficient statistic" approach We next compare our results to those obtained using the conventional approach in public finance. As emphasized by Chetty (2009), the first order conditions for optimal policy models in a wide variety of settings only involve some key behavioral elasticities at the optimal policy. The corresponding empirical estimates of optimal policy are based on estimates of these behavioral elasticities, but usually estimated at some policy level other than the optimal one. Under the assumptions of section 2, the marginal social return to an increase of t can be rewritten as

$$u'(t) = m(t) \cdot [(\lambda - 1) - t \cdot m'(t)/m(t)] \; = m(t) \cdot \left[(\lambda - 1) - \eta \cdot \frac{t}{1-t}\right], \tag{21}$$

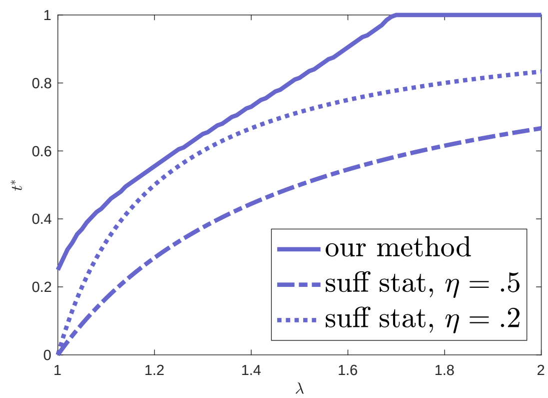
where η is the elasticity of health-care expenditures m with respect to copay $1-t$, $\eta := -\frac{\partial m(t)}{\partial (1-t)} \cdot \frac{1-t}{m(t)}$. Note that η is a function of t unless m is log-linear. Solving the first order condition $u'(t^*) = 0$ yields

$$t^* = \frac{1}{1 + \eta/(\lambda - 1)}. \tag{22}$$

The "sufficient statistic" approach then substitutes an estimate $\widehat{\eta}$ of η into equation (22). In order to estimate η, one could fit a linear regression of $\log(Y)$ on $\log(1-t)$ as well as the appropriate controls, and take the negative of the coefficient on $\log(1-t)$ as the estimate of η. This turns out not to be feasible in the present context, however, given that $t = 1$ for an important part of the experimental sample so that $\log(1-t)$ is not well defined. For these observations, the log-linear specification clearly makes no sense.

Various estimates for η based on the RAND experiment have been proposed in the literature, as discussed by Aron-Dine et al. (2013). The most famous estimate, constructed by the RAND investigators themselves, is given by $\widehat{\eta} = 0.2$. This estimate was constructed in a fairly complicated manner, based on so-called "arc-elasticities" for pairwise comparisons and averaging across these comparisons. Plugging this estimate into the sufficient-statistic formula yields $\tilde{t}^* = 1/1.4 \approx 0.7$, that is a suggested copay of approximately 30%. This is 12 percentage points higher than the optimal copay of 18% obtained using our method. Table 4 of Aron-Dine et al. (2013) presents various alternative estimates $\widehat{\eta}$, based on the more standard definition of an elasticity underlying our derivation of the sufficient statistic formula. Their estimates, omitting the free-care plan from calculations, are slightly larger than 0.5. Plugging this into the formula for $\tilde{t}^*$ yields $\tilde{t}^* \approx 1/2 = .5$ – that is a suggested copay of approximately 50%. This is 32 percentage points or almost 180 percent higher than the optimal copay of 18% obtained using our method.

Figure 2: The optimal policy t^* as a function of λ



Notes: This graph plots the optimal policy t^* as a function of distributive preference λ, estimated using our proposed method, using the sufficient statistic approach with the Aron-Dine et al. (2013) estimate of $\widehat{\eta} = 0.5$, and using the RAND investigators' estimate of $\widehat{\eta} = 0.2$.

