Player-Compatible Equilibrium*

Drew Fudenberg† Kevin He‡

First version: September 23, 2017
This version: December 24, 2017

Abstract

We define Player-Compatible Equilibrium or “PCE,” which imposes cross-player restrictions on magnitudes of the players’ “trembles” onto different actions. These restrictions are inspired by the idea that trembles correspond to deliberate experiments by inexperienced agents who are unsure of the prevailing distribution of strategies in opponent populations. We show that PCE selects the “intuitive” equilibria in a number of examples where trembling-hand perfect equilibrium (Selten, 1975) and proper equilibrium (Myerson, 1978) have no bite. We also provide a learning-based microfoundation for PCE in some easy-to-analyze classes of games. Finally, we conduct a lab experiment based on one of our examples and verify PCE leads to better predictions than other equilibrium concepts.

*We thank Dave Rand for useful conversations and comments. We are grateful to the staff and research assistants at Harvard Decision Science Laboratory, especially Alki Iliopoulou, Gabe Mansur, Christina Qiu, and Sylvie Stoloff. We thank National Science Foundation grant SES 1643517 for financial support.
†Department of Economics, MIT. Email: drew.fudenberg@gmail.com
‡Department of Economics, Harvard University. Email: hesichao@gmail.com
1 Introduction

Starting with Selten (1975), a number of papers have used the device of “trembles” or “mistakes” to generate restrictions on play at out-of-equilibrium information sets and hence refine the set of Nash equilibria. This paper introduces the concept of Player-Compatible Equilibrium (PCE), which uses trembles to capture the idea that departures from the equilibrium path correspond to deliberate experiments by agents who are uncertain about the prevailing distribution of strategies in the population. We restrict trembles based on the insight that different players have different incentives for experimenting with various strategies when learning the distribution of play. Intuitively, if one player could gain more than another from experimenting with a particular action whenever the two players have the same belief about the play of others, then we expect the player who has more to gain — the player who is more compatible with that action — to use it more often. Guided by this intuition, PCE imposes cross-player restrictions on the relative tremble probabilities onto different actions. By imposing these additional restrictions, we are able to select the “intuitive” equilibria in a number of examples where trembling-hand perfect equilibrium and even proper equilibrium (Myerson, 1978) have no bite. We say more about how these and other tremble-based equilibrium refinements relate to PCE in Sections 2 and 3 below; in particular we show that as intended in signalling games it implies the compatibility criterion of Fudenberg and He (2017a).

In addition to showing that PCE selects the “right” equilibria in our examples, we also provide a learning-based microfoundation for PCE in some easy-to-analyze classes of games in Section 4. There we consider a steady-state learning model in the style of Fudenberg and Levine (1993), Fudenberg and Levine (2006), Fudenberg and He (2017a,b), where agents arrive not knowing the distribution of play, and instead have a prior belief about what that distribution is. Here “trembles” arise endogenously from rational experimentation by young and inexperienced agents, and we show that the relative frequencies of experimentation by agents in different player roles satisfy cross-player compatibility restrictions similar to those that PCE imposes on tremble probabilities. As agents in the learning model become patient and long-lived, they will choose to experiment more and more, and we show they have enough data to learn about and best respond to a strategy distribution satisfying these compatibility restrictions. This leads aggregate play to approximate a PCE.

We feel that PCE’s learning foundation helps make it more compelling, but the concept is well-defined even in games where we have not explicitly characterized the implications of learning. Working directly with trembles instead of characterizing optimal experimentation permits a simpler and more tractable solution concept. Moreover, while we have only analyzed one specific learning model, we expect the restrictions of PCE to also emerge from other learning models and more general models of dynamic adjustment towards equilibrium.
We expect that the predictions of PCE will be verified experimentally in the lab, even though the learning model’s focus on arbitrarily long-lived players and perfectly rational experimentation do not apply there. We test our theory by conducting a lab experiment based on two versions of a link-formation game in Section 3.3. In contrast to existing equilibrium concepts, which all predict no difference in behavior between two versions of the game, PCE makes the comparative static prediction that more links will be formed in the version where players with low link-formation cost provide high benefits to others than in the version where low-cost players provide low benefits. Moreover, by making additional assumptions about the learners’ priors, we can predict that play will evolve differently in the two versions of the game. We pre-registered these predictions and found that they were mostly though not completely supported in a laboratory experiment, as we report in Section 5. In particular, we find that while initial play in the two versions of the link-formation game is statistically indistinguishable, the rates of link formation diverge with learning so that there is a significant difference in link-formation rates in the last quarter of the periods in the direction that PCE predicts.

1.1 Related Work

Tremble-based solution concepts date back to Selten (1975), who thanks Harsanyi for suggesting them. These solution concepts consider totally mixed strategy profiles where players do not play an exact best reply to the strategies of others, but may assign positive probability to some or all strategies that are not best replies. Different solution concepts in this class consider different kinds of “trembles,” but they all make predictions based on the limits of these non-equilibrium strategy profiles as the probability of trembling tends to zero. Since we compare PCE to these refinements below, we summarize them here for the reader’s convenience.

Perfect equilibrium (Selten, 1975), proper equilibrium (Myerson, 1978), approachable equilibrium (Van Damme, 1987), and extended proper equilibrium (Milgrom and Mollner, 2017b) are based on the idea that strategies with worse payoffs are played less often. An $\epsilon$-perfect equilibrium is a totally mixed strategy profile where every non-best reply has weight less than $\epsilon$. A limit of $\epsilon_t$-perfect equilibria where $\epsilon_t \to 0$ is called a trembling-hand perfect equilibrium. An $\epsilon$-proper equilibrium is a totally mixed strategy profile $\alpha$ where for every player $i$ and actions $a_i$ and $a'_i$, if $u_i(a_i, \alpha_{-i}) < u_i(a'_i, \alpha_{-i})$ then $\alpha_i(a_i) < \epsilon \cdot \alpha_i(a'_i)$. A limit of $\epsilon_t$-proper equilibria where $\epsilon_t \to 0$ is called a proper equilibrium; in this limit a more costly tremble is infinitely less likely than a less costly one, regardless of the cost difference. An approachable equilibrium is the limit of somewhat similar $\epsilon_t$-perfect equilibria, but where the players pay control costs to reduce their tremble probabilities. When these costs are “regular,” all of the trembles are of the same order. Because PCE does not require that the less likely trembles
are infinitely less likely than more likely ones, it is closer to approachable equilibrium than to proper equilibrium.

Proper equilibrium and approachable equilibrium do not impose cross-player restrictions on the relative probabilities of various trembles, so they have no bite in signalling games if each type of the sender is viewed as a different player. They do impose restrictions when applied to the ex-ante form of the game, i.e. at the stage before the sender has learned their type. However, as Cho and Kreps (1987) point out, evaluating the cost of mistakes at the ex-ante stage means that the interim losses are weighted by the prior distribution over sender types, so that less likely types are more likely to tremble. In addition, applying a different linear rescaling to each type’s utility function preserves every type’s preference over lotteries on outcomes, but changes the sets of proper and approachable equilibria, while such changes have no effect on the set of PCE. In light of these issues, when discussing tremble-based refinements in Bayesian games we will always apply them at the interim stage.

Like PCE, extended proper equilibrium places restrictions on the relative probabilities of tremble by different players, but it does so in a different way: An extended proper equilibrium is the limit of \((\beta, \epsilon_i)\)-proper equilibria, where \(\beta = (\beta_1, ... \beta_I)\) is a strictly positive vector of utility re-scaling, and \(\alpha_i(a_i) < \epsilon_i \cdot \alpha_j(a_j)\) if player \(i\)'s rescaled loss from \(a_i\) (compared to the best response) is less than \(j\)'s loss from \(a_j\). In a signalling game with only two possible signals, every Nash equilibrium that is strict for the sender is an extended proper equilibrium at the interim stage, because the utility rescalings for the types permits any ranking of their utility costs of deviating to the off-path signal. By contrast, Proposition 2 shows every PCE must satisfy the compatibility criterion of Fudenberg and He (2017a), which has bite even in binary signalling games such as the beer-quiche example of Cho and Kreps (1987). So an extended proper equilibrium need be a PCE, a fact that Example 1 further demonstrates in a game with larger action sets. Conversely, because extended proper equilibrium makes some trembles infinitely less likely than others, it can eliminate some PCE, as shown by example in Appendix A.2, which also restates the formal definition of extended proper equilibrium for ease of reference.

The strategic stability concept of Kohlberg and Mertens (1986) is also defined using trembles, but applies to components of Nash equilibria as opposed to single strategy profiles.

Finally, although it is not explicitly defined in terms of trembles, test-set equilibrium (Milgrom and Mollner, 2017a) is motivated by the idea that trembles onto an alternative best response to the equilibrium play is more likely than trembles onto an inferior response. This refinement is particularly useful in auction-like environments, where there are many weak Nash equilibria and all opponent deviations to alternative best responses generate the same

---

1 The test set \(T(\sigma)\) of strategy profile \(\sigma\) is the set of profiles that can be generated by replacing one player’s strategy in \(\sigma\) with an alternative best response. A Nash equilibrium \(\sigma\) is called a test-set equilibrium if each \(\sigma_i\) is undominated with respect to \(\Delta(A_{-i})\) and also undominated with respect to \(T(\sigma)\).
strategic incentives for any given player — in this case, the test set is “large”. But as we show in Example 1, PCE has bite in equilibria with “small” test sets, where most (e.g. all but 1) players strictly prefer their equilibrium action, and test-set equilibrium reduces to Nash.

The experiment we report on in Section 5 is related to experimental tests of other equilibrium refinements, such as the studies of Brandts and Holt (1992, 1993); Banks, Camerer, and Porter (1994) on whether observed play in signalling games satisfies the Intuitive Criterion (Cho and Kreps, 1987), as well as the work of Van Huyck et al. (1990) and on which equilibria tend to emerge in coordination games.

2 Player-Compatible Equilibrium

2.1 PCE in General Games

Consider a strategic-form game with finite number of players $i \in I$, finite strategy set $A_i$ and utility functions $u_i : A \to \mathbb{R}$ for each player $i$. We formalize the concept of “compatibility” between players and their actions in this general setting.

**Definition 1.** For player $i,j$ and actions $a_i^* \in A_i$, $a_j^* \in A_j$, say $i$ is more compatible with $a_i^*$ than $j$ is with $a_j^*$, abbreviated as $(a_i^*|i) \succ (a_j^*|j)$, if for every strictly mixed profile $\alpha_{-ij} \in \Delta(A_{-ij})$ such that

$$u_j(a_j^*, \hat{a}_i, \alpha_{-ij}) \geq \max_{a_j' \in A_j \setminus \{a_j^*\}} u_j(a_j', \hat{a}_i, \alpha_{-ij})$$

for some strictly mixed $\hat{a}_i \in \Delta(A_i)$, we have

$$u_i(a_i^*, \hat{a}_j, \alpha_{-ij}) > \max_{a_i' \in A_i \setminus \{a_i^*\}} u_i(a_i', \hat{a}_j, \alpha_{-ij}),$$

for every strictly mixed $\hat{a}_j \in \Delta(A_j)$.

In words, we start with some mixed strategy profile $\alpha_{-ij}$ for players other than $i$ and $j$. If there is some completion of the profile with a mixed action $\hat{a}_i \in \Delta(A_i)$ so that $a_i^*$ is weakly optimal for $j$ against the $-j$ profile $(\alpha_{-ij}, \hat{a}_i)$, then for every completion of $\alpha_{-ij}$ with a mixed action $\hat{a}_j \in A_j$ we have that $a_i^*$ is strictly optimal for $i$ against $(\alpha_{-ij}, \hat{a}_j)$. As this restatement makes clear, this property does not depend on the particular utility representations chosen.

