The goal of these notes is to give an outline of Johnson’s calculation of the Goodwillie derivatives of the identity functor on pointed spaces [Joh95]. Recall that the theory of Goodwillie calculus associates to a reduced homotopy functor $F : \text{Top}_* \to \text{Top}_*$ a tower of functors

$$
\begin{array}{c}
\vdots \\
\downarrow \\
P_n(F) \\
\downarrow \\
P_1(F) \\
\downarrow \\
P_0(F) \\
\end{array}
$$

where $P_n(F)$ is the universal $n$-excisive or $n$-polynomial approximation to $F$ [Goo03]. Setting $F = \text{id}$, a theorem of Goodwillie asserts that the natural map

$$
X \to \text{holim} \, P_n(\text{id})(X)
$$

is a weak equivalence for $X$ simply connected. Thus, it is of interest to understand the layers

$$
D_n(F) := \text{fib} \, (P_n(F) \to P_{n-1}(F)).
$$

This functor is an $n$-homogeneous functor, and these are completely classified.

**Theorem 1** (Goodwillie). The assignment $E \mapsto \Omega^\infty (E \wedge \Sigma_n (-)^\wedge n)$ extends to an equivalence of $\infty$-categories between $\Sigma_n$-spectra and degree $n$ homogeneous functors.

This theorem motivates the following definition.

**Definition 2.** The $n$th derivative of $F$ is the $\Sigma_n$-spectrum $\partial_n(F)$ such that

$$
D_n(F) \simeq \Omega^\infty (\partial_n(F) \wedge \Sigma_n (-)^\wedge n).
$$

Our goal is to understand the symmetric sequence $\{\partial_n(\text{id})\}_{n \geq 0}$. In order to do so, we require an algorithm for computing $\partial_n(F)$ in terms of $F$.

**Definition 3.** Let $I$ be a finite set, and write $P(I) = \{0, 1\}^I$ for the set of subsets of $I$, partially ordered by inclusion.

1. A (pointed) $I$-cube is a functor $\mathcal{X} : P(I) \to \text{Top}_*$.

*Date: 8 February 2018.*
(2) The total fiber of the $I$-cube $X$ is

$$\text{tfib}(X) := \text{fib}\left(X(\emptyset) \to \lim_{\emptyset \neq S \in P(I)} X(S) \right).$$

We write $[n] = \{1, \ldots, n\}$.

**Example 4.** A $[0]$-cube is a space, and the total fiber is the same space.

**Example 5.** A $[1]$-cube is a morphism, and the total fiber is its fiber.

**Example 6.** A $[2]$-cube is a commuting square, and the total fiber is the iterated fiber.

**Definition 7.** The $n$th cross effect of the functor $F : \text{Top} \to \text{Top}$ is the functor $\text{cr}_n : \text{Top} \to \text{Top}$ defined by the formula

$$\text{cr}_n(F)(X_1, \ldots, X_n) = \text{tfib}\left(S \mapsto \bigvee_{i \in S \subseteq [n]} X_i \right).$$

With this definition in hand, we can state the following useful recipe.

**Proposition 8.** Let $F : \text{Top} \to \text{Top}$ be a reduced homotopy functor. There are natural $\Sigma_n$-equivariant equivalences

$$\Omega^n \partial_n(F) \simeq \colim_{k_1, \ldots, k_n} \Omega^{k_1+\cdots+k_n} \text{cr}_n(F)(S^{k_1}, \ldots, S^{k_n}).$$

This formula should be compared to the usual formula for the linearization $\Omega^n G \Sigma^n$ of a functor $G$. Because of this parallel, we may at times refer to this construction as multilinearization.

Thus, our goal is to understand the multilinearization of the functor

$$\text{cr}_n(\text{id})(X_1, \ldots, X_n) = \text{tfib}\left(S \mapsto \bigvee_{i \in S \subseteq [n]} X_i \right).$$

In order to do so, it will be helpful to have a model for the total fiber.