Where do these differences come from? Note that we have (i) assumed the same model and the same social welfare function, (ii) have imposed the same value of $\lambda = 1.5$, and (iii) used the same data to estimate the behavioral relationship m. The difference stems from the way m, and thus u, are estimated. First, our method did not impose the log-linear functional form implicit in the constant-elasticity specification. In fact, η seems to vary significantly across t. Second, we estimate m in levels rather than in logs. This gives a larger weight to infrequent but high occurrences of large expenditures Y. Estimation in levels is appropriate according to the decision-theoretic setup. Third, we estimate m in a Bayesian way. This imposes some "shrinkage." Note, however, that we have used a prior which is relatively uninformative about the level and slope of m.

Varying λ The estimates of the optimal coinsurance rate discussed thus far are based on the normative choice of $\lambda = 1.5$. To explore the difference between our method and the sufficient statistic approach more systematically, Figure 2 plots the optimal policy t^* as a function of λ, estimated in three different ways; using our approach, using the sufficient statistic approach with the Aron-Dine et al. (2013) estimate of $\widehat{\eta} = 0.5$, and using the RAND investigators' estimate of $\widehat{\eta} = 0.2$. A higher value of λ (a higher preference for redistribution to the sick) implies a higher t^*, and so does a lower estimated elasticity $\widehat{\eta}$. Our approach consistently yields a higher t^* than the sufficient statistic approach, showing that our basic comparison is not specific to the value $\lambda = 1.5$.

Our method yields $t^* > 0$ even when $\lambda = 1$ because expenditures are estimated to be decreasing in t for small values of t, so that an additional US$ for the insured costs the insurance less than one US$. Note that values of $\lambda < 1$ would correspond to a preference for redistribution from the sick to the healthy.

5 Extensions

This section discusses several extensions of the setting introduced in Sections 2 and 3. First, we consider data where conditioning on covariates W_i is necessary for T_i to be independent of ϵ_i. We derive formulas for posterior expected social welfare for this case. Next, we briefly discuss non-normal outcomes Y. Then we consider optimal experimental design when the goal is to maximize social welfare, and assess the social value of adding experimental observations. Finally, we consider an alternative class of policy problems, where the goal is to maximize the average of some observable outcome net of the cost of inputs. The solution to this problem takes a form similar to the one we derived for the problem of optimal insurance, with different covariance functions $D(t)$ and $B(t)$.

Conditional independence We now discuss the generalization of the setting of Section 3 to the case where random assignment of T_i holds conditional on a vector of observable covariates W_i. Assume that we observe i.i.d. draws of (Y_i, T_i, W_i), that (as before) $Y_i = g(T_i, \epsilon_i)$, and that ϵ_i is independent of T_i given W_i. Define $k(t, w) = E[g(t, \epsilon_i)|W_i = w]$, assume

$$Y_i|T_i = t, W_i = w \sim N(k(t, w), \sigma^2), \tag{23}$$

and let

$$m(t) = E[g(t, \epsilon_i)] = \int k(t, w)dP_W(w). \tag{24}$$

Consider a prior for $k(\cdot, \cdot)$ of the form $k(\cdot, \cdot) \sim GP(\mu^k(\cdot), C^k(\cdot, \cdot))$, where now the mean function $\mu^k(\cdot)$ is a function of (t, w), and similarly for the covariance kernel $C^k(\cdot, \cdot)$. Consider furthermore a prior for P_W of the form $P_W \sim DP(\alpha, P_W^0)$, where $DP(\alpha, P_W^0)$ is the law of a Dirichlet process such that $E[P_W(\cdot)] = P_W^0(\cdot)$, and α is the "precision" of the prior. An introduction to Dirichlet priors can be found in Ghosh and Ramamoorthi (2003). Assume finally that the prior is such that $k(\cdot, \cdot)$ and P_W are independent.