**Remark 1.** If players $i$ and $j$ care a great deal about one another’s actions, then their best responses are unlikely to be determined by the play of the other players. Conversely, suppose the set of players $I$ can be partitioned as $I = I_1 \cup \ldots \cup I_K$, in such a way that whenever $i$
and \( j \) are in the same partition \( i, j \in I_k \), (1) they are non-interacting, meaning \( i \)'s payoff does not depend on the action of \( j \) and \( j \)'s payoff does not depend on the action of \( i \); (2) they have the same action set, \( A_i = A_j \). As a leading case, in a Bayesian game where each type is viewed as a different agent, the set of types for a given player role (e.g. the sender in a signalling game) define such a partition. In addition, Section 3 gives several examples of complete-information games with this kind of partitioning structure. In a partitioned game with \( i, j \in I_k \), we may write \( u_i(a_i, a_{-ij}) \) without ambiguity, since all augmentations of the strategy profile \( a_{-ij} \) with an action by player \( j \) lead to the same payoff for \( i \). When \( i, j \) are in the same partition and \( a^*_m \in A_i = A_j \), the definition for \( (a^*_m| i) \succ (a^*_m| j) \) reduces to: For every strictly mixed \( \alpha_{-ij} \in \Delta(A_{-ij}) \) such that

\[
u_j(a_m, \alpha_{-ij}) \geq \max_{a'_j \in A_j \setminus \{a_m\}} u_j(a'_j, \alpha_{-ij}),\]

we have

\[
u_i(a_m, \alpha_{-ij}) > \max_{a'_i \in A_i \setminus \{a_m\}} u_i(a'_i, \alpha_{-ij}).\]

That is, \( (a^*_m| i) \succ (a^*_m| j) \) if whenever \( a_m \) is even a weak best response for \( j \) against some opponents’ strategy profile, it is also a strict best response for \( i \) against the same strategy profile.

PCE is a tremble-based solution concept. It builds on and modifies Selten (1975)’s definition of trembling-hand perfect equilibrium as the limit of equilibria of perturbed games in which agents are constrained to tremble, so we begin by defining our notation for the trembles and the associated constrained equilibria.

**Definition 2.** A tremble profile \( \bar{\epsilon} \) assigns a positive number \( \bar{\epsilon}(a_i| i) > 0 \) to every player \( i \) and pure strategy \( a_i \). Given a tremble profile, write \( \Pi^\bar{\epsilon}_i \) for the set of \( \bar{\epsilon} \)-strategies of player \( i \), namely:

\[
\Pi^\bar{\epsilon}_i := \{ \alpha_i \in \Delta(A_i) \text{ s.t. } \alpha_i(a_i) \geq \bar{\epsilon}(a_i| i) \}. 
\]

We call \( \alpha^\circ \) an \( \bar{\epsilon} \)-equilibrium if for each \( i \),

\[
\alpha^\circ_i \in \arg \max_{a_i \in \Pi^\bar{\epsilon}_i} u_i(\alpha_i, \alpha^\circ_{-i}).
\]

Note that \( \Pi^\bar{\epsilon}_i \) is compact and convex. It is also non-empty when \( \bar{\epsilon} \) is close enough to 0. By standard results, whenever \( \bar{\epsilon} \) is small enough so that \( \Pi^\bar{\epsilon}_i \) is non-empty for each \( i \), an \( \bar{\epsilon} \)-equilibrium exists.

The key building block for PCE is \( \bar{\epsilon} \)-PCE, which roughly speaking can be viewed as an \( \bar{\epsilon} \)-equilibrium where the players’ actions are co-monotonic with respect to \( \succ \).
Definition 3. For tremble profile \( \epsilon \) and every \( \alpha \in \Pi_i^\epsilon \), define the set of player-compatible \( \epsilon \)-strategies for each player \( i \):

\[
\Pi_i^\epsilon(\alpha_{-i}) := \left\{ \alpha_i \in \Pi_i^\epsilon \text{ s.t. } \alpha_i(a_i) \geq \min \left[ \alpha_j(a_j), 1 - \sum_{a_i \neq a_i} \epsilon(a_i'|i) \right] \text{ whenever } (a_i|i) \succ (a_j|j) \right\}.
\]

A player-compatible \( \epsilon \)-equilibrium (or \( \epsilon \)-PCE) is strategy profile \( \alpha^o \) such that for all \( i \)

\[
\alpha_i^o \in \arg \max_{\alpha_i \in \Pi_i^\epsilon(\alpha_{-i})} u_i(\alpha_i, \alpha_{-i}^o).
\]

Relative to \( \epsilon \)-equilibrium, \( \epsilon \)-PCE imposes the additional restriction that whenever \((a_i^*|i) \succ (a_j^*|j)\), \( i \) will put at least as much weight on \( a_i^* \) as \( j \) does on \( a_j^* \), up to the constraint imposed by the minimum probabilities \( \epsilon(a_i'|i) \). In other words, when \((a_i^*|i) \succ (a_j^*|j)\) is the only compatibility relation, \( \alpha_i \in \Pi_i^\epsilon(\alpha_{-i}) \) if either \( \alpha_i(a_i^*) \geq \alpha_j(a_j^*) \) or \( \alpha_i(a_i^*) = \max_{a_i \in \Pi_i^\epsilon} \alpha_i(a_i^*) \).

As is usual for tremble-based equilibrium refinements, we now define PCE as the limit of a sequence of \( \epsilon \)-PCEs where \( \epsilon \to 0 \). In such a sequence, \( \liminf_{t \to \infty} \alpha^{(t)}(a_i^*)/\alpha^{(t)}(a_j^*) \geq 1 \) whenever \((a_i^*|i) \succ (a_j^*|j)\). This property is the driving force behind the PCE refinement.

Definition 4. A strategy profile \( \alpha^* \) is a player-compatible equilibrium (PCE) if there exists a sequence of tremble profiles \( \epsilon^{(t)} \to \overline{0} \) and an associated sequence of strategy profiles \( \alpha^{(t)} \), where each \( \alpha^{(t)} \) is an \( \epsilon^{(t)} \)-PCE, such that \( \alpha^{(t)} \to \alpha^* \).

Unlike \( \epsilon \)-equilibrium, which always exists provided \( \epsilon \) is close enough to \( \overline{0} \), \( \epsilon \)-PCE need not exist even for arbitrarily small \( \epsilon \). As a simple example, suppose there are 3 players where player 1 chooses a row, player 2 chooses a column, and player 3 chooses a matrix.

<table>
<thead>
<tr>
<th>Matrix A</th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top</td>
<td>1,1,0</td>
<td>1,4,0</td>
</tr>
<tr>
<td>Bottom</td>
<td>3,1,0</td>
<td>3,4,0</td>
</tr>
<tr>
<td>Matrix B</td>
<td>Left</td>
<td>Right</td>
</tr>
<tr>
<td>Top</td>
<td>3,3,1</td>
<td>3,2,1</td>
</tr>
<tr>
<td>Bottom</td>
<td>1,3,1</td>
<td>1,2,1</td>
</tr>
</tbody>
</table>

When player 3 chooses Matrix A, player 1 gets 1 from Top and 3 from Bottom. When player 3 chooses Matrix B, player 1 gets 3 from Top and 1 from Bottom. Player 2 always gets the same utility from Left as player 1 does from Top, but player 2 gets 1 more utility from Right than player 1 gets from Bottom. Therefore, \((\text{Right} \mid 2) \succ (\text{Bottom} \mid 1)\). Suppose \( \epsilon(\text{Bottom} \mid 1)=2h \) and all other minimum probabilities are \( h \) for any \( h > 0 \). In any \( \epsilon \)-PCE \( \alpha^o \), player 1 puts the minimum probability \( 2h \) on Bottom, and player 2 puts the minimum probability \( h \) on Right, but then 2’s play is not in \( \Pi_i^\epsilon(\alpha^o) \).

To prove the existence of PCE, we only need to find a convergent sequence of \( \epsilon \)-PCE for some sequence of \( \epsilon \to \overline{0} \). Intuitively, what went wrong in the above example is that the
tremble probabilities went against the compatibility structure of the game. We now show that $\vec{\epsilon}$-equilibria are also $\vec{\epsilon}$-PCE whenever the tremble profile $\vec{\epsilon}$ “respects the compatibility structure” in the following sense:

**Definition 5.** Tremble profile $\vec{\epsilon}$ respects the compatibility structure $\succ$ if $\vec{\epsilon}(a_i^*|i) \geq \vec{\epsilon}(a_j^*|j)$ for all $i, j, a_i^*, a_j^*$ with $(a_i^*|i) \succ (a_j^*|j)$.

This condition says the minimum weight $i$ could assign to $a_i^*$ is no smaller than the minimum weight $j$ could assign to $a_j^*$. It is equivalent to the inequality

$$\min_{\alpha_i \in \Pi_i^\epsilon} \alpha_i(a_i^*) \geq \min_{\alpha_j \in \Pi_j^\epsilon} \alpha_j(a_j^*).$$

**Lemma 1.** If $\vec{\epsilon}$ respects the compatibility structure $\succ$, then every $\vec{\epsilon}$-equilibrium is also an $\vec{\epsilon}$-PCE.

**Proof.** Let $\vec{\epsilon}$-equilibrium $\alpha^\circ$ be given. From the definition of $\vec{\epsilon}$-equilibrium, we have

$$\alpha_i^\circ \in \arg\max_{\alpha_i \in \Pi_i^\epsilon} u_i(\alpha_i, \alpha_{-i}^\circ).$$

Relative to $\Pi_i^\epsilon$, the $\Pi_i^\epsilon(\alpha_{-i}^\circ)$ imposes restrictions of the form

$$\alpha_i(a_i^*) \geq \min \left[ \alpha_j(a_j^*), 1 - \sum_{a_i' \neq a_i^*} \vec{\epsilon}(a_i'|i) \right]$$

whenever $(a_i^*|i) \succ (a_j^*|j)$. To show that $\alpha^\circ$ is also an $\vec{\epsilon}$-PCE, we argue that if $\vec{\epsilon}$ respects the compatibility structure $\succ$, then $\alpha^\circ$ satisfies these restrictions.

This is established in two cases.

**Case 1.** $\alpha_j^\circ(a_j^*) = \vec{\epsilon}(a_j^*|j)$. We immediately have $\alpha_i^\circ(a_i^*) \geq \vec{\epsilon}(a_i^*|i) \geq \vec{\epsilon}(a_j^*|j) = \alpha_j^\circ(a_j^*)$, where the second inequality comes from $\vec{\epsilon}$ respecting the compatibility structure.

**Case 2.** $\alpha_j^\circ(a_j^*) > \vec{\epsilon}(a_j^*|j)$. Since $\alpha^\circ$ is an $\vec{\epsilon}$-equilibrium, the fact that $j$ puts more than the minimum required weight on $a_j^*$ implies $a_j^*$ is at least a weak best response for $j$ against $(\alpha_{-ij}^\circ, \alpha_i^\circ)$, where both $\alpha_{-ij}^\circ$ and $\alpha_i^\circ$ are strictly mixed. The definition of $(a_i^*|i) \succ (a_j^*|j)$ then implies that $a_i^*$ must be a strict best response for $i$ against (the strictly mixed) $(\alpha_{-ij}^\circ, \alpha_j^\circ)$. In the $\vec{\epsilon}$-equilibrium, $i$ must assign as much weight to $a_i^\circ$ as possible, so that $\alpha_j^\circ(a_j^*) = 1 - \sum_{a_i' \neq a_i^\circ} \vec{\epsilon}(a_i'|i)$.

