Player-Compatible Equilibrium*

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Abstract

Player-Compatible Equilibrium (PCE) imposes cross-player restrictions on the magnitudes of the players’ “trembles” onto different strategies. These restrictions capture the idea that trembles correspond to deliberate experiments by agents who are unsure of the prevailing distribution of play. PCE selects intuitive equilibria in a number of examples where trembling-hand perfect equilibrium (Selten, 1975) and proper equilibrium (Myerson, 1978) have no bite. We show that rational learning and some near-optimal heuristics imply our compatibility restrictions in a steady-state setting and verify that our theoretical predictions also apply to finite-population simulations when all agents start learning at the same time.

Keywords: non-equilibrium learning, equilibrium refinements, trembling-hand perfect equilibrium, combinatorial bandits, Bayesian upper confidence bounds, Thompson sampling

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1 Introduction

Starting with Selten (1975), a number of papers have used the device of vanishingly small “trembles” or “mistakes” to refine the set of Nash equilibria. This paper introduces player-compatible equilibrium (PCE), which extends this approach by imposing cross-player restrictions on these trembles in a way that is invariant to the utility representations of players’ preferences over game outcomes. These cross-player restrictions are based on the concept of player compatibility, which says roughly that player $i$ is more compatible with strategy $s_i^*$ than player $j$ is with strategy $s_j^*$ if any play of the others that makes $s_j^*$ optimal for $j$ also makes $s_i^*$ optimal for $i$.

Section 2 defines PCE and proves that they exist in all finite games. Section 3 uses a series of examples to show that PCE can rule out seemingly implausible equilibria that other tremble-based refinements such as trembling-hand perfect equilibrium (Selten, 1975) and proper equilibrium (Myerson, 1978) cannot eliminate, and also notes that PCE satisfies the compatibility criterion of Fudenberg and He (2018) in signaling games.

The main conceptual novelty of PCE is the cross-player restrictions on trembles. Section 4 shows that these restrictions arise endogenously in three models of steady-state learning where agents are born into different player roles of a stage game and play it every period, facing a fixed but unknown distribution of opponents’ play. The learning problem features an exploration-exploitation trade-off, as the information an agent receives each period depends on her stage-game strategy. To specify how agents deal with this trade-off, we consider three distinct models of individual behavior. One is the classic model of rational Bayesians maximizing discounted expected utility under the belief that the environment (the aggregate strategy distribution in the population) is constant. The other two learning models are computationally simpler heuristics, namely Bayesian upper-confidence bounds (Kaufmann, Cappé, and Garivier, 2012), and Thompson sampling (Thompson, 1933). In all of these models, inexperienced agents respond to the exploration-exploitation trade-off by experimenting with non-optimal responses to others’ play, endogenously generating “trembles” onto off-path actions.

Using this setup, we study the implications of player compatibility on the relative magnitudes of these learning-based “trembles.” We focus on player roles $i$ and $j$ for whom the

\[\text{We say more about these models in Section 4. Briefly, upper confidence bound algorithms originated as computationally tractable algorithms for multi-armed bandit problems (Agrawal, 1995; Katehakis and Robbins, 1995). We consider a Bayesian version of the algorithm that keeps track of the learner’s posterior beliefs about the payoffs of different strategies, first analyzed by Kaufmann, Cappé, and Garivier (2012). Thompson sampling was motivated by the question of assigning patients to two treatments of unknown effectiveness. Recently, both academic researchers (Schwartz, Bradlow, and Fader, 2017; Agarwal, Long, Traupman, Xin, and Zhang, 2014) and firms, including Microsoft and Google, have applied Thompson sampling to deal with a range of active learning problems. See Francetich and Kreps (2018) for a discussion of these and other heuristics for active learning.}\]
stage game is \textit{isomorphically factorable}. Loosely speaking, a game is \textit{factorable} for a player if she can only learn about the payoff consequences of a particular strategy by play it — playing other strategies provides no relevant information. A factorable game is \textit{isomorphically factorable} for players $i$ and $j$ if there is an isomorphism between the information observed by $i$ and $j$. We show that in all three models of individual behavior, the relative frequencies of the experiments by agents in different player roles ranked by player compatibility satisfy the same cross-player restrictions that PCE imposes on tremble probabilities. Thus, we provide a learning microfoundation for the tremble restrictions. In the process, we develop an extension of the “coupling” argument used in Fudenberg and He (2018) to compare the optimal experimentation rates of Bayesian learners in signaling games.

We expect that the predictions of PCE will hold even when the population is not infinite and the environment is not stationary. To test this belief, we performed simulations of our three learning models on two versions of a strategic link-formation game from Example 3. Each player has a cost parameter that determines her private cost to maintaining each link and a benefit parameter that governs how much utility she provides to each linked partner. By varying whether the “high-value” players who provide a higher benefit to linked partners incur a higher or lower private link-formation cost than their “low-value” counterparts, we create two versions of the game that share the same set of Nash equilibria but exhibit opposite player-compatibility structures with respect to linking. We expect the reversed compatibility structures to induce different learning outcomes, and indeed the two versions of the game have different sets of PCE. By contrast, as far as we know no existing equilibrium concept returns two different solution sets when applied the two versions of this game.

In our simulations, the society consists of a total of 20 agents split into four player roles, who play the game over a finite horizon against randomly matched opponents each period. We find that aggregate play always converges to a PCE, and that when there are multiple PCE each can be selected depending on the agents’ priors.

\subsection{1.1 Related Work}

Tremble-based solution concepts date back to Selten (1975), who thanks Harsanyi for suggesting them. These solution concepts consider totally mixed strategy profiles where players do not play an exact best reply to the strategies of others, but may assign positive probability to some or all strategies that are not best replies. Different solution concepts in this class consider different kinds of “trembles,” but they all make predictions based on the limits of these non-equilibrium strategy profiles as the probability of trembling tends to zero. Since we compare PCE to these refinements below, we summarize them here for the reader’s convenience.

\textit{Perfect equilibrium} (Selten, 1975), \textit{proper equilibrium} (Myerson, 1978), \textit{approachable equi-}
librium (Van Damme, 1987), and extended proper equilibrium (Milgrom and Mollner, 2017) are based on the idea that strategies with worse payoffs are played less often. An $\epsilon$-perfect equilibrium is a totally mixed strategy profile where every non-best reply has weight less than $\epsilon$. A limit of $\epsilon_t$-perfect equilibria where $\epsilon_t \rightarrow 0$ is called a trembling-hand perfect equilibrium. An $\epsilon$-proper equilibrium is a totally mixed strategy profile $\sigma$ where for every player $i$ and strategies $s_i$ and $s'_i$, if $U_i(s_i, \sigma_{-i}) < U_i(s'_i, \sigma_{-i})$ then $\sigma_i(s_i) < \epsilon \cdot \sigma_i(s'_i)$. A limit of $\epsilon_t$-proper equilibria where $\epsilon_t \rightarrow 0$ is called a proper equilibrium; in this limit a more costly tremble is infinitely less likely than a less costly one, regardless of the cost difference. An approachable equilibrium is the limit of somewhat similar $\epsilon_t$-perfect equilibria, but where the players pay control costs to reduce their tremble probabilities. When these costs are “regular,” all of the trembles are of the same order. Because PCE does not require that the less likely trembles are infinitely less likely than more likely ones, it is closer to approachable equilibrium than to proper equilibrium. The strategic stability concept of Kohlberg and Mertens (1986) is also defined using trembles, but applies to components of Nash equilibria as opposed to single strategy profiles.

Proper equilibrium and approachable equilibrium do not impose cross-player restrictions on the relative probabilities of various trembles. For this reason, when each types of the sender is viewed as a different player these equilibrium concepts reduce to perfect Bayesian equilibrium in signaling games with two possible signals, such as the beer-quiche game of Cho and Kreps (1987). They do impose restrictions when applied to the ex-ante form of the game, i.e. at the stage before the sender has learned their type. However, as Cho and Kreps (1987) point out, evaluating the cost of mistakes at the ex-ante stage means that the interim losses are weighted by the prior distribution over sender types, so that less likely types are more likely to tremble. In addition, applying a different positive linear rescaling to each type’s utility function preserves every type’s preference over lotteries on outcomes, but changes the sets of proper and approachable equilibria, while such utility rescalings have no effect on the set of PCE. In light of these issues, when discussing tremble-based refinements in Bayesian games we will always apply them at the interim stage.

Like PCE, extended proper equilibrium places restrictions on the relative probabilities of tremble by different players, but it does so in a different way: An extended proper equilibrium is the limit of $(\beta, \epsilon_t)$-proper equilibria, where $\beta = (\beta_1, ... \beta_I)$ is a strictly positive vector of utility re-scaling, and $\sigma_i(s_i) < \epsilon_t \cdot \sigma_j(s_j)$ if player $i$’s rescaled loss from $s_i$ (compared to the best response) is less than $j$’s loss from $s_j$. In a signaling game with only two possible signals, every Nash equilibrium where each sender type strictly prefers not to deviate from her equilibrium signal is an extended proper equilibrium at the interim stage, because suitable utility rescalings for the types can lead to any ranking of their utility costs of deviating to the off-path signal. By contrast, Proposition 3 shows every PCE must satisfy the compatibility
criterion of Fudenberg and He (2018), which has bite even in binary signaling games such as the beer-quiche example of Cho and Kreps (1987). So an extended proper equilibrium need not be a PCE, a fact that Examples 1 and 2 further demonstrate. Conversely, because extended proper equilibrium makes some trembles infinitely less likely than others, it can eliminate some PCE, as shown by example in Online Appendix OA 3, which also restates the formal definition of extended proper equilibrium for ease of reference.

Although it is not explicitly defined in terms of trembles, test-set equilibrium (Milgrom and Mollner, 2018) is motivated by the idea that trembles onto an alternative best response to the equilibrium play is more likely than trembles onto an inferior response. This refinement is particularly useful in auction-like environments, where there are many weak Nash equilibria and all opponent deviations to alternative best responses generate the same strategic incentives for any given player — in this case, the test set is “large.” But as we show in Example 1, PCE has bite in equilibria with “small” test sets, where most (e.g. all but 1) players strictly prefer their equilibrium action, and test-set equilibrium reduces to Nash.

This paper is also related to recent work that provides explicit learning foundations for various equilibrium concepts, such as Battigalli, Cerreia-Vioglio, Maccheroni, and Marinacci (2016), Battigalli, Francetich, Lanzani, and Marinacci (2017), Esponda and Pouzo (2016), Lehrer (2012). Unlike those papers, we do not consider ambiguity aversion, misspecified priors, or model uncertainty. Instead we focus on the relative rates of experimentation under rational expected-utility maximization and related “near-optimal” heuristics. For this reason our analysis of learning is closer to Fudenberg and Levine (2006) and Fudenberg and He (2018). And in methodology the paper is related to other work on active learning and experimentation, such as Doval (2018), Francetich and Kreps (2018), Fryer and Harms (2017), Halac, Kartik, and Liu (2016), Klein and Rady (2011), and Strulovici (2010).

2 Player-Compatible Equilibrium

Consider a strategic-form game with finite number of players $i \in I$, finite strategy set $S_i$ with $|S_i| \geq 2$ and utility functions $U_i : S \rightarrow \mathbb{R}$ for each player $i$, where $S := \times_i S_i$. For each $i$, let $\Delta(S_i)$ denote the set of mixed strategies and $\Delta^0(S_i)$ the set of strictly mixed strategies, where every pure strategy in $S_i$ is assigned positive probability. For $K \subseteq I$, let $\Delta(S_K)$ represent the set of correlated strategies among players $K$, i.e. the set of distributions on strategy profiles of players in coalition $K$, $\times_{i \in K} S_i$. Let $\Delta^0(S_K)$ represent the interior of $\Delta(S_K)$, that is the set of full-support correlated strategies on $S_K$.

2 The test set $T(\sigma)$ of strategy profile $\sigma$ is the set of profiles that can be generated by replacing one player’s strategy in $\sigma$ with an alternative best response. A Nash equilibrium $\sigma$ is called a test-set equilibrium if each $\sigma_i$ is undominated with respect to $\Delta(S_{-i})$ and also undominated with respect to $T(\sigma)$.

3 Recall that a full-support correlated strategy assigns positive probability to every pure strategy profile.
We formalize the concept of “compatibility” between players and their strategies in this general setting, which will play a central role in the definition of PCE in determining cross-player restrictions on trembles.

**Definition 1.** For player \( i \neq j \) and strategies \( s_i^* \in S_i, s_j^* \in S_j \), say \( i \) is more compatible with \( s_i^* \) than \( j \) is with \( s_j^* \), abbreviated as \((s_i^*|i) \succeq (s_j^*|j)\), if for every correlated strategy \( \sigma_{-j} \in \Delta^o(S_{-j}) \) such that

\[
U_j(s_j^*, \sigma_{-j}) \geq \max_{s_j' \in S_j \setminus \{s_j^*\}} U_j(s_j', \sigma_{-j}),
\]

we have for every \( \sigma_{-i} \in \Delta^o(S_{-i}) \) satisfying \( \sigma_{-i}|S_{-ij} = \sigma_{-j}|S_{-ij} \),

\[
U_i(s_i^*, \sigma_{-i}) > \max_{s_i' \in S_i \setminus \{s_i^*\}} U_i(s_i', \sigma_{-i}).
\]

In words, if \( s_j^* \) is weakly optimal for the less-compatible \( j \) against the opponents’ correlated strategy \( \sigma_{-j} \), then \( s_i^* \) is strictly optimal for the more-compatible \( i \) against any correlated strategy by \( \sigma_{-i} \) that matches \( \sigma_{-j} \) in terms of the play of \(-ij\). As this restatement makes clear, the compatibility condition only depends on players’ preferences over probability distribution on \( S \), and not on the particular utility representations chosen.

Since \( \times_{k \in K} \Delta^o(S_k) \subseteq \Delta^o(S_K) \), fewer strategy-player pairs can be ranked under this definition than if only (uncorrelated) mixed strategy profiles were considered.\(^4\) We use the more stringent definition so that we can microfound our compatibility-based cross-player restrictions on a broader set of learning models.

The compatibility ordering \( \succeq \) need not be asymmetric, as \((s_i^*|i) \sim (s_j^*|j)\) if both actions are strictly dominated for their respective players (so that the “if” clause of the definition is never satisfied) or if both are strictly dominant. However, compatibility is asymmetric when neither of these edge cases applies.

**Proposition 1.** If \((s_i^*|i) \succeq (s_j^*|j)\), then at least one of the following is true: (i) \((s_j^*|j) \not\preceq (s_i^*|i)\); (ii) \(s_i^*\) is strictly dominated\(^5\) for \( i \) and \( s_j^*\) is strictly dominated for \( j \), whenever opponents play strictly mixed correlated strategies. (iii) \(s_i^*\) is strictly dominant for \( i \) and \( s_j^*\) is strictly dominant for \( j \), whenever opponents play strictly mixed correlated strategies.

**Proof.** See Appendix A.1.

\(^4\)Formally, this alternative definition would be “For every \( \sigma_{-ij} \in \times_{k \neq i,j} \Delta^o(S_k) \) such that \( U_j(s_j^*, \hat{\sigma}_i, \sigma_{-ij}) \geq \max_{s_j' \in S_j \setminus \{s_j^*\}} U_j(s_j', \hat{\sigma}_i, \sigma_{-ij}) \) for some \( \hat{\sigma}_i \in \Delta^o(S_i) \), we have for every \( \hat{\sigma}_j \in \Delta^o(S_j) \) that \( U_i(s_i^*, \hat{\sigma}_j, \sigma_{-ij}) > \max_{s_i' \in S_i \setminus \{s_i^*\}} U_i(s_i', \hat{\sigma}_j, \sigma_{-ij}) \).”

\(^5\)Recall that a strategy can be strictly dominated even though it is not strictly dominated by any pure strategy.
Remark 1. If players $i$ and $j$ care a great deal about one another’s strategies, then their best responses are unlikely to be determined only by the play of the third parties. In the other extreme, suppose the set of players $\mathbb{I}$ can be divided into $G$ mutually exclusive groups, $\mathbb{I} = \mathbb{I}_1 \cup \ldots \cup \mathbb{I}_G$, in such a way that whenever $i$ and $j$ belong to the same group $i, j \in \mathbb{I}_g$, (1) they are non-interacting, meaning $i$’s payoff does not depend on the strategy of $j$ and $j$’s payoff does not depend on the strategy of $i$; (2) they have the same strategy set, $\mathbb{S}_i = \mathbb{S}_j$.

As a leading case, in a Bayesian game where each type is viewed as a different player, the set of types for a given player role (e.g. the sender in a signaling game) satisfy these two conditions. In addition, Section 3 gives several examples of complete-information games with this kind of groups structure. In a game with groups structure with $i, j \in \mathbb{I}_g$, we may write $U_i(s_i, s_{-ij})$ without ambiguity, since all augmentations of the strategy profile $s_{-ij}$ with a strategy by player $j$ lead to the same payoff for $i$. When $i, j$ are in the same group and $s^*_g \in \mathbb{S}_i = \mathbb{S}_j$, the definition for $(s^*_g|i) \succ (s^*_g|j)$ reduces to: For every strictly mixed correlated strategy $\sigma_{-ij} \in \Delta^\circ(\mathbb{S}_{-ij})$ such that

$$U_j(s^*_g, \sigma_{-ij}) \geq \max_{s'_j \in \mathbb{S}_j \setminus \{s^*_g\}} U_j(s'_j, \sigma_{-ij}),$$

we have

$$U_i(s^*_g, \sigma_{-ij}) > \max_{s'_i \in \mathbb{S}_i \setminus \{s^*_g\}} U_i(s'_i, \sigma_{-ij}).$$

That is, $(s^*_g|i) \succ (s^*_g|j)$ if whenever $s^*_g$ is even a weak best response for $j$ against some opponents’ strategy profile, it is also a strict best response for $i$ against the same strategy profile.

While the player-compatibility condition is especially easy to state for non-interacting players, our learning foundation will also justify cross-player tremble restrictions for pairs of players $i, j$ whose payoffs do depend on each others’ strategies. Remark 3 gives an example of such a game.

We now move towards the definition of PCE. PCE is a tremble-based solution concept. It builds on and modifies Selten (1975)’s definition of trembling-hand perfect equilibrium as the limit of equilibria of perturbed games in which agents are constrained to tremble, so we begin by defining our notation for the trembles and the associated constrained equilibria.

Definition 2. A tremble profile $\epsilon$ assigns a positive number $\epsilon(s_i|i) > 0$ to every player $i$ and pure strategy $s_i$. Given a tremble profile $\epsilon$, write $\Pi^\epsilon_i$ for the set of $\epsilon$-strategies of player $i$, namely:

$$\Pi^\epsilon_i := \{\sigma_i \in \Delta(\mathbb{S}_i) \text{ s.t. } \sigma_i(s_i) \geq \epsilon(s_i|i) \ \forall s_i \in \mathbb{S}_i\}.$$
We call $\sigma^o$ an $\epsilon$-equilibrium if for each $i$,

$$\sigma^o_i \in \arg \max_{\sigma_i \in \Pi^o_i} U_i(\sigma_i, \sigma^o_{-i}).$$

Note that $\Pi^o_i$ is compact and convex. It is also non-empty when $\epsilon$ is close enough to 0. By standard results, whenever $\epsilon$ is small enough so that $\Pi^o_i$ is non-empty for each $i$, an $\epsilon$-equilibrium exists.

The key building block for PCE is $\epsilon$-PCE, which is an $\epsilon$-equilibrium where the tremble profile is “co-monotonic” with $\epsilon_0$.

**Definition 3.** Tremble profile $\epsilon$ is player compatible if $\epsilon(s^*_i|i) \geq \epsilon(s^*_j|j)$ for all $i, j, s^*_i, s^*_j$ such that $(s^*_i|i) \succeq (s^*_j|j)$. An $\epsilon$-equilibrium where $\epsilon$ is player compatible is called a player-compatible $\epsilon$-equilibrium (or $\epsilon$-PCE).

The condition on $\epsilon$ says the minimum weight $i$ could assign to $s^*_i$ is no smaller than the minimum weight $j$ could assign to $s^*_j$ in the constrained game,

$$\min_{\sigma_i \in \Pi^o_i} \sigma_i(s^*_i) \geq \min_{\sigma_j \in \Pi^o_j} \sigma_j(s^*_j).$$

This is a “cross-player tremble restriction,” that is, a restriction on the relative probabilities of trembles by different players. Note that it that it, like the compatibility order, depends on the players’ preferences over distributions on $S$ but not on the particular utility representation used. This invariance property distinguishes player-compatible trembles from other models of stochastic behavior such as the stochastic terms in logit best responses.

**Lemma 1.** If $\sigma^o$ is an $\epsilon$-PCE and $(s^*_i|i) \succeq (s^*_j|j)$, then

$$\sigma^o_i(s^*_i) \geq \min \left[\sigma^o_j(s^*_j), 1 - \sum_{s'_i \neq s^*_i} \epsilon(s'_i|i) \right].$$

**Proof.** Suppose $\epsilon$ is player-compatible and let $\epsilon$-equilibrium $\sigma^o$ be given. For $(s^*_i|i) \succeq (s^*_j|j)$, suppose $\sigma^o_j(s^*_j) = \epsilon(s^*_j|j)$. Then we immediately have $\sigma^o_i(s^*_i) \geq \epsilon(s^*_i|i) \geq \epsilon(s^*_j|j) = \sigma^o_j(s^*_j)$, where the second inequality comes from $\epsilon$ being player compatible. On the other hand, suppose $\sigma^o_j(s^*_j) > \epsilon(s^*_j|j)$. Since $\sigma^o$ is an $\epsilon$-equilibrium, the fact that $j$ puts more than the minimum required weight on $s^*_j$ implies $s^*_j$ is at least a weak best response for $j$ against $(\sigma_{-ij}, \sigma^o_i)$, where both $\sigma_{-ij}$ and $\sigma^o_i$ are strictly mixed. The definition of $(s^*_i|i) \succeq (s^*_j|j)$ then implies that $s^*_i$ must be a strict best response for $i$ against (the strictly mixed) $(\sigma_{-ij}, \sigma^o_i)$, which has the same marginal on $S_{-ij}$ as $(\sigma_{-ij}, \sigma^o_i)$. In the $\epsilon$-equilibrium, $i$ must assign as much weight to $s^*_i$ as possible, so that $\sigma^o_i(s^*_i) = 1 - \sum_{s'_i \neq s^*_i} \epsilon(s'_i|i)$. Combining these two cases establishes the desired result. 

\[\square\]
While an $\epsilon$-equilibrium always exists provided $\epsilon$ is close enough to 0, these $\epsilon$-equilibria need not satisfy the conclusion of Lemma 1 when the tremble profile $\epsilon$ is not player compatible. We illustrate this in Example 7 of the Online Appendix.

As is usual for tremble-based equilibrium refinements, we now define PCE as the limit of a sequence of $\epsilon$-PCE where $\epsilon \to 0$.

**Definition 4.** A strategy profile $\sigma^*$ is a player-compatible equilibrium (PCE) if there exists a sequence of player-compatible tremble profiles $\epsilon^{(t)} \to 0$ and an associated sequence of strategy profiles $\sigma^{(t)}$, where each $\sigma^{(t)}$ is an $\epsilon^{(t)}$-PCE, such that $\sigma^{(t)} \to \sigma^*$.

The cross-player restrictions embodied in player-compatible trembles translate into analogous restrictions on PCE, which allows us to quickly rule out non-PCE equilibria in practice.