Under these assumptions, the posterior expectation of $m(t)$ is equal to $\widehat{m}(t) = \int \widehat{k}(t, w)d\widehat{P}_W(w)$, where $\widehat{k}$ and $\widehat{P}_W$ are the corresponding posterior expectations. The posterior expectation of $k(t, w)$ is given by

$$\begin{aligned}
\widehat{k}(t, w) &= \mu^k(t, w) + C^k(t, w) \cdot \left[\boldsymbol{C}^k + \sigma^2 \boldsymbol{I}\right]^{-1} \cdot (\boldsymbol{Y} - \boldsymbol{\mu}^k), && \text{where} \\
\mu_i^k &= E[k(T_i)|\boldsymbol{T}, \boldsymbol{W}] = \mu^k(T_i, W_i), \\
C_{i,j}^k &= \mathrm{Cov}(k(T_i, W_i), k(T_j, W_j)|\boldsymbol{T}, \boldsymbol{W}) = C((T_i, W_i), (T_j, W_j)), && \text{and} \\
C_i^k(t, w) &= \mathrm{Cov}(k(t, w), k(T_i, W_i)|\boldsymbol{T}, \boldsymbol{W}) = C^k((t, w), (T_i, W_i)).
\end{aligned} \tag{25}$$

The posterior expectation of $dP_W(w)$ is equal to

$$d\widehat{P}_W(w) = \frac{\alpha}{\alpha + n} dP_W^0 + \frac{n}{\alpha + n} dP_W^n, \tag{26}$$

where P_W^n is the empirical distribution of W_i in the sample. Combining these results, we get

$$\begin{aligned}
\widehat{m}(t) =& \widehat{\mu}(t) + \widehat{C}(t) \cdot \left[\boldsymbol{C}^k + \sigma^2 \boldsymbol{I}\right]^{-1} \cdot (\boldsymbol{Y} - \boldsymbol{\mu}^k), && \text{where} \\
\widehat{\mu}(t) :=& \frac{\alpha}{\alpha+n} \int \mu^k(t,w) dP_W^0(w) + \frac{1}{\alpha+n} \sum_i \mu^k(t, W_i), && \text{and} \\
\widehat{C}(t) :=& \frac{\alpha}{\alpha+n} \int C^k(t,w) dP_W^0(w) + \frac{1}{\alpha+n} \sum_i C^k(t, W_i).
\end{aligned} \tag{27}$$

Similarly, for social welfare we get

$$\begin{aligned}
\widehat{u}(t) =& \widehat{\nu}(t) + \widehat{\boldsymbol{D}}(t) \cdot \left[\boldsymbol{C}^k + \sigma^2 \boldsymbol{I}\right]^{-1} \cdot (\boldsymbol{Y} - \boldsymbol{\mu}^k), && \text{where} \\
\widehat{\nu}(t) :=& \lambda \int_0^t \widehat{\mu}(x) dx - t \cdot \widehat{\mu}(t), && \text{and} \\
\widehat{\boldsymbol{D}}(t) :=& \lambda \cdot \int_0^t \widehat{C}(s, t') ds - t \cdot \widehat{C}(t, t').
\end{aligned} \tag{28}$$

Non-normal residuals So far, it was assumed that the outcomes Y_i are conditionally normally distributed. This seems a reasonable approximation in the context of our application. When outcomes are not normally distributed, there are various possible ways to generalize our setting, including the following two.

First, one could specify an appropriate alternative model for the outcome Y_i given T_i and W_i. Williams and Rasmussen (2006) discusses this in detail for the the case of binary outcomes, for instance. This approach has the advantage that it remains fully in the Bayesian paradigm, with its desirable decision theoretic properties. It has the disadvantage that the mapping from data to estimates becomes nonlinear and less transparent. In this case computation of $\widehat{m}$ and $\widehat{u}$ generally requires numerical simulation.

Alternatively, one could use the exact same estimators for $m(\cdot)$, $u(\cdot)$, and $u'(\cdot)$ which we have been using, but re-interpret them as posterior best linear predictors rather than posterior expectations. This has the advantage of maintaining the transparent and simple mapping from data to estimates. This is also in line with common empirical practice. Most non-parametric regression estimators are linear in the outcomes Y, and ordinary least squares regressions are commonly fit in settings with non-normal outcomes. This approach has the disadvantage that it lacks the decision theoretic justifications of the fully Bayesian approach.

