BRAID GROUPS AND BRAID THOM SPECTRA

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1. Context and mission statement

Recall that the goal of this seminar is to prove the immersion conjecture.

Theorem 1 (R. Cohen [Coh85])**.** *A compact smooth n-manifold M immerses in $\mathbb{R}^{2n-\alpha(n)}$.*

In light of the Hirsch–Smale theorem as discussed in Emily's and Danny's talks, this result is equivalent to the existence of a *formal* immersion, which is to say that the stable normal bundle of M is represented by a bundle of dimension $n-\alpha(n)$. Thus, the theorem is equivalent to the lifting problem

$$\begin{array}{ccc} & & BO(n-\alpha(n)) \\ & \overset{\exists}{\nearrow} & \downarrow \\ M & \xrightarrow{\nu_M} & BO. \end{array}$$

We will solve this lifting problem in a sequence of approximations. So far, we have achieved two of these approximations, the first being the up-to-bordism statement proven in Jun Hou's talk. The second is the following cohomology level statement, which follows from the theorem of Massey [Mas60] proven in Bena's talk:

$$\begin{array}{ccc} H^*(BO)/I_n & \dashleftarrow & H^*(BO(n-\alpha(n))) \\ \downarrow & \nwarrow & \uparrow \\ H^*(M) & \xleftarrow{\nu_M^*} & H^*(BO). \end{array}$$

Here, and for the rest of this talk, all cohomology is implicitly taken with coefficients in $\mathbb{F}_2$.

Our next approximation will be a version at the level of Thom spectra. We will define a spectrum MO/I_n over MO realizing the quotient by I_n under the Thom isomorphism Φ, and we will situate it in a commutative diagram of the following form [BP77]:

$$\begin{array}{ccc} MO/I_n & \dashrightarrow & MO(n-\alpha(n)) \\ \uparrow & \searrow & \downarrow \\ T\nu_M & \longrightarrow & MO. \end{array}$$

In order to begin this task, let us recall two more facts that we have seen. First, from Brown–Peterson [BP64] via Bena's talk, we have complete knowledge of the ideal in question:

$$I_n = \sum_{j > \lceil \frac{n-i}{2} \rceil} H^i(BO) Sq^j.$$

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Second, from Thom [Tho54] via Jun Hou's talk, we know that

$$H^*(MO) \cong \bigoplus_w \Sigma^{|w|} A,$$

where w ranges over all tuples $(j_1, \ldots, j_r)$ with $j_i + 1$ not a binary power. Combining these facts, it follows that

$$H^*(MO)/\Phi(I_n) \cong \bigoplus_{|w| \leq n} \Sigma^{|w|} A/J_{\lceil \frac{n-|w|}{2} \rceil}, \qquad J_k := A\left\{\chi(Sq^j) : j > k\right\}.$$

Therefore, our immediate goal will be to realize the A-modules A/J_k as Thom spectra.

2. Vector bundles on configuration spaces

If X is a topological space, the ordered *configuration space* of k points in X is

$$\mathrm{Conf}_k(X) = \{(x_1, \ldots, x_k) : x_i \neq x_j \text{ if } i \neq j\},$$

and the corresponding unordered configuration space is the quotient $B_k(X) = \mathrm{Conf}_k(X)/\Sigma_k$. If X is a connected, aspherical manifold of dimension greater than 1, a theorem of Fadell–Neuwirth [FN62a] implies that each configuration space is again connected and aspherical, hence a classifying space for its fundamental group. In the case $X = \mathbb{C}$, a theorem of Fox–Neuwirth [FN62b] identifies this fundamental group.

Definition 2 (Artin [Art47])**.** The *braid group* on k strands is

$$\pi_1(B_k(\mathbb{C})) \cong \beta_k := \langle \sigma_1, \ldots, \sigma_{k-1} : \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1\rangle.$$

There is a canonical vector bundle γ_k over $B_k(\mathbb{C})$, which may either be constructed explicitly as

$$\gamma_k : \mathrm{Conf}_k(\mathbb{C}) \times_{\Sigma_k} \mathbb{R}^k \xrightarrow{\mathbb{R}^k \to \mathrm{pt}} B_k(\mathbb{C})$$

or described via its classifying map as induced on classifying spaces by the composite

$$\beta_k \to \Sigma_k \to O(k),$$

where the first map sends σ_i to the transposition $(i, i+1)$, and the second map is the permutation representation. We can now state the main result of the talk.