**Proposition 2.** For any PCE $\sigma^*$, player $k$, and strategy $\bar{s}_k$ such that $\sigma^*_k(\bar{s}_k) > 0$, there exists a sequence of strictly mixed strategy profiles $\sigma^{(t)}_{-k} \to \sigma^*_{-k}$ such that (i) for every pair $i, j \neq k$ with $(s^*_i|i) \succeq (s^*_j|j)$,

$$\liminf_{t \to \infty} \frac{\sigma^{(t)}_i(s^*_i)}{\sigma^{(t)}_j(s^*_j)} \geq 1;$$

and (ii) $\bar{s}_k$ is a best response for $k$ against every $\sigma^{(t)}_{-k}$ in the sequence.

That is, treating each $\sigma^{(t)}_{-k}$ as a strictly mixed approximation to $\sigma^*_{-k}$, in a PCE each player $k$ essentially best responds to opponent play that respects player compatibility.

This result follows from Lemma 1, which shows every $\epsilon$-PCE respects player compatibility up to the “adding up constraint” that probabilities on different actions must sum up to 1 and $i$ must place probability no smaller than $\epsilon(s^*_i|i)$ on actions $s'_i \neq s^*_i$. The “up to” qualification disappears in the $\epsilon^{(t)} \to 0$ limit because the required probabilities on $s'_i \neq s^*_i$ tend to 0.

**Proof.** By Lemma 1, for every $\epsilon^{(t)}$-PCE we get

$$\frac{\sigma^{(t)}_i(s^*_i)}{\sigma^{(t)}_j(s^*_j)} \geq \min \left[ \frac{\sigma^{(t)}_j(s^*_j)}{\sigma^{(t)}_j(s^*_j)}, \frac{1 - \sum_{s'_i \neq s^*_i} \epsilon^{(t)}(s'_i|i)}{\sigma^{(t)}_j(s^*_j)} \right] = \min \left[ 1, \frac{1 - \sum_{s'_i \neq s^*_i} \epsilon^{(t)}(s'_i|i)}{\sigma^{(t)}_j(s^*_j)} \right] \geq 1 - \sum_{s'_i \neq s^*_i} \epsilon^{(t)}(s'_i|i).$$

This says

$$\inf_{t \geq T} \frac{\sigma^{(t)}_i(s^*_i)}{\sigma^{(t)}_j(s^*_j)} \geq 1 - \sup_{t \geq T, s'_i \neq s^*_i} \sum \epsilon^{(t)}(s'_i|i).$$
For any sequence of trembles such that $\epsilon^{(t)} \to 0$,

$$\lim_{T \to \infty} \sup_{t \geq T} \sum_{s'_i \neq s^*_i} \epsilon^{(t)}(s'_i|s_i) = 0,$$

so

$$\liminf_{t \to \infty} \frac{\sigma_i^{(t)}(s^*_i)}{\sigma_j^{(t)}(s^*_j)} = \lim_{T \to \infty} \left\{ \inf_{t \geq T} \frac{\sigma_i^{(t)}(s^*_i)}{\sigma_j^{(t)}(s^*_j)} \right\} \geq 1.$$

This shows that if we fix a PCE $\sigma^*$ and consider a sequence of player-compatible trembles $\epsilon^{(t)}$ and $\epsilon^{(t)} - \text{PCE} \sigma^{(t)} \to \sigma^*$, then each $\sigma^{(t)}_{-k}$ satisfies $\liminf_{t \to \infty} \frac{\sigma_i^{(t)}(s^*_i)}{\sigma_j^{(t)}(s^*_j)} \geq 1$ whenever $i, j \neq k$ and $(s^*_i|i) \succeq (s^*_j|j)$. Furthermore, from $\sigma^*_k(\tilde{s}_k) > 0$ and $\sigma^{(t)}_k \to \sigma^*_k$, we know there is some $T_1 \in \mathbb{N}$ so that $\sigma^{(t)}_k(\tilde{s}_k) > \sigma^*_k(\tilde{s}_k)/2$ for all $t \geq T_1$. We may also find $T_2 \in \mathbb{N}$ so that $\epsilon^{(t)}(\tilde{s}_k|k) < \sigma^*_k(\tilde{s}_k)/2$ for all $t \geq T_2$, since $\epsilon^{(t)} \to 0$. So when $t \geq \max(T_1, T_2)$, $\sigma^{(t)}_k$ places strictly more than the required weight on $\tilde{s}_k$, so $\tilde{s}_k$ is at least a weak best response for $k$ against $\sigma^{(t)}_{-k}$. Now the subsequence of opponent play $(\sigma^{(t)}_{-k})_{t \geq \max(T_1, T_2)}$ satisfies the requirement of this lemma. $\square$

Since PCE is defined as the limit of $\epsilon-$equilibria for a restricted class of trembles, PCE form a subset of trembling-hand perfect equilibria; the next result shows this subset is not empty. It uses the fact that the tremble profiles with the same lower bound on the probability of each action satisfy the compatibility condition in any game.

**Theorem 1.** PCE exists in every finite strategic-form game.

**Proof.** Consider a sequence of tremble profiles with the same lower bound on the probability of each action, that is $\epsilon^{(t)}(s_i|i) = \epsilon^{(t)}$ for all $i$ and $s_i$, and with $\epsilon^{(t)}$ decreasing monotonically to 0 in $t$. Each of these tremble profiles is player-compatible (regardless of the compatibility structure $\succeq$) and there is some finite $T$ large enough that $t \geq T$ implies an $\epsilon^{(t)}$-equilibrium exists, and some subsequence of these $\epsilon^{(t)}$-equilibria converges since the space of strategy profiles is compact. By definition these $\epsilon^{(t)}$-equilibria are also $\epsilon^{(t)}$-PCE, which establishes existence of PCE. $\square$

**Remark 2.** The proof of Theorem 1 constructs a PCE using a sequence of uniform trembles converging to 0. In addition to the cross-player restrictions of the compatibility condition, these uniform trembles impose the same lower bound on the tremble probabilities for all strategies of each given player. By slightly modifying the proof to use tremble profiles that satisfy $\epsilon(s^*_i|i) = C\epsilon(s^*_j|j)$ whenever $(s^*_i|i) \succeq (s^*_j|j)$ for some fixed $C > 1$, we can establish the existence of another kind of tremble-based equilibrium, expressed as the limit of epsilon-equilibria where $i$ plays $s^*_i$ at least $C$ times as often as $j$ plays $s^*_j$ whenever $(s^*_i|i) \succeq (s^*_j|j)$ but not conversely. We view the uniform tremble profile and the “$C$-multiples” tremble profile
as convenient proof techniques for existence, not as a theoretical foundation of PCE. Indeed, neither sort of tremble profile seems likely to emerge endogenously from agents’ deliberate choices in any natural learning model. But as we show in Section 4, a variety of distinct learning models lead to other, more complicated patterns of experimentation that respect the compatibility structure and select PCE outcomes. This suggests PCE is a fairly weak solution concept that should be expected to apply broadly. For that reason it is perhaps surprising that PCE does have bite and empirical validity in some cases of interest, as we will discuss in Sections 3.

♦

3 Examples of PCE

In this section, we study examples of games where PCE rules out unintuitive Nash equilibria. We will also use these examples to distinguish PCE from existing refinements.

3.1 Bayesian Games

We introduce the setup of Bayesian games and show how they fit into the general setup of Section 2.

A Bayesian game consists of \( N \) players \( \{1, \ldots, N\} =: [N] \). Each player \( n \) has a finite type space \( \Theta_n \) and \( \lambda \in \Delta(\Theta) \) is the prior distribution over type profiles, \( \Theta := \times_n \Theta_n \). We assume each type of player \( n \) has the same set of feasible extensive-form strategies, which we denote \( S_n \). Let the utility function of \( n \) be \( u_n : S \times \Theta \to \mathbb{R} \) where \( S = \times_n S_n \) is the space of extensive-form strategy profiles.

To relate this to the general notation of Section 2, we adopt the “interim” interpretation of Bayesian games and consider each type as a separate player. This makes no difference for the set of Nash equilibria, but it does matter for PCE and for the learning foundations of equilibrium. In learning models, the interim interpretation corresponds to a setting where the type of each agent (or each participant in a laboratory experiment) is fixed once and for all, while the ex ante interpretation corresponds to a setting where the types are chosen at random each period.\(^6\)

Let \( \mathbb{I} \) be the set of all types across all Bayesian game players \( n \in [N] \), that is \( \mathbb{I} = \cup_{n \in [N]} \Theta_n \). The pure-strategy set of a player \( i = \theta_n \) is \( S_i = S_n \). The utility function of player \( i \) is given by

\[
U_i(s_i, s_{-i}) := \mathbb{E}_{\tilde{\theta}_{-n} \mid \theta_n} \left[ u_n \left( (s_{\theta_n}, s_{\tilde{\theta}_{-n}}), (\theta_n, \tilde{\theta}_{-n}) \right) \right],
\]

where the expectation is taken over \( \tilde{\theta}_{-n} \), the type profile of \( -n \) given that \( n \) has type \( \theta_n \).

\(^6\)See Dekel, Fudenberg, and Levine (2002) for a theoretical discussion of the differences between these settings, and Fudenberg and Vespa (2018) for a test of that theory in the lab.
As in Remark 1, two types of the same player are non-interacting because they are never simultaneously present. Indeed the RHS of the above expression does not contain \( s_{\theta_n}' \) for any \( \theta_n' \in \Theta_n \setminus \{\theta_n\} \), which means \( U_{\theta_n} \) does not depend on the strategies played by other types of the same Bayesian-game player \( n \).

Following the general notation, a mixed strategy of \( i = \theta_n \) is \( \sigma_i \in \Delta(S_{\theta_n}) = \Delta(S_n) \), but alternatively we also write \( \sigma_n(\cdot|\theta_n) \).

We now turn to a Bayesian game of simultaneous moves where PCE formalizes an intuitive restriction on “trembles” that is not captured by trembling-hand perfect equilibrium, proper equilibrium, extended proper equilibrium, or test-set equilibrium.

**Example 1.** There are two equally likely states of the world, \( \theta_I \) and \( \theta_{II} \). Player 1 (P1) knows the state, but Player 2 (P2) does not, and the two players move simultaneously. P1 has three possible actions: H, T, and Out, and P2 chooses between H and T. If P1 plays Out, then P1 gets 3 and P2 gets 0 regardless of P2’s action. If P1 plays H or T, then P2’s objective is to match P1’s play in state \( \theta_I \) but mismatch it in state \( \theta_{II} \). For P1, H is a better response than T if P2 plays H with high probability; otherwise, T is a better response than H. Here is the payoff matrix:

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
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<tbody>
<tr>
<td>H</td>
<td>10,11</td>
<td>-100,0</td>
</tr>
<tr>
<td>T</td>
<td>5,1</td>
<td>-5,10</td>
</tr>
<tr>
<td>Out</td>
<td>3,0</td>
<td>3,0</td>
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</tbody>
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<table>
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<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>9,1</td>
<td>-101,10</td>
</tr>
<tr>
<td>T</td>
<td>6,11</td>
<td>-4,0</td>
</tr>
<tr>
<td>Out</td>
<td>3,0</td>
<td>3,0</td>
</tr>
</tbody>
</table>

From the interim perspective, this is a 3-player game with \( I = \{\theta_I, \theta_{II}, P2\} \) consisting of the two types of P1 and the one type of P2.

Consider the trembling-hand perfect equilibrium \( \sigma \) where both types of P1 always play Out and P2 plays T, which is the limit of the \( \epsilon \)-equilibria where \( \theta_I \) and \( \theta_{II} \) play H, T, and Out with probabilities \( (\epsilon^2, \epsilon, 1-\epsilon-\epsilon^2) \) and \( (\epsilon, \epsilon^2, 1-\epsilon-\epsilon^2) \) respectively, under the minimum trembles \( \epsilon(H|\theta_I) = \epsilon(T|\theta_{II}) = \epsilon^2 \) and \( \epsilon(T|\theta_I) = \epsilon(H|\theta_{II}) = \epsilon \). In these \( \epsilon \)-equilibria, \( \theta_I \) is much more likely to play T than \( \theta_{II} \), while \( \theta_{II} \) is much more likely to play H than \( \theta_I \). Moreover, this strategy profile is an (interim) extended proper equilibrium.\(^7\)

\(^7\)For every given small \( \epsilon > 0 \), consider the strategy profile \( \sigma' \) where the 0 probabilities in \( \sigma \) are replaced by \( \sigma_1(H|\theta_I) = \epsilon^4 \), \( \sigma_1(T|\theta_I) = \epsilon^2 \), \( \sigma_1(H|\theta_{II}) = \epsilon^5 \), \( \sigma_1(T|\theta_{II}) = \epsilon^3 \), and \( \sigma_2(H) = \epsilon \). Let the utility scalings be \( \beta_{\theta_I} = 1 \), \( \beta_{\theta_{II}} = 2 \), and \( \beta_{P2} = 10^{-4} \). It is clear that when P2 assigns probability close to 1 to T, Out is the unique best response for each type of P1, and moreover for sufficiently small \( \epsilon > 0 \), the unique best response for P2 is T. Since the \( \sigma_1(T|\theta_I) \) tremble is more likely than any other trembles of P1 by an order of at least \( 1/\epsilon \), conditional on \( \text{P1 not playing Out} \), it is almost certain that P1 is \( \theta_I \) and played T, so T is a strict best response for P2. We can verify that given the utility scalings, more costly mistakes by any type are played no more than \( \epsilon \) times as often as less costly mistakes for any other type. Thus \( \sigma' \) is a \((\beta, \epsilon)\)-extended proper equilibrium, and so \( \sigma \) is extended proper.
Also, $\sigma$ is a test-set equilibrium. To see this, note that both $\theta_I$ and $\theta_{II}$ have strict incentives in the equilibrium, so their equilibrium strategies are trivially undominated in the test set. But these strict incentives for $\theta_I$ and $\theta_{II}$ imply that the test set for P2 is the singleton set consisting of only the equilibrium strategy profile, so P2’s strategy is also undominated in the test set.

However, it seems intuitive that $\theta_I$ should tremble to $H$ more than $\theta_{II}$ does, while $\theta_{II}$ should tremble to $T$ more than $\theta_I$ does, because regardless of P2’s play, $\theta_I$ gets 1 more than $\theta_{II}$ does when playing $H$ while $\theta_{II}$ gets 1 more than $\theta_I$ does when playing $T$. This intuition is formalized in the notion of type compatibility, as we have $\Theta \cup \{\theta\}$.

$\Theta \cup \{\theta\}$ does when playing $T\theta$.

With each sender type viewed as a different player, this game has $|\Theta| + 1$ players, $I = \Theta \cup \{2\}$, where the strategy set of each sender type $\theta$ is $S_\theta = S$ while the strategy set of the
receiver is $S_2 = A^S$, the set of signal-contingent plans. So a mixed strategy of $\theta$ is a possibly mixed signal choice $\sigma_1(\cdot|\theta) \in \Delta(S)$, while a mixed strategy $\sigma_2 \in \Delta(A^S)$ of the receiver is a mixed plan about how to respond to each signal.

In a signaling game, it is clear that for $s^* \in S$ and for two types $\theta, \theta'$, $(s^*|\theta) \succeq (s^*|\theta')$ if and only if for every strictly mixed receiver play $\sigma_2$,

$$u_1(s^*, \sigma_2; \theta) \geq \max_{s' \neq s^*} u_1(s', \sigma_2; \theta)$$

implies

$$u_1(s^*, \sigma_2; \theta') > \max_{s' \neq s^*} u_1(s', \sigma_2; \theta'),$$

which is the definition of compatibility used in Fudenberg and He (2018).

**Proposition 3.** In a signaling game, every PCE $\sigma^*$ is a Nash equilibrium satisfying the compatibility criterion of Fudenberg and He (2018).

*Proof.* See Appendix A.2. \qed

This proposition in particular implies that in the beer-quiche game of Cho and Kreps (1987), the quiche-pooling equilibrium is not a PCE, as it does not satisfy the compatibility criterion.

### 3.2 Restaurant Game

Here we give an example of a complete-information game where PCE differs from other solution concepts.

**Example 2.** There are three players in the game: a food critic (P1), a regular diner (P2), and a restaurant (P3). Simultaneously, the critic and the diner decide whether to go eat at the restaurant (In) or not (Out). A customer choosing Out always gets 0 payoff. If both customers choose Out, the game ends immediately and the restaurant also gets 0 payoff. If at least one customer shows up at the restaurant, then the restaurant must decide whether to produce low-quality (L) or high-quality (H) food. Choosing L yields a profit of +2 per customer while choosing H yields a profit of +1 per customer. In addition, if the food critic is present, she will write a review based on food quality, which affects the restaurant’s payoff by $\pm 2$. The restaurant cannot distinguish between the two kinds of customers and must serve the same quality of food to all patrons. Finally, each customer gets a payoff of $x < -1$ from

---

8 Type compatibility requires testing against all opponent strategies and not just that are strictly mixed, so it is possible that $(s^*|\theta) \succeq (s^*|\theta')$ but $\theta$ is not more type-compatible than $\theta'$ with $s^*$. However, if $\theta$ is more type-compatible than $\theta'$ with $s^*$ then $(s^*|\theta) \succeq (s^*|\theta')$. 

---
consuming low-quality food and a payoff of \( y > 0 \) from consuming high-quality food, while the critic gets an additional +1 payoff from going to the restaurant and writing a review (regardless of food quality). This situation is depicted in the game tree below.

The strategy profile \((\text{Out1, Out2, L})\) is an extended proper equilibrium, sustained by the restaurant’s belief that when it gets to play, it is far more likely that the diner deviated to patronizing the restaurant than the critic.\(^9\) However, it is easy to verify that \((\text{In1}|\text{Critic}) \succ (\text{In2}|\text{Diner})\), and whenever \(\sigma_1^{(t)}(\text{In1})/\sigma_2^{(t)}(\text{In2}) > \frac{1}{3}\), the restaurant strictly prefers \(H\) over \(L\). Thus by Proposition 2, there is no PCE where the restaurant plays \(L\) with positive probability.

\(\diamondsuit\) Remark 3. In the restaurant game, the critic and the diner are non-interacting players in the sense of Remark 1. Suppose we slightly modify the payoffs of these two customers so that each suffers a congestion cost of \(-0.5\) if they both show up at the restaurant. The new payoffs are given in the figure below.

\(^9\)It is easy to see that by scaling the critic’s payoff by a large positive constant, it can be “more costly” for the critic to deviate to \(\text{In1}\) than for the diner to deviate to \(\text{In2}\), so that we can sustain this belief in an extended proper equilibrium.
Though the critic and the diner are no longer non-interacting players, the analysis of Example 2 continues to apply. For any $\sigma_{-2}$ of strictly mixed correlated play by the critic and the restaurant that makes the diner indifferent between $\text{In}_2$ and $\text{Out}_2$, we must have $U_1(\text{In}_1, \sigma_{-1}) \geq 0.5$ for any $\sigma_{-1}$ that agrees with $\sigma_{-2}$ in terms of the restaurant’s play. This is because the critic’s utility is minimized when the diner chooses $\text{In}_2$ with probability 1, but even then the critic gets 0.5 higher utility from going to a crowded restaurant than the diner gets from going to an empty restaurant, holding fixed food quality at the restaurant. This shows $(\text{In}_1|\text{Critic}) \succsim (\text{In}_2|\text{Diner})$ even with the modified payoffs.

3.3 Link-formation game

Example 3. There are 4 players in the game, split into two sides: North and South. The players are named North-1, North-2, South-1, and South-2, abbreviated as N1, N2, S1, and S2 respectively.

These players engage in a strategic link-formation game. Each player simultaneously takes an action: either Inactive or Active. An Inactive player forms no links. An Active player forms a link with every Active player on the opposite side. (Two players on the same side cannot form links.) For example, suppose N1 plays Active, N2 plays Active, S1 plays Inactive, and S2 plays Active. Then N1 creates a link to S2, N2 creates a link to S2, S1 creates no links, and S2 creates links to both N1 and N2.
Each player \( i \) is characterized by two parameters: cost \((c_i)\) and quality \((q_i)\). Cost refers to the private cost that a player pays for each link she creates. Quality refers to the benefit that a player provides to others when they link to her. A player who forms no links gets a payoff of 0. In the above example, the payoff to North-1 is \( q_{S2} - c_{N1} \) and the payoff to South-2 is \( (q_{N1} - c_{S2}) + (q_{N2} - c_{S2}) \).

We consider two versions of this game, shown below. In the anti-monotonic version on the left, players with a higher cost also have a lower quality. In the co-monotonic version on the right, players with a higher cost also have a higher quality. There are two pure-strategy Nash outcomes for each version: all links form or no links form. Standard refinements (extended proper equilibrium, proper equilibrium, trembling-hand perfect equilibrium, \( p \)-dominance, Pareto efficiency, strategic stability, pairwise stability) all make matching predictions in both games. However, “all links form” is the unique PCE outcome in the anti-monotonic case, while both “all links” and “no links” are PCE outcomes under co-monotonicity.

<table>
<thead>
<tr>
<th>player</th>
<th>cost</th>
<th>quality</th>
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<tbody>
<tr>
<td>North-1</td>
<td>14</td>
<td>30</td>
</tr>
<tr>
<td>North-2</td>
<td>19</td>
<td>10</td>
</tr>
<tr>
<td>South-1</td>
<td>14</td>
<td>30</td>
</tr>
<tr>
<td>South-2</td>
<td>19</td>
<td>10</td>
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<td>14</td>
<td>10</td>
</tr>
<tr>
<td>South-2</td>
<td>19</td>
<td>30</td>
</tr>
</tbody>
</table>

Proposition 4. Each of the following refinements selects the same subset of pure Nash equilibria when applied to the anti-monotonic and co-monotonic versions of the link-formation game: extended proper equilibrium, proper equilibrium, trembling-hand perfect equilibrium, \( p \)-dominance, Pareto efficiency, strategic stability, and pairwise stability. Moreover the link-formation game is not a potential game.

Proof. See Appendix A.3.

As we report in Section 5, for a number of natural (and not necessarily optimal) models of individual learning behavior, the long-run learning outcomes in these two versions of the link-formation game differs in the way that PCE suggests.
4 Learning Foundations

The key distinction between PCE and other tremble-based solution concepts is that PCE imposes cross-player restrictions on tremble probabilities in a way that is invariant to the utility representation. In a game with \((s^*_i|i) \succeq (s^*_j|j)\), player-compatible trembles satisfy \(\epsilon(s^*_i|i) \geq \epsilon(s^*_j|j)\), so that each \(\epsilon\)-equilibrium under a player-compatible tremble profile has \(i\) playing \(s^*_i\) more frequently than \(j\) playing \(s^*_j\) (up to the adding up constraint).

We now provide a learning foundation for this cross-player restriction on trembles. We study the learning problem of an agent who plays the same stage game each period in the role of player \(i\), against a fixed but unknown aggregate distribution of play by the agents in the other player roles. The agent faces a trade-off between gathering information (exploration) and maximizing her myopic payoff given her current beliefs (exploitation), because her own play influences the information she observes about the play of the others.

We define a class of “factorable games” where this trade-off is clear, and then compare the induced behavior of agents in roles \(i\) and \(j\) when \((s^*_i|i) \succeq (s^*_j|j)\). The definition of player compatibility says \(i\) finds \(s^*_i\) strictly optimal whenever \(j\) finds \(s^*_j\) weakly optimal provided that they have the same beliefs about the play of others. However, this does not immediately imply that their learning behavior should be similarly comparable when facing the same opponents’ play, because even if agents in roles \(i\) and \(j\) start with the same prior belief about the play of others in period 1, their beliefs may diverge as they undertake different experiments and thus get different information about the play of others. Nevertheless, in Section 4.3 we embed the agents in a steady-state learning model, and prove that under three different models of individual behavior, an agent in role \(i\) on average plays \(s^*_i\) more frequently than an agent in role \(j\) plays \(s^*_j\).