Optimal experimental design and optimal sample size The decision problem considered thus far was to pick a policy t maximizing expected social welfare $\widehat{u}$ given experimental data $\boldsymbol{Y}, \boldsymbol{T}$. We can now take a step back and ask how to optimally design such experiments in order to maximize ex-ante expected welfare. And, taking one more step back, we can ask what the optimal sample size is, or equivalently, how to gauge the social value of an additional experimental observation.

An experimental design is a vector of policy levels $\boldsymbol{T} = (T_1, \ldots, T_n)$, assigned to a random sample of units $i = 1, \ldots, n$. The optimal design maximizes ex-ante expected welfare. Ex ante welfare, as a function of

$\boldsymbol{T}$, is defined assuming that the policy t is chosen as $\widehat{t^*} = t^*(\boldsymbol{Y}, \boldsymbol{T})$ once the experiment is completed. Define

$$\begin{aligned}\widehat{v}(\boldsymbol{T}) &:= E[\max_t \widehat{u}(t)|\boldsymbol{T}] = E[\widehat{u}(\widehat{t^*})|\boldsymbol{T}] \\ &= E[\max_t \nu(t) + \boldsymbol{D}(t) \cdot \left[\boldsymbol{C} + \sigma^2 \boldsymbol{I}\right]^{-1} \cdot (\boldsymbol{Y} - \boldsymbol{\mu})|\boldsymbol{T}],\end{aligned} \tag{29}$$

where $\boldsymbol{Y}|\boldsymbol{T} \sim N\left(\boldsymbol{\mu}, \boldsymbol{C} + \sigma^2 \boldsymbol{I}\right)$. The optimal experimental design $\boldsymbol{T}^*$ satisfies $\boldsymbol{T}^* \in \operatorname{argmax}_{\boldsymbol{T}} \widehat{v}(\boldsymbol{T})$. The dependence of $\widehat{v}(\boldsymbol{T})$ on $\boldsymbol{T}$ is implicit, through the dependence of $\boldsymbol{D}$, $\boldsymbol{C}$, and the distribution of $\boldsymbol{Y}$ on the design points T_i. $\widehat{v}(\boldsymbol{T})$ can be evaluated using simulation, and solutions to the maximization problem can be found numerically. Analytic characterizations are available in an older version of this manuscript [REFERENCE OMITTED FOR BLIND REVIEW].

Consider now the value of adding observations to our sample, and the value of the whole experiment. Both are characterized by the following value function for experiments of size n, assuming that both the experimental design $\boldsymbol{T}$ and the policy t are chosen optimally,

$$\widehat{v}(n) := \max_{\boldsymbol{T}} \widehat{v}(\boldsymbol{T}) = \max_{\boldsymbol{T}} E[\max_t \widehat{u}(t)|\boldsymbol{T}] = E[\widehat{u}(\widehat{t^*})|\boldsymbol{T} = \boldsymbol{T}^*]. \tag{30}$$

The value of adding an observation to the sample is given by $\widehat{v}(n+1) - \widehat{v}(n)$. The value of the whole experiment is given by $\widehat{v}(n) - \widehat{v}(0)$, where $\widehat{v}(0) = \max_t E[u(t)]$ is the prior expected maximum of u. The optimal sample size satisfies $n^* = \operatorname{argmax}_n \left(\widehat{v}(n) - \sum_{i=1}^n c(i)\right)$. Here $c(i)$ is the cost of an additional unit of observation at sample size i.

Production objective So far we have considered optimal policy problems of a form common in public finance, where social welfare reflects a trade-off between public revenues and the welfare (utility) of transfer recipients or tax payers. Welfare is estimated indirectly in these settings, since utility is not observable.