These two cases show that when $\vec{\epsilon}$ respects the compatibility structure, every $\vec{\epsilon}$-equilibrium $\alpha^\circ$ satisfies all the additional constraints of $\vec{\epsilon}$-PCE.

The next result uses the fact that the tremble profiles with the same lower bound on the probability of each action satisfy the compatibility condition in any game.
Proposition 1. PCE exists in every finite strategic-form game.

Proof. Consider a sequence of tremble profiles with the same lower bound on the probability of each action, that is $\mathbf{\epsilon}^{(t)}(a_i|i) = \epsilon^{(t)}$ for all $i$ and $a_i$, and with $\epsilon^{(t)}$ decreasing monotonically to 0. Each of these tremble profiles respects every compatibility structure $\succ$, and there is some finite $T$ large enough that $t \geq T$ implies an $\mathbf{\epsilon}^{(t)}$-equilibrium exists. By Lemma 1, these $\mathbf{\epsilon}^{(t)}$-equilibria are also $\mathbf{\epsilon}$-PCE, and some subsequence of them converges since the space of strategy profiles is compact.

2.2 PCE in Signalling Games

We now relate PCE to compatibility criterion from Fudenberg and He (2017a), which only applies to signalling games. We will see that PCE implies the compatibility criterion in this class of games. In this subsection we will use notation closely mirroring the conventions in signalling games, but it can be readily recast into the notation of Section 2.1.

A signalling game is a two-player Bayesian game, where the sender (player 1 or P1) knows her type $\theta \in \Theta$ and the receiver (player 2 or P2) only knows the prior distribution on types, $\lambda \in \Delta(\Theta)$. After learning her type, the sender sends a signal $s \in S$ to the receiver. Then, the receiver responds with an action $a \in A$. A pair of behavior strategies $(\pi_1, \pi_2)$ specifies $\pi_1(s|\theta)$ as the probability of sending signal $s$ as type $\theta$ and $\pi_2(a|s)$ as the probability of responding to signal $s$ with action $a$. Utilities $u_i(\theta, s, a)$ depend on sender’s type, the signal, and the action.

To relate this notation to that we used in the general setup, we view different types of P1 as different players, so that the set of players is $I = \Theta \cup \{2\}$ where “2” refers to the receiver. We have $A_\theta = S$ for each sender type and $A_2 = A^S$, the set of signal-contingent plans for the receiver. A behavior strategy $\pi_1(\cdot|\theta) \in \Delta(S) = \Delta(A_\theta)$ of type $\theta$ corresponds to $\alpha_\theta$ in the notation of Section 2.1. Similarly, a behavior strategy of the receiver, $\pi_2 \in \Delta(A)^S$, is an element of $\Delta(A_2) = \Delta(A^S)$ and would have been denoted as $\alpha_2$ in the previous subsection. By Kuhn’s theorem, it suffices to consider these behavior strategies for the receiver instead of mixed strategies.

In these games, different types of the sender are non-interacting players in the sense of Remark 1. Moreover, $(s|\theta) \succ (s|\theta')$ if and only if for every strictly mixed receiver play $\pi_2$,

$$u_1(\theta, s, \pi_2(\cdot|s)) \geq \max_{s' \neq s} u_1(\theta, s', \pi_2(\cdot|s'))$$

implies

$$u_1(\theta', s, \pi_2(\cdot|s)) > \max_{s' \neq s} u_1(\theta', s', \pi_2(\cdot|s')),$$

which is the definition of compatibility used in Fudenberg and He (2017a).
Proposition 2. In a signalling game, every PCE is a Nash equilibrium satisfying the compatibility criterion, as defined in Fudenberg and He (2017a).

Proof. Since every PCE is a trembling-hand perfect equilibrium and since this latter solution concept refines Nash, \( \pi^* \) is a Nash equilibrium.

Now since \( \pi^* \) is a PCE, there must exist a sequence of tremble profiles \( \overrightarrow{\epsilon(t)} \to 0 \) and a sequence of associated \( \overrightarrow{\beta(t)} \)-PCEs such that \( \pi(t) \to \pi^* \). Suppose \( \hat{s} \) is off-path in \( \pi^* \). Then there exists some \( T \) so that for all \( t \geq T \), \( \pi_1(t)(\hat{s}|\theta) < \frac{1}{2} \) for every \( \theta \). Also, since \( \overrightarrow{\epsilon(t)} \to 0 \), we may choose \( T \) large enough that \( 1 - \sum_{s' \neq \hat{s}} \overrightarrow{\epsilon(t)}(\hat{s}|\theta) > \frac{1}{2} \) whenever \( t \geq T \). But since each \( \pi(t) \) is an \( \overrightarrow{\beta(t)} \)-PCE, we know \( \pi_1(t)(\hat{s}|\theta') \geq \min \left[ \pi_1(t)(\hat{s}|\theta''), 1 - \sum_{s' \neq \hat{s}} \overrightarrow{\epsilon}(\hat{s}|\theta') \right] \) for each \( t \) whenever \( (\hat{s}|\theta') \succ (\hat{s}|\theta'') \). The choice of \( T \) implies that whenever \( t \geq T \), \( \pi_1(t)(\hat{s}|\theta') \geq \min \left[ \pi_1(t)(\hat{s}|\theta''), 1 - \sum_{s' \neq \hat{s}} \overrightarrow{\epsilon}(\hat{s}|\theta') \right] = \pi_1(t)(\hat{s}|\theta'') \). This implies that for large \( t \), the receiver’s posterior likelihood ratio for \( \theta'' \) compared to \( \theta' \) after seeing \( \hat{s} \) cannot exceed the prior likelihood ratio, so that the receiver’s posterior belief after signal \( \hat{s} \) falls in the set \( P(\hat{s}, \pi^*) \) as defined in Fudenberg and He (2017a). We may view \( \pi_2(t) \) as a mixture over \( \sigma_2 \in A^S = A_2 \). For every \( t \) and every \( \sigma_2 \in A^S \) where \( \sigma_2(s') \notin \text{BR}(P(s', \pi^*), s') \), we must have \( \pi_2(t)(\sigma_2) = \epsilon(t)(\text{receiver}, \sigma_2) \), since the receiver could strictly improve his payoff by picking a different action that best responds to \( \pi_1(t) \) at \( s' \). Therefore as \( \epsilon(t) \to 0 \), the probability that \( \pi(t) \) assigns to responses outside of \( \text{BR}(P(s', \pi^*), s') \) goes to 0.

Note that this proposition shows that in the beer-quiche game of Cho and Kreps (1987), the quiche-pooling equilibrium is not a PCE, as it does not satisfy the compatibility criterion.

The compatibility criterion has a learning foundation: It is a necessary condition for an equilibrium to be patiently stable in the steady-state learning models of Fudenberg and He (2017a). PCE has the same learning foundation as the compatibility criterion when specialized to signalling games, but applies to a wider class of games. In general, we view PCE as a solution concept capturing in reduced form the idea that different players have different incentives for playing various strategies when they are learning prevailing distribution of strategies, where players more compatible with a given strategy are more likely to tremble onto it during learning. In Section 4, we will present specific examples where learning eliminates non-PCE equilibria.

3 Examples of PCE

In this section, we study examples of games where PCE rules out unintuitive Nash equilibria. We will also use these examples to distinguish PCE from existing refinements.
3.1 Bayesian Games

In a Bayesian games, two types of the same player role are never simultaneously present. As such, their actions cannot affect each others’ payoffs and they constitute a leading example of non-interacting players from Remark 1. This subsection explores PCE in Bayesian games in greater depth.

Consider a Bayesian game where $\Theta_i$ is player $i$'s finite type space and $\lambda \in \Delta(\Theta)$ is the prior distribution over types. Write $A_i$ for the set of pure actions available to player $i$, so a behavior strategy for player $i$ is $\sigma_i : \Theta_i \to \Delta(A_i)$. The utility function of $i$ is $u_i : A \times \Theta \to \mathbb{R}$.

For $\sigma_{-i}$ a strategy profile of $-i$, write $u_i(a_i, \sigma_{-i}; \theta_i)$ to mean the expected payoff of $i$ when playing action $a_i$ as type $\theta_i$. More precisely,

$$u_i(a_i, \sigma_{-i}; \theta_i) := \mathbb{E}_{\theta_{-i}}[u_i(a_i, \sigma_{-i}(\theta_{-i}); \theta_i, \theta_{-i})]$$

where expectation is taken with respect to the $\lambda$-conditional distribution of $\theta_{-i}$ given that $i$ has type $\theta_i$.

To relate this to the general setup, consider each type as a different player, so that $I = \cup_i \Theta_i$ and $A_{\theta_i} = A_i$. Two types of the same player role are non-interacting, and it is easy to see that type $\theta'_i$ is more compatible with action $a_i$ than $\theta''_i$ if and only if for every strictly mixed strategy $\sigma_{-i}$ such that

$$u_i(a_i, \sigma_{-i}; \theta'_i) \geq \max_{a'_i \neq a_i} u_i(a'_i, \sigma_{-i}; \theta''_i),$$

we have

$$u_i(a_i, \sigma_{-i}; \theta'_i) > \max_{a'_i \neq a_i} u_i(a'_i, \sigma_{-i}; \theta'_i)$$

We now turn to a Bayesian game of simultaneous moves where PCE formalizes an intuitive restriction on “trembles” that is not captured by trembling-hand perfect equilibrium or proper equilibrium.

**Example 1.** There are two states of the world, $\theta_I$ and $\theta_{II}$, equally likely. Player 1 (P1) knows the state, but Player 2 (P2) does not, and the players move simultaneously. P1 has three possible actions: H, T, and Out, and P2 chooses between H and T. If P1 plays Out, both players get payoff 3 regardless of P2’s action. If P1 plays H or T, then P2’s objective is to match P1’s play in state $\theta_I$ but mismatch it in state $\theta_{II}$. For P1, H is a better response than T if P2 plays H with high probability; otherwise, T is a better response than H. Here is the payoff matrix.
Consider the Nash equilibrium $\sigma$ where both types of P1 always play Out and P2 plays $T$, which is rationalized by the perverse belief that $\theta_I$ is much more likely to play $T$ than $\theta_{II}$, while $\theta_{II}$ is much more likely to play $H$ than $\theta_I$. Under this belief, conditional on P1 playing $H$, it is very likely that the state of the world is $\theta_{II}$, so P2 wants to mismatch by playing $T$. When P1 plays $T$, it is very likely the state is $\theta_I$, so P2 wants to match by playing $T$. Since $T$ is optimal in either state, it is the unique best response for P2 under this belief.