4.1 Factorable Games and Isomorphic Factoring

Fix an extensive-form game \(\Gamma\) as the stage game. The collection of information sets of player \(i\) is written as \(\mathcal{I}_i\), with generic element \(I\). At each \(I \in \mathcal{I}_i\), player \(i\) chooses an action \(a_I\) from the set of possible actions \(A_I\). So an extensive-form pure strategy of \(i\) specifies an action to play at every information set \(I \in \mathcal{I}_i\). We denote by \(S_i\) the set of all such strategies.

Let \(Z\) represent the set of terminal nodes of \(\Gamma\). For each player \(i\), there is a terminal node partition \(\Upsilon_i = (\Upsilon^1_i, ..., \Upsilon^L_i)\) over \(Z\). When the game is played, agents in the role of \(i\) are only told which element \(\Upsilon_i \in \Upsilon_i\) is reached, as in Fudenberg and Kamada (2015, 2018). So, the information that agents observe at the end of a match may or may not reveal exactly what the other players did. We assume a player always knows her own payoff, so that any two terminal nodes in the same partition element \(\Upsilon_i\) give the same utility to \(i\). Write \(\Upsilon_i(s_i, (a_I)_{I \in \mathcal{I}_i}) \in \Upsilon_i\) for the partition element reached when \(i\) uses strategy \(s_i\) and
the profile of actions \((a_I)_{I \in \mathcal{I}_{-i}}\) is played at opponents’ information sets.

We now introduce the idea of “factorability,” which we will use to focus on a subclass of games that exhibit a sharp trade-off between exploration and exploitation.

**Definition 5.** The game \(\Gamma\) with terminal node partitions \(\Upsilon\) is **factorable** for \(i\) if for each \(s_i \in S_i\) there is a (possibly empty) subset of the of opponents’ information sets, \(F_i[s_i] \subseteq \mathcal{I}_{-i}\) such that: (1) for each \(s_i\) and each pair of opponents’ action profiles \((a_I)_{I \in \mathcal{I}_{-i}}\), \((a_I')_{I \in \mathcal{I}_{-i}}\), \(\Upsilon_i(s_i, (a_I)_{I \in \mathcal{I}_{-i}}) = \Upsilon_i(s_i, (a_I')_{I \in \mathcal{I}_{-i}})\) if and only if \((a_I)_{I \in F_i[s_i]} = (a_I')_{I \in F_i[s_i]}\); (2) \(F_i[s_i] \cap F_i[s_i'] = \emptyset\) for \(s_i \neq s_i'\).

Factorability is a restriction on the terminal node partition \(\Upsilon_i\). Since we require that \(i\)’s payoff is measurable with respect to \(\Upsilon_i\), factorability also indirectly imposes restrictions on the payoff structure of the game tree \(\Gamma\). When \((\Gamma, \Upsilon)\) is factorable for \(i\), we refer to \(F_i[s_i]\) as the \(s_i\)-**relevant information sets**. In a general extensive-form game with terminal node partitions, the partition \(i\) reaches after playing strategy \(s_i\) can depend on the profile of opponent actions at all opponent information sets in the game tree. Condition (1) of factorability says that \(\Upsilon_i(s_i, (a_I)_{I \in \mathcal{I}_{-i}})\) is a one-to-one function of the components \((a_I)_{I \in F_i[s_i]}\), but not a function of the other components. That is, the partition reached only depends on opponents’ moves at the \(s_i\)-relevant information sets, and this dependence is one-to-one. Since \(i\)’s payoff is measurable with respect to \(\Upsilon_i\), condition (1) requires that \(i\)’s utility when playing \(s_i\) only depends on opponents’ actions at the \(s_i\)-relevant information sets. Condition (2) implies that \(i\) does not learn about the payoff consequence of \(s_i'\) through playing a different strategy \(s_i\) when \(i\)’s prior is that opponents’ play on different information sets are independent.

Factorable games feature a very clear exploration-exploitation trade-off: Choosing strategy \(s_i\) reveals all the payoff-relevant opponent actions for \(s_i\) in that period, and provides no information about the payoff consequences of any other strategy \(s_i' \neq s_i\); since there is no intersection between the \(s_i\)-relevant information sets and the \(s_i'\)-relevant ones.

Note that if \(F_i[s_i]\) is empty, then \(s_i\) is a kind of “opt out” action in a factorable game. That is, \(i\) gets the same utility regardless of anyone else’s play anywhere in the game tree after choosing \(s_i\), but \(i\) also gets no information about the payoff consequences of any of their strategies from choosing \(s_i\).

We now augment examples from Section 3 with terminal node partitions to illustrate factorable games.

**Example 4.** Consider the restaurant game from Example 2 under the terminal node partition in which for each of Critic (P1) and Diner (P2), playing \(\text{In}\) reveals the resulting terminal node, while playing \(\text{Out}\) gives no information about either opponent’s play. Formally, this means \(\Upsilon_1\) consists of five partition elements: the four terminal vertices reachable after \(\text{In1}\).
each belongs to a partition element of size one, while the remaining three terminal vertices in the game tree make up the fifth partition element. \( \Upsilon_2 \) consists of five analogously-defined partition elements. Let \( F_i[\text{In } i] \) consist of the two information sets of \(-i\) and let \( F_i[\text{Out } i] \) be the empty set for each \( i \in \{1, 2\} \). In the diagram below, the three blue colored terminal nodes are in the same partition element for the Critic, while the three red colored terminal nodes are in the same partition element for the Diner.

The restaurant game with this terminal node partition structure is factorable for the Critic and the Diner: Since \( i \)'s payoff is the same regardless of the others’ play after choosing \( \text{Out } i \), Condition (1) is satisfied. Also, \( F_i[\text{In } i] \cap F_i[\text{Out } i] = \emptyset \) so Condition (2) is satisfied as well. Note however that the game is not factorable with the discrete terminal node partition, as then when the Critic plays \( \text{Out} 1 \), she learns the restaurant’s food quality from the terminal node if the the Diner chooses \( \text{In} 2 \), which is payoff-relevant information for her other strategy \( \text{In} 1 \). Factorability rules out this kind of free information, so that when we analyze the non-equilibrium learning problem we know that each agent can only learn a strategy’s payoff consequences by playing it herself.

The next example concerns the link-formation game.

**Example 5.** Consider the link-formation game from Example 3. Since the game involves 4 players simultaneously making binary choices, there will be 16 terminal vertices in the extensive-form representation of the game, one for each possible action profile. For player \( N_1 \), we define a terminal node partition consisting of 9 elements: for each \( s_{N_2}, s_{S_1}, s_{S_2} \in \{\text{Active, Inactive}\} \), there is a partition element \( \{(\text{Active, } s_{N_2}, s_{S_1}, s_{S_2})\} \) consisting of the single terminal node where \( N_1 \) plays \( \text{Active} \), \( N_2 \) plays \( s_{N_2} \), \( S_1 \) plays \( s_{S_1} \), and \( S_2 \) plays \( s_{S_2} \). Finally, there is a 9th partition element containing the remaining 8 possible action profiles of the game, all involving \( N_1 \) playing \( \text{Inactive} \). The idea is that \( N_1 \) observes the play of \( N_2 \),
S1, and S2 by playing Active, while N1 observes nothing by playing Inactive. Analogously, we can define the partition structure for all other players in the game as well. It is now easy to see that this partition structure is factorable for each player $i$, where $F_i[\text{Active}]$ consists of the information sets of the other three agents and $F_i[\text{Inactive}]$ is empty. This partition structure is factorable because each player gets a sure payoff of 0 after choosing Inactive.

More generally, $(\Gamma, \Upsilon)$ is factorable for $i$ whenever it is a binary participation game for $i$.

**Definition 6.** $(\Gamma, \Upsilon)$ is a binary participation game for $i$ if the following are satisfied.

1. $i$ has a unique information set with two actions, without loss labeled In and Out.
2. All complete paths in the game tree $\Gamma$ (i.e. paths from the root to a terminal vertex) pass through $i$’s information set.
3. All complete paths where $i$ plays In pass through the same information sets.
4. Terminal vertices associated with $i$ playing Out all have the same payoff for $i$, and all belong to the same partition element $\Upsilon_i^{\text{Out}} \in \Upsilon_i$.
5. Terminal vertices associated with $i$ playing In are discretely partitioned by $\Upsilon_i$, that is for each such terminal vertex $Z$ there exists a $\Upsilon_i^Z \in \Upsilon_i$ with $\Upsilon_i^Z = \{Z\}$.

If $(\Gamma, \Upsilon)$ is a binary participation game for $i$, then let $F_i[\text{In}]$ be the common collection of $-i$ information sets encountered in all complete paths passing through $i$ playing In. Let $F_i[\text{Out}]$ be the empty set. We see that $(\Gamma, \Upsilon)$ is factorable for $i$. Clearly $F_i[\text{In}] \cap F_i[\text{Out}] = \emptyset$, so Condition (2) of factorability is satisfied. When $i$ chooses the strategy In, the tree structure of $\Gamma$ implies different profiles of play on $F_i[\text{In}]$ must lead to different terminal nodes, and the discrete partitioning of such terminal nodes means Condition (1) of factorability is satisfied for strategy In. When $i$ plays Out, $i$ gets the same payoff and same observation regardless of the others’ play, so Condition (1) of factorability is satisfied for strategy Out.

The restaurant game with Example 4’s partition structure is a binary participation game for the critic and the diner. (Note that if both customers play Out, then the restaurant’s information set is never reached. However, the definition of binary participation game only requires that all complete paths passing through $i$ playing In run through the same collection of information sets.) The link-formation game with Example 5’s partition structure is a binary participation game for every player.

**Example 6.** To give a different class of examples of factorable games, consider a game of signaling to one or more audiences, under the discrete terminal node partition. To be precise, Nature moves first and chooses a type for the sender, drawn according to $\lambda \in \Delta(\Theta)$, where
Θ is a finite set. The sender then chooses a signal \( s \in S \), observed by all receivers \( r_1, \ldots, r_n \). Each receiver then simultaneously chooses an action. The profile of receiver actions, together with the sender’s type and signal, determine payoffs for all players. Viewing different types of senders as different players, this game is factorable for all sender types. This is because for each type \( i \) we have \( F_i[s] \) is the set of \( n_r \) information sets by the receivers after seeing signal \( s \).

Before we turn to compare the learning behavior of agents \( i \) and \( j \), we must deal with one final issue. To make sensible comparisons between strategies \( s_i^* \) and \( s_j^* \) of two different players \( i \neq j \) in a learning setting, we must make assumptions on their informational value about the play of others: namely, the information \( i \) gets from choosing \( s_i^* \) must be sufficiently similar to the information that \( j \) gets from choosing \( s_j^* \).

**Definition 7.** When \((\Gamma, \Upsilon)\) is factorable for both \( i \) and \( j \), the factoring is *isomorphic* for \( i \) and \( j \) if there exists a bijection \( \varphi : S_i \to S_j \) such that \( F_i[s_i] \cap I_{-ij} = F_j[\varphi(s_i)] \cap I_{-ij} \) for every \( s_i \in S_i \).

This says the \( s_i \)-relevant information sets (for \( i \)) are the same as the \( \varphi(s_i) \)-relevant information sets (for \( j \)), insofar as the actions of \(-ij\) are concerned. For example, the restaurant game with the terminal partition given in Example 4 is isomorphically factorable for the critic and the diner (under the isomorphism \( \varphi(\text{In1}) = \text{In2}, \varphi(\text{Out1}) = \text{Out2} \)) because \( F_1[\text{In1}] = F_2[\text{In2}] = \) the singleton set containing the unique information set of the restaurant. As another example, all signaling games (with possibly many receivers as in Example 6) are isomorphically factorable for the different types of the sender. Similarly, the link-information game is isomorphically factorable for pairs \((N1, N2)\), and \((S1, S2)\), but note that it is not isomorphically factorable for \((N1, S1)\).

### 4.2 Three Learning Models

In all of learning models we consider, an agent is born into player role \( i \) and maintains this role throughout her lifetime. She has a geometrically distributed lifetime with \( 0 \leq \gamma < 1 \) probability of survival between periods. Each period, the agent chooses a strategy \( s_i \in S_i \), observes the realized partition \( \Upsilon_i \in \Upsilon_i \), and collects the associated payoffs. Then, with probability \( \gamma \), she continues into the next period and plays the stage game again. With complementary probability, she exits the system. Thus each period the agent observes an element of \( S_i \times \Upsilon_i \), consisting of her own strategy that period and the resulting partition element. The set of possible histories at the end of period \( t \) is therefore \((S_i \times \Upsilon_i)^t\).

**Definition 8.** The set of all finite histories of all lengths for \( i \) is \( Y_i := \cup_{t \geq 0} (S_i \times \Upsilon_i)^t \). For a history \( y_i \in Y_i \) and \( s_i \in S_i \), the subhistory \( y_{i,s_i} \) is the (possibly empty) subsequence of \( y_i \) where the agent played \( s_i \).
When \((\Gamma, \mathcal{Y})\) is factorable for \(i\), the one-to-one mapping from opponent’s actions on the \(s_i\)-relevant information sets to \(\mathcal{Y}_i\) required by Condition (1) of Definition 5 implies that we may think of a one-period history in which \(i\) plays \(s_i\) as an element in \(\{s_i\} \times (\times_{I \in F_i[s_i]} A_I)\) instead of an element in \(\{s_i\} \times \mathcal{Y}_i\) as in Definition 8. This convention will make it easier to compare histories belonging to different players.

**Notation 1.** A history \(y_i\) will also refer to an element of \(\bigcup_{t \geq 0} \left( \bigcup_{s_i \in S_i} \left\{ s_i \right\} \times (\times_{I \in F_i[s_i]} A_I) \right) t\).

**Definition 9.** A learning rule \(r_i : Y_i \to \Delta(S_i)\) specifies a (possibly mixed) strategy in the game \(\Gamma\) after each history.

We now consider three different specifications of the agents’ policy functions when the information structure is factorable.

### 4.2.1 Expected Utility Maximization and the Gittins index

Suppose the learning agents are Bayesians who maximize expected discounted utility, and that the prior beliefs of agents in the role of player \(i\) are represented by a density \(g_i\) on \(\times_{I \in \mathcal{I}_{-i}} \Delta(A_I)\) that can be written as the product of full-support marginal densities on \(\Delta(A_I)\) across different \(I \in \mathcal{I}_{-i}\). Thus, the agent believes that play at each opponent information set \(I\) is generated from some mixture in \(\Delta(A_I)\) each period, with the moves at different information sets generated independently, whether the information sets belong to the same player or to different ones. Furthermore, the agent’s beliefs about these mixtures are independent.\(^{10}\) In a signaling game, for example, this means that the senders only update their beliefs about the receiver response to a given signals \(s\) based on the responses received to that signal, and that their beliefs about this response do not depend on the responses they have observed to other signals \(s' \neq s\).

In addition to the survival chance \(0 \leq \gamma < 1\) between periods, agents further discount future payoff according to their patience \(0 \leq \delta < 1\), so they have an overall effective discount factor of \(0 \leq \delta \gamma < 1\).

When \(\Gamma\) is factorable for \(i\), the dynamic optimization problem facing \(i\) is analogous to a multi-armed bandit problem with independent arms, since \(i\) cannot learn about the payoff consequences of strategy \(s'_i \neq s_i\) through playing \(s_i\). The optimal policy \(r_{G,i}\) therefore involves playing the strategy \(s_i\) with the highest Gittins index after each history \(y_i\). More precisely, \(r_{G,i}\) should be written as \(r^{g_i,\delta \gamma}_{G,i}\) to emphasize its dependence on the prior \(g_i\) and the effective discount factor \(\delta \gamma\).

\(^{10}\)As Fudenberg and Kreps (1993) point out, an agent who believes two opponents are randomizing independently may nevertheless have subjective correlation in her uncertainty about the randomizing probabilities of these opponents.
4.2.2 Bayesian upper-confidence bound

The Bayesian upper confidence bound (Bayes-UCB) procedure was first proposed by Kaufmann, Cappé, and Garivier (2012) as a computationally tractable algorithm for dealing with the exploration-exploitation trade-off in bandit problems. The agent starts with a prior belief about the payoff consequence of each arm. In each period $t$, Bayes-UCB computes the $q(t)$-quantile of the posterior payoff distribution of each arm based on the belief at that time, then pulls the arm with the highest such quantile value. Thus the Bayes-UCB procedure is a form of index policy.\footnote{Both Kaufmann, Cappé, and Garivier (2012) and Kaufmann (2018) show that the appropriate (but not problem-specific) specification of the function $q$ makes it asymptotically optimal for various reward structures.}

We adopt a variant of Bayes-UCB that makes use of the game tree structure instead of treating each strategy $s_i \in S_i$ as an arm of an abstract bandit problem. By factorability, player $i$’s payoff following $s_i$ depends on the profile of actions at the $s_i$-relevant information sets $F_i[s_i]$, so we consider an agent who ranks opponents’ mixed actions from least favorable to most favorable for her at each $s_i$-relevant information set $I$, then uses her current beliefs to compute her expected payoff from $s_i$ when opponents use the “$q(t)$ quantile mixed action” at each $I$. For such a ranking to be well-defined, opponents’ actions in different $I' \neq I'' \in F_i[s_i]$ must have no interaction effect on $i$’s payoff. To make this precise, we restrict to factorable games that satisfy an additive separability assumption.

**Definition 10.** Game $(\Gamma, \Upsilon)$ is additively separable for $i$ if there is a collection of auxiliary functions $u_{s_i,I} : A_I \to \mathbb{R}$ such that $U_i(s_i, (a_I)_I \in F_i[s_i]) = \sum_{I \in F_i[s_i]} u_{s_i,I}(a_I)$.

Games which are not additively separable for $i$ are ones where opponents’ actions on $F_i[s_i]$ interact in some way to determine $i$’s payoff following $s_i$. Additive separability is trivially satisfied whenever $|F_i[s_i]| \leq 1$ for each $s_i$, so that there is at most one $s_i$-relevant information set for each strategy $s_i$ of $i$. It is also satisfied in the link-formation game in Example 5 even though here $|F_i[\text{Active } i]| = 2$, as each agent computes her payoff by summing her linking costs/benefits with respect to each opponent on the other side. Finally, this condition is also satisfied in the restaurant game in Example 4, for even though $F_i[\text{In } i]$ contains two information sets, only play by the Restaurant affects $i$’s payoff, so we can let $u_{\text{In}_i,I}$ for $I$ corresponding to the choice of the other customer be identically 0.

We consider a procedure that, for each $I \in F_i[s_i]$, computes the $q$-th quantile of $u_{s_i,I}(a_I)$ under $i$’s belief about $-i$’s play — the “upper confidence bound” of the contribution from opponent’s play on $I$ towards $i$’s utility — then sums these quantiles to return an index of the strategy $s_i$. Performing this procedure for each $s_i \in S_i$, the agent then chooses the strategy with the highest total.
Definition 11. Let prior $g_i$ and quantile-choice function $q : \mathbb{N} \rightarrow [0,1]$ be given for $i$. For each $s_i \in S_i$ and $y_{i,I}$ a subhistory of play on $I \in F_i[s_i]$, let $\tilde{a}_I(y_{i,I})$ be the $A_I$-valued random variable distributed according to the posterior belief $g_i(\cdot|y_{i,I})$ about $-i$’s play on $I$, and let $\tilde{u}_{s_i,I}(y_{i,I}) := u_{s_i,I}(\tilde{a}_I(y_{i,I}))$. For each $x \in [0,1]$ and real-valued random variable $\tilde{u}$, let $Q(\tilde{u};x)$ be the $x$-th quantile of $\tilde{u}$. The Bayes-UCB index for $s_i$ after history $y_i$ (relative to $g_i$ and $q$) is

$$\sum_{I \in F_i[s_i]} Q(\tilde{u}_{s_i,I}(y_{i,I}); q(#(s_i|y_i)))$$

where $#(s_i|y_i)$ is the number of times $s_i$ has been used in history $y_i$.

The Bayes-UCB policy $\sigma_{UCB,i}$ prescribes choosing the strategy with the highest Bayes-UCB index after every history. More precisely, the policy should be denoted as $\sigma_{UCB,i}^{g_i,q}$ to emphasize its dependence on prior $g_i$ and quantile-choice function $q$.

This procedure embodies a kind of wishful thinking for $q \geq 0.5$. The agent optimistically evaluates the payoff consequence of each $s_i$ under the assessment that opponents will play a favorable response to $s_i$ at each of the $s_i$-relevant information sets, where greater $q$ corresponds to greater optimism in this evaluation procedure. Indeed, if $q$ approaches $1$ for every $i$, the Bayes-UCB procedure approaches picking the action with the highest potential payoff.

If $F_i[s_i]$ consists of only a single information set of for every $s_i$, then the procedure we define is the standard Bayes-UCB policy. In general, our procedure differs from the usual Bayes-UCB procedure, which would instead compute

$$Q \left( \sum_{I \in F_i[s_i]} \tilde{u}_{s_i,I}(y_{i,I}); q(#(s_i|y_i)) \right).$$

Instead, our procedure computes the sum of the quantiles, which is easier than computing the quantile of the sum, a calculation that requires taking the convolution of the associated distributions.

We use this simpler procedure for the same reason that quantiles are added in recent extensions of non-Bayesian UCB algorithms to combinatorial bandits problems. A combinatorial bandit consist of a set of basic arms, each with an unknown distribution of rewards, together with a collection of subsets of basic arms called super arms. Each period, the agent must choose a super arm, which results in pulling all of the basic arms in that subset and obtaining a utility based on the outcomes of these pulls. In many such problems, the computational costs of directly analyzing the super arms are too high. Recently, Gai, Krishnamachari, and Jain (2012) and Chen, Wang, and Yuan (2013), among others, have studied algorithms that separately compute an index\textsuperscript{12} for each basic arm, then choosing

\textsuperscript{12}The non-Bayesian UCB index of a basic arm is an “optimistic” estimate of its mean reward that combines
the super arm maximizing the expected one-period payoff treating the indices as the true mean rewards of the basic arms.\textsuperscript{13} This amounts to assigning an index to each super arm by adding up the indices of its associated basic arms, when the utility from a super arm is the sum of the rewards generated by its basic arms.

To translate into our language, the learning problem facing each agent is a combinatorial bandit problem, where each basic arm corresponds to a $-i$ information set and the super arms are identified with strategies $s_i \in S_i$. The subset of basic arms in $s_i$ are the $s_i$-relevant information sets, $F_i[s_i]$, and adding up the $q$–th quantiles of utility contributions from different $I \in F_i[s_i]$ is more tractable than applying the usual Bayes-UCB directly to each strategy $s_i$, which makes the heuristic we study more cognitively plausible.\textsuperscript{14}

### 4.2.3 Thompson sampling

First proposed by Thompson (1933), Thompson sampling (TS) is a probability-matching random heuristic for general active-learning problems. Faced with a set of strategies $s_i \in S_i$, each leading to an expected payoff that depends on an unknown state of the world, TS prescribes drawing a simulated state from the belief distribution on states, then playing the myopically optimal action in the simulated state. Put another way, TS uses a mixed strategy where each pure strategy is played with the probability that it is in fact the objectively optimal one, under the current belief about the true state. Following Chapelle and Li (2011)’s recent demonstration of TS’s performance, the heuristic has been enjoying a renaissance in the computer science community. It has been applied to dynamic learning problems where calculating the optimal policy is computationally infeasible, including Internet display advertising (Schwartz, Bradlow, and Fader, 2017; Agarwal, Long, Traupman, Xin, and Zhang, 2014).