Another important class of policy problems is based on objectives defined in terms observable outcomes. Such problems can be described in the language of production functions. As an example, consider an educational setting, where i indexes schools, and Y_i measures long-run student outcomes of interest (or proxies for these long-run outcomes such as test scores). The vector $T \in \mathbb{R}^{d_t}$ is equal to the level of educational inputs such as teachers per student (class size), teacher salaries (affecting self-selection into teaching), school facilities, extra tutoring, length of the school year, etc.

Average student outcomes in school i are determined by the "educational production function" $Y_i = g(T_i, \epsilon_i)$ where ϵ_i denotes unobserved inputs such as students' family backgrounds. The policy maker's objective is to maximize average (expected) outcomes $E[Y_i]$ across schools, net of the cost of inputs. The unit-price of input j is given by p_j. The policy maker's willingness to pay for a unit-increase in Y is given by λ. This yields the objective function $u(t) = \lambda \cdot m(t) - p \cdot t$, where we define $m(t) = E[g(t, \epsilon_i)]$, as before.

Given the assumptions of Section 3 (experimental assignment of T_i, normal residuals, Gaussian process prior for $m(\cdot)$), the posterior mean for u is given by

$$\begin{aligned}
\widehat{u}(t) &= \nu(t) + \boldsymbol{D}(t) \cdot \left[\boldsymbol{C} + \sigma^2 \boldsymbol{I}\right]^{-1} \cdot (\boldsymbol{Y} - \boldsymbol{\mu}), && \text{where now} \\
\nu(t) &= \lambda \cdot \mu(t) - p \cdot t && \text{and} \\
D(t, t') &= \lambda \cdot \boldsymbol{C}(t, t'). && (31)
\end{aligned}$$

and the optimal policy satisfies the first order condition $\widehat{u}'(\widehat{t^*}) = \nu'(\widehat{t^*}) + \boldsymbol{B}(\widehat{t^*}) \cdot \left[\boldsymbol{C} + \sigma^2 \boldsymbol{I}\right]^{-1} \cdot (\boldsymbol{Y} - \boldsymbol{\mu}) = 0$, as before, where now $B(t, t') = \lambda \cdot \frac{\partial}{\partial t} \boldsymbol{C}(t, t')$.

Examples of experimental evidence on the role of educational inputs can be found in Fryer (2014), Angrist and Lavy (1999), Krueger (1999), and Rivkin et al. (2005). Further examples for such choice-of-inputs problems can be found in the experimental development economics literature; cf. the survey in Banerjee and Duflo (2009). The profit maximization problem of the firm, as treated in standard microeconomic theory (cf. Mas-Colell et al., 1995, chapter 5), can be described in these terms as well.

6 Conclusion

This paper combines insights from the theory of optimal taxation and insurance with insights from machine learning and nonparametric Bayesian decision theory. This paper proposes a framework based on a standard social welfare function, (quasi-)experimental policy variation, and Gaussian process priors, which leads to tractable, explicit expressions characterizing the optimal policy choice. Applying the proposed method to data from the RAND health insurance experiment we find values for the optimal policy choice that are substantially different from those obtained using the standard "sufficient statistic" approach.

This paper points toward a large area of potential applications for machine learning methods in informing policy. Most commonly, machine learning methods are devised to solve problems of prediction. Relative to pure prediction problems, two additional conceptual layers enter the problem of optimal policy choice. First, we need some form of exogenous variation to arrive at causal estimates, so that we can interpret predictions as counterfactual average outcomes. Second, we need some basis for normative evaluations of these counterfactual outcomes. One possible normative basis is the class of social welfare functions which are considered in this paper.

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Appendix

This appendix provides additional background and technical details to supplement our main discussion.

A The envelope theorem

A key step in the derivation of the social welfare function in equation (4) is the assumption that individuals' behavioral responses do not affect private welfare. This assumption is justified by the envelope theorem. There are many versions of this theorem, this section reviews a basic version. For further discussion see Mas-Colell et al. (1995), Milgrom and Segal (2002), and Chetty (2009).