This strategy profile is an (interim) extended proper equilibrium (so it is also a proper equilibrium and a trembling-hand perfect equilibrium at the interim stage). To see why, for every given small $\epsilon > 0$, consider the strategy profile $\sigma'$ where the 0 probabilities in $\sigma$ are replaced by $\sigma_1(\theta_I)(H) = \epsilon^4$, $\sigma_1(\theta_I)(T) = \epsilon^2$, $\sigma_1(\theta_{II})(H) = \epsilon^5$, $\sigma_1(\theta_{II})(T) = \epsilon^3$, and $\sigma_2(H) = \epsilon$. Let the utility scalings be $\beta_{\theta_I} = 1$, $\beta_{\theta_{II}} = 2$, and $\beta_{P2} = 10^{-4}$. It is clear that when P2 assigns probability close to 1 to $T$, Out is the unique best response for each type of P1. Also, we claim that for sufficiently small $\epsilon > 0$, the unique best response for P2 is $T$. Observe that P2’s decision between $H$ and $T$ hinges on his belief about the state of the world in the event that P1 plays $T$: When P1 plays Out, P2 is indifferent between $H$ and $T$. Since the $\sigma_1(\theta_I)(T)$ tremble is more likely than any other trembles of P1 by an order of at least $1/\epsilon$, we know that conditional on P1 not playing Out, it is almost certain that the state of the world is $\theta_I$ and that P1 played $T$. Therefore, $T$ is a strict best response for P2. We can verify that given the utility scalings, more costly mistakes by any type are played no more than $\epsilon$ times as often as less costly mistakes for any other type. Thus $\sigma'$ is a ($\beta, \epsilon$)-extended proper equilibrium, and so $\sigma$ is extended proper.

This is also a test-set equilibrium. To see this, note that both $\theta_I$ and $\theta_{II}$ have strict incentives in the equilibrium, so their equilibrium strategies are trivially undominated in the test set. But these strict incentives for $\theta_I$ and $\theta_{II}$ imply that the test set for P2 is the singleton set consisting of only the equilibrium strategy profile, so P2’s strategy is also undominated in the test set.

However, it seems intuitive that $\theta_I$ should tremble to $H$ more than $\theta_{II}$ does, while $\theta_{II}$ should tremble to $T$ more than $\theta_I$ does. This is because regardless of P2’s play, $\theta_I$ gets 1 more than $\theta_{II}$ does when playing $H$ while $\theta_{II}$ gets 1 more than $\theta_I$ does when playing $T$. This intuition is formalized in the notion of type compatibility, as we have $(H|\theta_I) \succ (H|\theta_{II})$ and $(T|\theta_{II}) \succ (T|\theta_I)$. Capturing this intuition requires cross-type restrictions on the relative probabilities of trembles, as in PCE.

PCE rules out this unintuitive equilibrium. To see this, in every $\varepsilon$-PCE where the mini-
The minimum probability on each action for each player is below 0.05, the fact that \((H|\theta_I) > (H|\theta_{II})\) implies

\[
\sigma_1(\theta_I)(H) \geq \min [\sigma_1(\theta_{II})(H), 1 - \epsilon(T|\theta_I) - \epsilon(Out|\theta_I)] \geq \min [\sigma_1(\theta_{II})(H), 0.9] \geq 0.9 \cdot \sigma_1(\theta_{II})(H)
\]

and a symmetric argument shows \(\sigma_1(\theta_{II})(T) \geq 0.9 \cdot \sigma_1(\theta_I)(T)\), so the expected payoff for P2 of playing \(H\) is:

\[
\frac{1}{2} \cdot [11 \cdot \sigma_1(\theta_I)(H) + 1 \cdot \sigma_1(\theta_I)(T)] + \frac{1}{2} \cdot [1 \cdot \sigma_1(\theta_{II})(H) + 11 \cdot \sigma_1(\theta_{II})(T)]
\]

\[
\geq \frac{1}{2} \cdot [1 \cdot \sigma_1(\theta_I)(H) + 10 \cdot \sigma_1(\theta_{II})(H)] + \frac{1}{2} \cdot [(10 \cdot \sigma_1(\theta_I)(T) + 1 \cdot \sigma_1(\theta_{II})(T)]
\]

where in the first inequality we made use of \(\sigma_1(\theta_I)(H) \geq 0.9 \sigma_1(\theta_{II})(H)\) and \(\sigma_1(\theta_{II})(T) \geq 0.9 \cdot \sigma_1(\theta_I)(T)\). Note that the last line is the expected payoff of P2 playing \(T\). So P2 will assign minimum weight possible to \(T\) in every \(\overline{\epsilon}\)-PCE where \(\overline{\epsilon}\) is close enough to \(\overline{0}\). But that means \(H\) is strictly optimal for both types of P1 in such an \(\overline{\epsilon}\)-PCE, implying both types of P1 must put probability 1 on \(H\) in every PCE.

PCE can also lead to new refinements in network-formation games of incomplete information. We give a very simple example below.

### 3.2 Selten’s Horse

Here we give an example of a complete-information game where PCE differs from extended proper equilibrium.

**Example 2.** Here is a three-player extensive-form game that modifies the Selten’s horse, making it more amenable to the learning analysis we do in Section 4.
In this game, P1 and P2 each chooses between Down and Across (labeled Down1, Across1 for P1, Down2, Across2 for P2). The payoff from choosing Across is always 0. The payoff from choosing Down depends on P3’s play. For P1, the payoffs when P3 chooses L, M and R are \(x, y, w\) respectively. For P2, these payoffs are \(x, z, w\). We impose \(x < 0\), \(w > 0\), \(y > z\).

If both P1 and P2 choose Across, P3 gets 0. Otherwise, P3’s payoff depends on which opponents play Down and the action P3 takes. If only P1 plays Down, P3’s best response is R. If only P2 plays Down, P3’s best response is L. If they both play Down, P3’s payoff is defined as the average payoff from the previous cases, with R as the best response. (R is P3’s best response whenever P1 plays Down at least \(\frac{2}{3}\) as frequently as P2 does.)

The strategy profile (Across1, Across2, L) is an extended proper equilibrium, sustained by P3’s belief that when she gets to play, it is more likely that P2 deviated to playing down than P1.\(^2\) However, it is easy to verify that (Down1|P1) \(\succ\) (Down2|P2). This means in every \(\bar{\epsilon}\)–PCE where all minimum probabilities are below \(\frac{1}{3}\), P1 plays Down1 at least \(\frac{2}{3}\) as frequently as P2 plays Down2, so P3 will put as much probability as possible on R in every such \(\bar{\epsilon}\)–PCE. This shows that (Across1, Across2, L) is not a PCE.

\(\diamondsuit\)

### 3.3 Link-formation game

**Example 3.** There are 4 players in the game, split into two sides: North and South. The players are named North-1, North-2, South-1, and South-2.

These players engage in a strategic link-formation game. Each player simultaneously takes an action: either Inactive or Active. An Inactive player forms no links. An Active player forms a link with every Active player on the opposite side. (Two players on the same

\(^2\)It is easy to see that by scaling P1’s payoff by a large positive constant, it can be “more costly” for P1 to deviate to d1 than for P2 to deviate to d2, so that we can sustain this belief in an extended proper equilibrium.
side cannot form links.) For example, suppose North-1 plays Active, North-2 plays Active, South-1 plays Inactive, and South-2 plays Active. Then North-1 creates a link to South-2, North-2 creates a link to South-2, South-1 creates no links, and South-2 creates links to both North-1 and North-2.

![Diagram showing link formation between players.]

Each player $i$ is characterized by two parameters: cost ($c_i$) and quality ($q_i$). Cost refers to the private cost that a player pays for each link she creates. Quality refers to the benefit that a player provides to others when they link to her. A player who forms no links gets a payoff of 0. In the above example, the payoff to North-1 is $q_{South-2} - c_{North-1}$ and the payoff to South-2 is $(q_{North-1} - c_{South-2}) + (q_{North-2} - c_{South-2})$.

We consider two versions of this game, shown below. In the anti-monotonic version on the left, players with a higher cost also have a lower quality. In the co-monotonic version on the right, players with a higher cost also have a higher quality. There are two pure-strategy Nash outcomes for each version: all links form or no links form. Standard refinements (extended proper equilibrium, proper equilibrium, trembling-hand perfect equilibrium, $p$-dominance, Pareto efficiency, strategic stability, pairwise stability) all make matching predictions in both games. However, “all links form” is the unique PCE outcome in the anti-monotonic case, while both “all links” and “no links” are PCE outcomes under co-monotonicity.

<table>
<thead>
<tr>
<th>player</th>
<th>cost</th>
<th>quality</th>
<th>player</th>
<th>cost</th>
<th>quality</th>
</tr>
</thead>
<tbody>
<tr>
<td>North-1</td>
<td>14</td>
<td>30</td>
<td>North-1</td>
<td>14</td>
<td>10</td>
</tr>
<tr>
<td>North-2</td>
<td>19</td>
<td>10</td>
<td>North-2</td>
<td>19</td>
<td>30</td>
</tr>
<tr>
<td>South-1</td>
<td>14</td>
<td>30</td>
<td>South-1</td>
<td>14</td>
<td>10</td>
</tr>
<tr>
<td>South-2</td>
<td>19</td>
<td>10</td>
<td>South-2</td>
<td>19</td>
<td>30</td>
</tr>
</tbody>
</table>

**Proposition 3.** None of the following refinements both (i) eliminates the “no links” equilibrium outcome in the link-formation game when qualities are anti-monotonic with costs, and (ii) allows the “no links” equilibrium outcome when qualities are comonotonic with costs: extended proper equilibrium, proper equilibrium, trembling-hand perfect equilibrium,
p-dominance, Pareto efficiency, strategic stability, and pairwise stability. Moreover, the link-formation game is not a potential game.

The proof is in Appendix A.1.

As we reported in Section 5, our experimental evidence shows that play in these two treatments differs in the way that PCE suggests.

4 The Learning Foundation of PCE

Section 3 studied some examples where PCE refines away “unintuitive” Nash equilibria. We now illustrate the idea that PCE’s restrictions on trembles encode a reduced-form learning model by examining a steady-state learning model and instantiating it with some examples from Section 3 as the stage game. We show that when the stage game is suitably augmented through either adding an appropriate feedback structure for learning or adding a chance of error in implementing the players’ intended play, the steady-state learning model delivers the same refinement as PCE.

The steady-state learning model closely follows the model in Fudenberg and He (2017a) and can be found in Appendix A.3. Briefly, society contains a continuum of agents, who are born into player roles and have geometrically distributed lifetimes with $\gamma$ chance of survival between periods. Each period, agents are randomly and anonymously matched to play a fixed stage game, with the goal of maximizing expected discounted payoffs from the game under effective discount factor $\delta \gamma$. Agents believe there is a constant but unknown prevailing distribution of play in each opponent population. Agents are born with a prior $g$ over these aggregate plays, independent over different opponent populations, and learn over time using only their own personal experience. ³We focus on the social steady states of the learning system, which are time-invariant distributions of play in the society. The set of steady state strategy profiles under parameters $g, \delta, \gamma$ is denoted $\Psi^*(g, \delta, \gamma)$ and we consider the set of patiently stable strategy profiles, defined as $\lim_{\delta \to 1} \lim_{\gamma \to 1} \Psi^*(g, \delta, \gamma)$.

From the viewpoint of learning theory, the description of the game also needs to specify what the players observe at the end of a single play of the game. The default assumption is that all players observe the terminal node, but in many settings some players may observe less than this - they may only observe an element of some terminal-node partition. We now consider two augmented versions of this example where our learning model leads to the same equilibrium refinement as PCE.