In our setting, each “state” is an opponent behavior strategy profile, i.e. some $\alpha_{-i} \in \times_{I \in \mathcal{I}_{-i}} \Delta(A_I)$ specifying a mixture over actions at every $-i$ information set in the game tree. A prior distribution $g_i$ over $\times_{I \in \mathcal{I}_{-i}} \Delta(A_I)$ gives $i$’s initial belief. After each history $y_i$, a learning agent in the role of $i$ draws a behavior strategy $\alpha_{-i}$ from the posterior distribution $g_i(\cdot | y_i)$, and then plays a myopic best response to this draw. Equivalently, her learning rule

\begin{itemize}
  \item[\textsuperscript{13}] Kveton, Wen, Ashkan, and Szepesvari (2015) have established tight $O(\sqrt{n \log n})$ regret bounds for this kind of algorithm across $n$ periods.
  \item[\textsuperscript{14}] Establishing performance bounds for this index-summing version of Bayes-UCB in combinatorial bandits seems to be an open problem.
\end{itemize}
 specified a mixed strategy \( r_{TS,i}(y_i) \in \Delta(S_i) \) after history \( y_i \), which satisfies

\[
r_{TS,i}(y_i)(s_i) = g_i \left( \alpha_{-i} \in \times_{I \neq i} \Delta(A_I) : s_i \in \arg \max_{s'_i \in S_i} U_i(s'_i, \alpha_{-i}) \right) | y_i \).
\]

### 4.3 Aggregate Responses Respect Player Compatibility

#### 4.3.1 Aggregate Responses

The next definition lets us aggregate a learning agent’s stochastic behavior across different periods into a measure of the average lifetime that the agent spends playing each strategy.

Fix a learning rule \( r_i \) for agent \( i \) and a mixed strategy \( \sigma_{-i} \in \times_{k \neq i} \Delta(S_k) \) for \( -i \). Suppose in every period, \( i \) chooses a strategy by applying \( r_i \) to her history so far, while play of her opponents is drawn from \( \sigma_{-i} \), independently across periods. This environment generates a stochastic process \( X^t_i \) describing \( i \)’s strategy in period \( t \), where the randomness comes from random realizations of \( -i \)’s play drawn from \( \sigma_{-i} \) each period.

**Definition 12.** Let \( X^t_i \) be the \( S_i \)-valued random variable representing \( i \)’s play in period \( t \).

The aggregate response of \( i \) against \( \sigma_{-i} \) under learning rule \( r_i \) is \( \phi_i(\cdot ; r_i, \sigma_{-i}) : S_i \to [0, 1] \), where for each \( s_i \in S_i \) we have

\[
\phi_i(s_i ; r_i, \sigma_{-i}) := (1 - \gamma) \sum_{t=1}^{\infty} \gamma^{t-1} \cdot \mathbb{P}_{r_i,\sigma_{-i}} \{ X^t_i = s_i \},
\]

where the subscripts on \( \mathbb{P}_{r_i,\sigma_{-i}} \) emphasize that the distribution of \( X^t_i \) depends on the learning rule \( r_i \) and on \( \sigma_{-i} \).

We may interpret \( \phi_i(\cdot ; r_i, \sigma_{-i}) \) as a mixed strategy for \( i \) representing \( i \)’s lifetime average play, where her strategies in different periods are weighted by the relative probabilities of surviving into those periods. The mixed strategy \( \phi_i(\cdot ; r_i, \sigma_{-i}) \) has a population interpretation as well. Suppose there is a continuum of agents in the society, each engaged in their own copy of the learning problem above. In each period, enough new agents are added to the society to exactly balance out the share of agents who exit between periods. Then \( \phi_i(\cdot ; r_i, \sigma_{-i}) \) describes the distribution on \( S_i \) we would find if we sample an agent uniformly at random from the society and ask them which \( s_i \in S_i \) they plan on playing today.

We will demonstrate that if \( (s^*_i|i) \gtrdot (s^*_j|j) \), then for any opponents’ play and for learning policies discussed before, we have \( \phi_i(s^*_i ; r_i, \sigma_{-i}) \geq \phi_j(s^*_j ; r_i ; \sigma_{-i}) \). This provides a microfoundation for the compatibility-based cross-player restrictions on trembles in Definition 3.

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\( ^{15} \)This is well-defined in generic games where the set of strictly mixed opponent profiles that induce indifference between two actions in \( S_i \) has Lebesgue measure 0.
**Definition 13.** Agents \( i \) and \( j \) in the learning model have *equivalent regular priors* if their prior beliefs about the play of \( -i \) and \( -j \) are given by prior densities \( g_i : \times_{I \in \mathcal{I}_{-i}} \Delta(A_I) \to \mathbb{R} \) and \( g_j : \times_{I \in \mathcal{I}_{-j}} \Delta(A_I) \to \mathbb{R} \) that have full support and satisfy \( g_i|_{F_i[s_i] \cap F_j[\varphi(s_i)]} = g_j|_{F_i[s_i] \cap F_j[\varphi(s_i)]} \) for each \( s_i \in \mathcal{S}_i \).

Our goal in this section is to prove the next theorem.

**Theorem 2.** Suppose \((\Gamma, \Upsilon)\) has an isomorphic factoring for \( i \) and \( j \) and that \((s_i^*|i) \gtrsim (s_j^*|j)\) with \( \varphi(s_i^*) = s_j^* \). Consider two learning agents in the roles of \( i \) and \( j \) with equivalent regular priors.\(^{16}\) For any \( 0 \leq \gamma < 1 \) and any mixed strategy profile \( \sigma \), we have \( \phi_i(s_i^*; r_i, \sigma_{-i}) \geq \phi_j(s_j^*; r_j, \sigma_{-j}) \) under any of the following conditions:

- \( r \) is the Gittins index \( r_G \), and the two agents have the same patience \( 0 \leq \delta < 1 \).
- \( r \) is the Bayes-UCB \( r_{B-UCB} \), and \( i \) and \( j \) have the same quantile-choice function \( q_i = q_j = q : \mathbb{N} \to (0, 1) \), provided \( \Gamma \) is additively separable for both \( i \) and \( j \) and that \( u_{s_i, I} \) and \( u_{s_j, I} \) rank mixed strategies at information set \( I \) in the same way for every \( s_j = \varphi(s_i) \) and every \( I \in F_i[s_i] \cap F_j[s_j] \).
- \( r \) is the Thompson sampling rule \( r_{TS} \) and \( |\mathcal{S}_i| = |\mathcal{S}_j| = 2 \).

**Remark 4.** This result provides learning foundations for PCE in a number of games, including Examples 4 and 5.

### 4.3.2 Index policies

To prove the first two parts of Theorem 2, we will use the fact that the Gittins index and the Bayes-UCB are index policies in the following sense:

**Definition 14.** When \( \Gamma \) is factorable for \( i \), a policy \( r_i : Y_i \to \mathcal{S}_i \) is an *index policy* if there exist functions \((\iota_{s_i})_{s_i \in \mathcal{S}_i} \) with each \( \iota_{s_i} \) mapping subhistories of \( s_i \) to real numbers, such that \( r_i(y_i) \in \arg\max_{s_i \in \mathcal{S}_i} \{ \iota_{s_i}(y_i, s_i) \} \).

If an agent uses an index policy, we can think of her behavior in the following way. At each history, she computes an index for each strategy \( s_i \in \mathcal{S}_i \) based on the subhistory of those periods where she chose \( s_i \). Then, she chooses the strategy with the highest index. The Gittins index (computed from a given regular prior) is a leading example of an index policy, but this class is quite broad as we will see below.

\(^{16}\)The theorem easily generalizes to the case where \( i \) starts with one of \( L \geq 2 \) possible priors \( g_i^{(1)}, \ldots, g_i^{(L)} \) with probabilities \( p_1, \ldots, p_L \) and \( j \) starts with priors \( g_j^{(1)}, \ldots, g_j^{(L)} \) with the same probabilities, and each \( g_i^{(l)}, g_j^{(l)} \) is a pair of equivalent regular priors for \( 1 \leq l \leq L \).
The next result shows that for two index-policy learners \( i \) and \( j \), if their index functions \((t_{s_i})_{s_i \in S_i}\) and \((t_{s_j})_{s_j \in S_j}\) satisfy a condition analogous to the player-compatibility condition for strategies \( s_i^* \in S_i \) and \( s_j^* \in S_j \) after equivalent histories, then their aggregate responses against any play of \(-ij\) will respect compatibility. We must now define this notion of equivalence. Formally speaking, even under an isomorphic factoring with \( \varphi(s_i^*) = s_j^* \), the subhistories of \((i,s_i^*)\) do not belong to the same space as the subhistories of \((j,s_j^*)\), as the former might contain actions taken by \( j \) while the latter might contain actions taken by \( i \).

**Definition 15.** For \((\Gamma, \Upsilon)\) isomorphically factorable for \( i \) and \( j \) with \( \varphi(s_i) = s_j \), \( i \)'s subhistory \( y_{i,s_i} \) is **third-party equivalent** to \( j \)'s subhistory \( y_{j,s_j} \), written as \( y_{i,s_i} \sim y_{j,s_j} \), if they contain the same sequence of observations about the actions of \(-ij\).

Recall that, by Notation 1, we can think of each subhistory \( y_{i,s_i} \) as a sequence of elements in \( \times_{I \in F_i[s_i]} A_I \) and each subhistory \( y_{j,s_j} \) as a sequence of elements in \( \times_{I \in F_j[s_j]} A_I \). By isomorphic factorability, the \( s_i \)-relevant information sets for \( i \) are the same as the \( s_j \)-relevant information sets for \( j \), up to excluding each others’ play. That is \( F_i[s_i] \cap \mathcal{I}_{-ij} = F_j[s_j] \cap \mathcal{I}_{-ij} \). Third-party equivalence of \( y_{i,s_i} \) and \( y_{j,s_j} \) says \( i \) has played \( s_i \) as many times as \( j \) has played \( s_j \), and that the sequence of \(-ij\)'s actions that \( i \) encountered when experimenting with \( s_i \) are the same as those that \( j \) encountered from experimenting with \( s_j \).

The next proposition uses a coupling argument to identify a class of pure-strategy index policies in the learning problem that endogenously generate more trembles onto \( s_i^* \) than onto \( s_j^* \). Specifically, this result holds for any index policies for \( i \) and \( j \) such that at each history \( y_j \) for \( j \) where \( s_j^* \) has the highest index, no strategy \( s'_i \neq s_i^* \) can have the highest index for \( j \) at any history third-party equivalent to \( y_j \). The proof uses this hypothesis to deduce a general conclusion about the aggregate responses of these agents in the learning problem, where the two agents typically do not have third-party equivalent histories in any given period.

**Proposition 5.** Suppose \( r_i, r_j \) are pure-strategy index policies. Suppose \((\Gamma, \Upsilon)\) has isomorphic factoring for \( i \) and \( j \) with \( \varphi(s_i^*) = s_j^* \). Suppose that for every \( s'_i \neq s_i^* \) and \( s_j = \varphi(s'_i) \), at any histories \( y_i, y_j \) with \( y_{i,s_i^*} \sim y_{j,s_j^*} \) and \( y_{i,s'_i} \sim y_{j,s_j} \), if \( s_j^* \) has the weakly highest index for \( j \), then \( s'_i \) does not have the weakly highest index for \( i \). Then provided \( i \) and \( j \) have equivalent regular priors, \( \phi_i(s^*_i; r_i, \sigma_{-i}) \geq \phi_j(s^*_j; r_j, \sigma_{-j}) \) for any \( 0 \leq \gamma < 1 \) and \( \sigma \in \times_k \Delta(S_k) \).

**Proof.** Let \( 0 \leq \delta, \gamma < 1 \) be fixed. This proof is similar to the coupling argument in the proof of Fudenberg and He (2018)'s Lemma 2. We will highlight here the salient differences but skip over details that are similar to the other proof.

A pre-programmed response path \( \mathfrak{A} = (a_{t,I})_{t \in \mathcal{T}, I \in \mathcal{I}} \) is an element in \( \times_{I \in \mathcal{I}} (A^*_I) \). For each such path and learning rule \( r_i \) for player \( i \), imagine running the rule against the data-generating process such that the \( k \)-th time \( i \) plays \( s_i \), \( i \) will observe the action \( a_k,I \in A_I \) being played for \( I \in F_i[s_i] \).
Given a learning rule \( r_i \), each \( \mathfrak{A} \) induces a deterministic infinite history of \( i \)'s strategies \( y_i(\mathfrak{A}, r_i) \in (\mathcal{S}_i)^\infty \) when \( r_i \) is played against it. Lemma OA.1 in the Online Appendix shows there is a distribution \( \eta \) over these pre-programmed response paths, so that for any player \( i \), any rule \( r_i \), and any strategy \( s_i \in \mathcal{S}_i \), we have

\[
\phi_i(s_i; r_i, \sigma_{-i}) = (1 - \gamma) \mathbb{E}_{\mathfrak{A} \sim \eta} \left[ \sum_{t=1}^{\infty} \gamma^{t-1} \cdot (y_t(\mathfrak{A}, r_i) = s_i) \right],
\]

where \( y_t(\mathfrak{A}, r_i) \) refers to the \( t \)-th period strategy in the infinite history of strategies \( y_i(\mathfrak{A}, r_i) \).

Given this result, to prove that \( \phi_i(s_i^*; r_i, \sigma_{-i}) \geq \phi_j(s_j^*; r_j, \sigma_{-j}) \), it suffices to show that for every \( \mathfrak{A} \), the period where \( s_i^* \) is played for the \( k \)-th time in induced history \( y_i(\mathfrak{A}, r_i) \) happens earlier than the period where \( s_j^* \) is played for the \( k \)-th time in history \( y_j(\mathfrak{A}, r_j) \).

We provide the argument for this when \( k = 1 \), which forms the base case of an inductive argument. The general inductive argument is the same as in the proof of Fudenberg and He (2018) Lemma 2.

Let some \( \mathfrak{A} \) be fixed. If \( j \) never plays \( s_j^* \), we are done. Otherwise, for each \( s_j^* \neq s_j^* \), \( s_j^* = \varphi(s_j^* \prime) \), write \( N_{\varphi(s_j^*)} < \infty \) for the number of times that strategy \( s_j^* \) is played in \( y_j(\mathfrak{A}, r_j) \) before \( s_j^* \) is played for the first time. To show that \( i \) plays \( s_i^* \) earlier than \( j \) plays \( s_j^* \) for the first time, it suffices to show that for every \( s_i^* \neq s_i^* \), \( i \) spent no more than \( N_{\varphi(s_i^*)} \) periods playing \( s_i^* \) before playing \( s_i^* \) for the first time in history \( y_i(\mathfrak{A}, r_i) \).

By way of contradiction, suppose there is some \( s_i^* \neq s_i^* \) that gets played more than \( N_{\varphi(s_i^*)} \) times before \( s_i^* \) is played for the first time. Find the subhistory \( y_i \) of \( y_i(\mathfrak{A}, r_i) \), that leads to \( s_i^* \) being played for the \((N_{\varphi(s_j^*)} + 1)\)-th time, and find the subhistory \( y_j \) of \( y_j(\mathfrak{A}, r_j) \) that leads to \( j \) playing \( s_j^* \) for the first time. Note that \( y_i, s_i^* \sim y_j, s_j^* \) vacuously, since \( i \) has never played \( s_i^* \) in \( y_i \) and \( j \) has never played \( s_j^* \) in \( y_j \). Also, \( y_i, s_i^* \sim y_j, s_j^* \) since \( i \) has played \( s_i^* \) for \( N_{\varphi(s_i^*)} \) times and \( j \) has played \( s_j^* \) for the same number of times, while the definition of pre-programmed response sequence implies they would have seen the same history of play on the common information sets \( F_i[s_i^*] \cap F_j[s_j^*] \). This satisfies the definition of third-party equivalence of histories.

Since \( r_j(y_j) = s_j^* \) and \( r_j \) is an index rule, \( s_j^* \) must have weakly the highest index at \( y_j \).

By hypotheses of this lemma, \( s_i^* \) must not have the weakly highest index at \( y_i \). And yet \( r_i(y_i) = s_i^* \), contradicting the definition of an index rule.

Proposition 5 provides a bridge between player compatibility and relative experimentation frequencies. Appendix B shows that when \( (s_i^*) | i \gtrsim (s_j^*) | j \), the Gittins index and the Bayes-UCB index both satisfy the hypothesis of Proposition 5 (provided the additional regularity conditions of Theorem 2 hold). This proves Theorem 2's first two claims and provides two learning models that microfound PCE’s tremble restrictions.\(^{17}\) Since player compat-

\(^{17}\)Other natural index rules that we do not analyze explicitly here also serve as microfoundations of our
ibility is defined in the language of best responses against opponents’ strategy profiles in the stage game, the key step in showing that an index learning rule satisfies the hypothesis of Proposition 5 is to reformulate the index as the expected utility of using each strategy against a profile of opponents’ play. For the Gittins index, this profile is the “synthetic” opponent strategy profile constructed from the best stopping rule in the auxiliary optimal-stopping problem defining the index. This is similar to the construction of Fudenberg and He (2018), but in the more general setting of this paper the arguments becomes more subtle, as the induced synthetic strategy may be correlated (even though players have independent prior beliefs). For Bayes-UCB, under the assumptions of Theorem 2, the agent may rank opponents’ mixed strategies on each $I \in F_i[s_i]$ from least favorable to most favorable. The Bayes-UCB index of $s_i$ with quantile $q$ is, roughly speaking, equivalent to the expected utility of $s_i$ when opponents play mixed actions ranked at the $q$-th quantile in terms of $i$’s payoff under $i$’s current belief about opponents’ play.

Appendix B also shows that Thompson sampling generates player compatible trembles under the assumptions of Theorem 2, providing another learning foundation for the tremble restrictions. Unlike the two learning models discussed above, Thompson sampling is not an index strategy, as it is a stochastic function of the agent’s history. To understand the intuition, think of the case where $i$ and $j$ are non-interacting. When $(s_i^* | i) \succ (s_j^* | j)$, any sample of $-ij$’s play that makes $s_j^*$ weakly optimal for $j$ also makes $s_i^*$ strictly optimal for $i$, so there are more samples for which $s_i^*$ will be chosen for $i$ than $s_j^*$ will be chosen for $j$. So when two Thompson samplers $i$ and $j$ have the same beliefs about the play of $-ij$, $i$ is more likely to play $s_i^*$ than $j$ is to play $s_j^*$. Again, a coupling argument extends this into a conclusion about their aggregate experimentation frequencies.

5 Simulations

To test the robustness of our predictions, we simulate societies of 20 agents playing the link-formation game from Example 4 for a fixed number of periods using the two computationally efficient heuristics from Section 4.2, namely Bayes-UCB and Thompson sampling. (We also simulate the exactly optimal rational experimentation rule, but as this proves intractable for horizons over 50 periods we report those results in Online Appendix OA 4.) The setup of the simulations differs in two key ways from the learning framework of Section 4: Agents have a known, finite lifespan instead of a geometrically distributed one, and they all start together and end at the same time, so their environment is not stationary. To make the simulations closer to the typical real-world environment, we also introduced a 5% chance of implementation error into the Bayes-UCB simulations, so that for each agent in each period, cross-player restrictions on trembles, provided they satisfy Proposition 5 whenever $(s_i^* | i) \succ (s_j^* | j)$. 


the intended play is implemented 95% of the time, but the opposite strategy is chosen 5% of the time.\textsuperscript{18}

Within each society, 5 agents are assigned to each of the N1, N2, S1, and S2 roles, with the roles held fixed for life. In each period, each agent is randomly matched with 3 other agents in the other 3 roles, and agents simultaneously choose a strategy based on their personal history and their learning rule. An agent choosing \textbf{Inactive} receives no information about the play of others. An agent choosing \textbf{Active} observes the play of the three other agents in her match, but not the play of other agents in the society.

Each agent believes she is facing a stationary distribution of play from each opponent population (even though this assumption is false). In each simulation, all agents whose roles belong to the same side (i.e. both North or both South) share the same prior beliefs about the distribution of play in the two opposing player populations. We consider four sets of prior specifications, in increasing order of the myopic expected payoff of choosing \textbf{Active}: (i) \textit{very pessimistic}, with $\bar{p} \sim \text{Beta}(1, 5)$ and $\bar{p} \sim \text{Beta}(5, 1)$, where $\bar{p}$ is the probability that a random player from the high-quality opponent population chooses \textbf{Active}, while $p$ is the probability that a random player from the low-quality opponent population chooses \textbf{Active}; (ii) \textit{pessimistic}, with $\bar{p} \sim \text{Beta}(1, 2)$ and $\bar{p} \sim \text{Beta}(2, 1)$; (iii) \textit{uniform}, with $\bar{p} \sim \text{Beta}(1, 1)$ and $\bar{p} \sim \text{Beta}(1, 1)$; (iv) \textit{optimistic}, with $\bar{p} \sim \text{Beta}(2, 1)$ and $\bar{p} \sim \text{Beta}(1, 2)$.

For each of the learning rules and for each of the four sets of priors, we simulated 1000 societies playing each of the two versions of the link-formation game, with the implementation error for Bayes-UCB set to 5%.\textsuperscript{19} Figure 1 shows the average rates of link formation in different periods, with the blue curves showing the results for the anti-monotonic version of the game, the red curves showing the results for the co-monotonic version. The shaded areas show one standard deviation around the mean, as different simulated societies had different learning dynamics due to random matching, the stochastic nature of the agents’ policies, and the random implementation errors.

In the anti-monotonic version of the game, aggregate play converges towards the all-links equilibrium for all priors and all learning rules, as PCE predicts. The same is true in the co-monotonic version of the game for Bayes-UCB agents when the prior is sufficiently optimistic. When the prior is more pessimistic, the link-formation rate may decrease over time. That is, as we vary priors, the long-run link-formation outcomes can converge towards all of the PCE predictions in each of the two versions of the game. The dependence of the long-run outcome on the prior is natural when there are multiple PCE, as PCE is designed to capture the possible long-run effects of learning for fairly arbitrary priors.

\textsuperscript{18}We did not add this error to Thompson sampling, as it is already stochastic.

\textsuperscript{19}The Online Appendix contains additional simulations for error rates of 1% and 10%. The results for 10% error rate are similar to the results for 5% error rate. With a 1% error rate, the asymptotic behavior is again similar but in some cases convergence is much slower.
Figure 1: The dynamics of link-formation rates for societies of Bayesian UCB and Thompson sampling agents. Dashed lines show the theoretical maximum and minimum link-formation rates subject to the implementation errors.
The different learning dynamics in the two versions of the game come from different best responses to player-compatible trembles: In both versions and under each of the learning rules, low-cost players are more compatible with Active, and in the simulations they are more likely to play that strategy. These low-cost players are high-quality in the anti-monotonic version of the game, but low-quality in the co-monotonic version of the game. In the event that their matched opponents also choose Active and observe the choice of the low-cost players, the quality of these low-cost players encourage their matched opponents to play Active again in the anti-monotonic version, but discourage choosing Active in the co-monotonic version. This result remains robust if we replace implementation errors with a small fraction of commitment agents who play the same strategy regardless of their histories, as shown in the Online Appendix.

To test the robustness of our findings to varying levels of experimentation, we fixed the pessimistic priors and simulated the Bayes-UCB heuristic with five different (time-invariant) quantile levels: \( q \in \{0.5, 0.6, 0.7, 0.8, 0.9\} \). The results are in Figure 2. The anti-monotonic version leads to the all-links equilibrium in the long run for all these quantile levels. But except when \( q = 0.9 \), the link-formation rate in the co-monotonic version of the game tends to decrease towards the no-links equilibrium.