Let t be a (policy) parameter, for instance the share of health care expenditures covered by insurance, and let x be a vector of individual choices, such as the choice of when to visit a doctor or hospital, etc. Suppose an individual maximizes $\upsilon(x,t)$ subject to $x \in \mathscr{X}$, given t. The set $\mathscr{X}$ captures all constraints faced by the individual. Let $x(t)$ be the individual's choice given t, where we assume that she maximizes her utility, $x(t) \in \operatorname{argmax}_{x\in\mathscr{X}} \upsilon(x,t)$. The individual's welfare (maximum utility) is given by

$$V(t) = \upsilon(x(t),t) = \sup_{x\in\mathscr{X}} \upsilon(x,t). \tag{32}$$

Let $x^* = x(t)$ for some fixed t, and define

$$\begin{aligned}\tilde{V}(s) &= V(s) - \upsilon(x^*,s) = \upsilon(x(s),s) - \upsilon(x(t),s)\\ &= \sup_{x\in\mathscr{X}} \upsilon(x,s) - \upsilon(x^*,s).\end{aligned} \tag{33}$$

This definition immediately implies $\tilde{V}(s) \geq 0$ for all s and $\tilde{V}(t) = 0$. If $\tilde{V}$ is differentiable at t, it follows that $\tilde{V}'(t) = 0$, so that

$$V'(t) = \frac{\partial}{\partial s}\upsilon(x^*,s)|_{s=t}, \tag{34}$$

where x^* does not depend on s on the right hand side. Behavioral changes are thus irrelevant for the welfare impact of a marginal policy change. General conditions to guarantee differentiability of V are difficult to obtain; sufficient conditions are discussed in Milgrom and Segal (2002). Note, however, that differentiability of x and in particular continuity of the feasible set $\mathscr{X}$ are not required.

In the context of our health insurance application, the choice vector x might include behavioral margins such as labor supply, preventative health behavior, whether to visit a doctor, which doctor to visit, etc. For a given choice vector x, the coinsurance rate t then determines how much money the individual has available for consumption other than health care. An individual's utility υ_i depends on all her choices and her consumption. The envelope theorem tells us that the effect of a policy change on utility υ_i is the same as the effect of the hypothetical increase in her consumption that would result holding her current choices fixed. This effect can be calculated mechanically, multiplying current health care expenditures by the change in t.

B General policy problem

Sections 2 and 5 discussed two common classes of policy problems in economics. These are special cases of a more general class of policy problems, where we can write the social welfare function in the form

$$u = Lm + u_0, \tag{35}$$

for a known function u_0 on $\mathscr{T} \subset \mathbb{R}^{d_t}$ and a linear operator L mapping the set of continuously differentiable functions m defined on $\mathscr{T}$ into itself. The linear operator might be defined using operations such as integration, multiplication by known functions, etc. Maintaining the same assumptions as before on experimental data and the policy maker's prior, in particular $m \sim GP(\mu(.), C(., .))$, where μ and C are defined on $\mathscr{T}$ again, and assuming the necessary continuity and differentiability conditions, we get posterior expectations of the form

$$\begin{aligned}
\widehat{m}(t) &= \mu(t) + \boldsymbol{C}(t) \cdot \left[\boldsymbol{C} + \sigma^2 \boldsymbol{I}\right]^{-1} \cdot (\boldsymbol{Y} - \boldsymbol{\mu}) \\
\widehat{u}(t) &= \nu(t) + \boldsymbol{D}(t) \cdot \left[\boldsymbol{C} + \sigma^2 \boldsymbol{I}\right]^{-1} \cdot (\boldsymbol{Y} - \boldsymbol{\mu}) \\
\widehat{u}'(t) &= \nu'(t) + \boldsymbol{B}(t) \cdot \left[\boldsymbol{C} + \sigma^2 \boldsymbol{I}\right]^{-1} \cdot (\boldsymbol{Y} - \boldsymbol{\mu})
\end{aligned} \tag{36}$$