Theorem 3. *There is an isomorphism of A-modules*

$$H^*(T\gamma_k) \cong A/J_{\lceil \frac{k}{2} \rceil}.$$

There are obvious inclusions $\beta_1 \to \beta_2 \to \cdots \to \beta_\infty$, where the latter is defined to be the colimit. These maps are compatible with the classifying maps of the various γ_k, and there results a stable vector bundle γ_∞ over $B\beta_\infty$.

Corollary 4 (Mahowald [Mah77], F. Cohen [Coh78])**.** *There is an equivalence $T\gamma_\infty \simeq H\mathbb{F}_2$.*

Remark 5. The mod 2 Steenrod problem, originally solved in the affirmative by Thom, asks whether an arbitrary homology class in a space is represented by a map from a compact manifold. Corollary 4 asserts that, in fact, this manifold may be taken to *braid oriented*, in particular admitting a flat connection, and that the choice of such a braid oriented manifold is essentially unique.

Remark 6. Oddly enough, the complexification $\gamma_k \otimes \mathbb{C}$ is trivial for every k. Indeed, this bundle is the tangent bundle of the complex manifold $B_k(\mathbb{C})$, which embeds as an open subset of $\mathbb{C}^k$ via the map the configuration $\{z_1, \ldots, z_k\}$ to the (ordered!) collection of coefficients of the polynomial $\prod_i (z - z_i)$.

3. (Co)homology of braid groups

The main input to the proof of Theorem 3 will be an understanding of the (co)homology of the spaces $B_k(\mathbb{C})$. We follow Fuks [Fuk70] in our calculation; an alternative approach is due to F. Cohen [CLM76].

3.1. Additive structure.

It will be convenient to pass through the Poincaré duality isomorphism

$$H^i(B_k(\mathbb{C})) \cong \widetilde{H}_{2k-i}(B_k(\mathbb{C})^+),$$

as the compactification admits a natural decomposition into cells, which are often called Fox–Neuwirth cells [FN62b].

Any configuration of k points in the plane may be described by giving the real parts of the points together with, for each such real part, a configuration of points in the imaginary line. More formally, an ordered partition $k = \lambda_1 + \cdots + \lambda_r$ determines a subspace

$$\begin{aligned} B_k(\mathbb{C})^+ \supseteq e(\lambda) &\cong B_r(\mathbb{R}) \times B_{\lambda_1}(i\mathbb{R}) \times \cdots \times B_{\lambda_r}(i\mathbb{R}) \\ &\cong \mathring{\Delta}^r \times \mathring{\Delta}^{\lambda_1} \times \cdots \times \mathring{\Delta}^{\lambda_r}, \end{aligned}$$

which is an open cell of dimension $k+r$. Some of the boundary components of this cell correspond to two distinct real parts colliding and hence consist of other cells; the rest, which correspond to configurations colliding or escaping to infinity, we attach to the point at infinity. From this description, one deduce the following formula:

Lemma 7. *The differential in the reduced cellular chain complex of $B_k(\mathbb{C})^+$ is determined by the formula*

$$\partial e(m_1, \ldots, m_r) = \sum_{i=1}^{r-1} \binom{m_i + m_{i+1}}{m_i} e(m_1, \ldots, m_i + m_{i+1}, \ldots, m_r).$$

We will say that a cellular chain c is *symmetric* if the coefficients of $e(\lambda)$ and of $e(\sigma \cdot \lambda)$ in c are equal for every $\sigma \in \Sigma_r$. We will say that c is *binary* if every λ is a partition into binary powers whenever the coefficient of $e(\lambda)$ is nonzero.

Lemma 8. *Every class in $\widetilde{H}_*(B_k(\mathbb{C})^+)$ is represented by a unique symmetric binary chain.*

Proof. First, we claim that every symmetric chain c is a cycle. Consider the coefficient of $e(m_1, \ldots, m_i + m_{i+1}, \ldots m_r)$ in ∂c, where c is a symmetric chain. If $m_i = m_{i+1}$, then this coefficient is $\binom{2m_i}{m_i}$ is even, and, if $m_i \neq m_{i+1}$, then the assumption that c is symmetric implies that the coefficient is even. Thus, the inclusion of the symmetric binary chains into the cellular chains is a chain map, where the source is regarded as a chain complex with trivial differential.