**Notation 9.** Let $I$ be a finite set. For $S \subseteq I$, we write

$$[0, 1]^S := \{t \in [0, 1]^I : t_i = 0 \text{ if } i \notin S\}.$$

We further write

$$\partial_1[0, 1]^S := \{t \in [0, 1]^S : t_i = 1 \text{ for some } i \in S\}.$$

Evidently, there is an inclusion $S \subseteq T$ of subsets of $I$ if and only if there is an inclusion $[0, 1]^S \subseteq [0, 1]^T$ of subspaces of $[0, 1]^I$; thus, we obtain a functor $[0, 1]^\bullet : P(I) \to \text{Top}$. The same remarks apply to the subspaces $\partial_1[0, 1]^S$, and we have the following generalization of the standard formula for the homotopy fiber of a map.

**Lemma 10.** Let $X$ be an $I$-cube. There is a pullback diagram

$$\begin{array}{ccc}
\text{tfib}(X) & \longrightarrow & \text{Nat}([0, 1]^\bullet, X) \\
\downarrow & & \downarrow \\
\text{pt} & \longrightarrow & \text{Nat}(\partial_1[0, 1]^\bullet, X),
\end{array}$$

where the bottom map is induced by the inclusion of the basepoint.

In other words, a point in the total fiber of $X$ is a collection of maps $\{f_S : [0, 1]^S \to X(S)\}_{S \subseteq I}$ such that
(1) for each $S \subseteq T \subseteq I$, the diagram

$$
\begin{array}{ccc}
[0, 1]^S & \xrightarrow{f_S} & X(S) \\
\downarrow & & \downarrow \\
[0, 1]^T & \xrightarrow{f_T} & X(T)
\end{array}
$$

commutes, and

(2) $f_S(t)$ is the basepoint in $X(S)$ whenever some $t_i = 1$.

**Definition 11.** For a nonempty finite set $I$ and an element $i \in I$ and an $I$-cube $X$, the comparison map for $X$ is the map

$$
tfib(X) \to \text{Map}_+(\left[0, 1\right]^{\left|I\right| - 1} \bigwedge_{i \in I} X(I_i))
$$

defined by evaluation at $I \setminus \{i\}$ for $i \in I$ (note that $[0, 1]^{I \setminus \{i\}}$ is an $(|I| - 1)$-dimensional cube.

Applying this construction to the $n$-cube of Definition 7, we obtain the clockwise composite in the commuting diagram

$$
\begin{array}{ccc}
\text{cr}_n(id)(X_1, \ldots, X_n) & \xrightarrow{\phi} & \text{Map}_+(\left[0, 1\right]^{n(n-1)} \bigwedge_{i=1}^n X_i) \\
\downarrow & & \downarrow \\
\text{Map}_+(\Delta_n, \bigwedge_{i=1}^n X_i) & \xrightarrow{\phi} & \text{Map}_+(\left[0, 1\right]^{n(n-1)} \bigwedge_{i=1}^n X_i),
\end{array}
$$

where $\Delta_n$ is defined as a quotient of the form

$$
\Delta_n := \left[0, 1\right]^{n(n-1)} \bigg/ \bigcup_{1 \leq i < j \leq n} W_{ij}
$$

In order to describe the subspaces in question, it will be convenient to think of $\left[0, 1\right]^{n(n-1)}$ as the space of matrices $t = (t_{ij})_{1 \leq i, j \leq n}$ with $t_{ij} \in [0, 1]$ and $t_{ii} = 0$; here, the $i$th row $(t_{ij})_{1 \leq j \leq n}$ contains the coordinates of the $i$th $(n - 1)$-dimensional cube $[0, 1]^{\{1, \ldots, i-1, i+1, \ldots, n\}}$. With this notation in mind, we define

$$
Z = \left\{ t \in [0, 1]^{n(n-1)} : t_{ij} = 1 \text{ for some } 1 \leq i, j \leq n \right\}
$$

$$
W_{ij} = \left\{ t \in [0, 1]^{n(n-1)} : t_{ik} = t_{jk} \text{ for all } 1 \leq k \leq n \right\}.
$$

Since it is immediate from Lemma 10 that any $f : [0, 1]^{n(n-1)} \to \bigwedge_{i=1}^n X_i$ in the image of the comparison map sends $Z$ to the basepoint, all that remains in constructing the map $\phi$ is to check the following.