where

$$\begin{aligned}
\nu(t) &= (L\mu)(t) + u_0(t), \\
D(t, t') &= \operatorname{Cov}(u(t), m(t')) = L_x C(x, t'), \\
B(t, t') &= \operatorname{Cov}\left(\frac{\partial}{\partial t} u(t), m(t')\right) = \frac{\partial}{\partial t} D(t, t') = \frac{\partial}{\partial t} L_x C(x, t').
\end{aligned} \tag{37}$$

In these equations we write $L_x C(x, t')$ to emphasize that this expression applies the linear operator L to $C(x, t')$ as a function of x for fixed t'.

With this more general formulation, we see immediately how our baseline application extends to more general policy problems. This includes in particular the case where t is a multidimensional vector, including for instance tax rates for several tax brackets, or features such as maximum deductibles in insurance plans. This also includes the case where λ is allowed to vary with t, so that $u(t) = \int_0^t \lambda(x) m(x) dx - t \cdot m(t)$.

C Explicit covariance kernels

Consider the optimal insurance problem of Section 2, where $u(t) = \lambda \int_0^t m(x) dx - t \cdot m(t)$, and a covariance kernel for the prior on $m(\cdot)$ of the form

$$C(t_1, t_2) = v_0 + v_1 \cdot t_1 t_2 + \tfrac{1}{\varphi(0)} \cdot \varphi\left(\frac{t_1 - t_2}{l}\right), \tag{38}$$

where φ is the standard normal pdf and l is a parameter determining the length scale of the kernel. Denote the standard normal cdf by Φ. We neglect covariates here for clarity of exposition; otherwise this is the covariance kernel used in our application.

For this setting, we can provide explicit expressions for the covariance functions D and B,

$$\begin{aligned}
D(t,x) &= \lambda \cdot \int_0^t C(x',x)dx' - t \cdot C(t,x) \\
&= (\lambda-1)v_0 t + (\lambda/2-1)v_1 x t^2 \\
&+ \frac{1}{\varphi(0)} \cdot \left[\lambda l \cdot \left(\Phi\left(\frac{t-x}{l}\right) - \Phi\left(\frac{-x}{l}\right)\right) - t \cdot \varphi\left(\frac{t-x}{l}\right)\right], \qquad \text{and} \qquad (39) \\
B(t,x) &= (\lambda-1)\cdot C(t,x) - t \cdot \tfrac{\partial}{\partial t} C(t,x) \\
&= (\lambda-1)v_0 + (\lambda-2)v_1 t x + \frac{1}{\varphi(0)} \cdot \left[(\lambda-1)\cdot\varphi\left(\frac{t-x}{l}\right) - \frac{t}{l}\cdot\varphi'\left(\frac{t-x}{l}\right)\right] \\
&= (\lambda-1)v_0 + (\lambda-2)v_1 t x + \frac{\varphi\left(\frac{t-x}{l}\right)}{\varphi(0)} \cdot \left[(\lambda-1) + \frac{t\cdot(t-x)}{l^2}\right]. \qquad (40)
\end{aligned}$$

We finally get

$$\begin{aligned}
\mathrm{Var}(u'(t)) &= \mathrm{Var}((\lambda-1)\cdot m(t) - t\cdot m'(t)) = \\
&= (\lambda-1)^2 \cdot C(t,t) - 2(\lambda-1)\cdot t \cdot \tfrac{\partial}{\partial t'} C(t,t')|_{t'=t} + t^2 \cdot \frac{\partial^2}{\partial t' \partial t} C(t,t')|_{t'=t} \\
&= (\lambda-1)^2 v_0 + (\lambda-2)^2 t^2 v_1 + (\lambda-1)^2 + \frac{t^2}{l^2}. \qquad (41)
\end{aligned}$$

The latter expression is useful for the construction of credible sets; cf. Section 3.