Second, we note that binary cells do not appear with nonzero coefficient in the boundary of any chain, since $\binom{2^r}{i}$ is even for $0 < i < 2^r$. It follows that the chain map of the previous paragraph is injective on homology.

Third, we note that any binary cycle is symmetric, which follows from the fact that $\binom{2^a+2^b}{2^a}$ is odd by Lucas' theorem.

Thus, in order to show that the chain map of the first paragraph is surjective on homology, implying the lemma, it suffices to show that every cycle c is homologous to a binary cycle. We omit the somewhat involved combinatorics of this argument—see [Fuk70, 4.7]. □

Thus, the rank of $H^i(B_k(\mathbb{C}))$ is equal to the number of ways of writing k as an (unordered!) sum of $k - i$ binary powers. We write $\langle 2^{\ell_1}, \ldots, 2^{\ell_r} \rangle$ with $\ell_1 \geq \cdots \geq \ell_r$ for the cohomology class Poincaré dual to $\sum_\sigma e(2^{\ell_{\sigma(1)}}, \ldots, 2^{\ell_{\sigma(r)}})$.

3.2. **Stabilization and Hopf structure.** There is a group homomorphism $\beta_k \times \beta_\ell \to \beta_{k+\ell}$ for each k and ℓ, which can be described geometrically in terms of the map on configuration spaces

$$\begin{gathered} B_k(\mathbb{C}) \times B_\ell(\mathbb{C}) \to B_{k+\ell}(\mathbb{C}) \\ (\{z_1, \ldots, z_k\}, \{w_1, \ldots, w_\ell\}) \mapsto \{z_1, \ldots, z_k, w_1 + a, \ldots, w_\ell + a\}, \end{gathered}$$

where a is one plus the difference between the centers of mass of the configurations plus the sum of the respective largest deviations from the centers of mass. In particular, taking $\ell = 1$ and $w = \{0\}$, we obtain the *stabilization map*

$$\sigma_k : B_k(\mathbb{C}) \to B_{k+1}(\mathbb{C}).$$

Lemma 9. *The map $\sigma_k^* : H^i(B_{k+1}(\mathbb{C})) \to H^i(B_k(\mathbb{C}))$ is given by the formula*

$$\sigma_k^* \langle 2^{\ell_1}, \ldots, 2^{\ell_{k-i}} \rangle = \begin{cases} \langle 2^{\ell_1}, \ldots, 2^{\ell_{k-1-i}} \rangle & \ell_{k-i} = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If c is a cycle in $B_k(\mathbb{C})$ then

$$\sigma_k(c) \cap e(\lambda_1, \ldots, \lambda_r) = \begin{cases} \sigma_k(c \cap e(\lambda_1, \ldots, \lambda_{r_1})) & \lambda_r = 1 \\ \varnothing \quad \text{otherwise.} \end{cases}$$

□

Corollary 10. *The map $\sigma_k^* : H^i(B_{k+1}(\mathbb{C})) \to H^i(B_k(\mathbb{C}))$ is surjective, and it is an isomorphism if either k is even or $i \leq k/2$.*

Proof. The first claim is immediate from Lemma 9, the second and third follow from the additional observation that 2^0 appears in any representation of an odd number as a sum of binary powers and in any representation of k as a sum of more than $k/2$ binary powers. □

It also follows that $H^*(\beta_\infty)$ has an additive basis given by all expressions of the form $\langle 2^{\ell_1}, \ldots, 2^{\ell_r} \rangle$ with $\ell_1 \geq \cdots \geq \ell_r > 0$, and the kernel of the map $H^*(\beta_\infty) \to H^*(\beta_k)$ is the submodule of elements of *weight* greater than k, where the weight of a basis element is the sum of its entries.