³These assumptions are sufficient for the examples discussed in this section. Our work on signalling games uses additional independence assumptions to characterize the patiently stable sets. That work also shows that the same conclusions emerge from related models such as when some of the players have access to the experiences of some others.
4.1 Selten’s Horse with Terminal-Node Partitions

Consider the learning model with Example 2 as the stage game, and suppose that in each period, if P1 chooses \textbf{Across1} then she does not observe what and P2 and P3 played. Otherwise, she observes P2 and P3’s actions. The same goes when P2 chooses \textbf{Across2}. To interpret, the \textbf{Across} action corresponds to walking away from the current match, which precludes learning the actions of the matched opponents. This feedback structure is an example of the terminal-node partitions studied in Fudenberg and Kamada (2015, 2016). Note that it allows the players to observe their own payoff after each period, because a player’s payoff after choosing the outside-option \textbf{Across} is invariant to the actions of her opponents. Assume that P1 and P2 have the same prior over P3’s action.

In this learning model, the dynamic optimization problem facing each P1 (and P2) is formally equivalent to a two-armed bandit. Playing \textbf{Across} is the same as pulling the safe arm that always pays 0. Playing \textbf{Down} is the same as pulling the risky arm that pays a random amount depending on whether P3 plays L, M, or R, where the (time-invariant) probabilities that P3 plays these actions is the parameter of the risky arm that P1 and P2 wish to learn.

**Proposition 4.** Selten’s Horse with terminal-node partitions, P3 plays L with probability 1 in every patiently stable strategy profile.

**Proof.** Applying the same comparative Gittins-index argument as Lemma 1 of Fudenberg and He (2017a), it is clear that for every discount factor \(\delta\) and agent’s continuation probability \(\gamma\), every regular prior,\(^4\) and every steady state profile \(\alpha\) corresponding to these parameters, we have \(\alpha_1(\text{Down1}) \geq \alpha_2(\text{Down2})\) due to \((\text{Down1}|P1) > (\text{Down2}|P2)\).

Next, let the profile of regular priors \((g_1, g_2, g_3)\) for the three player populations be fixed. For every \(\xi > 0\), we show that in every patiently stable strategy profile, P3 plays L with probability less than \(\xi\).

By Theorem 2 of Fudenberg, He, and Imhof (2017), there exists a constant \(N\) so that in every steady state and every \(n\) with (i) \(\alpha_1(\text{Down1}) \geq \alpha_2(\text{Down2})\), and (ii) \(n \cdot \alpha_1(\text{Down1}) \geq N\), a P3 aged \(n\) or older has probability at least \(1 - \xi/2\) of holding a belief that leads to her playing R at the information set (i.e. essentially believing it is more likely that P1 gave her the move than P2 gave her the move). At the same time, \textbf{Down1} can potentially lead to the best payoff in the game for P1. This means for every \(T\) we may find a large enough \(0 < \beta(T) < 1\) so that whenever \(\gamma \delta \geq \beta(T)\), P1’s Gittins index for \textbf{Down1} always exceeds that for \textbf{Across1} in the first \(T\) periods of life regardless of history. This shows

\(^4\)“Regular” here means that the priors have a continuous density that is non-doctrinaire in the sense of assigning non-zero probability to every open ball, and in addition does not vanish any faster than polynomially on the boundary of the probability simplex - see Fudenberg, He, and Imhof (2017). One example is Dirichlet priors, which do vanish on the boundary.
\( \alpha_1(\text{Down1}) \geq 1 - \gamma^T \) in a every steady state with \( \delta \gamma \geq \tilde{\beta}(T) \). In such a steady state, if given that P3’s age \( n \) is such that \( n \cdot (1 - \gamma^T) > N \), then there is probability at least \( 1 - \xi/2 \) that this P3 plays \( \text{L} \).

To finish the argument, we show that the age requirement implicit in \( n \cdot (1 - \gamma^T) > N \) is satisfied for all except a \( \xi/2 \) fraction of P3’s. Find a large enough \( T > 0 \) and \( 0 < \gamma < 1 \) so that \( \frac{(1 - \gamma^T) \cdot (\xi/2)}{1 - \gamma} \) is only made larger by increasing \( \gamma \). Now whenever \( \delta \geq \sqrt{\tilde{\beta}(T)} \) and \( \gamma \geq \gamma \), we will have \( \tilde{\varphi}(\text{Down1}) \geq \frac{\xi/2}{1 - \gamma} \cdot (1 - \gamma^T) > N \). But when \( n = \frac{\xi/2}{1 - \gamma} \), the fraction of P3 younger than \( n \) is \( 1 - \left( \gamma \frac{\xi/2}{1 - \gamma} \right) \approx \xi/2 \). This shows that in every steady state with \( \delta \geq \sqrt{\tilde{\beta}(T)} \) and \( \gamma \geq \gamma \), \( \alpha_3(\text{R}) \geq 1 - \xi \). In particular, this will hold in every \( \delta \)-stable strategy profile where \( \delta \geq \sqrt{\tilde{\beta}(T)} \). Hence, it will also hold in any patiently stable strategy profile. As the choice of \( \xi > 0 \) was arbitrary, this shows that P3 plays \( \text{L} \) with probability 1 in every patiently stable strategy profile.

\[ \square \]

### 4.2 Selten’s Horse with Uniform Errors

Again instantiate the learning model with Example 2 as the stage game, but now suppose that at the end of each period, P3’s strategy is always revealed to P1 and P2, regardless of their play. Also assume that for P1 and P2, there is some probability \( \epsilon > 0 \) that the intended play is not implemented. That is, there is \( 1 - \epsilon \) chance that the intended play is implemented, \( \epsilon \) chance that an action is chosen uniformly at random instead. In addition, assume that players only observe the realized action of their opponents, and not the intended ones, so that intentions are imperfectly observed.

**Proposition 5.** In Selten’s Horse with uniform errors, P3 intends to play \( \text{L} \) with probability 1 in every patiently stable strategy profile.

**Proof.** In this model, the intended play of P1 and P2 myopically best responds to their belief each period. Directly from the definition of compatibility, at any history where P2 intends to play \( \text{Down2} \), P1 would also intend to play \( \text{Down1} \). This gives \( \alpha_1(\text{Down1}) \geq \alpha_2(\text{Down2}) \) in every steady state, since P1 intends to play \( \text{Down} \) more often than P2 and the two players have the same probability of implementation error.

Again invoking Theorem 2 of Fudenberg, He, and Imhof (2017), there exists a constant \( N \) so that for every steady state \( \psi \) and every \( n \) with (i) \( \alpha_1(\text{Down1}) \geq \alpha_2(\text{Down2}) \), and (ii) \( n \cdot \alpha_1(\text{Down1}) \geq N \), a P3 aged \( n \) or older has probability at least \( 1 - \xi/2 \) of holding a belief that makes her play \( \text{R} \) at the information set. The random error in implementation implies that \( \tilde{\varphi}_1(\text{Down1}) \geq \epsilon/2 \) in every steady state. As we take \( \gamma \to 1 \), P3’s who are younger than
account for no more than \(\xi/2\) fraction of the population.\(^5\)

Since \(\xi > 0\) is arbitrary, we conclude that in every patiently stable strategy profile, \(P_3\) intends to play \(L\) with probability 1.

4.3 Link-Formation Game

Next, we apply the steady-state learning model to Example 3, augmenting it with either a terminal-node partitions feedback structure or uniform errors. In each case, we show that the “no links form” equilibrium outcome is not patiently stable when \((c_i)\) is anti-monotonic with \((b_i)\).

**Link-Formation Game with terminal-node partitioning:** North-1 and North-2 always observe the action of every player. But, South-1 and South-2 only observe the action of every player when they play Active. If a southern player chooses Inactive, they get a payoff of 0 and do not observe the play of any other player.

**Proposition 6.** Consider the steady-state learning model instantiated with the anti-monotonic version of the link-formation game as the stage game, with terminal-node partitioning. Then, all-links form is the unique patiently stable outcome.

**Proof.** From the perspective of a southern player, the optimization problem is a two-armed bandit with the “safe arm” Inactive always paying 0 and the “risky arm” Active paying a random amount whose expectation depends on the probabilities that North-1 and North-2 play Active. The comparative Gittins-index argument as Lemma 1 of Fudenberg and He (2017a) shows that in every steady state, \(\alpha_{\text{South-1}}(\text{Active}) \geq \alpha_{\text{South-2}}(\text{Active})\).

Since the best response of every northern player to a strategy profile satisfying

\[
\alpha_{\text{South-1}}(\text{Active}) \geq \alpha_{\text{South-2}}(\text{Active})
\]

is Active, we can invoke Theorem 2 of Fudenberg, He, and Imhof (2017) in the same way as in the proof of Proposition 4 to show that in every patiently stable strategy profile, each northern player chooses Active with probability 1. However, such a strategy profile must also be Nash equilibrium. The only Nash equilibrium with this behavior by northern players is the equilibrium where everyone plays Active.

---

\(^5\)This argument does not work if \(\epsilon = 0\). For example, if priors \(g_1\) and \(g_2\) assign high probability to \(P_3\) playing \(L\) with probability near 1, while \(g_3\) assigns probability near 1 to strategies for which \(P_2\) is more likely to to play \(\text{Down2}\) than \(P_1\) to play \(\text{Down1}\), then there is a steady state where \(P_1\) and \(P_2\) play \(\text{Across}\) every period, \(P_3\) play \(L\). Here \(\alpha_1(\text{Down1}) = 0\), so the data size requirement in Theorem 2 of Fudenberg, He, and Imhof (2017) is not satisfied, as that result requires that the expected number of observed experiments from the more compatible type needs to exceed a prior-dependent constant.
Link-Formation Game with uniform errors: Actions of every kingdom are always observed. However, with some probability \( \epsilon > 0 \), a uniformly random action instead of the intended action is implemented. Intention is not observable.

**Proposition 7.** Consider the steady-state learning model instantiated with the anti-monotonic version of the link-formation game with uniform errors as the stage game. The only patiently stable strategy profile involves every player intending to play \textbf{Active} with probability 1.

*Proof.* Same as the proof of Proposition 5. \hfill \qed

5 An Experimental Test of PCE

5.1 Ex-ante predictions

PCE make different predictions in the anti-monotonic and co-monotonic versions of the link-formation game (Example 3): “all links form” is the unique PCE outcome in the anti-monotonic version, while both “all links form” and “no links form” are PCE outcomes in the co-monotonic version. This suggests that after a period of learning, the “all links form” outcome should be more likely in the anti-monotonic version. By contrast, Proposition 3 shows that existing refinement concepts predict no difference in the behavior in these two versions of the game.

![Figure 1](image-url)

Figure 1: Players’ myopic best responses depend on their beliefs about the play of high-quality and low-quality opponents. In Region 1, all players have \textbf{Active} as the best response. In Region 2, the best response is \textbf{Active} for low-cost players and \textbf{Inactive} for high-cost players. In Region 3, the best response is \textbf{Inactive} for all players.
To understand why these two treatments induce different learning outcomes, consider the subjects’ myopic best actions as a function of their beliefs about opponents’ play, which was depicted in Figure 1. In both treatments, a low-cost player has *Active* as her best response whenever her belief falls in Region 1 or Region 2, but a high-cost player only has *Active* as his best response when his belief falls in Region 1. In the anti-monotonic treatment, distributions of opponents’ play above the 45 degree line are consistent with player compatibility, whereas in the co-monotonic treatment the distributions below the 45 degree line are consistent. When subjects’ observations are consistent with player compatibility in the anti-monotonic treatment, this implies that learning should lead to an increase in the link formation rate, since subjects’ beliefs eventually fall in Region 1. But in the co-monotonic treatment, the predictions of learning are *a priori* ambiguous: Observations consistent with player compatibility may discourage everyone from playing *Active* (Region 3), encourage only the low-quality players to choose *Active* and further depress the returns to *Active* in the future (Region 2), or encourage everyone to play *Active* (Region 1). An analyst who not only believes in player compatibility, but also believes that initial play in the co-monotonic game would fall “far enough” below the 45 degree line can therefore make the prediction that the rate of link formation should decrease over time in the co-monotonic treatment.