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20 Francetich and Kreps (2018) use simulations to assess the efficiency of various learning heuristics but unlike us they do not compare experimentation rates under different payoff functions.

21 We consider societies of 160 agents and include one commitment agent per strategy for each player role. Commitment agents play the same strategy each period, regardless of their histories. The remaining agents (i.e. 95% of the population) follow the Bayes-UCB heuristic with quantile level 0.6 without any implementation errors. When agents start with uniform priors, we again get divergence in link-formation rates between the anti-monotonic and co-monotonic versions of the game. With more pessimistic or more optimistic priors, either learning never takes off due to high-cost agents never experimenting, or society starts near the all-links equilibrium and stays there forever.

22 As \( q \) increases the agents are more likely to play the action with the highest possible payoffs, so with a pessimistic prior they experiment more.
Figure 2: The dynamics of link-formation rates for societies of Bayesian UCB agents, under different quantile levels. Dashed lines show the theoretical maximum and minimum link-formation rates subject to the implementation errors.

6 Concluding Discussion

PCE makes two key contributions. First, it generates new and sensible restrictions on equilibrium play by imposing cross-player restrictions on the relative probabilities that different players assign to certain strategies — namely, those strategy pairs $s_i, s_j$ ranked by the compatibility relation $(s_i|i) \succeq (s_j|j)$. As we have shown through examples, this distinguishes PCE from other refinement concepts, and allows us to make comparative statics predictions.
in some games where other equilibrium refinements do not.

Second, PCE shows how the the device of restricted “trembles” can capture some of the implications of non-equilibrium learning. As we saw, PCE’s cross-player restrictions arise endogenously in different models of how agents learn: the standard model of Bayesian agents maximizing their expected discounted lifetime utility, as well as heuristics for dealing with exploration-exploitation trade-offs like Bayesian upper confidence bounds and Thompson sampling, which are not exactly optimal but have good performance guarantees. We conjecture that the result that \( i \) is more likely to experiment with \( s_i \) than \( j \) with \( s_j \) when \( (s_i|i) \succeq (s_j|j) \) applies in other natural models of learning or dynamic adjustment, such as those considered by Francetich and Kreps (2018), and that it may be possible to provide foundations for PCE in other and perhaps larger classes of games.

The strength of the PCE refinement depends on the completeness of the compatibility order \( \succeq \), since \( \epsilon \)-PCE imposes restrictions on \( i \) and \( j \)’s play only when the relation \( (s_i|i) \succeq (s_j|j) \) holds. Definition 1 supposes that player \( i \) thinks all mixed strategies of other players are possible, as it considers the set of all strictly mixed correlated strategies \( \sigma_{-i} \in \Delta^\circ(S_{-i}) \). If the players have some prior knowledge about their opponents’ utility functions, player \( i \) might deduce a priori that the other players will only play strategies in some subset \( \mathcal{A}_{-i} \) of \( \Delta^\circ(S_{-i}) \). As we show in Fudenberg and He (2017), in signaling games imposing this kind of prior knowledge leads to a more complete version of the compatibility order. It may similarly lead to a more refined version of PCE.

In this paper we have only provided learning foundations for factorable games, in which player \( i \) can only learn about the consequences of strategy \( s_i \) by playing it. In more general extensive-form games two complications arise. First, player \( i \) may have several actions that lead to the same information set of player \( j \), which makes the optimal learning strategy more complicated. Second, player \( i \) may get information about how player \( j \) plays at some information sets thanks to an experiment by some other player \( k \), so that player \( i \) has an incentive to free ride. We plan to deal with these complications in future work.

Moreover, we conjecture that in games where actions have a natural ordering, learning rules based on the idea that nearby strategies induce similar responses can provide learning foundations for refinements in which players tremble more onto nearby actions, as in Simon (1987). More speculatively, the interpretation of trembles as arising from learning may provide learning-theoretic foundations for equilibrium refinements that restrict beliefs at off-path information sets in general extensive-form games, such as perfect Bayesian equilibrium (Fudenberg and Tirole, 1991; Watson, 2017), sequential equilibrium (Kreps and Wilson, 1982) and its extension to games with infinitely many actions (Simon and Stinchcombe, 1995; Myerson and Reny, 2018).
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Appendix

A Omitted Proofs from the Main Text

A.1 Proof of Proposition 1

Proposition 1: If \((s_i^*|i) \succeq (s_j^*|j)\), then at least one of the following is true: (i) \((s_j^*|j) \not\succeq (s_i^*|i)\); (ii) \(s_i^*\) is strictly dominated (possibly by a mixed strategy) for \(i\) and \(s_j^*\) is strictly dominated (possibly by a mixed strategy) for \(j\), provided opponents play strictly mixed correlated strategies. (iii) \(s_i^*\) is strictly dominant for \(i\) and \(s_j^*\) is strictly dominant for \(j\), provided opponents play strictly mixed correlated strategies.

Proof. Assume \((s_i^*|i) \succeq (s_j^*|j)\) and that neither (ii) nor (iii) holds. We show that these assumptions imply \((s_j^*|j) \not\succeq (s_i^*|i)\).

Partition the set \(\Delta^o(S_{-j})\) into three subsets, \(\Pi^+ \cup \Pi^0 \cup \Pi^-\), with \(\Pi^+\) consisting of representing \(\sigma_{-j} \in \Delta^o(S_{-j})\) that make \(s_j^*\) strictly better than the best alternative pure strategy, \(\Pi^-\) the elements of \(\Delta^o(S_{-j})\) that make \(s_j^*\) indifferent to the best alternative, and \(\Pi^-\) the elements that make \(s_j^*\) strictly worse. If \(\Pi^0\) is non-empty, then there is some \((\sigma_{-j}) \in \Pi^0\) such that

\[
\max_{s_j \in S_j \setminus \{s_j^*\}} u_j(s_j^*, \sigma_{-j}) = u_j(s_j^*, \sigma_{-j}).
\]

By \((s_i^*|i) \succeq (s_j^*|j)\), we get

\[
u_i(s_i^*, \hat{\sigma}_{-i}) > \max_{s_i \in S_i \setminus \{s_i^*\}} u_i(s_i^*, \hat{\sigma}_{-i}),
\]

for every \(\hat{\sigma}_{-i} \in \Delta^o(S_{-i})\) such that \(\sigma_{-j}|_{S_{-ij}} = \sigma_{-j}|_{S_{-ij}}\), so we do not have \((s_j^*|j) \succeq (s_i^*|i)\).

Also, if both \(\Pi^+\) and \(\Pi^-\) are non-empty, then \(\Pi^0\) is non-empty. This is because both \(\sigma_{-j} \mapsto u_j(s_j^*, \sigma_{-j})\) and \(\sigma_{-j} \mapsto \max_{s_j' \in S_j \setminus \{s_j^*\}} u_j(s_j', \sigma_{-j})\) are continuous functions. If \(u_j(s_j^*, \sigma_{-j}) - \max_{s_j' \in S_j \setminus \{s_j^*\}} u_j(s_j', \sigma_{-j}) > 0\) and also \(u_j(s_j^*, \sigma'_{-j}) - \max_{s_j' \in S_j \setminus \{s_j^*\}} u_j(s_j', \sigma'_{-j}) < 0\), then some mixture between \(\sigma_{-j}\) and \(\sigma'_{-j}\) must belong to \(\Pi^0\).

So we have shown that if either \(\Pi^0\) is non-empty or both \(\Pi^+\) and \(\Pi^-\) are non-empty, then \((s_j^*|j) \not\succeq (s_i^*|i)\).

If only \(\Pi^+\) is non-empty, then \(s_j^*\) is strictly dominant for \(j\). Together with \((s_i^*|i) \succeq (s_j^*|j)\), would imply that \(s_i^*\) is strictly dominant for \(i\), which would contradict the assumption that (iii) does not hold.

Finally suppose that only \(\Pi^-\) is non-empty, so that for every \(\sigma_{-j} \in \Delta^o(S_{-j})\) there exists a strictly better pure response than \(s_j^*\) against \(\sigma_{-j}\), then there exists a mixed strategy \(\sigma_j\) for \(j\) that strictly dominates \(s_j^*\) against all correlated play in \(\Delta^o(S_{-j})\). This shows \(s_j^*\) is strictly
dominated for \( j \) provided \(-j\) play a strictly mixed profile. If (ii) does not hold, then there is a \( \sigma_{-i} \in \Delta^\circ(S_{-i}) \) against which \( s^*_i \) is a weak best response, so the fact that \( s^*_j \) is not a strict best response against any \( \sigma_{-j} \in \Delta^\circ(S_{-j}) \) means \( (s^*_j|j) \not\succ (s^*_j|i) \). \( \square \)

A.2 Proof of Proposition 3

**Proposition 3:** In a signaling game, every PCE \( \sigma^* \) is a Nash equilibrium satisfying the compatibility criterion, as defined in Fudenberg and He (2018).

**Proof.** Since every PCE is a trembling-hand perfect equilibrium and since this latter solution concept refines Nash, \( \sigma^* \) is a Nash equilibrium.

To show that it satisfies the compatibility criterion, we need to show that \( \sigma^*_2 \) assigns probability 0 to plans in \( A^S \) that do not best respond to beliefs in the set \( P(s, \sigma^*) \) as defined in Fudenberg and He (2018). For any plan assigned positive probability under \( \sigma^*_2 \), by Proposition 2 we may find a sequence of strictly mixed signal profiles \( \sigma^{(t)}_1 \) of the sender, so that whenever \( (s|\theta) \gtrsim (s|\theta') \) we have \( \liminf_{t \to \infty} \sigma^{(t)}_1(s|\theta)/\sigma^{(t)}_1(s|\theta') \geq 1 \). Write \( q^{(t)}(\cdot|s) \) as the Bayesian posterior belief about sender’s type after signal \( s \) under \( \sigma^{(t)}_1 \), which is well defined because each \( \sigma^{(t)}_1 \) is strictly mixed. Whenever \( (s|\theta) \gtrsim (s|\theta') \), this sequence of posterior beliefs satisfies \( \liminf_{t \to \infty} q^{(t)}(\theta|s)/q^{(t)}(\theta'|s) \geq \lambda(\theta)/\lambda(\theta') \), so if the receiver’s plan best responds to every element in the sequence, it also best responds to an accumulation point \( (q^\infty(\cdot|s))_{s \in S} \) with \( q^\infty(\theta|s)/q^\infty(\theta'|s) \geq \lambda(\theta)/\lambda(\theta') \) whenever \( (s|\theta) \gtrsim (s|\theta') \). Since the player compatibility definition used in this paper is slightly stronger than the type compatibility definition that the set \( P(s', \sigma^*) \) is based on, the plan best responds to \( P(s', \sigma^*) \) after every signal \( s \). \( \square \)

A.3 Proof of Proposition 4

**Proposition 4:** Each of the following refinements selects the same subset of pure Nash equilibria when applied to the anti-monotonic and co-monotonic versions of the link-formation game: extended proper equilibrium, proper equilibrium, trembling-hand perfect equilibrium, \( p \)-dominance, Pareto efficiency, strategic stability, and pairwise stability. Moreover the link-formation game is not a potential game.

Step 1. Extended proper equilibrium, proper equilibrium, and trembling-hand perfect equilibrium allow the “no links” equilibrium in both versions of the game. For \( (q_i) \) anti-monotonic with \( (c_i) \), for each \( \epsilon > 0 \) let \( N1 \) and \( S1 \) play Active with probability \( \epsilon^2 \), \( N2 \) and \( S2 \) play Active with probability \( \epsilon \). For small enough \( \epsilon \), the expected payoff of Active for player \( i \) is approximately \( (10 - c_i)\epsilon \) since terms with higher order \( \epsilon \) are negligible. It is clear that this payoff is negative for small \( \epsilon \) for every player \( i \), and that under the utility re-scalings \( \beta_{N1} = \beta_{S1} = 10, \beta_{N2} = \beta_{S2} = 1 \), the loss to playing Active
smaller for N2 and S2 than for N1 and S1. So this strategy profile is a \((\beta, \epsilon)\)-extended proper equilibrium. Taking \(\epsilon \to 0\), we arrive at the equilibrium where each player chooses Inactive with probability 1.

**Proof.** For the version with \((q_i)\) co-monotonic with \((c_i)\), consider the same strategies without re-scalings, i.e. \(\beta = 1\). Then already the loss to playing Active smaller for N2 and S2 than for N1 and S1, making the strategy profile a \((1, \epsilon)\)-extended proper equilibrium.

These arguments show that the “no links” equilibrium is an extended proper equilibrium in both versions of the game. Every extended proper equilibrium is also proper and trembling-hand perfect, which completes the step.

**Step 2.** \(p\)-dominance eliminates the “no links” equilibrium in both versions of the game. Regardless of whether \((q_i)\) are co-monotonic or anti-monotonic with \((c_i)\), under the belief that all other players choose Active with probability \(p\) for \(p \in (0, 1)\), the expected payoff of playing Active (due to additivity across links) is \((1 - p) \cdot 0 + p \cdot (10 + 30 - 2c_i) > 0\) for any \(c_i \in \{14, 19\}\).

**Step 3.** Pareto eliminates the “no links” equilibrium in both versions of the game. It is immediate that the no-links equilibrium outcome is Pareto dominated by the all-links equilibrium outcome under both parameter specifications, so Pareto efficiency would rule it out whether \((c_i)\) is anti-monotonic or co-monotonic with \((q_i)\).

**Step 4.** Strategic stability (Kohlberg and Mertens, 1986) eliminates the “no links” equilibrium in both versions of the game. First suppose the \((c_i)\) are anti-monotonic with \((q_i)\). Let \(\eta = 1/100\) and let \(\epsilon' > 0\) be given. Define \(\epsilon_{N1}(\text{Active}) = \epsilon_{S1}(\text{Active}) = 2\epsilon'\), \(\epsilon_{N2}(\text{Active}) = \epsilon_{S2}(\text{Active}) = \epsilon'\) and \(\epsilon_i(\text{Inactive}) = \epsilon'\) for all players \(i\). When each \(i\) is constrained to play \(s_i\) with probability at least \(\epsilon_i(s_i)\), the only Nash equilibrium is for each player to choose Active with probability \(1 - \epsilon'\). (To see this, consider N2’s play in any such equilibrium \(\sigma\). If N2 weakly prefers Active, then N1 must strictly prefer it, so \(\sigma_{N1}(\text{Active}) = 1 - \epsilon' \geq \sigma_{N2}(\text{Active})\). On the other hand, if N2 strictly prefers Inactive, then \(\sigma_{N2}(\text{Active}) = \epsilon' < 2\epsilon' \leq \sigma_{N1}(\text{Active})\). In either case, \(\sigma_{N1}(\text{Active}) \geq \sigma_{N2}(\text{Active})\).

When both North players choose Active with probability \(1 - \epsilon'\), each South player has Active as their strict best response, so \(\sigma_{S1}(\text{Active}) = \sigma_{S2}(\text{Active}) = 1 - \epsilon'\). Against such a profile of South players, each North player has Active as their strict best response, so \(\sigma_{N1}(\text{Active}) = \sigma_{N2}(\text{Active}) = 1 - \epsilon'\).

Now suppose the \((c_i)\) are co-monotonic with \((q_i)\). Again let \(\eta = 1/100\) and let \(\epsilon' > 0\) be given. Define \(\epsilon_{N1}(\text{Active}) = \epsilon_{S1}(\text{Active}) = \epsilon', \epsilon_{N2}(\text{Active}) = \epsilon'/1000\), \(\epsilon_{S2}(\text{Active}) = \epsilon'\) and \(\epsilon_i(\text{Inactive}) = \epsilon'\) for all players \(i\). Suppose by way of contradiction there is a Nash equilibrium \(\sigma\) of the constrained game which is \(\eta\)-close to the Inactive equilibrium. In such an equilibrium, N2 must strictly prefer Inactive, otherwise N1 strictly prefers Active so \(\sigma\) could not be \(\eta\)-close to the Inactive equilibrium. Similar argument shows that S2 must
strictly prefer \textbf{Inactive}. This shows N2 and S2 must play \textbf{Active} with the minimum possible probability, that is, \( \sigma_{N2}(\text{Active}) = \epsilon' / 1000 \) and \( \sigma_{S2}(\text{Active}) = \epsilon' \). This implies that, even if \( \sigma_{N1}(\text{Active}) \) were at its minimum possible level of \( \epsilon' \), S1 would still strictly prefer playing \textbf{Inactive} because S1 is 1000 times as likely to link with the low-quality opponent as the high-quality opponent. This shows \( \sigma_{S1}(\text{Active}) = \epsilon' \). But when \( \sigma_{S1}(\text{Active}) = \sigma_{S2}(\text{Active}) = \epsilon' \), N1 strictly prefers playing \textbf{Active}, so \( \sigma_{N1}(\text{Active}) = 1 - \epsilon' \). This contradicts \( \sigma \) being \( \eta \)-close to the no-links equilibrium.

\textbf{Step 5.} Pairwise stability (Jackson and Wolinsky, 1996) does not apply to this game. This is because each player chooses between either linking with every player on the opposite side who plays \textbf{Active}, or linking with no one. A player cannot selectively cut off one of her links while preserving the other.

\textbf{Step 6.} The game does not have an ordinal potential, so refinements of potential games (Monderer and Shapley, 1996) do not apply. To see that this is not a potential game, consider the anti-monotonic parametrization. Suppose a potential \( P \) of the form \( P(a_{N1}, a_{N2}, a_{S1}, a_{S2}) \) exists, where \( a_i = 1 \) corresponds to \( i \) choosing \textbf{Active}, \( a_i = 0 \) corresponds to \( i \) choosing \textbf{Inactive}. We must have

\[ P(0, 0, 0, 0) = P(1, 0, 0, 0) = P(0, 0, 1), \]

since a unilateral deviation by one player from the \textbf{Inactive} equilibrium does not change any player’s payoffs. But notice that \( u_{N1}(1, 0, 0, 1) - u_{N1}(0, 0, 0, 1) = 10 - 14 = -4 \), while \( u_{S2}(1, 0, 0, 1) - u_{S2}(1, 0, 0, 0) = 30 - 19 = 11 \). If the game has an ordinal potential, then both of these expressions must have the same sign as \( P(1, 0, 0, 1) - P(1, 0, 0, 0) = P(1, 0, 0, 1) - P(0, 0, 0, 1) \), which is not true. A similar argument shows the co-monotonic parametrization does not have a potential either.

\section*{B Learning Models and Player Compatible Trembles}

Omitted proofs from this section can be found in the Online Appendix.

\subsection*{B.1 The Gittins Index}

Let survival chance \( \gamma \in [0, 1) \) and patience \( \delta \in [0, 1) \) be fixed. Let \( \nu_{s_i} \in \times_{I \in F_i} \Delta(\Delta(A_I)) \) be a belief over opponents’ mixed actions at the \( s_i \)-relevant information sets. The Gittins index of \( s_i \) under belief \( \nu_{s_i} \) is given by the maximum value of the following auxiliary optimization problem:
\[ \sup_{\tau \geq 1} \mathbb{E}_{\nu_{s_i}} \left\{ \sum_{t=1}^{\tau} (\delta \gamma)^{t-1} \cdot u_i(s_i, (a_I(t))_{I \in F_{[s_i]}}) \right\}, \]

where the supremum is taken over all positive-valued stopping times \( \tau \geq 1 \). Here \((a_I(t))_{I \in F_{[s_i]}}\) means the profile of actions that \(-i\) play on the \(s_i\)-relevant information sets the \(t\)-th time that \(i\) uses \(s_i\) — by factorability, only these actions and not actions elsewhere in the game tree determine \(i\)'s payoff from playing \(s_i\). The distribution over the infinite sequence of profiles \((a_I(t))_{t=1}^{\infty}\) is given by \(i\)'s belief \(\nu_{s_i}\), that is, there is some fixed mixed action in \(\times_{I \in F_{[s_i]}} \Delta(A_I)\) that generates profiles \((a_I(t))\) i.i.d. across periods \(t\). The event \(\{\tau = T\}\) for \(T \geq 1\) corresponds to using \(s_i\) for \(T\) times, observing the first \(T\) elements \((a_I(t))_{t=1}^{T}\), then stopping.

Write \(V(\tau; s_i, \nu_{s_i})\) for the value of the above auxiliary problem under the (not necessarily optimal) stopping time \(\tau\). The Gittins index of \(s_i\) is \(\sup_{\tau > 0} V(\tau; s_i, \nu_{s_i})\). We begin by linking \(V(\tau; s_i, \tau_{s_i})\) to \(i\)'s stage-game payoff from playing \(s_i\). From belief \(\nu_{s_i}\) and stopping time \(\tau\), we will construct the correlated profile \(\alpha(\nu_{s_i}, \tau) \in \Delta^o(\times_{I \in F_{[s_i]}} A_I)\), so that \(V(\tau; s_i, \nu_{s_i})\) is equal to \(i\)'s expected payoff when playing \(s_i\) while opponents play according to this correlated profile on the \(s_i\)-relevant information sets.

**Definition A.1.** A full-support belief \(\nu_{s_i} \in \times_{I \in F_{[s_i]}} \Delta(\Delta(A_I))\) for player \(i\) together with a (possibly random) stopping rule \(\tau > 0\) together induce a stochastic process \((\bar{a}_{(-i), t})_{t \geq 1}\) over the space \(\times_{I \in F_{[s_i]}} A_I \cup \{\varnothing\}\), where \(\bar{a}_{(-i), t} \in \times_{I \in F_{[s_i]}} A_I\) represents the opponents’ actions observed in period \(t\) if \(\tau \geq t\), and \(\bar{a}_{(-i), t} = \varnothing\) if \(\tau < t\). We call \(\bar{a}_{(-i), t}\) player \(i\)'s internal history at period \(t\) and write \(\mathbb{P}_{(-i)}\) for the distribution over internal histories that the stochastic process induces.

Internal histories live in the same space as player \(i\)'s actual experience in the learning problem, represented as a history in \(Y_i\). The process over internal histories is \(i\)'s prediction about what would happen in the auxiliary problem (which is an artificial device for computing the Gittins index) if he were to use \(\tau\).

Enumerate all possible profiles of moves at information sets \(F_{[s_i]}\) as \(\times_{I \in F_{[s_i]}} A_I = \{a_{(-i)}^{(1)}, \ldots, a_{(-i)}^{(K)}\}\), let \(p_{t,k} := \mathbb{P}_{(-i)}[\bar{a}_{(-i), t} = a_{(-i)}^{(k)}]\) for \(1 \leq k \leq K\) be the probability under \(\nu_{s_i}\) of seeing the profile of actions \(a_{(-i)}^{(k)}\) in period \(t\) of the stochastic process over internal histories, \((\bar{a}_{(-i), t})_{t \geq 0}\), and let \(p_{t,0} := \mathbb{P}_{(-i)}[\bar{a}_{(-i), t} = \varnothing]\) be the probability of having stopped before period \(t\).

**Definition A.2.** The synthetic correlated profile at information sets in \(F_{[s_i]}\) is the element of \(\Delta^o(\times_{I \in F_{[s_i]}} A_I)\) (i.e. a correlated random action) that assigns probability \(\frac{\sum_{t=1}^{\infty} \beta^{t-1} p_{t,k}}{\sum_{t=1}^{\infty} \beta^{t-1}(1-p_{t,0})}\) to the profile of actions \(a_{(-i)}^{(k)}\). Denote this profile by \(\alpha(\nu_{s_i}, \tau)\).

Note that the synthetic correlated profile depends on the belief \(\nu_{a_i}\), stopping rule \(\tau\), and effective discount factor \(\beta\). Since the belief \(\nu_{s_i}\) has full support, there is always a
positive probability assigned to observing every possible profile of actions on $F_i[s_i]$ in the first period, so the synthetic correlated profile is strictly mixed. The significance of the synthetic correlated profile is that it gives an alternative expression for the value of the auxiliary problem under stopping rule $\tau$.