Now, the pairings above fit into a commuting diagram

$$\begin{array}{ccc} \beta_k \times \beta_\ell & \longrightarrow & \beta_{k+\ell} \\ \downarrow & & \downarrow \\ \beta_\infty \times \beta_\infty & \dashrightarrow & \beta_\infty. \end{array}$$

The geometric description shows that the bottom map is associative and commutative up to homotopy, and $H^*(\beta_\infty)$ obtains in this way the structure of a (bicommutative) Hopf algebra. By considering intersections as in Lemma 9, one arrives at the following formula.

Lemma 11. *The comultiplication of $H^*(\beta_\infty)$ is given by the formula*

$$\langle 2^{\ell_1}, \ldots, 2^{\ell_r} \rangle \mapsto \sum \langle 2^{\ell_{i_1}}, \ldots, 2^{\ell_{i_j}} \rangle \otimes \langle 2^{\ell_{i_{j+1}}}, \ldots, 2^{\ell_{i_r}} \rangle,$$

where the sum is taken over all decompositions of $\{1, \ldots, r\}$ into two ordered subsets.

We draw the following important conclusion.

Corollary 12. *There is an algebra isomorphism*

$$H_*(\beta_\infty) \cong \mathbb{F}_2[x_k : k \geq 1],$$

where x_k is the basis element dual to $\langle 2^k \rangle$.

Remark 13. Corollary 12 asserts that the dual Hopf algebra $H_*(\beta_\infty)$ is formally isomorphic *as an algebra* to the dual Steenrod algebra, and we will see that this fact is no coincidence. On the other hand, this isomorphism is certainly not an isomorphism of Hopf algebras; for example, the dual Steenrod algebra is not cocommutative.

3.3. **Cohomology ring and Stiefel–Whitney classes.** In order to identify the structure of the cohomology ring, we require one geometric computation.

Lemma 14. *The generator $\langle 2^i\rangle \in H^{2^i-1}(B_{2^i}(\mathbb{C}))$ is indecomposable.*

Proof. Define a submanifold $M_i \subseteq B_{2^i}(\mathbb{C})$ recursively as follows: M_1 consists of configurations of two antipodal points on the unit circle, and M_i is constructed by allowing appropriately scaled copies of M_{i-1} to orbit one another antipodally on the unit circle (see [Fuk70, Fig. 3] for an illustration of this space). The submanifold M_i intersects $e(2^i)$ transversely in a single point, so it suffices to show that the cohomology of this M_i is indecomposable in top degree. But $M_i = S^1 \times_{C_2} (M_{i-1} \times M_{i-1})$ is a twisted product, so the claim follows from the cohomology calculation of Jun Hou's lecture. □

Corollary 15. *The ring structure on $H^*(\beta_\infty)$, and hence on $H^*(B_k(\mathbb{C}))$, is given by the formula*

$$\left\langle \underbrace{2^m,\dots,2^m}_{a_m},\dots,\underbrace{2,\dots,2}_{a_1} \right\rangle \left\langle \underbrace{2^m,\dots,2^m}_{b_m},\dots,\underbrace{2,\dots,2}_{b_1} \right\rangle = \binom{a_m+b_m}{a_m}\cdots\binom{a_1+b_1}{a_1} \left\langle \underbrace{2^m,\dots,2^m}_{a_m+b_m},\dots,\underbrace{2,\dots,2}_{a_1+b_1} \right\rangle.$$

Proof. Lemma 14 implies that x_k is primitive in $H_*(\beta_\infty)$. Applying the dual cup product to $x_1^{a_1+b_1}\cdots x_m^{a_m+b_m}$ and using this fact yields the claim. □

Although we will not use this caculation, it is interesting to compute the Stiefel–Whitney classes of the bundle γ_k.

Proposition 16. *The duality isomorphism $H^i(B_k(\mathbb{C})) \xrightarrow{\simeq} \widetilde{H}_{2n-i}(B_k(\mathbb{C})^+)$ sends $w_i(\gamma_k)$ to the class represented by the sum of all cells of dimension $2n-i$.*

Proof. In general, $w_i(\gamma)$ is Poincaré dual to the locus of linear dependence of a collection of $k-i+1$ generic smooth sections of γ. We define sections s_j of γ_k for $1 \le j \le k-i+1$ by the formula

$$s_j(\{z_1,\dots,z_k\}) = \left\{ \left(z_1, \Re(z_1)^{j-1}\right),\dots,\left(z_k,\Re(z_k)^{j-1}\right)\right\}.$$