The different learning dynamics in these two versions of the link-formation game echo Brandts and Holt (1992)’s “normal type-dependence” and “reverse type-dependence” games, which were pairs of signalling games with the same equilibria that nevertheless induce two opposite separating behaviors from senders under a “large” set of beliefs about receivers’ play. Much as the two opposite dependencies of messages on sender types induce different learning outcomes in Brandts and Holt (1992), we expect the opposite dependencies of action on quality in our two treatments to do the same thing here.\(^6\)

Based on PCE, learning theory, and our prior guesses about what initial play would look like, we posted the following preregistered predictions on the website AsPredicted.org\(^7\):

1. In every treatment and in every period, there will be more instances of low-cost players choosing *Active* than high-cost players choosing *Active*. This hypothesis follows from the fact that low cost players are more compatible with *Active*, and our basic premise that players who are more compatible with an action play it more often.

2. In the last quarter of the periods, the link-formation rate in the anti-monotonic treatment would be higher than in the co-monotonic treatment. This prediction comes from the fact that PCE unambiguously predicts the all-links outcome in the anti-monotonic treatment but allows both the all-links and no-links outcomes in the co-monotonic treatment.

---

\(^6\)Note that the Brandts and Holt (1992) type-dependence property only holds for a “large” subset of beliefs as opposed to all of them.

\(^7\)Available at: [https://aspredicted.org/we4x7.pdf](https://aspredicted.org/we4x7.pdf)
treatment and the idea that we are most confident in the predictions of PCE (or any equilibrium for that matter) when players have had a chance to learn.

3. More than 50% of the links would be formed in the last quarter of the periods in the anti-monotonic treatment. This prediction complements the previous one and tests how well the unique PCE prediction in the anti-monotonic version matches the data.

4. The link-formation rate would be increasing over periods in the anti-monotonic treatment but decreasing over periods in the co-monotonic treatment. This prediction was based on the idea that observing player-compatible choices in previous periods should lead to an increase of link-formation rate in the anti-monotonic version, and our guess that initial play in the co-monotonic would fall in Region 2 of Figure 1.

5.2 Experimental Procedure

Each experimental session had 20 or 24 subjects from the Harvard Decision Science Lab (HDSL) subject pool. For 4 of the sessions, subjects were Harvard College students. For the other 4 sessions, anyone in the HDSL pool could participate. The version of the link-formation game was the cross-sessions treatment: in 4 of the sessions, subjects played the anti-monotonic link-formation game, while in the other 4 they played the co-monotonic link-formation game.

Each subject was assigned a role: North-1, North-2, South-1, or South-2. This role was fixed throughout the experiment. Subjects played the link-formation game for 20 periods, with random anonymous matching each period. At the end of each period, they observed the actions of other players in their match. They also had access to the complete history of play in their previous matches. Subjects were paid at least $10 at the completion of the experiment, with up to $10 more in bonus payments depending linearly on the sum of their payoffs across 20 rounds of the game. On average, each subject earned about $16 for this 40-minute study.

At the beginning of each session, the subjects were given a printout with instructions and rules of the game. The experimenter went over the instructions of the link-formation game and the details of the matchmaking process on a large screen and answered any questions. Then, the subjects answered two comprehension questions to check their understanding of the rules. They saw an explanation for the correct answers to the comprehension questions before starting the first period of the experiment.
5.3 Experimental results

We will discuss all of our pre-registered analyses, as well as some secondary analyses that we performed after seeing the data. The key finding is that while initial play is similar across the two treatments, after learning the rates of link formation in the two treatments diverge. Learning increases link-formation rate 2.64 times as much in the anti-monotonic treatment as in the co-monotonic treatment.

**Prediction 1:** Low cost players choose Active more often than high cost players do.

Out of the 40 subgroups formed by two treatments and 20 periods per session, this prediction was correct in 38 of the groups. We pre-registered the procedure of dividing the data into 40 subsamples by period and treatment, then applying the Mann-Whitney test to compare the low-cost and high-cost players’ behavior within each subsample. As shown in Table 4 in the Online Appendix, dividing the data set into 40 subgroups made each subgroup so small that most of these comparisons were not statistically significant. However, a joint test rejects the null hypothesis that the low-cost and high-cost players behave the same way in all 40 subgroups. We conducted another test that divides the data into 4 subgroups, based on treatment and the two halves of the periods. In each of the resulting 4 subsamples, we see significantly higher rate of playing Active among the low-cost players than high-cost players ($p < 0.01$).

<table>
<thead>
<tr>
<th></th>
<th>anti-monotonic</th>
<th>low-cost player</th>
<th>high-cost player</th>
</tr>
</thead>
<tbody>
<tr>
<td>first half of periods</td>
<td>93.70% ± 1.13%</td>
<td>85.65% ± 1.64%</td>
<td></td>
</tr>
<tr>
<td>second half of periods</td>
<td>98.48% ± 0.57%</td>
<td>90.87% ± 1.34%</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>anti-monotonic</th>
<th>co-monotonic</th>
<th>p-values</th>
</tr>
</thead>
<tbody>
<tr>
<td>first quarter of periods</td>
<td>74.35% ± 2.77%</td>
<td>68.70% ± 2.77%</td>
<td>0.1069</td>
</tr>
<tr>
<td>last quarter of periods</td>
<td>88.70% ± 1.96%</td>
<td>74.13% ± 2.73%</td>
<td>&lt; 10^{-4}</td>
</tr>
<tr>
<td>p-values</td>
<td>&lt; 10^{-4}</td>
<td>0.1853</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 1: Frequency of playing Active for the two treatments and in two halves of the session. The cells show average frequency and one standard error.

**Prediction 2:** In the last quarter of the periods, the link-formation rate in the anti-monotonic treatment would be higher than in the co-monotonic treatment.

Table 2: Rates of link formation. The cells show point estimate and one standard error. The p-values of pairwise comparisons were calculated as a (two-tailed) Mann-Whitney test.
In the data, the rates of link formation in these two subgroups were 89% and 74% respectively, and the difference is strongly significant \( p < 10^{-4} \) using pre-registered Mann-Whitney test).

**Prediction 3:** More than 50% of the links would be formed in the last quarter of the periods under the anti-monotonic treatment. This obviously also holds \( p \approx 0 \) using the pre-registered significance test), since the point estimate of 88.70% is more than 19 standard errors away from 50%.

**Prediction 4:** The link-formation rate will be increasing over periods in the anti-monotonic treatment but decreasing over periods in the co-monotonic treatment. Contrary to this prediction, link-formation rate increased in both treatments with learning. However, this increase was 2.64 times as large in the anti-monotonic treatment, which was the source of divergence in play after learning for the two treatments, and the learning effect on link-formation rate for the co-monotonic treatment was not statistically significant \( p = 0.1852 \) under Mann-Whitney test).

### 6 Concluding Discussion

We have provided a learning foundation for PCE based on a particular steady-state learning model with forward-looking Bayesian agents. We conjecture there exist other natural learning models and more general models of dynamic adjustment that can provide alternative micro-foundations for PCE in other and perhaps broader classes of games. We leave this as future work.

On a related note, our learning model assumes that agents correct maximize their expected discounted utility. We do not attempt to model errors in computing conditional probabilities or in trading off exploitation and exploration when learning requires experimentation. We do not incorporate ambiguity attitudes or behavioral preferences such as reference points into our model. While we do not expect a model of rational Bayesian actors to precisely describe most people’s behavior, we still expect the qualitative predictions of PCE to apply. This is because we suspect these complications will not reverse the basic comparative static result that the more “compatible” player will use a given action more often in the adjustment process leading to equilibrium. Indeed, this suspicion was confirmed in our experiment.

### References


A Appendix

A.1 Omitted Proofs

Proposition 3: None of the following refinements both (i) eliminates the “no links” equilibrium outcome in the link-formation game when qualities are anti-monotonic with costs, and (ii) allows the “no links” equilibrium outcome when qualities are comonotonic with costs: extended proper equilibrium, proper equilibrium, trembling-hand perfect equilibrium, $p$-dominance, Pareto efficiency, strategic stability, and pairwise stability. Moreover the link-formation game is not a potential game.

Proof. We analyze the solution concepts one at a time.

1. To see that the Nash equilibrium where every player chooses “Inactive” is an extended proper equilibrium when $(q_i)$ are anti-monotonic with $(c_i)$, for each $\epsilon > 0$ let N1 and S1 play trade with probability $\epsilon^2$, N2 and S2 play trade with probability $\epsilon$. For small enough $\epsilon$, the expected payoff of “Active” for player $i$ is approximately $(10 - c_i)\epsilon$ since terms with higher order $\epsilon$ are negligible. It is clear that this payoff is negative for small $\epsilon$ for every player $i$, and that under the utility re-scalings $\beta_{N1} = \beta_{S1} = 10$, $\beta_{N2} = \beta_{S2} = 10$, the loss to playing “Active” smaller for N2 and S2 than for N1 and S1. So this strategy profile is an $(\beta, \epsilon)$-extended proper equilibrium. Taking $\epsilon \to 0$, we arrive at the equilibrium where each player chooses “Inactive” with probability 1. Every extended proper equilibrium is also proper and trembling-hand perfect.

2. The $p$-dominance refinement does not eliminate the no-links equilibrium when $(q_i)$ are anti-monotonic with $(c_i)$, because under the belief that all other players choose
“Active” with probability $p$, the expected payoff of playing “Active” (due to additivity across trading partners) is $\left(1 - p\right) \cdot 0 + p \cdot \left(10 + 30 - 2c_i\right) > 0$ for any $c_i \in \{14, 19\}$.

3. It is immediate that the no-links equilibrium outcome is Pareto dominated by the all-links equilibrium outcome under both parameter specifications, so Pareto efficiency would rule it out whether $(c_i)$ is anti-monotonic or comonotonic with $(q_i)$.  

4. Strategic stability (Kohlberg and Mertens, 1986) eliminates the Inactive equilibrium (as a singleton stable set) under both parametrizations. To see why, first consider the case where $(c_i)$ are anti-monotonic with $(q_i)$. View each strategy profile as a point in $[0, 1]^4$, with each coordinate referring to the probability that each player chooses Active, so that the Inactive equilibrium is the point $(0, ..., 0)$. Fix $\eta = 1/100$ say. For every choice of $\epsilon' > 0$, we define : $\epsilon_N(Active) = \epsilon_S(Active) = 2\epsilon'$, $\epsilon_{N2}(Active) = \epsilon_{S2}(Active) = \epsilon'$ and $\epsilon_i(Inactive) = \epsilon'$ for all players $i$. We show that in the game where each player $i$ is constrained to play strategy $s_i$ with probability at least $\epsilon_i(s_i)$, the only Nash equilibrium is for each player to choose :Active with probability $1 - \epsilon'$. To see this, consider the choice of $N2$ in any such equilibrium $\sigma$. If $N2$ weakly prefers playing Active, then $N1$ must strictly prefer playing it, so we get $\sigma_{N1}(Active) = 1 - \epsilon' \geq \sigma_{N2}(Active)$. On the other hand if $N2$ strictly prefers Inactive, then $\sigma_{N2}(Active) = \epsilon' < \sigma_{N1}(Active)$, since $N1$ is constrained to play Active with minimum probability $2\epsilon'$. So in either case, $\sigma_{N1}(Active) \geq \sigma_{N2}(Active)$. Against such a strategy profile of the northern players, each southern player has Active as a strict best response, so they all give it the maximum allowed probability of $1 - \epsilon'$. But when all southern players play Active with the same probability, all northern players also have Active as a strict best response showing that “play Active as much as possible” is the only Nash equilibrium of the constrained game. This equilibrium is more than $\eta$ away from the Inactive equilibrium, which must not be strategically stable as a singleton.