Lemma A.1.

$$V(\tau; s_i, \nu_{s_i}) = U_i(s_i, \alpha(\nu_{s_i}, \tau))$$

The proof is the same as in Fudenberg and He (2018) and is omitted.

Notice that even though $i$ starts with the belief that opponents randomize independently at different information sets, and also holds an independent prior belief, $V(\tau; s_i, \nu_{s_i})$ is in general equal to the payoff of playing $s_i$ against a correlated profile. To see an example, suppose $F_i[s_i]$ consists of two information sets, one for each of two players $k_1 \neq k_2$, whose choice between Heads and Tails. Agent $i$’s prior belief is that each of $k_1, k_2$ is either always playing Heads or always playing Tails, with each of the 4 possible combinations of strategies given 25% prior probability. Now consider the stopping rule where $i$ stops if $k_1$ and $k_2$ play differently in the first period, but continues for 100 more periods if they play the same action in the first period. Then the procedure defined above generates a distribution over pairs of Heads and Tails that is mostly given by play in periods 2 through 100, which is either (Heads, Heads) or (Tails, Tails), each with 50% probability. In other words, it is the stopping rule $\tau$ that creates the correlation.

Consider now the situation where $i$ and $j$ share the same beliefs about play of $-ij$ on the common information sets $F_i[s_i] \cap F_j[s_j] \subseteq \mathcal{I}_{-ij}$. For any pure-strategy stopping time $\tau_j$ of $j$, we define a random stopping rule of $i$, the mimicking stopping time for $\tau_j$. Lemma A.2 will establish that the mimicking stopping time generates a synthetic correlated profile that matches the corresponding profile of $\tau_j$ on $F_i[s_i] \cap F_j[s_j]$.

The key issue in this construction is that $\tau_j$ maps $j$’s internal histories to stopping decisions, which does not live in the same space as $i$’s internal histories. In particular, $\tau_j$ makes use of $i$’s play to decide whether to stop. For $i$ to mimic such a rule, $i$ makes use of external histories, which include both the common component of $i$’s internal history on $F_i[s_i] \cap F_j[s_j]$, as well as simulated histories on $F_j[s_j] \setminus (F_i[s_i] \cap F_j[s_j])$.

For $\Gamma$ isomorphically factorable for $i$ and $j$ with $\varphi(s_i) = s_j$, we may write $F_i[s_i] = F^C \cup F^j$ with $F^C \subseteq \mathcal{I}_{-ij}$ and $F^j \subseteq \mathcal{I}_i$. Similarly, we may write $F_j[s_j] = F^C \cup F^i$ with $F^i \subseteq \mathcal{I}_i$. (So, $F^C$ is the common information sets that are payoff-relevant for both $s_i$ and $s_j$.) Whenever $j$ plays $s_j$, he observes some $(a_{(C)}, a_{(i)}) \in (\times_{I \in F^C A_I}) \times (\times_{I \in F^i A_I})$, where $a_{(C)}$ is a profile of actions at information sets in $F^C$ and $a_{(i)}$ is a profile of actions at information sets in $F^i$. So, a pure-strategy stopping rule in the auxiliary problem defining $j$’s Gittins index for $s_j$ is a function $\tau_j : \bigcup_{I \supseteq 1} (\times_{I \in F^C A_I}) \times (\times_{I \in F^i A_I})_I \to \{0, 1\}$ that maps finite histories of observations to stopping decisions, where “0” means continue and “1” means stop.
Definition A.3. Player \(i\)'s mimicking stopping rule for \(\tau_j\) draws \(\alpha^i \in \times_{t \in F^i} \Delta(A_I)\) from \(j\)'s belief \(\nu_{s_j}\) on \(F^i\), and then draws \((a_{(i),t})_{t \geq 1}\) by independently generating \(a_{(i),t}\) from \(\alpha^i\) each period. Conditional on \((a_{(i),t})\), \(i\) stops according to the rule \((\tau_i|(a_{(i),t}))((a_{(C),t}, a_{(j),t})_{t=1}^T) := \tau_j((a_{(C),t}, a_{(i),t})_{t=1}^T)\).\(^{23}\)

That is, the mimicking stopping rule involves ex-ante randomization across pure-strategy stopping rules \(\tau_i|(a_{(i),t})_{t=1}^\infty\). First, \(i\) draws a behavior strategy on the information set \(F^i\) according to \(j\)'s belief about \(i\)'s play. Then, \(i\) simulates an infinite sequence \((a_{(i),t})_{t=1}^\infty\) of \(i\)'s play using this drawn behavior strategy and follows the pure-strategy stopping rule \(\tau_i|(a_{(i),t})_{t=1}^\infty\).

As in Definition A.1, the mimicking strategy and \(i\)'s belief \(\nu_{s_i}\) generates a stochastic process \((\tilde{a}_{(j),t}, \tilde{a}_{(C),t})_{t \geq 1}\) of internal histories for \(i\) (representing actions on \(F_i[s_i]\) that \(i\) anticipates seeing when he plays \(s_i\)). It also induces a stochastic process \((\tilde{e}_{(i),t}, \tilde{e}_{(C),t})_{t \geq 1}\) of "external histories" defined in the following way:

Definition A.4. The stochastic process of external histories \((\tilde{e}_{(i),t}, \tilde{e}_{(C),t})_{t \geq 1}\) is defined from the process of internal histories \((\tilde{a}_{(j),t}, \tilde{a}_{(C),t})_{t \geq 1}\) that \(\tau_i\) generates and given by: (i) if \(\tau_i < t\), then \((\tilde{e}_{(i),t}, \tilde{e}_{(C),t}) = \emptyset\); (ii) otherwise, \(\tilde{e}_{(C),t} = \tilde{a}_{(C),t}\), and \(\tilde{e}_{(i),t}\) is the \(t\)-th element of the infinite sequence \((a_{(i),t})_{t=1}^\infty\) that \(i\) simulated before the first period of the auxiliary problem.

Write \(\mathbb{P}_e\) for the distribution over the sequence of of external histories generated by \(i\)'s mimicking stopping time for \(\tau_j\), which is a function of \(\tau_j, \nu_{s_j}\), and \(\nu_{s_i}\).

To understand the distinction between internal and external histories, note that the probability of \(i\)'s first-period internal history satisfying \((\tilde{a}_{(j),1}, \tilde{a}_{(C),1}) = (\tilde{a}_{(j),1}, \tilde{a}_{(C),1})\) for some fixed values \((\tilde{a}_{(j),1}, \tilde{a}_{(C),1}) \in \times_{t \in F_i[s_i]} A_I\) is given by the probability that a mixed play \(\alpha_{-i}\) on \(F_i[s_i]\), drawn according to \(i\)'s belief \(\nu_{s_i}\), would generate the profile of actions \((\tilde{a}_{(j),1}, \tilde{a}_{(C),1})\). On the other hand, the probability of \(i\)'s first-period external history satisfying \((\tilde{e}_{(i),1}, \tilde{e}_{(C),1}) = (\tilde{a}_{(i),1}, \tilde{a}_{(C),1})\) for some fixed values \((\tilde{a}_{(i),1}, \tilde{a}_{(C),1}) \in \times_{t \in F_j[s_j]} A_I\) also depends on \(j\)'s belief \(\nu_{s_j}\), for this belief determines the distribution over \((a_{(i),t})_{t=1}^\infty\) drawn before the start of the auxiliary problem.

To see the distribution of the external history in more detail, note that for any \(\tilde{a}_{(C)} \in \times_{t \in F^C} A_I\) and \(\tilde{a}_{(i)} \in \times_{t \in F^i} A_I\), the probability that the first period of \(i\)'s external history is \((\tilde{a}_{(i),1}, \tilde{a}_{(C),1})\) is

\[
\mathbb{P}_e[(\tilde{e}_{(i),1}, \tilde{e}_{(C),1}) = (\tilde{a}_{(i),1}, \tilde{a}_{(C),1})] = \int \alpha_i(\tilde{a}_{(i)}) d\nu_{s_j}(\alpha_i) \cdot \int \alpha_{-i}(\tilde{a}_{(C)}) d\nu_{s_i}(\alpha_{-i}),
\]

where the multiplication comes from the fact that \(i\)'s simulation of \((a_{(i),t})_{t=1}^\infty\) happens before the
auxiliary problem starts. (This expression depends on \( \nu_{s_j} \) because the mimicking stopping rule simulates \( (a^i_t)_{t=1}^\infty \) based on \( j \)'s belief.)

Also, for any \( \bar{a}'(C) \in \times_{I \in FC} A_I \) and \( \tilde{a}'(i) \in \times_{I \in FA_I} \),

\[
\mathbb{P}_e[\bar{e}_{(i),1}, \bar{e}(C), 1, \bar{e}_{(i),2}, \bar{e}(C), 2] = (\bar{a}_{(i)}, \bar{a}(C), \bar{a}_{(i)}, \bar{a}'(C))]
\]
\[
= \mathbb{P}_e[\tau_i \geq 2|\bar{e}_{(i),1}, \bar{e}(C), 1] = (\bar{a}_{(i)}, \bar{a}(C))] \cdot \int \alpha_i(\bar{a}_{(i)}) \cdot \alpha_i(\bar{a}'(i)) d\nu_{s_j}(\alpha_i) \cdot \int \alpha_{-i}(\bar{a}(C)) \cdot \alpha_{-i}(\bar{a}'(C)) d\nu_{s_j}(\alpha_{-i})
\]
\[
= (1 - \tau_j(\bar{a}_{(i)}, \bar{a}(C))) \cdot \int \alpha_i(\bar{a}_{(i)}) \cdot \alpha_i(\bar{a}'(i)) d\nu_{s_j}(\alpha_i) \cdot \int \alpha_{-i}(\bar{a}(C)) \cdot \alpha_{-i}(\bar{a}'(C)) d\nu_{s_j}(\alpha_{-i}).
\]

If \( \bar{e}_{(i),1} = \bar{a}_{(i)} \), then \( i \) must be using a pure-strategy stopping rule \((\tau_i|a^i_t)\) that stops after period 1 if and only if \( \tau_j(\bar{a}_{(i)}, \bar{a}(C), 1) = 1 \). So the conditional probability term on the second line is either 0 or 1; for the second-period external history not to be \( \emptyset \), we must have \( \tau_j(\bar{a}_{(i)}, \bar{a}(C)) = 0 \). In the final line, the multiplication in the first integrand reflects the fact that elements of the sequence \((a^i_t)\) are independently generated from the simulated \( \alpha_i \sim \nu_{s_j} \), and the multiplication in the second integrand comes from \( i \)'s knowledge that plays on the information sets \( FC \) are independently generated across periods from some fixed \( \alpha_{-i} \).

When using the mimicking stopping time for \( \tau_j \) in the auxiliary problem, \( i \) expects to see the same distribution of \( -ij \)'s play before stopping as \( j \) does when using \( \tau_j \), on the information sets that are both \( s_i \)-relevant and \( s_j \)-relevant. This is formalized in the next lemma.

**Lemma A.2.** For \((\Gamma, \mathcal{Y})\) isomorphically factorable for \( i \) and \( j \) with \( \varphi(s_i) = s_j \), suppose \( i \) holds belief \( \nu_{s_i} \) over play in \( F_I[s_i] \) and \( j \) holds belief \( \nu_{s_j} \) over play in \( F_J[s_j] \), such that \( \nu_{s_i}|_{F_j[s_i] \cap F_j[s_j]} = \nu_{s_j}|_{F_j[s_i] \cap F_j[s_j]} \), that is the two sets of beliefs match when marginalized to the common information sets in \( \mathcal{I}_{-ij} \). Let \( \tau_i \) be \( i \)'s mimicking stopping time for \( \tau_j \). Then, the synthetic correlated profile \( \alpha(\nu_{s_j}, \tau_j) \) marginalized to the information sets of \( -ij \) is the same as \( \alpha(\nu_{s_i}, \tau_i) \) marginalized to the same information sets.

**Proposition A.1.** Suppose \( \Gamma \) isomorphically factorable for \( i \) and \( j \) with \( \varphi(s^*_i) = s^*_j \), \( \varphi(s'_i) = s'_j \), where \( s^*_i \neq s'_i \) and \( (s^*_i|i) \gtrsim (s'_i|j) \). Suppose \( i \) holds belief \( \nu_{s_i} \in \times_{I \in F_j[s_i]} \Delta(\Delta(A_I)) \) about opponents' play after each \( s_i \) and \( j \) holds belief \( \nu_{s_j} \in \times_{I \in F_j[s_j]} \Delta(\Delta(A_I)) \) about opponents' play after each \( s_j \), such that \( \nu_{s_i}|_{F_j[s_i] \cap F_j[s_j]} = \nu_{s_j}|_{F_j[s_i] \cap F_j[s_j]} \) and \( \nu_{s_j}|_{F_j[s_i] \cap F_j[s'_j]} = \nu_{s'_j}|_{F_j[s_i] \cap F_j[s'_j]} \). If \( s^*_j \) has the weakly highest Gittins index for \( j \) under effective discount factor \( 0 \leq \delta \gamma < 1 \), then \( s'_i \) does not have the weakly highest Gittins index for \( i \) under the same effective discount factor.

**Proof.** We begin by defining a profile of strictly mixed correlated actions at information sets \( \bigcup_{s_j \in S_j} F_j[s_j] \subseteq \mathcal{I}_{-j} \), namely a collection of strictly mixed correlated profiles \( (\alpha_{F_j[s_j]})_{s_j \in S_j} \) where \( \alpha_{F_j[s_j]} \in \Delta^2(\times_{I \in F_j[s_j]} A_I) \). For each \( s_j \neq s'_j \) the profile \( \alpha_{F_j[s_j]} \) is the synthetic correlated
profile \(\alpha(\nu_{s_j}, \tau_{s_j}^*)\) as given by Definition A.2, where \(\tau_{s_j}^*\) is an optimal pure-strategy stopping time in \(j\)'s auxiliary stopping problem involving \(s_j\). For \(s_j = s_j'\), the correlated profile \(\alpha_{F_j}[s_j']\) is instead the synthetic correlated profile associated with the mimicking stopping rule for \(\tau_{s_j}'\), i.e. agent \(i\)'s pure-strategy optimal stopping time in \(i\)'s auxiliary problem for \(s_j'\).

Next, define a profile of strictly mixed correlated actions at information sets \(\cup_{s_i \in S_i} F_i[s_i] \subseteq \mathcal{I}_{-i}\) for \(i\)'s opponents. For each \(s_i \notin \{s_i^*, s_i'\}\), just use the marginal distribution of \(\alpha_{F_j}[\varphi(s_i)]\) constructed before on \(F_i[s_i] \cap \mathcal{I}_{-ij}\), then arbitrarily specify play at \(j\)'s information sets contained in \(F_i[s_i]\), if any. For \(s_i'\), the correlated profile is \(\alpha(\nu_{s_i'}, \tau_{s_i}^*)\), i.e. the synthetic move associated with \(i\)'s optimal stopping rule for \(s_i'\). Finally, for \(s_i^*\), the correlated profile \(\alpha_{F_i}[s_i^*]\) is the synthetic correlated profile associated with the mimicking stopping rule for \(\tau_{s_i}^*\).

From Lemma A.2, these two profiles of correlated actions agree when marginalized to information sets of \(-ij\). Therefore, they can be completed into strictly mixed correlated strategies, \(\sigma_{-i}\) and \(\sigma_{-j}\) respectively, such that \(\sigma_{-i}|_{\mathcal{I}_{-ij}} = \sigma_{-j}|_{\mathcal{I}_{-ij}}\). For each \(s_j \neq s_j'\), the Gittins index of \(s_j\) for \(j\) is \(U_j(s_j, \sigma_{-j})\). Also, since \(\alpha_{F_i}[s_i']\) is the mixed profile associated with the suboptimal mimicking stopping time, \(U_j(s_j', \sigma_{-j})\) is no larger than the Gittins index of \(s_j'\) for \(j\). By the hypothesis that \(s_j^*\) has the weakly highest Gittins index for \(j\), we get

\[
U_j(s_j^*, \sigma_{-j}) \geq \max_{s_j \neq s_j^*} U_j(s_j, \sigma_{-j}).
\]

By the definition of \((s_i^*|i) \succ (s_j^*|j)\) we must also have

\[
U_i(s_i^*, \sigma_{-i}) > \max_{s_i \neq s_i^*} U_i(s_i, \sigma_{-i}),
\]

so in particular \(U_i(s_i^*, \sigma_{-i}) > U_i(s_i'|\sigma_{-i})\). But \(U_i(s_i^*, \sigma_{-i})\) is no larger than the Gittins index of \(s_i^*\), for \(\alpha_{F_i}[s_i^*]\) is the synthetic strategy associated with a suboptimal mimicking stopping time. As \(U_i(s_i'|\sigma_{-i})\) is equal to the Gittins index of \(s_i'\), this shows \(s_i'\) cannot have even weakly the highest Gittins index at history \(y_i\), for \(s_i^*\) already has a strictly higher Gittins index than \(s_i'\) does. \(\square\)

We combine Lemma 5 with Proposition A.1 to establish the first statement of Theorem 2.

**Corollary A.1.** The Gittins index learning rules \(r_{i,G}\) and \(r_{j,G}\) satisfy the hypotheses of Proposition 5 when \((s_i^*|i) \succ (s_j^*|j)\), provided \(i\) and \(j\) have the same patience \(\delta\), survival chance \(\gamma\), and equivalent regular priors.

**Proof.** Equivalent regular priors require that priors are independent and that \(i\) and \(j\) share the same prior beliefs over play on \(F^* := F_i[s_i^*] \cap F_j[s_j^*]\) and over play on \(F' := F_i[s_i'] \cap F_j[s_j']\).
Thus after histories $y_i, y_j$ such that $y_{i,s_i'} \sim y_{j,s_j'}$ and $y_{i,s_i'} \sim y_{j,s_j'}$, $\nu_{s_i'}|_{F^*} = \nu_{s_j'}|_{F^*}$ and $\nu_{s_i'}|_{F'} = \nu_{s_j'}|_{F'}$, so the hypotheses of Proposition A.1 are satisfied. \hfill \Box

### B.2 Bayes-UCB

We start with a lemma that shows the Bayes-UCB index for a strategy $s_i$ is equal to $i$’s payoff from playing $s_i$ against a certain profile of mixed actions on $F_i[s_i]$, where this profile depends on $i$’s belief about actions on $F_i[s_i]$, the quantile $q$, and how $u_{s_i,I}$ ranks mixed actions in $\Delta(A_I)$ for each $I \in F_i[s_i]$.

**Lemma A.3.** Let $n_{s_i}$ be the number of times $i$ has played $s_i$ in history $y_i$ and let $q_{s_i} = q(n_{s_i}) \in (0, 1)$. Then the Bayes-UCB index for $s_i$ and given quantile-choice function $q$ after history $y_i$ is equal to $U_i(s_i, (\bar{\alpha}_I)_{I \in F_i[s_i]})$ for some profile of mixed actions where $\bar{\alpha}_I \in \Delta^q(A_I)$ for each $I$. Furthermore, $\bar{\alpha}_I$ only depends on $q_{s_i}$, $g_i(\cdot | y_{i,I})$ i’s posterior belief about play on $I$, and how $u_{s_i,I}$ ranks mixed strategies in $\Delta(A_I)$.

**Proof.** For each $I \in F_i[s_i]$, the random variable $\bar{u}_{s_i,I}(y_{i,I})$ only depends on $y_{i,I}$ through the posterior $g_i(\cdot | y_{i,I})$. Furthermore, $Q(\bar{u}_{s_i,I}(y_{i,I}); q_{s_i})$ is strictly between the highest and lowest possible values of $u_{s_i,I}(\cdot)$, each of which can be attained by some pure action on $A_I$, so there is a strictly mixed $\bar{\alpha}_I \in \Delta^q(A_I)$ such that $Q(\bar{u}_{s_i,I}(y_{i,I}); q_{s_i}) = u_{s_i,I}(\bar{\alpha}_I)$. Moreover, if $u_{s_i,I}$ and $u'_{s_i,I}$ rank mixed strategies on $\Delta(A_I)$ in the same way, there are $a \in \mathbb{R}$ and $b > 0$ so that $u'_{s_i,I} = a + bu_{s_i,I}$. Then $Q(\bar{u}'_{s_i,I}(y_{i,I}); q_{s_i}) = a + bQ(\bar{u}_{s_i,I}(y_{i,I}); q_{s_i})$, so $\bar{\alpha}_I$ still works for $u'_{s_i,I}$. \hfill \Box

The second statement of Theorem 2 follows as a corollary.

**Corollary A.2.** The Bayes-UCB rule $r_i,_{UCB}$ and $r_j,_{UCB}$ satisfy the hypotheses of Proposition 5 when $(s_i^*|i) \succeq (s_j^*|j)$, provided the hypotheses of Theorem 2 are satisfied.

The idea is that when $i$ and $j$ have matching beliefs, by Lemma A.3 we may calculate their Bayes-UCB indices for different strategies as their myopic expected payoff of using these strategies against some common opponents’ play. This is similar to the argument for the Gittins index, up to replacing the synthetic correlated profile with the mixed actions defined in Lemma A.3. Applying the definition of compatibility, we can deduce that when $(s_i^*|i) \succeq (s_j^*|j)$ and $\varphi(s_i^*) = s_j^*$, if $s_j^*$ has the highest Bayes-UCB index for $j$ then $s_i^*$ must have the highest Bayes-UCB index for $i$.

### B.3 Thompson sampling

Under the assumption that $|S_i| = |S_j| = 2$, enumerate $S_i = \{s_i^1, s_i^2\}$, $S_j = \{s_j^1, s_j^2\}$ and without loss of generality suppose the isomorphic factoring is such that $\varphi(s_i^1) = s_j^1$, $\varphi(s_i^2) =
By isomorphic factoring, for \( h \in \{1, 2\} \), we write \( F^{C,h} = F_i[s^h_i] \cap F_j[s^h_j] \subseteq I_{-ij} \), \( F^{i,h} = F_i[s^h_i] \subseteq I_i \), and \( F^{j,h} = F_j[s^h_j] \subseteq I_j \). So, \( F^{C,h} \cup F^{i,h} \) are the \( s^h_i \)-relevant information sets for \( i \) and \( F^{C,h} \cup F^{j,h} \) are the \( s^h_j \)-relevant information sets for \( j \).

We first establish a lemma: suppose \( (s^1_i | i) \preceq (s^1_j | j) \). If there is some mixed play on \( F^{C,1} \), \( F^{C,2} \), \( F^{i,1} \), \( F^{i,2} \) that makes \( a^1_j \) weakly optimal for \( j \), then the same mixed play on \( F^{C,1} \), \( F^{C,2} \) paired with any mixed play on \( F^{j,1} \), \( F^{j,2} \) must make \( a^1_i \) strictly optimal for \( i \).

**Lemma A.4.** Suppose \( (s^1_i | i) \preceq (s^1_j | j) \). If \( \alpha^h_{-ij} \in \times I \Delta^0(A_I) \), \( \alpha^h_i \in \times I \Delta^0(A_I) \), \( h \in \{1, 2\} \) are such that \( U_j(s^1_j, (\alpha^1_{-ij}, \alpha^1_i)) \geq U_j(s^2_j, (\alpha^2_{-ij}, \alpha^2_i)) \), then for any \( \alpha^h_{ij} \in \times I \Delta^0(A_I) \), \( h \in \{1, 2\} \), we have \( U_i(s^1_i, (\alpha^1_{-ij}, \alpha^1_i)) > U_i(s^2_i, (\alpha^2_{-ij}, \alpha^2_i)) \).