These sections are linearly dependent at the configuration $\{z_1,\dots,z_k\}$ if and only if there is a nonzero (real) polynomial of degree $k-i$ vanishing on the set $\{\Re(z_j)\}_{j=1}^k$. Such a polynomial exists precisely when this set has cardinality at most $k-i$, which is to say that the points of the configuration are distributed among $k-i$ vertical lines. □

Corollary 17. *The homomorphism $H^*(BO(k)) \to H^*(B_k(\mathbb{C}))$ induced by γ_k is surjective.*

Proof. First, we note that $H^*(BO) \to H^*(\beta_\infty)$ is surjective by induction on degree. Indeed, it suffices to check that each $\langle 2^k\rangle$ lies in the image of this map, but Proposition 16 implies that the sum of all basis element of degree 2^k-1 lies in the image, and all of these basis elements but $\langle 2^k\rangle$ are decomposable and so lie in the image by induction. The claim now follows from the commuting diagram

$$\begin{array}{ccc} H^*(BO) & \twoheadrightarrow & H^*(\beta_\infty) \\ \downarrow & & \downarrow\!\!\downarrow \\ H^*(BO(k)) & \longrightarrow & H^*(B_k(\mathbb{C})) \end{array}$$

□

Using this fact, one could go on to determine the structure of $H^*(B_k(\mathbb{C}))$ explicitly as a module over the Steenrod algebra.

4. Braid Thom spectra

We now assemble the remainder of the proof of Theorem 3. To begin we recall that

$$A^* \cong \mathbb{F}_2[\xi_i : i \geq 1],$$

where ξ_i is the ith Milnor generator, i.e., it is the dual basis element to $Sq^{2^{i-1}} \cdots Sq^1$. Thus, setting $t_i = \chi^*(\xi_i)$, it follows that A^* is equally a polynomial algebra on the t_i. Imitating our earlier calculation, we define a *weight* grading on A^* by requiring that

- $\mathrm{wt}(1) = 0$,
- $\mathrm{wt}(t_i) = 2^i$, and
- $\mathrm{wt}(\alpha\beta) = \mathrm{wt}(\alpha) + \mathrm{wt}(\beta)$.

Lemma 18. *The submodule of A^* of weight at most k is precisely $(A/J_{\lceil \frac{k}{2} \rceil})^*$.*

Proof. Recall that $J_{\lceil \frac{k}{2} \rceil} = A\left\{\chi(Sq^j) : 2j > k\right\}$. Since χ is an anti-automorphism, and since the elements Sq^I with $I = (i_1, \ldots, i_r)$ admissible form a basis for A, it follows that the elements $\chi(Sq^I)$ with I admissible and $2i_1 > k$ form a basis for $J_{\lceil \frac{k}{2} \rceil}$. Recall that the excess of I is $e(I) = i_1 - i_2 - \cdots i_r$, and the degree of I is $i_1 + i_2 \cdots + i_r$; therefore,

$$2i_i = e(I) + |I|.$$

The lemma will follow upon verifying that, if α is a monomial in the generators t_i, then $\mathrm{wt}(\alpha)$ is equal to $e(I) + |I|$, where Sq^I is the basis element dual to α, which follows from the fact that this equality holds for each t_i. □

In order to conclude the statement of the theorem, it suffices to show that $x_i \mapsto t_i$ in the top composite of the solid diagram

$$\begin{array}{ccccccccccc} \mathbb{F}_2[x_i] & \overset{\sim}{=\!=} & H_*(\beta_\infty) & \xrightarrow{\simeq} & H_*(T\gamma_\infty) & \longrightarrow & H_*(MO) & \longrightarrow & A^* & \overset{\sim}{=\!=} & \mathbb{F}_2[t_i] \\ \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\ \mathbb{F}_2[x_i]_k & \overset{\sim}{=\!=} & H_*(B_k(\mathbb{C})) & \xrightarrow{\simeq} & H_*(T\gamma_k) & & \dashrightarrow & & (A/J_{\lceil \frac{k}{2} \rceil})^* & \overset{\sim}{=\!=} & \mathbb{F}_2[t_i]_k, \end{array}$$

for given this it will follow that the dashed filler exists and is an isomorphism.