Now consider case where $(c_i)$ are comonotonic with $(q_i)$. Again let $\eta = 1/100$ and let $\epsilon' > 0$ be given. Define $\epsilon_{N1}(Active) = \epsilon_S(Active) = \epsilon'$, $\epsilon_{N2}(Active) = \epsilon'/1000$, $\epsilon_{S2}(Active) = \epsilon'$ and $\epsilon_i(Inactive) = \epsilon'$ for all players $i$. Suppose by way of contradiction there is a Nash equilibrium $\sigma$ of the constrained game which is $\eta$-close to the Inactive equilibrium. In such an equilibrium, $N2$ must strictly prefer Inactive, otherwise $N1$ strictly prefers Active so $\sigma$ could not be $\eta$-close to the Inactive equilibrium. Similar argument shows that $S2$ must strictly prefer Inactive. This shows $N2$ and $S2$ must play Active with minimum possible probability, that is $\sigma_{N2}(Active) = \epsilon'/1000$ and $\sigma_{S2}(Active) = \epsilon'$. This implies that, even if $\sigma_{N1}(Active)$ were at its minimum possible level of $\epsilon'$, $S1$ would still strictly prefer playing Inactive because $S1$ is 1000 times as likely to link with the low-quality opponent as the high-quality opponent. This shows
\[ \sigma_{S1}(\text{Active}) = \epsilon'. \] But when \( \sigma_{S1}(\text{Active}) = \sigma_{S2}(\text{Active}) = \epsilon' \), \( N1 \) strictly prefers playing \text{Active}, so \( \sigma_{N1}(\text{Active}) = 1 - \epsilon' \). This contradicts \( \sigma \) being \( \eta \)-close to autarky.

5. Pairwise stability (Jackson and Wolinsky, 1996) does not apply to this game, since each player chooses between either linking with every player on the opposite side who plays “Active”, or linking with no one. A player cannot selectively cut off one of her links while preserving the other.

6. The game does not have an ordinal potential, so refinements of potential games (Monderer and Shapley, 1996) do not apply. To see that this is not a potential game, consider the anti-monotonic parametrization suppose a potential \( P \) of the form \( P(a_{N1}, a_{N2}, a_{S1}, a_{S2}) \) exists, where \( a_i \in \{1, 0\} \) denotes whether player \( i \) chooses “Active”. We must have

\[
P(0, 0, 0, 0) = P(1, 0, 0, 0) = P(0, 0, 0, 1),
\]

since a unilateral deviation by one player from the inactive equilibrium does not change any player’s payoffs. But notice that \( u_{N1}(1, 0, 0, 1) - u_{N1}(0, 0, 0, 1) = 10 - 14 = -4 \), while \( u_{S2}(1, 0, 0, 1) - u_{S2}(1, 0, 0, 0) = 30 - 19 = 11 \). If the game has an ordinal potential, then both of these expressions must have the same sign as \( P(1, 0, 0, 1) - P(1, 0, 0, 0) = P(1, 0, 0, 1) - P(0, 0, 0, 1) \), which is not true. A similar argument shows the co-monotonic parametrization does not have a potential either.

\[ \Box \]

### A.2 PCE and Extended Proper Equilibrium

**Definition 6.** Fix an \( n \)-player strategic-form game and for each mixed strategy profile \( \alpha \), and let \( L_i(a_i|\alpha) \) represent the expected loss for \( i \) from playing action \( a_i \) instead of the best response to \( \alpha_{-i} \). For \( \beta \in \mathbb{R}^n_{++} \) and \( \epsilon > 0 \), a \((\beta, \epsilon)\)-extended proper equilibrium is a totally mixed strategy profile \( \alpha \) such that \( \beta_i L_i(a_i|\alpha) > \beta_j L_j(a_j|\alpha) \) implies \( \alpha_i(a_i) \leq \epsilon \cdot \alpha_j(a_j) \) for all \( i, j, a_i \in A_i, a_j \in A_j \). Strategy profile \( \alpha^* \) is an extended proper equilibrium if there exist \( \beta \in \mathbb{R}^n_{++} \) and sequences \((\epsilon_t), (\alpha_t)\) so that \( \epsilon_t > 0 \) for each \( t \) and \( \epsilon_t \to 0 \), each \( \alpha_t \) is a \((\beta, \epsilon_t)\)-extended proper equilibrium, and \( \alpha_t \to \alpha^* \).

Extended proper equilibrium requires that more costly mistakes are infinitely less likely. By contrast, PCE only places the directional restriction that the less compatible player trembles with smaller probability, without further magnitude restrictions. This means extended proper equilibrium may impose restrictions in some games where PCE has no bite.

**Example 4.** Consider a three-player game where Row chooses a row, Column chooses a
column, and Geo chooses a matrix. The payoff to Geo is always 0. The payoffs to Row and Column are listed in the tables below.

<table>
<thead>
<tr>
<th></th>
<th>West</th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>1,1</td>
<td>1,1</td>
<td></td>
</tr>
<tr>
<td>Down</td>
<td>0,1</td>
<td>0,0</td>
<td></td>
</tr>
</tbody>
</table>

The strategy profile (Up, Left, West) is not an extended proper equilibrium, because Column would deviate to Right against a tremble where Row’s costly deviation to Down is much rarer than Geo’s costless deviation to East. However, it is a PCE. For example, take a sequence of trembles $\epsilon(t)$ where the minimum probability on each action is $1/t$. By Lemma 1, every $\epsilon(t)$-equilibrium in this sequence is an $\epsilon(t)$-PCE, and we see that Column is indifferent between Left and Right if Row deviates to Down exactly as much as Geo deviates to East.

### A.3 Steady-State Learning

We summarize a steady-state learning model based on Fudenberg and He (2017a), applied to a strategic form game with general feedback structure as in Battigalli, Cerreia-Vioglio, Maccheroni, and Marinacci (2015). We will describe the model tersely and readers should refer to Fudenberg and He (2017a) for motivation behind the model.

#### A.3.1 The stage game

Consider a strategic-form game with finite number of players $i \in I$, finite strategy sets $A_i$ and utility functions $u_i : A \rightarrow \mathbb{R}$ for each player $i$. Write $\Pi_i$ for the set of all mixed strategies of $i$, with typical element $\alpha_i \in \Pi_i$. For each $i$, there is a finite set of consequences $X_i$ and a function $y_i : A \rightarrow X_i$ mapping strategy profiles to distributions over consequences. A consequence structure is factored if each $u_i$ only depends on $a_i$ and $x_i$, that is to say we can write $u_i(a_i, a_{-i}) = \tilde{u}_i(a_i, x_i(a))$ for every strategy profile $a$. In a factored consequence structure, a player’s payoff contains no additional information about others’ play beyond the information in the consequence.

Fudenberg and Levine (1993) consider the special case where the game is an extensive-form game, $X_i$ is the set of terminal vertices of the game tree for every $i$, and $x_i(a)$ is the terminal vertex reached by pure strategy profile $a$. In Example 2, a possible consequence structure is $X_i = \{L, M, R, \emptyset\}$ for $i = 1, 2$. If player $i$ chooses across, $x_i(a) = \emptyset$ regardless of $a_{-i}$, so $i$ gets no information about others’ play. If player $i$ chooses down, $x_i(a) = a_3$ and $i$ observes the play of P3. This is a factored consequence structure because playing across gives 0 payoff to $i$ regardless of how others play. When $i$ plays down, only the play of P3 affects $i$’s payoffs, since P1 and P2 are non-interacting players.
A.3.2 Learning by individual agents

Time is discrete and all agents are rational Bayesians with geometrically distributed lifetimes. They survive between periods with probability \( 0 \leq \gamma < 1 \) and further discount future utility flows by \( 0 \leq \delta < 1 \), so their objective is to maximize the expected value of \( \sum_{t=0}^{\infty} (\gamma \delta)^t \cdot u_t \). Here, \( 0 \leq \gamma \delta < 1 \) is the effective discount factor, and \( u_t \) is the payoff \( t \) periods from today.

At birth, each agent is assigned a role \( i \in I \) in the stage game. Agents know their role, which is fixed for life. Every period, each agent is randomly and anonymously matched with opponents in roles \(-i\) to play the stage game, and the game’s outcome determines the agent’s payoff that period. At the end of each period, agent \( i \) observe \( x_i(a) \), where \( a \) is the strategy profile played in her match. Agents do not observe the identity, age, or past experiences of their opponents, nor do agents observe \( a_{-i} \) except the information contained in the consequence \( x_i(a) \). Agents update their beliefs and play the signalling game again with new random opponents next period, provided they are still alive.

Agents believe they face a fixed but unknown distribution of opponents’ aggregate play, so they believe that their observations will be exchangeable. The environment will indeed be constant in the steady states that we analyze.

Formally, each agent in player role \( i \) is born with a prior density function over the aggregate mixed strategy of the \(-i\) populations, \( g_i : \prod_{-i} \rightarrow \mathbb{R}_+ \), which integrates to 1. We write \( g_i^{(j)} \) for the marginal of \( g_i \) on \( \Pi_j \).

We now state a regularity assumption on the agents’ priors that will be maintained throughout.

**Definition 7.** A prior \( g = (g_i) \) is regular if

(a). [independence] \( g_i(\alpha_{-i}) = \prod_{j \neq i} g_i^{(j)}(\alpha_j) \) for each \( i \)

(b). [nice] For each \( i \neq j \), there are positive constants \( c_{\alpha_j}^{(i,j)} \) such that

\[
\alpha_j \mapsto \frac{g_i^{(j)}(\alpha_j)}{\prod_{a_j \in A_j} \alpha_j(a_j)^{c_{\alpha_j}^{(i,j)}-1}}
\]

is uniformly continuous and bounded away from zero on the relative interior of \( \Pi_j \).

Independence ensures that \( i \) does not learn how \( j \) plays by observing the behavior of some other \( k \neq j \). The non-doctrinaire nature of \( g \) implies that the agents never see an observation that they assigned zero prior probability, so that they have a well-defined optimization problem after any history.

The set of histories for an age \( t \) player \( i \) \( Y_i[t] := (A_i \times X_i)^t \), where each period the history records the action played and the consequence observed. The set of all histories for player \( i \)
is the union $Y_i := \bigcup_{t=0}^{\infty} Y_i[t]$. The dynamic optimization problem of $i$ has an optimal policy function $\sigma_i : Y_i \rightarrow A_i$, where $\sigma_i(y_i)$ is the strategy that a player $i$ with history $y_i$ would send the next time she plays the stage game.

### A.3.3 Random Matching and Aggregate Play

The society contains a unit mass of agents in each player role $i \in I$. Each agent has $0 \leq \gamma < 1$ chance of surviving at the end of each period and complementary chance $1 - \gamma$ of dying. To preserve population sizes, $(1 - \gamma)$ new player $i$’s are born each period. Each period, agents in the society are matched uniformly at random to play the stage game.