**Proof.** Since the random actions \( \alpha^h_{-ij} \), \( \alpha^h_i \) are strictly mixed, there exists a strictly mixed strategy profile \( \sigma_{-j} \in \times I \Delta^0(S_j) \) so that \( U_j(s^1_j, (\alpha^1_{-ij}, \alpha^1_i)) = U_j(s^1_j, \sigma_{-j}) \) and \( U_j(s^2_j, (\alpha^2_{-ij}, \alpha^2_i)) = U_j(s^2_j, \sigma_{-j}) \). By the definition of compatibility, for any \( \sigma_j \in \Delta^0(S_j) \), we have \( U_i(s^1_i, (\sigma_{-ij}, \sigma_j)) > U_i(s^2_i, (\sigma_{-ij}, \sigma_j)) \). Moreover, for any strictly mixed action profiles \( \alpha^h_{ij} \in \times I \Delta^0(A_I) \), \( h \in \{1, 2\} \), we can find some \( \sigma_j \in \Delta^0(S_j) \) generating it.

The final statement of Theorem 2 follows from Lemma A.4.

**Corollary A.3.** Suppose \((\Gamma, \Upsilon)\) has an isomorphic factoring for \( i \) and \( j \) and that \((s^*_i | i) \preceq (s^*_j | j)\) with \( \varphi(s^*_i) = s^*_j \), and \(|S_i| = |S_j| = 2\). Consider two learning agents in the roles of \( i \) and \( j \) using the Thompson sampling heuristic, whose priors satisfy Theorem 2. For any \( 0 \leq \gamma < 1 \) and any mixed strategy profile \( \sigma \), we have \( \phi_i(s^*_i; r_i, \sigma_{-i}) \geq \phi_j(s^*_j; r_j, \sigma_{-j}) \).

The argument first establishes that if \( i \) and \( j \) have matching beliefs about the play of third parties \( -ij \) and \((s^1_i | i) \preceq (s^1_j | j)\), then under Thompson sampling \( i \) plays \( s^1_i \) with greater probability than \( j \) plays \( s^1_j \). This is because, by Lemma A.4, any sample of \( -j \)'s play that makes \( s^1_j \) optimal for \( j \) also makes \( s^1_i \) optimal for \( i \) (after replacing the play of \( i \) in the \(-j \) sample with any play by \( j \)). Then, using an inductive argument and coupling, we show that the frequency with which \( i \) plays \( s^1_i \) by period \( t \) first-order stochastically dominates the analogous frequency with which \( j \) plays \( s^1_j \) by period \( t \).
Online Appendix

OA 1 Proofs Omitted from the Appendix

OA 1.1 Proof of Lemma A.2

Lemma A.2: For \((\Gamma, \Upsilon)\) isomorphically factorable for \(i\) and \(j\) with \(\varphi(s_i) = s_j\), suppose \(i\) holds belief \(\nu_{s_i}\) over play in \(F_i[s_i]\) and \(j\) holds belief \(\nu_{s_j}\) over play in \(F_j[s_j]\), such that \(\nu_{s_i}|_{F_i[s_i] \cap F_j[s_j]} = \nu_{s_j}|_{F_i[s_i] \cap F_j[s_j]}\), that is the two sets of beliefs match when marginalized to the common information sets in \(\mathcal{I}_{-ij}\). Let \(\tau_i\) be \(i\)'s mimicking stopping time for \(\tau_j\). Then, the synthetic correlated profile \(\alpha(\nu_{s_j}, \tau_j)\) marginalized to the information sets of \(-ij\) is the same as \(\alpha(\nu_{s_i}, \tau_i)\) marginalized to the same information sets.

Proof. Let \((\bar{a}_{(i,t)}, \bar{a}_{(C,t)})_{t \geq 1}\) and \((\bar{e}_{(i,t)}, \bar{e}_{(C,t)})_{t \geq 1}\) be the stochastic processes of internal and external histories for \(\tau_i\), with distributions \(\mathbb{P}_{-i}\) and \(\mathbb{P}_e\). Enumerate possible profiles of actions on \(F_C\) as \(\times_{t \in F_C} A_{t} = \{a_{(1)}(c), \ldots, a_{(kC)}(c)\}\), possible profiles of actions on \(F_j\) as \(\times_{t \in F_j} A_t = \{a_{(j)}(i), \ldots, a_{(j)}(K_i)\}\), and possible profiles of actions on \(F_i\) as \(\times_{t \in F_i} A_t = \{a_{(i)}(1), \ldots, a_{(i)}(K_i)\}\).

Write \(p_{t, (k_i, k_C)} := \mathbb{P}_{-i}[\bar{a}_{(i,t)}, \bar{a}_{(C,t)} = (a_{(j)}^{(k_i)}, a_{(C)}^{(k_i)})]\) for \(k_j \in \{1, ..., K_j\}\) and \(k_C \in \{1, ..., K_C\}\). Also write \(q_{t, (k_i, k_C)} := \mathbb{P}_e[\bar{e}_{(i,t)}, \bar{e}_{(C,t)} = (a_{(i)}^{(k_i)}, a_{(C)}^{(k_i)})]\) for \(k_i \in \{1, ..., K_i\}\) and \(k_C \in \{1, ..., K_C\}\). Let \(p_{t, (0, 0)} = q_{t, (0, 0)} := \mathbb{P}_{-i}[\tau_i < t] = \mathbb{P}_e[\tau_i < t]\) be the probability of having stopped before period \(t\).

The distribution of external histories that \(i\) expects to observe before stopping under belief \(\nu_{s_i}\) when using the mimicking stopping rule \(\tau_i\) is the same as the distribution of internal histories that \(j\) expects to observe when using stopping rule \(\tau_j\) under belief \(\nu_{s_j}\). This is because of the hypothesis \(\nu_{s_i}|_{F_C} = \nu_{s_j}|_{F_C}\), together with fact that \(i\) simulates the data-generating process on \(F^i\) by drawing a mixed action \(\alpha^i\) according to \(j\)'s belief \(\nu_{s_j}|_{F^j}\). This means for every \(k_i \in \{1, ..., K_i\}\) and every \(k_C \in \{1, ..., K_C\}\),

\[
\frac{\sum_{t=1}^{\infty}(\delta \gamma)^{t-1} q_{t, (k_i, k_C)}}{\sum_{t=1}^{\infty}(\delta \gamma)^{t-1}(1 - q_{t, (0, 0)})} = \alpha(\nu_{s_j}, \tau_j)(a_{(i)}^{(k_i)}, a_{(C)}^{(k_i)}).
\]

For a fixed \(k_C \in \{1, ..., K_C\}\), summing across \(k_i\) gives

\[
\frac{\sum_{t=1}^{\infty}(\delta \gamma)^{t-1} \sum_{k_i=1}^{K_i} q_{t, (k_i, k_C)}}{\sum_{t=1}^{\infty}(\delta \gamma)^{t-1}(1 - q_{t, (0, 0)})} = \alpha(\nu_{s_j}, \tau_j)(a_{(C)}^{(k_C)}).
\]

By definition, the processes \((\bar{a}_{(i,t)}, \bar{a}_{(C,t)})_{t \geq 0}\) and \((\bar{e}_{(i,t)}, \bar{e}_{(C,t)})_{t \geq 0}\) have the same marginal
distribution on the second dimension:

\[ \sum_{k_i=1}^{K_i} q_{t,(k_i,kC)} = \mathbb{P}_{-1}[(\tilde{a}(C),t) = (a^{(kC)}_i)] = \sum_{k_j=1}^{K_j} p_{t,(k_j,kC)}. \]

Making this substitution and using the fact that \( p_{t,(0,0)} = q_{t,(0,0)}, \)

\[ \frac{\sum_{t=1}^{\infty} (\delta \gamma)^{t-1} \sum_{k_j=1}^{K_j} p_{t,(k_j,kC)}}{\sum_{t=1}^{\infty} (\delta \gamma)^{t-1} (1 - p_{t,(0,0)})} = \alpha(\nu_{s_j}, \tau_j)(a^{(kC)}_i) = \alpha(\nu_{s_j}, \tau_j)(a^{(kC)}_i). \]

But by the definition of synthetic correlated profile, the LHS is \( \sum_{k_j=1}^{K_j} \alpha(\nu_{s_j}, \tau_i)(a^{(k_j)}_i, a^{(kC)}_i) = \alpha(\nu_{s_j}, \tau_i)(a^{(kC)}_i). \)

Since the choice of \( a^{(kC)}_i \in \times_{I \in FC} A_I \) was arbitrary, we have shown that the synthetic profile \( \alpha(\nu_{s_j}, \tau_j) \) of the original stopping rule \( \tau_j \) and the one associated with the mimicking strategy of \( i, \alpha(\nu_{s_j}, \tau_i) \), coincide on \( FC \).

**OA 1.2 Proof of Corollary A.2**

**Corollary A.2:** The Bayes-UCB rule \( r_{i,UCB} \) and \( r_{j,UCB} \) satisfy the hypotheses of Proposition 5 when \( (s^*_i|i) \succeq (s^*_j|j) \), provided the hypotheses of Theorem 2 are satisfied.

**Proof.** Consider histories \( y_i, y_j \) with \( y_{i,s^*_i} \sim y_{j,s^*_j} \) and \( y_{i,s'_i} \sim y_{j,s'_j} \). By Lemma A.3, there exist \( \tilde{a}_{i}^{\sim} \in \Delta^0(A_I) \) for every \( I \in \cup_{s_i \in S_i} F_i[s_i] \) and \( \tilde{a}_{j}^{\sim} \in \Delta^0(A_I) \) for every \( I \in \cup_{s_j \in S_j} F_j[s_j] \) so that \( t_{i,s_i}(y_i) = U_i(s_i, (\alpha_{i}^{-1})_I | F_i[s_i]) \) and \( t_{j,s_j}(y_j) = U_j(s_j, (\alpha_{j}^{-1})_I | F_j[s_j]) \) for all \( s_i, s_j \), where \( t_{i,s_i}(y_i) \) is the Bayes-UCB index for \( s_i \) after history \( y_i \) and \( t_{j,s_j}(y_j) \) is the Bayes-UCB index for \( s_j \) after history \( y_j \).

Because \( y_{i,s^*_i} \sim y_{j,s^*_j} \) and \( y_{i,s'_i} \sim y_{j,s'_j} \), \( y_i \) contains the same number of \( s^*_i \) experiments as \( y_j \) contains \( s^*_j \), and \( y_i \) contains the same number of \( s'_i \) experiments as \( y_j \) contains \( s'_j \). Also by third-party equivalence and the fact that \( i \) and \( j \) start with the same beliefs on common relevant information sets, they have the same posterior beliefs \( g_i(\cdot|y_{i,t}), g_j(\cdot|y_{j,t}) \) for any \( I \in F_i[s^*_i] \cap F_j[s^*_j] \) and \( I \in F_i[s'_i] \cap F_j[s'_j] \). Finally, the hypotheses of Theorem 2 say that on any \( I \in F_i[s^*_i] \cap F_j[s^*_j], u_{s^*_i,t} \) and \( u_{s^*_j,t} \) have the same ranking of mixed actions, while on any \( I \in F_i[s'_i] \cap F_j[s'_j], u_{s'_i,t} \) and \( u_{s'_j,t} \) have the same ranking of mixed actions. So, by Lemma A.3, we may take \( \tilde{a}_{i}^{\sim} = \tilde{a}_{j}^{\sim} \) for all \( I \in F_i[s^*_i] \cap F_j[s^*_j] \) and \( I \in F_i[s'_i] \cap F_j[s'_j] \).

Find some \( \sigma_{-j} = (\sigma_{-ij}, \sigma_j) \in \times_{k \neq j} \Delta^0(S_k) \) so that \( \sigma_{-j} \) generates the random actions \( \tilde{a}_{j}^{\sim} \) on every \( I \in \cup_{s_j \in S_j} F_j[s_j] \). Then we have \( t_{j,s_j}(y_j) = U_j(s_j, \sigma_{-j}) \) for every \( s_j \in S_j \). The fact that \( s^*_j \) has weakly the highest index means \( s^*_j \) is weakly optimal against \( \sigma_{-j} \). Now take \( \sigma_{-i} = (\sigma_{-ij}, \sigma_j) \) where \( \sigma_j \in \Delta^0(S_j) \) is such that it generates the random actions
Proof.

\(\alpha^{-i}\) on \(F_i[s_i^*] \cap \mathcal{I}_j\) and \(F_i[s_i'] \cap \mathcal{I}_j\). But since \(\alpha^{-i} = \alpha^{-j}\) for all \(I \in F_i[s_i^*] \cap F_j[s_j^*]\) and \(I \in F_i[s_i'] \cap F_j[s_j']\), \(\sigma_i\) generates the random actions \((\alpha^{-i})\) on all of \(F_i[s_i^*]\) and \(F_i[s_i']\), meaning \(\nu_i(s_i^*|y_i) = U_i(s_i^*, \sigma_i)\) and \(\nu_i(s_i'|y_i) = U_i(s_i', \sigma_i)\). The definition of compatibility implies \(U_i(s_i^*, \sigma_i) > U_i(s_i', \sigma_i)\), so \(\nu_i(s_i^*|y_i) > \nu_i(s_i'|y_i)\). This shows \(s_i^*\) does not have weakly the highest Bayes-UCB index, since \(s_i^*\) has a strictly higher one.

\(\square\)

**OA 1.3  Proof of Corollary A.3**

**Corollary A.3:** Suppose \((\Gamma, \Upsilon)\) has an isomorphic factoring for \(i\) and \(j\) and that \((s_i^*|i) \succeq (s_j^*|j)\) with \(\varphi(s_i^*|i) = s_j^*\) and \(|\mathcal{S}_i| = |\mathcal{S}_j| = 2\). Consider two learning agents in the roles of \(i\) and \(j\) using the Thompson sampling heuristic, whose priors satisfy Theorem 2. For any \(0 < \gamma < 1\) and any mixed strategy profile \(\sigma\), we have \(\phi_i(s_i^*; r_i, \sigma_i) \geq \phi_j(s_j^*; r_j, \sigma_j)\).

**Proof.** Suppose \((s_i^*|i) \succeq (s_j^*|j)\). Fix \(0 < \gamma < 1\) and any mixed strategy profile \(\sigma\). We now show \(\phi_i(s_i^*; r_{TS,i}, \sigma_i) \geq \phi_j(s_j^*; r_{TS,j}, \sigma_j)\).

**Step 1: Behavior under matching beliefs.** We show that after histories \(y_i, y_j\) satisfying \(y_i|s_i' \sim y_j|s_j^*\) and \(y_i|s_i' \sim y_j|s_j^*\), we have \(r_{TS,i}(y_i)(s_i^*) \geq r_{TS,j}(y_j)(s_j^*)\).

Since the prior densities \(g_i, g_j\) are independent about opponents’ play at different information sets, the posterior beliefs inherit this property. We may then write \(i\)'s posterior belief about play at relevant information sets as \(\nu_i = \nu_i,FC \times \nu_i,F1\), where \(\nu_i,FC \in \times_{I \in FC,1 \cup FC,2} \Delta(\Delta(A_I))\) and \(\nu_i,F1 \in \times_{I \in F1,1 \cup F1,2} \Delta(\Delta(A_I))\). Similarly, \(j\)'s posterior belief is \(\nu_j = \nu_j,FC \times \nu_j,F1\), where \(\nu_j,FC \in \times_{I \in FC,1 \cup FC,2} \Delta(\Delta(A_I))\) and \(\nu_j,F1 \in \times_{I \in F1,1 \cup F1,2} \Delta(\Delta(A_I))\).

Again from the independence of \(g_i, g_j\), each player only updates belief about play at a given information set using observations of play on the same information set. Since priors \(g_i, g_j\) agree about play on \(FC,1 \cup FC,2\), from third-party equivalence of \(y_i, y_j\), we get \(\nu_i,FC = \nu_j,FC\).

Let \(A_{-j} \subseteq \times_{I \in FC,1 \cup FC,2} \Delta^0(\Delta(A_I))\) be those strictly mixed behavior strategies of \(-j\) on \(F_j[s_i^*] \cup F_j[s_j^*]\) that make \(s_j^*\) weakly optimal for \(j\). That is for each \(\alpha_{-j} \in A_{-j}\), we have \(U_j(s_j^*, \alpha_{-j}) \geq U_j(s_j^*, \alpha_{-j})\). We may write each \(\alpha_{-j} \in A_{-j}\) as \(\alpha_{-j} = (\alpha_{-ij}, \alpha_i)\) where \(\alpha_{-ij} \in \times_{I \in FC,1 \cup FC,2} \Delta^0(\Delta(A_I))\) and \(\alpha_i \in \times_{I \in F1,1 \cup F1,2} \Delta^0(\Delta(A_I))\). From the definition of Thompson sampling,

\[r_{TS,i}(y_j)(s_i^*) \leq \nu_j(A_{-j}),\]

since \(s_i^*\) is only played when the simulated “state of the world” (i.e. behavior strategy of \(-j\)) makes \(s_i^*\) at least weakly optimal.\(^{24}\) Now consider the projection \(A_{-ij} \subseteq \times_{I \in FC,1 \cup FC,2} \Delta^0(\Delta(A_I))\), constructed from \(A_{-j}\) so that \(\alpha_{-ij} \in A_{-ij}\) if and only if \((\alpha_{-ij}, \alpha_i)\) \(A_{-j}\) for at least one \(\alpha_i \in \times_{I \in F1,1 \cup F1,2} \Delta^0(\Delta(A_I))\). Clearly \(\nu_j(A_{-j}) \leq \nu_j,FC(A_{-ij})\).

\(^{24}\)We may ignore any non-strictly-mixed play by \(-j\) that make \(s_i^*\) optimal, since the set of all such play has probability 0 under \(\nu_j\).
We now show \( r_{TS,i}(y_t)(s^1_i) \geq \nu_{i,\text{FC}}(A_{-ij}) \). This is true because for any \( \alpha_{-ij} \in A_{-ij} \) and \( \alpha_j \in \times_{I \in F_j, i \cup F_j, 2} \Delta^\circ(A_I) \), by Lemma A.4, \( s^1_j \) is strictly better than \( s^2_j \) against \((\alpha_{-ij}, \alpha_j)\). So each element of \( A_{-ij} \times (\times_{I \in F_j, i \cup F_j, 2} \Delta^\circ(A_I)) \) makes \( s^1_i \) strictly optimal, and the \( \nu_i \) weight assigned to this class of “states of the world” is \( \nu_{i,\text{FC}}(A_{-ij}) \). When any of these states is simulated, Thompson sampling requires that \( s^1_i \) is played with probability 1.

This, together with the fact that \( \nu_{j,\text{FC}} = \nu_{i,\text{FC}} \), establishes Step 1.

**Step 2: Coupling beliefs.** We again make use of pre-programmed response sequences, i.e. paths \( \mathfrak{A} = (a_{t,i})_{t \geq 1, i \in I} \). Unlike in the case of rational experimenters where each \( \mathfrak{A} \) induced a deterministic path of play for each player, for Thompson samplers each \( \mathfrak{A} \) induces a random path of play. This is because the agents’ policies \( r_{TS,i}, r_{TS,j} \) now specify mixed strategies after all histories. Let \( \sigma_i(\mathfrak{A}) \in \Delta(\mathfrak{S}_i) \) refer to the distribution over \( \{s^1_i, s^2_i\} \) induced by \( \mathfrak{A} \), among player \( i \)'s with age \( t \). If we can show that for each \( \mathfrak{A} \), we have \( \sum_{t=0}^\infty \gamma_t \cdot \sigma_i(\mathfrak{A}) \geq \sum_{t=0}^\infty \gamma_t \cdot \sigma_j(\mathfrak{A}) \) then we are done, as we have

\[
\phi_i(s_i; \sigma_i, \alpha_{-i}) = \int \left\{ (1 - \gamma) \sum_{t=1}^\infty \gamma^{t-1} \cdot 1\{S_t^i = s_i | \mathfrak{A}\} \right\} d\eta(\mathfrak{A}) = \int \left\{ (1 - \gamma) \sum_{t=1}^\infty \gamma^{t-1} \cdot \sigma_i(\mathfrak{A})(s_i) \right\} d\eta(\mathfrak{A})
\]

for every \( s_i \in \mathfrak{S}_i \). Here, \( S_t^i \) is the random variable corresponding to \( i \)'s play when her age is \( t \), and \( \eta \) is the distribution of pre-programmed response sequences that \( \sigma \) generates, as defined in Lemma OA.1. Thus it suffices to show that \( \sum_{t=1}^\infty \gamma^{t-1} \cdot \sigma_i(\mathfrak{A}) \geq \sum_{t=1}^\infty \gamma^{t-1} \cdot \sigma_j(\mathfrak{A}) \) for each \( \mathfrak{A} \).

**Step 3: First-order stochastic dominance in the cumulative frequency of playing action 1.** Let \( \mathfrak{A} \) be fixed. Define random variable \( X_i^t = \sum_{t=1}^\infty 1\{S^t_i = s^1_i\} \), \( X_j^t = \sum_{t=1}^\infty 1\{S^t_j = s^1_j\} \) as the cumulative counts of the number of times that \( i \) and \( j \) have played strategies \( s^1_j \) and \( s^1_i \) by the end of period \( t \).