The idea for verifying this claim is that, although x_i is indecomposable, it becomes decomposable after adding more structure. Indeed, as we saw in the proof of Lemma 14, x_i arises from the fundamental class of a manifold M_i embedded via ι_i, say, and the following diagram commutes:

$$\begin{array}{ccc} S^1 \times_{C_2} (M_i \times M_i) & & \\ \big\downarrow{\scriptstyle S^1\times_{C_2}(\iota_i\times\iota_i)} & \searrow^{\iota_{i+1}} & \\ S^1 \times_{C_2} (B_{2^i}(\mathbb{C}) \times B_{2^i}(\mathbb{C})) & \longrightarrow & B_{2^{i+1}}(\mathbb{C}), \end{array}$$

where the bottom map is obtained by placing the center of mass of the respective configurations antipodally on the unit circle after scaling appropriately.

We conclude that x_{i+1} arises as a "twisted product" of x_i with itself, and this twisted product is a remnant of the E_2-algebra structure on $\bigsqcup_{k\geq 0} B_k(\mathbb{C})$; indeed, this space is the free E_2-algebra on a point. Moreover, since $H\mathbb{F}_2$ is also an E_2-algebra—in fact, an E_∞-algebra—one also has such a twisted product on the dual Steenrod algebra, which has been calculated.

Proposition 19 (Steinberger [Ste86, Thm. 2.2], Nishida [Nis68])**.** *The generator t_{i+1} is the twisted product of t_i with itelf.*

Remark 20. What we are calling a twisted product is otherwise known as a $\cup_1$-product, and the dictionary between $\cup_i$-products and Dyer–Lashoff operations is that the Dyer–Lashoff operation Q^r evaluated on an element of degree p is the $\cup_{r-p}$-product of the element with itself. Taking $p = 2^i - 1$ and $r = 2^i$, Proposition 19 is the equation $Q^{2^i}\chi(\xi_i) = \chi(\xi_{i+1})$, which is Steinberger's computation.

Proof of Theorem 3. First, we check that $x_1 \mapsto t_1$. For this, we note that there is a C_2-equivariant homotopy equivalence

$$\begin{aligned} \mathrm{Conf}_2(\mathbb{C}) &\xrightarrow{\sim} S^1 \\ (z_1, z_2) &\mapsto \frac{z_1 - z_2}{\|z_1 - z_2\|}, \end{aligned}$$

from which we obtain the horizontal equivalences in the commuting diagram

$$\begin{array}{ccc} \mathrm{Conf}_2(\mathbb{C}) \times_{C_2} \mathbb{R}^2 & \xrightarrow{\sim} & S^1 \times_{C_2} \mathbb{R}^2 \\ \big\downarrow{\scriptstyle \gamma_2} & & \big\downarrow{\scriptstyle \mathrm{triv}\oplus\mathrm{sgn}} \\ B_2(\mathbb{C}) & \xrightarrow{\sim} & \mathbb{RP}^1. \end{array}$$

Therefore, the classifying map $B_2(\mathbb{C}) \to BO$ of γ_2 is homotopic to the inclusion of $\mathbb{RP}^1$, and the claim follows.

Finally, we note we have a homotopy commutative diagram

$$\begin{array}{ccc} \bigsqcup_{k\geq 0} B_k(\mathbb{C}) \xrightarrow{\mathbb{C}\subseteq\mathbb{R}^\infty} \bigsqcup_{k\geq} B_k(\mathbb{R}^\infty) \simeq \bigsqcup_{k\geq} B\Sigma_k & \longrightarrow & \bigsqcup_{k\geq 0} BO(k) \\ \big\uparrow & & \big\downarrow \\ B_k(\mathbb{C}) & \xrightarrow{\gamma_k} & BO \times \mathbb{Z} \end{array}$$

in which the upper left horizontal arrow is a map of E_2-algebras and the upper right horizontal arrow and righthand vertical arrow are maps of E_∞-algebras, and in particular of E_2-algebras. It now follows from formal properties of the Thom spectrum functor that the twisted products on either side are compatible, so $x_i \mapsto t_i$, as desired. $\square$

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