D Equivalent kernel

By symmetry and unimodality of the posterior under our assumptions, the posterior expectation $\widehat{m}(t) = E[Y|T=t]$ can be written as a maximum a posteriori, that is, as the solution to the penalized regression

$$\widehat{m} = \underset{l(.)}{\mathrm{argmin}} \left[\frac{1}{\sigma^2} \cdot \sum_i (Y_i - l(T_i))^2 + \|l-\mu\|_C^2\right], \qquad (42)$$

where $\|m-\mu\|_C^2$ is a penalty term. The norm $\|m\|_C^2$ is the reproducing kernel Hilbert space norm corresponding to the covariance kernel C. It is defined as the norm corresponding to an inner product on the space of all linear combinations of functions of the form $C(t,.)$ and their limits, where $\langle C(t_1,.), C(t_2,.)\rangle = C(t_1,t_2)$, cf. Wahba (1990) and van der Vaart and van Zanten (2008). By equation (9), the posterior expectation can also be written in the form $\widehat{m}(t) = w_0(t) + \frac{1}{n}\sum_i w(t,T_i)\cdot Y_i$ for some weight function w. The weight function $w(.,T_i)$ thus corresponds to the estimate of $\widehat{m}$ we would obtain if we had $Y_i = n$ and $Y_j = 0$ for $j \neq i$, and if we replace μ by 0. Representation (42) then implies

$$\begin{aligned} w(.,T_i) &= \underset{l(.)}{\operatorname{argmin}} \left[\sum_{j \neq i} l(T_i)^2 + (n - l(T_i))^2 + \sigma^2 \cdot \|l\|_C^2 \right] \\ &= \underset{l(.)}{\operatorname{argmin}} \left[\frac{1}{2} \int l(t)^2 dF_n(t) + \frac{\sigma^2}{2n} \cdot \|l\|_C^2 - l(T_i) \right], \end{aligned} \tag{43}$$

where F_n is the empirical distribution function of T. If we replace the empirical distribution F_n by the population distribution F in this expression, we get an approximation of w by the solution to the minimization problem

$$\overline{w}(.,t') = \underset{l(.)}{\operatorname{argmin}} \left[\frac{1}{2} \int l(t)^2 dF(t) + \frac{\sigma^2}{2n} \cdot \|l\|_C^2 - l(t') \right]. \tag{44}$$

The solution to this latter minimization problem is called the equivalent kernel (cf. Silverman, 1984; Sollich and Williams, 2005; Williams and Rasmussen, 2006, chapter 7). The equivalent kernel does not depend on the data, but it does depend on the sample size n which scales the penalty term $\|m\|_C^2$. The validity of this approximation hinges on the uniform closeness of $\int m(t)^2 dF_n(t)$ and $\int m(t)^2 dF(t)$.

We can feed this equivalent kernel approximation into our policy problem, to get an approximation to posterior expected social welfare in terms of a weighted average of outcomes with deterministic weights. For the general policy problem of Appendix B, where $u = Lm + u_0$, this yields $\widehat{u}(t) \approx \tilde{u}_0(t) + \frac{1}{n} \sum_i v(t, T_i) \cdot Y_i$, where $v(\cdot, T_i) = L\overline{w}(., t')$. This approximation points toward a derivation of the frequentist properties of $\widehat{u}(\cdot)$ and of $\widehat{t^*}$. In particular, if $\widehat{m}$ is consistent at a fast enough rate and some conditions on the weight functions hold, then the central limit theorem suggests $\widehat{u}(t) \sim^A N\left(u(t), \frac{1}{n} \operatorname{Var}(v(t, T_i) \cdot Y_i)\right)$. A Taylor expansion around the optimum suggests $\widehat{t^*} \sim^A N\left(t^*, \frac{1}{n \cdot u''(t^*)^2} \operatorname{Var}(\partial_t v(t, T_i) \cdot Y_i)\right)$.