**Claim 3(a):** For every \( t \geq 1 \), \( X_i^t \) FOSD \( X_j^t \). This is shown by induction on the ages of the agents. New agents \( i \) and \( j \) without any history trivially satisfy third-party equivalence. By **Step 1**, this shows \( \mathbb{P}[X_i^t = 1] \geq \mathbb{P}[X_j^t = 1] \), establishing FOSD in the case of \( t = 1 \). Now assume by induction that \( X_i^t \) FOSD \( X_j^t \) for all \( t \leq M \). To show that \( X_i^{M+1} \) FOSD \( X_j^{M+1} \), we need to show that:

**Claim 3(b):** \( \mathbb{P}[X_i^{M+1} \geq x] \geq \mathbb{P}[X_j^{M+1} \geq x] \) for all \( x \in \{0, 1, ..., M+1\} \). This is clear for \( x = 0 \) so let some \( x > 0 \) be fixed. To have played strategy 1 at least \( x \) times by the end of period \( M + 1 \), an agent needs to have either played it \( x \) times already by the end of period \( M \), or to have played it \( x - 1 \) times and then play it one more time during period \( M + 1 \). That is,

\[
\mathbb{P}[X_i^{M+1} \geq x] = \mathbb{P}[X_i^M \geq x] + \mathbb{P}[X_i^M = x - 1] \cdot r_{TS,i}(y_i^{x-1,M-x+1})(s^1_i)
\]

\((1)\)

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where \( y_i^{x-1,M-x+1} \) refers to the history of observations from playing strategy \( s_i^1 \) a total of \( x-1 \) times and strategy \( s_i^2 \) a total of \( M-x+1 \) times. (By the definition of response path \( \mathfrak{A} \), no matter in what order these strategies were played the history would contain the same information about play at information sets in \( F_i[s_i^1] \) and \( F_i[s_i^2] \), so \( r_{TS,i}(y_i^{x-1,M-x+1}) \) is well defined.) An analogous expression holds for \( j \):

\[
\mathbb{P}[X_{j}^{M+1} \geq x] = \mathbb{P}[X_{j}^{M} = x-1] \cdot r_{TS,j}(y_{j}^{x-1,M-x+1})(s_j^1) .
\] (2)

Note that \( y_i^{x-1,M-x+1} \) and \( y_j^{x-1,M-x+1} \) are third-party equivalent since play on the common \(-ij\) information sets, \( F_{C,1} \) and \( F_{C,2} \), are generated by the same response path \( \mathfrak{A} \), and both histories observe play on \( F_{C,1} \) for a total of \( x-1 \) times, observe play on \( F_{C,2} \) for a total of \( M-x+1 \) times. We may now write using Equation (1),

\[
\begin{align*}
\mathbb{P}[X_{i}^{M+1} \geq x] & = \mathbb{P}[X_{i}^{M} = x-1] \cdot r_{TS,i}(y_{i}^{x-1,M-x+1})(s_i^1) \\
& \geq \mathbb{P}[X_{i}^{M} = x-1] \cdot r_{TS,i}(y_{i}^{x-1,M-x+1})(s_i^1) \\
& \quad - (1 - r_{TS,i}(y_{i}^{x-1,M-x+1})(s_i^1)) \cdot (\mathbb{P}[X_{i}^{M} \geq x] - \mathbb{P}[X_{j}^{M} \geq x]) \\
& = \mathbb{P}[X_{i}^{M} \geq x] + \left( \mathbb{P}[X_{i}^{M} = x-1] + \mathbb{P}[X_{i}^{M} \geq x] - \mathbb{P}[X_{j}^{M} \geq x] \right) \cdot r_{TS,i}(y_{i}^{x-1,M-x+1})(s_i^1) \\
& = \mathbb{P}[X_{i}^{M} \geq x] + \left( \mathbb{P}[X_{i}^{M} = x-1] - \mathbb{P}[X_{j}^{M} \geq x] \right) \cdot r_{TS,i}(y_{i}^{x-1,M-x+1})(s_i^1) \\
& \geq \mathbb{P}[X_{i}^{M} \geq x] + \left( \mathbb{P}[X_{j}^{M} = x-1] - \mathbb{P}[X_{j}^{M} \geq x] \right) \cdot r_{TS,i}(y_{i}^{x-1,M-x+1})(s_i^1) \\
& = \mathbb{P}[X_{i}^{M} \geq x] + \left( \mathbb{P}[X_{j}^{M} = x-1] \right) \cdot r_{TS,i}(y_{i}^{x-1,M-x+1})(s_i^1),
\end{align*}
\]

where the first inequality uses the inductive hypothesis \( X_{i}^{M} \) FOSD \( X_{j}^{M} \) to deduce

\[
\mathbb{P}[X_{i}^{M} \geq x] - \mathbb{P}[X_{j}^{M} \geq x] \geq 0,
\]

while the second inequality uses the same hypothesis to get \( \mathbb{P}[X_{i}^{M} \geq x-1] \geq \mathbb{P}[X_{j}^{M} \geq x-1] \).

By third-party equivalence of \( y_i^{x-1,M-x+1} \) and \( y_j^{x-1,M-x+1} \) and Step 1, \( r_{TS,i}(y_i^{x-1,M-x+1})(s_i^1) \geq r_{TS,j}(y_j^{x-1,M-x+1})(s_j^1) \). This shows \( \mathbb{P}[X_{i}^{M+1} \geq x] \geq \mathbb{P}[X_{j}^{M+1} \geq x] \), using Equation (2). The proof of Claim 3(b) is complete and we have shown \( X_{i}^{M+1} \) FOSD \( X_{j}^{M+1} \).

By induction, \( X_{i}^{t} \) FOSD \( X_{j}^{t} \) for all \( t \geq 1 \). The proof of Claim 3(a) is complete.

**Step 4: Translating into aggregate response.**

The FOSD established in Step 3 implies that, \( \mathbb{E}[X_{i}^{t}] \geq \mathbb{E}[X_{j}^{t}] \) for each \( t \). We have by telescoping:

\[
\sum_{t=1}^{\infty} (\gamma^{t-1} - \gamma^{t}) \cdot \mathbb{E}[X_{i}^{t}] = \sum_{t=1}^{\infty} \gamma^{t-1} \cdot \sigma_{i}^{t}(\mathfrak{A}).
\]

So, \( \sum_{t=1}^{\infty} \gamma^{t-1} \cdot \sigma_{i}^{t}(\mathfrak{A}) \geq \sum_{t=1}^{\infty} \gamma^{t-1} \cdot \sigma_{j}^{t}(\mathfrak{A}) \), which completes the proof by Step 2. \(\square\)
OA 2  Additional Results and Examples

OA 2.1  An $\epsilon$-equilibrium that Violates the Conclusion of Proposition 2

Lemma 1 gives a condition satisfied by all $\epsilon$-equilibria where $\epsilon$ satisfies player compatibility. The following example shows that when $\epsilon$ violates player compatibility, the conclusion of Lemma 1 need not hold.

Example 7. There are 3 players: player 1 chooses a row, player 2 chooses a column, and player 3 chooses a matrix.

<table>
<thead>
<tr>
<th></th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top</td>
<td>1,1,0</td>
<td>1,4,0</td>
</tr>
<tr>
<td>Bottom</td>
<td>3,1,0</td>
<td>3,4,0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top</td>
<td>3,3,1</td>
<td>3,2,1</td>
</tr>
<tr>
<td>Bottom</td>
<td>1,3,1</td>
<td>1,2,1</td>
</tr>
</tbody>
</table>

When player 3 chooses Matrix A, player 1 gets 1 from Top and 3 from Bottom. When player 3 chooses Matrix B, player 1 gets 3 from Top and 1 from Bottom. Player 2 always gets the same utility from Left as player 1 does from Top, but player 2 gets 1 more utility from Right than player 1 gets from Bottom. Therefore, $\mathbf{(Right \mid P2)} \succeq \mathbf{(Bottom \mid P1)}$.

Suppose $\epsilon$($\text{Bottom \mid P1}$)$=2h$ and all other minimum probabilities are $h$ for any $0 < h < \frac{1}{2}$. In any $\epsilon$-equilibrium $\sigma^o$, $\sigma_3^o(\text{Matrix B}) > \frac{1}{2}$, so player 1 puts the minimum probability $2h$ on Bottom, and player 2 puts the minimum probability $h$ on Right. This violates the conclusion of lemma 1.

♦

OA 2.2  Pre-Programmed Response Paths

The next result shows that playing against opponent strategies drawn i.i.d. from $\sigma_{-i}$ each period generates the same experimentation frequencies as playing against a pre-programmed response paths drawn according to a certain distribution $\eta$ at the start of the learner’s life, then held fixed forever. The $\eta$ is the same for all agents and does not depend on their (possibly stochastic) learning rules. In the main text, this result is used to couple together the learning problems of two agents $i \neq j$ and compare their experimentation frequencies.

Lemma OA.1. In a factorable game, for each $\sigma \in \times_k \Delta(S_k)$, there is a distribution $\eta$ over pre-programmed response paths, so that for any player $i$, any possibly random rule $r_i : Y_i \rightarrow \Delta(S_i)$, and any strategy $s_i \in S_i$, we have

$$\phi_i(s_i; r_i, \sigma_{-i}) = (1 - \gamma) E_{\mathfrak{A} \sim \eta} \left[ \sum_{t=1}^{\infty} \gamma^{t-1} \cdot (y^t_i(\mathfrak{A}, r_i) = s_i) \right],$$

56
where \( y_i^t(\mathbf{a}, r_i) \) refers to the \( t \)-th period history in \( y_i(\mathbf{a}, r_i) \).

**Proof.** In fact, we will prove a stronger statement: we will show there is such a distribution that induces the same distribution over period-\( t \) histories for every \( i \), every learning rule \( r_i \), and every \( t \).

Think of each pre-programmed response path \( \mathbf{a} \) as a two-dimensional array, \( \mathbf{a} = (a_{t,I})_{t \in N, I \in \mathcal{I}} \). For non-negative integers \( (N_I)_I \in \mathcal{I} \), each profile of sequences of actions \( ((a_{h_{t,I}}^{N_I})_{h_{I} = 1})_{I \in \mathcal{I}} \) where \( a_{h_{t,I}} \in A_I \) defines a “cylinder set” of pre-programmed response paths with the form:

\[
\{ \mathbf{a} : a_{t,I} = a_{h_{t,I}} \text{ for each } I \in \mathcal{I}, 1 \leq h_{t} \leq N_{I} \}.
\]

That is, the cylinder set consists of those response paths whose first \( N_{I} \) elements for information set \( I \) match a given sequence, \( (a_{h_{t,I}}^{N_I})_{h_{I} = 1} \). (If \( N_{I} = 0 \), then there is no restriction on \( a_{t,I} \) for any \( t \).) We specify the distribution \( \eta \) by specifying the probability it assigns to these cylinder sets:

\[
\eta \{ (a_{h_{t,I}}^{N_I})_{h_{I} = 1} \}_{I \in \mathcal{I}} = \prod_{I \in \mathcal{I}} \prod_{h_{I} = 1}^{N_{I}} \sigma(s : s(I) = a_{h_{t,I}}),
\]

where we have abused notation to write \( ((a_{h_{t,I}}^{N_I})_{h_{I} = 1})_{I \in \mathcal{I}} \) for the cylinder set satisfying this profile of sequences, and we have used the convention that the empty product is defined to be 1. Recall that a strategy profile \( s \) in the extensive-form game specifies an action \( s(I) \in A_I \) for every information set \( I \) in the game tree. The probability that \( \eta \) assigns to the cylinder set involves multiplying the probabilities that the given mixed strategy \( \sigma \) leads to such a pure-strategy profile \( s \) so that \( a_{h_{t,I}} \) is to be played at information set \( I \), across all such \( a_{h_{t,I}} \) restrictions defining the cylinder set.

We establish the claim by induction on \( t \) for period-\( t \) history. In the base case of \( t = 1 \), we show playing against a response path drawn according to \( \eta \) and playing against a pure strategy\(^{25}\) drawn from \( \sigma_{-i} \in \times_{k \neq i} \Delta(S_k) \) generate the same period-1 history. Fixing a learning rule \( r_i : Y_i \to \Delta(S_i) \) of \( i \), the probability of \( i \) having the period-1 history \( (s_i^{(1)}, (a_I^{(1)})_{I \in F_i[s_i^{(1)}]}) \in Y_i[1] \) in the random-matching model is \( r_i(\emptyset)(s_i^{(1)}) \cdot \sigma(s : s(I) = a_I^{(1)}) \) for all \( I \in F_i[s_i^{(1)}] \). That is, \( i \)'s rule must play \( s_i^{(1)} \) in the first period of \( i \)'s life, which happens with probability \( r_i(\emptyset)(s_i^{(1)}) \). Then, \( i \) must encounter such a pure strategy that generates the required profile of moves \( (a_I^{(1)})_{I \in F_i[s_i^{(1)}]} \) on the \( s_i^{(1)} \)-relevant information sets, which has probability \( \sigma(s : s(I) = a_I^{(1)}) \) for all \( I \in F_i[s_i^{(1)}] \). The probability of this happening against a response path drawn

\(^{25}\)In the random matching model agents are facing a randomly drawn pure strategy profile each period (and not a fixed behavior strategy): they are matched with random opponents, who each play a pure strategy in the game as a function of their personal history. From Kuhn’s theorem, this is equivalent to facing a fixed profile of behavior strategies.
from \( \eta \) is

\[
\begin{align*}
    r_i(0)(s_i^{(1)}) \cdot \eta(\mathcal{A}: a_{I} = a_{I}^{(1)} \text{ for all } I \in F_i[s_i^{(1)}])
    &= r_i(0)(s_i^{(1)}) \cdot \prod_{I \in F_i[s_i^{(1)}]} \sigma(s : s(I) = a_{I}^{(1)}) \\
    &= r_i(0)(s_i^{(1)}) \cdot \sigma(s : s(I) = a_{I}^{(1)} \text{ for all } I \in F_i[s_i^{(1)}]),
\end{align*}
\]

where the second line comes from the probability \( \eta \) assigns to cylinder sets, and the third line comes from the fact that \( \sigma \in \times_k \Delta(S_k) \) involves independent mixing of pure strategies across different players.

We now proceed with the inductive step. By induction, suppose random matching and the \( \eta \)-distributed response path induce the same distribution over the set of period-\( T \) histories, \( Y_i[T] \), where \( T \geq 1 \). Write this common distribution as \( \phi_{i,T}^{RM} = \phi_{i,T}^{\eta} = \phi_{i,T} \in \Delta(Y_i[T]) \). We prove that they also generate the same distribution over length \( T + 1 \) histories.

Suppose random matching generates distribution \( \phi_{i,T+1}^{RM} \in \Delta(Y_i[T+1]) \) and the \( \eta \)-distributed response path generates distribution \( \phi_{i,T+1}^{\eta} \in \Delta(Y_i[T+1]) \). Each length-\( T + 1 \) history \( y_i[T+1] \in Y_i[T+1] \) may be written as \((y_i[T], (s_i^{(T+1)}, (a_I^{(T+1)})_{I \in F_i[s_i^{(T+1)}]})\), where \( y_i[T] \) is a length-\( T \) history and \((s_i^{(T+1)}, (a_I^{(T+1)})_{I \in F_i[s_i^{(T+1)}]})\) is a one-period history corresponding to what happens in period \( T + 1 \). Therefore, we may write for each \( y_i[T+1] \),

\[
\phi_{i,T+1}^{RM}(y_i[T+1]) = \phi_{i,T}^{RM}(y_i[T]) \cdot \phi_{i,T+1|T}^{RM}((s_i^{(T+1)}, (a_I^{(T+1)})_{I \in F_i[s_i^{(T+1)}]})|y_i[T]),
\]

and

\[
\phi_{i,T+1}^{\eta}(y_i[T+1]) = \phi_{i,T}^{\eta}(y_i[T]) \cdot \phi_{i,T+1|T}^{\eta}((s_i^{(T+1)}, (a_I^{(T+1)})_{I \in F_i[s_i^{(T+1)}]})|y_i[T]),
\]

where \( \phi_{i,T+1|T}^{RM} \) and \( \phi_{i,T+1|T}^{\eta} \) are the conditional probabilities of the form “having history \((s_i^{(T+1)}, (a_I^{(T+1)})_{I \in F_i[s_i^{(T+1)}]})\) in period \( T + 1 \), conditional on having history \( y_i[T] \in Y_i[T] \) in the first \( T \) periods.” If such conditional probabilities are always the same for the random-matching model and the \( \eta \)-distributed response path model, then from the hypothesis \( \phi_{i,T}^{RM} = \phi_{i,T}^{\eta} \), we can conclude \( \phi_{i,T+1}^{RM} = \phi_{i,T+1}^{\eta} \).

By argument exactly analogous to the base case, we have for the random-matching model

\[
\phi_{i,T+1|T}^{RM}((s_i^{(T+1)}, (a_I^{(T+1)}))|y_i[T]) = r_i(y_i(T))(s_i^{(T+1)}) \cdot \sigma(s : s(I) = a_{I}^{(T+1)} \text{ for all } I \in F_i[s_i^{(T+1)}]),
\]

since the matching is independent across periods.

But in the \( \eta \)-distributed response path model, since a single response path is drawn once and fixed, one must compute the conditional probability that the drawn \( \mathcal{A} \) is such that the response \((a_I^{(T+1)})_{I \in F_i[s_i^{(T+1)}]}\) will be seen in period \( T + 1 \), given the history \( y_i[T] \) (which is
informative about which response path $i$ is facing).

For each $I \in \mathcal{I}_{-i}$, let the non-negative integer $N_I$ represent the number of times $i$ has observed play on the information set $I$ in the history $y_i[T]$. For each $I$, let $(a_{h_I}, I)_{h_I=1}^{N_I}$ represent the sequence of opponent actions observed on $I$ in chronological order. The history $y_i[T]$ so far shows $i$ is facing a response sequence in the cylinder set consistent with $((a_{h_I}, I)_{h_I=1}^{N_I})_{I \in \mathcal{I}}$. If $\mathfrak{A}$ is to respond to $i$’s next play of $s_i^{(T+1)}$ with $a_i^{(T+1)}$ on the $s_i^{(T+1)}$-relevant information sets, then $\mathfrak{A}$ must belong to a more restrictive cylinder set, satisfying the restrictions:

$$
((a_{h_I}, I)_{h_I=1}^{N_I})_{I \in \mathcal{I}\setminus F_i[s_i^{(T+1)}]}; (a_{h_I}, I)_{h_I=1}^{N_I+1})_{I \in F_i[s_i^{(T+1)}]},
$$

where for each $I \in F_i[s_i^{(T+1)}]$, $a_{N_i+1}, i = a_i^{(T+1)}$. The conditional probability is then given by the ratio of $\eta$-probabilities of these two cylinder sets, which from the definition of $\eta$ must be $\prod_{I \in F_i[s_i^{(T+1)}]} \sigma(s : s(I) = a_i^{(T+1)})$. As before, the independence of $\sigma$ across players means this is equal to $\sigma(s : s(I) = a_i^{(T+1)})$ for all $I \in F_i[s_i^{(T+1)}]$. □

### OA 3 PCE and Extended Proper Equilibrium

Here we compare PCE with Milgrom and Mollner (2017)’s extended proper equilibrium.

**Definition OA.5.** Fix an $n$-player strategic-form game where $i$’s pure strategy set is $S_i$. For each mixed strategy profile $\sigma$, let $L_i(s_i | \sigma)$ represent the expected loss for $i$ from playing strategy $s_i$ instead of the best response to $\sigma_{-i}$. For $\beta \in \mathbb{R}^n_{++}$ and $\epsilon > 0$, a $(\beta, \epsilon)$-extended proper equilibrium is a totally mixed strategy profile $\sigma$ such that $\beta_i L_i(s_i | \sigma) > \beta_j L_j(s_j | \sigma)$ implies $\sigma_i(s_i) \leq \epsilon \cdot \sigma_j(s_j)$ for all $i, j \in S_i, s_j \in S_j$. Strategy profile $\sigma^*$ is an extended proper equilibrium if there exist $\beta \in \mathbb{R}^n_{++}$ and sequences $(\epsilon_t), (\sigma_t)$ so that $\epsilon_t > 0$ for each $t$, $\epsilon_t \to 0$, each $\sigma_t$ is a $(\beta, \epsilon_t)$-extended proper equilibrium, and $\sigma_t \to \sigma^*$.

Extended proper equilibrium requires that more costly mistakes are infinitely less likely, where “more costly” is defined relative to a vector $\beta$ of utility rescaling. By contrast, PCE only imposes the restriction that the less compatible player trembles onto a given strategy with smaller probability, without further magnitude restrictions. Neither solution concept is nested in the other. We have shown a number of examples in Section 3 where PCE refines away extended proper equilibria. But conversely, extended proper equilibrium may impose restrictions in some games where PCE has no bite.

**Example 8.** Consider a three-player game where Row chooses a row, Column chooses a column, and Geo chooses a matrix. The payoff to Geo is always 0. The payoffs to Row and Column are listed in the tables below.
The strategy profile (Up, Left, West) is not an extended proper equilibrium, because Column would deviate to Right against a tremble where Row’s costly deviation to Down is much rarer than Geo’s costless deviation to East. However, it is a PCE. For example, take a sequence of trembles $\epsilon^{(t)}$ where the minimum probability on each action is $1/t$. Such uniform trembles always satisfy player compatibility, so every $\epsilon^{(t)}$-equilibrium in this sequence is an $\epsilon^{(t)}$-PCE, and we see that Column is indifferent between Left and Right if Row deviates to Down exactly as much as Geo deviates to East.

### OA 4 Additional Simulations

Our simulations in Section 5 show that link formation rate always converges towards the unique PCE in the anti-monotonic version of the link-formation game, and that starting with different priors can select either of the two PCE outcomes in the co-monotonic version of the game. This section of the Online Appendix examines the robustness of these conclusions to using the exactly optimal experimentation strategy and to different implementation error rates.

First, we consider rational agents who maximize the sum of their expected utility across 50 periods with no discounting. We numerically solve out the rational learning rule using backwards induction and simulate agents who use this rule subject to a 5% chance of implementation error. Backward induction with a longer time horizon proves intractable, as the number of contingencies grows exponentially in horizon length.\(^{26}\) The agents take the possibility of their own errors into account when formulating the rational learning rule. For simplicity, we still assume agents have Beta priors over the probabilities that opponents play Active, even though opponents can choose Active only with probabilities between 5% and 95%. The results are in Figure OA.1.

For all four sets of priors, a society playing the anti-monotonic version of the game converges towards the all-links equilibrium. With very pessimistic priors, the link-formation rate for rational agents by period 50 is higher than that of either Bayes-UCB or Thompson sampling agents with the same prior and in the same period. The link-formation rate in the co-monotonic version of the game converges towards the no-links equilibrium for very pessimistic and pessimistic priors, but stays near the all-links equilibrium for uniform priors.

\(^{26}\)In fact, the computational intractability of the rational rule motivated heuristics like the Bayes-UCB and Thompson sampling.
Figure OA.1: The dynamics of link-formation rates for societies of rational agents, across 50 periods. Dashed lines show the theoretical maximum and minimum link-formation rates subject to the implementation errors.
and optimistic priors. The rational learning rule specifies more experimentation with \textbf{Active} for all players compared to the Bayes-UCB and Thompson sampling, which explains these differences.

Figures OA.2 and OA.3 show simulation results for rational and Bayes-UCB agents with 1\% and 10\% implementation errors, respectively. The results for 10\% implementation error are very similar to those for 5\% implementation error. With 1\% implementation error, societies playing the anti-monotonic version of the game converge much more slowly towards the all-links equilibrium. Nevertheless, Figure OA.4 verifies that, when the Bayes-UCB society is simulated for 10,000 periods, they do eventually converge to this unique PCE.

So far, we have modeled noise in the learning environment as every agent having a small probability of an implementation error each period. Alternatively, we can introduce a small fraction of "commitment" agents into the society, who play the same action every period regardless of their history. The rest of the agents follow their intended learning rule exactly and do not encounter implementation errors. In Figure OA.5, we simulate societies of 160 agents with 40 agents in each player role. There is one agent committed to playing the strategy \textbf{Active} every period and another one committed to playing \textbf{Inactive} every period in each of the 4 player roles. The remaining agents are normal and following the Bayes-UCB heuristic with a quantile level 0.6. When the prior is too pessimistic, either the high-cost normal agents or all agents do not experiment with \textbf{Active} in period 1. Since there are no implementation errors, they keep their prior beliefs forever and learning never takes off. When the prior is highly optimistic, play starts near the all-links equilibrium and stays there. The most interesting dynamics happen with uniform priors, where all societies start near the all-links equilibrium but the link-formation rates for the anti-monotonic and co-monotonic versions of the game diverge with learning. This is because when a normal, high-cost agent is matched with a high-quality commitment agent playing \textbf{Inactive} in early periods, she becomes discouraged by this observation and switches to playing \textbf{Inactive} forever. In the co-monotonic version of the game, the switcher has high-quality herself, so the pool of high-quality \textbf{Inactive} agents grows through contagion. In the anti-monotonic version of the game, the switcher has low-quality, so her switching improves the average quality among agents still playing \textbf{Active}. This countervailing force quickly stabilizes the link-formation rate after a small drop in the early periods.
Figure OA.2: The dynamics of link-formation rates for societies of rational and Bayes-UCB agents, with 1% implementation error. Dashed lines show the theoretical maximum and minimum link-formation rates subject to the implementation errors.
Figure OA.3: The dynamics of link-formation rates for societies of rational and Bayes-UCB agents, with 10% implementation error. Dashed lines show the theoretical maximum and minimum link-formation rates subject to the implementation errors.
Figure OA.4: The dynamics of link-formation rates for societies of Bayes-UCB agents with 1% implementation error, simulated for 10,000 periods.
Figure OA.5: The dynamics of link-formation rates for societies of 160 agents, with 4 agents (one per player role) committed to playing Active every period, and another 4 agents committed to playing Inactive every period.