A state $\psi$ of the learning model is described by the mass of agents with each possible history. We write it as

$$\psi \in \times_{i \in I} \Delta(Y_i).$$

We refer to the $i$-th component of a state $\psi$ by $\psi_i$. Given the agents’ optimal policies, each possible history for an agent completely determines how that agent will play in their next match. Given the policy profile $(\sigma_i)$, each state $\psi$ induces an aggregate mixed strategy $\sigma_i(\psi_i) \in \Delta(A_i)$ for each player $i$ population, where we extend the domain of $\sigma_i$ from $Y_i$ to distributions on $Y_i$ in the natural way.

We will study the steady states of this learning model. A steady state is a state $\psi$ that reproduces itself indefinitely when agents use their optimal policies. The set of all steady states for regular prior $g$ and $0 \leq \delta, \gamma < 1$ is denoted $\Psi^*(g, \delta, \gamma)$. The set of steady state strategy profiles is $\Pi^*(g, \delta, \gamma) := \{\sigma(\psi^*) : \psi^* \in \Psi^*(g, \delta, \gamma)\}$. For each $0 \leq \delta < 1$, a strategy profile $\pi^*$ is $\delta$-stable under $g$ if there is a sequence $\gamma_k \rightarrow 1$ and an associated sequence of steady state strategy profiles $\pi^{(k)} \in \Pi^*(g, \delta, \gamma_k)$, such that $\pi^{(k)} \rightarrow \pi^*$. Strategy profile $\pi^*$ is patiently stable under $g$ if there is a sequence $\delta_k \rightarrow 1$ and an associated sequence of strategy profiles $\pi^{(k)}$ where each $\pi^{(k)}$ is $\delta_k$-stable under $g$ and $\pi^{(k)} \rightarrow \pi^*$. Strategy profile $\pi^*$ is patiently stable if it is patiently stable under some regular prior $g$.

### B Online Appendix

#### B.1 Pre-Registration Document
1) Have any data been collected for this study already?
No, no data have been collected for this study yet.

2) What's the main question being asked or hypothesis being tested in this study?
We study a link-formation game involving 4 players, split into two slides: North and South. These 4 players are named North-1, North-2, South-1, South-2.
Each player simultaneously plays an action: active or inactive. A player who plays inactive forms no links. A player who plays active forms a link to every active player on the opposite side. Each player is characterized by two parameters: cost and quality. Cost describes the player's private cost for forming each link. Quality is the benefit that the player provides to everyone who links to him/her. When player i forms a link to player j, the value of this link is (quality of j) - (cost of i). A player who forms no links gets 0 payoff. A player who forms one or more links gets payoff equal to sum of the values of the link(s) he/she forms.

In each lab session, subjects will be assigned to one of the four player roles. Then, they will play 20 periods of this link-formation game as this assigned role, matching with random opponents each period. The sum of the payoffs across 20 periods determine subjects' bonus payment at the conclusion of the experiment, in addition to a guaranteed completion payment.

We have two treatments that differ in the specifications of players' cost and quality parameters, to be described in "Conditions" below. The two treatments are called anti-monotonic and co-monotonic.

Hypothesis 1: In each treatment and in each period, North-1 and South-1 (who have a lower cost) choose "active" more often than North-2 and South-2 (who have a higher cost).
Hypothesis 2: More than 50% of the feasible links will be formed in the last 5 periods of the anti-monotonic treatment.
Hypothesis 3: Fewer links will be formed in the last 5 periods of the co-monotonic treatment than the anti-monotonic treatment.
Hypothesis 4: In the co-monotonic treatment, the fraction of links formed is lower in the last 5 periods than in the first 5 periods.

3) Describe the key dependent variable(s) specifying how they will be measured.
1. Frequency of playing "active" for participants in different player roles and in different treatments.
2. Fraction of links formed. In each game, up to 4 pairs of links can be formed: North-1 <-> South-1, North-1 <-> South-2, North-2 <-> South-1, North-2 <-> South-2. We will measure the fraction of these feasible links that form in different periods and for different treatments. This variable is used in Hypotheses 2, 3, and 4.

4) How many and which conditions will participants be assigned to?
There are two treatments: anti-monotonic and co-monotonic. They refer to whether the quality parameter ranks players in the opposite way or in the same way as the cost parameter.

In both treatments, North-1 and South-1 have a cost of 14, while North-2 and South-2 have a cost of 19.
Anti-monotonic treatment: North-1 and South-1 have a quality of 30, North-2 and South-2 have a quality of 10.
Co-monotonic treatment: North-1 and South-1 have a quality of 10, North-2 and South-2 have a quality of 30.

Each subject will participate in only one treatment.

5) Specify exactly which analyses you will conduct to examine the main question/hypothesis.
For Hypothesis 1, we will run a Mann-Whitney test comparing the played actions of lower-cost players and higher-cost players in each treatment.

For Hypothesis 2, among the games in the last 5 periods of anti-monotonic treatment sessions, we will calculate the sample mean and standard deviation of fraction of links formed. We will calculate (sample mean - 50%) / standard error.

For Hypotheses 3 and 4, we will again calculate statistical significance using Mann-Whitney test. For example, for Hypothesis 3, the outcome of each period maps to an integer between 0 and 4, the number of pairs of links formed. We will use Mann-Whitney test to compare the number of links formed in the last 5 games of co-monotonic treatments versus last 5 games of anti-monotonic treatments.

6) Any secondary analyses?
None.
### B.2 Details of the Experiment

Here are the treatments and subject pools used in the 8 sessions. We intended to recruit 24 participants for each session. However, for two of the sessions involving Harvard College students, we only managed to recruit 20 participants. Since these two sessions belonged to two different treatments, the total numbers of participants in the two treatments are still balanced.

<table>
<thead>
<tr>
<th>Experiment date</th>
<th>Treatment</th>
<th>Type of Subjects</th>
<th>Number of Participants</th>
</tr>
</thead>
<tbody>
<tr>
<td>October 12</td>
<td>anti-monotonic</td>
<td>Harvard College students</td>
<td>24</td>
</tr>
<tr>
<td>October 13</td>
<td>co-monotonic</td>
<td>Harvard College students</td>
<td>20</td>
</tr>
<tr>
<td>October 20</td>
<td>co-monotonic</td>
<td>Harvard College students</td>
<td>24</td>
</tr>
<tr>
<td>October 26</td>
<td>anti-monotonic</td>
<td>General HDSL pool</td>
<td>24</td>
</tr>
<tr>
<td>November 1</td>
<td>co-monotonic</td>
<td>General HDSL pool</td>
<td>24</td>
</tr>
<tr>
<td>November 2</td>
<td>co-monotonic</td>
<td>General HDSL pool</td>
<td>24</td>
</tr>
<tr>
<td>November 3</td>
<td>anti-monotonic</td>
<td>General HDSL pool</td>
<td>24</td>
</tr>
<tr>
<td>November 16</td>
<td>anti-monotonic</td>
<td>Harvard College students</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 3: The experimental sessions.

The next table records the frequencies of participants choosing **Inactive** in different periods of the two treatments.
<table>
<thead>
<tr>
<th>Period</th>
<th>Low-cost, inactive player</th>
<th>High-cost, inactive player</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7</td>
<td>12</td>
<td>0.1011</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>9</td>
<td>0.2016</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>4</td>
<td>0.5032</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>6</td>
<td>0.1493</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>7</td>
<td>0.0925</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>6</td>
<td>0.0716</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>5</td>
<td>0.0474</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>5</td>
<td>0.0114</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>4</td>
<td>0.2031</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>8</td>
<td>0.0074</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>2</td>
<td>0.2840</td>
</tr>
<tr>
<td>12</td>
<td>2</td>
<td>5</td>
<td>0.1221</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>4</td>
<td>0.0866</td>
</tr>
<tr>
<td>14</td>
<td>0</td>
<td>4</td>
<td>0.0215</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
<td>3</td>
<td>0.1572</td>
</tr>
<tr>
<td>16</td>
<td>1</td>
<td>5</td>
<td>0.0474</td>
</tr>
<tr>
<td>17</td>
<td>0</td>
<td>4</td>
<td>0.0215</td>
</tr>
<tr>
<td>18</td>
<td>1</td>
<td>5</td>
<td>0.0474</td>
</tr>
<tr>
<td>19</td>
<td>0</td>
<td>5</td>
<td>0.0114</td>
</tr>
<tr>
<td>20</td>
<td>0</td>
<td>5</td>
<td>0.0114</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Period</th>
<th>Low-cost, inactive player</th>
<th>High-cost, inactive player</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>13</td>
<td>0.1094</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>14</td>
<td>0.0224</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>11</td>
<td>0.0038</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>6</td>
<td>0.5027</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>7</td>
<td>0.3854</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>6</td>
<td>0.3773</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>4</td>
<td>0.7497</td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td>5</td>
<td>0.7332</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>7</td>
<td>0.1706</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>4</td>
<td>0.2031</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>2</td>
<td>0.284</td>
</tr>
<tr>
<td>12</td>
<td>3</td>
<td>3</td>
<td>0.5036</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>2</td>
<td>0.284</td>
</tr>
<tr>
<td>14</td>
<td>2</td>
<td>2</td>
<td>0.5044</td>
</tr>
<tr>
<td>15</td>
<td>4</td>
<td>5</td>
<td>0.3664</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
<td>8</td>
<td>0.0074</td>
</tr>
<tr>
<td>17</td>
<td>4</td>
<td>8</td>
<td>0.1103</td>
</tr>
<tr>
<td>18</td>
<td>4</td>
<td>9</td>
<td>0.0692</td>
</tr>
<tr>
<td>19</td>
<td>2</td>
<td>14</td>
<td>0.0005</td>
</tr>
<tr>
<td>20</td>
<td>2</td>
<td>14</td>
<td>0.0005</td>
</tr>
</tbody>
</table>

Table 4: Total number of instances of subjects playing Inactive in different periods and under different treatments, across all sessions. The p-values are obtained from applying a one-sided Mann-Whitney test (with the alternative hypothesis that high-cost players play Inactive more) applied to each of the 40 subsamples, divided up by period and treatment.

As a joint test of behavioral difference between low-cost and high-cost players across all periods and treatments, we fit two nested models of the probability of playing Active. In the full model,

$$\Pr[A_{i,c,j,t} = \text{Active}] = \Phi(\beta_{j,t} + 1\{c = \text{high}\} \cdot \delta_{j,t})$$

where $$A_{i,c,j,t} \in \{\text{Active, Inactive}\}$$ is the action choice of player $$i$$ with cost $$c$$ in treatment $$j$$, period $$t$$, $$\beta_{j,t}$$ is the fixed effect of (treatment $$j$$, period $$t$$), $$\delta_{j,t}$$ is the interaction effect of having high cost in treatment $$j$$, period $$t$$, and $$\Phi$$ is the Gaussian distribution function. In the restricted model, we suppose $$\delta_{j,t} = 0$$ for each $$(j, t)$$, so that

$$\Pr[A_{i,c,j,t} = \text{Active}] = \Phi(\beta_{j,t})$$

Relative to the reduced model, the full model adds 40 degrees of freedom and reduces sum
of squared residuals by 114.31 ($p < 10^{-8}$ under the Chi-squared test). The result is similar if we estimate the analogous linear probability models instead of probit models, resulting in an $F$-statistic of 2.7807 ($p < 10^{-7}$).