**Lemma 12.** If $f : [0, 1]^{n(n-1)} \to \bigwedge_{i=1}^n X_i$ lies in the image of the comparison map, then $f$ sends $W_{ij}$ to the basepoint for any $1 \leq i < j \leq k$.  

Proof. In the solid diagram

\[
\begin{array}{c}
[0, 1]^{n(n-1)} \\
W_{ij} \downarrow \\
[0, 1]^{n-1} \leftarrow [0, 1]^{n-2} \rightarrow [0, 1]^{n-1} \\
\downarrow f_i \quad \downarrow f_{ij} \quad \downarrow f_j \\
X_i \quad X_i \lor X_j \quad X_j
\end{array}
\]

the squares commute by the assumption that \( f \) lies in the image of the comparison map. The dashed filler exists by the definition of \( W_{ij} \), and it follows that, for \( t \in W_{ij} \), the points \( f_i(t) \in X_i \) and \( f_j(t) \in X_j \) are retracts of the same point in \( X_i \lor X_j \), so each is the respective basepoint. □

Thus, the map \( \psi : cr_n(id)(X_1, \ldots, X_n) \rightarrow \text{Map}_s(\Delta_n, \bigwedge_{i=1}^n X_i) \) is defined. Note, moreover, that \( \Delta_n \) is closed in \([0, 1]^{n(n-1)}\) under the action of \( \Sigma_n \) on the rows, and the map \( \psi \) is \( \Sigma_n \)-equivariant.

On the face of it, this map would seem to discard a great deal of information about the cross effect, but it turns out that only the “first order” information it captures can influence the respective multilinearizations.

**Theorem 13 (Johnson).** The map \( \psi \) induces an equivalence after multilinearization.

**Corollary 14.** There is an equivalence of \( \Sigma_n \)-spectra

\[ \partial_n(id) \simeq \mathbb{D}(\Sigma^\infty \Delta_n), \]

where \( \mathbb{D} = \text{Sp}(-, S) \) denotes the Spanier-Whitehead dual.

**Proof.** Since \( \psi \) is \( \Sigma_n \)-equivariant, Theorem 13 and Proposition 8 supply the equivariant equivalence of infinite loop spaces

\[
\Omega^\infty \partial_n(id) \simeq \text{colim}_{k_1, \ldots, k_n} \Omega^{k_1 + \cdots + k_n} \text{Map}_s(\Delta_n, S^{k_1 + \cdots + k_n})
\]

\[
\simeq \text{colim}_k \text{Map}_s(\Sigma^k \Delta_n, \Sigma^k S^0)
\]

\[
\simeq \Omega^\infty \text{Sp}(\Sigma^\infty \Delta_n, S).
\]

□

In order to prove this theorem, we require a criterion for recognizing such maps.

**Lemma 15.** Let \( \psi : F \rightarrow G \) be a natural transformation between functors of \( n \) variables. If \( \psi_{(X_1, \ldots, X_n)} \) is \(((n+1)k - c)\)-connected whenever each \( X_i \) is \( k \)-connected, then \( \psi \) induces a weak equivalence after multilinearization.

**Proof.** The hypothesis on the \( X_i \) implies that each \( \Sigma^\ell X_i \) is \((k + \ell)\)-connected, which implies, using the hypothesis on \( \psi \), that \( \Omega^\alpha \psi_{(\Sigma^\ell X_1, \ldots, \Sigma^\ell X_n)} \) is \(((n+1)(k + \ell) - c - n\ell)\)-connected. Since this number tends to infinity with \( n \), and since spheres are compact, it follows that the induced map on colimits is an equivalence. □

**Corollary 16.** If \( \Omega \psi_{(\Sigma X_1, \ldots, \Sigma X_n)} \) is \(((n+1)k - c)\)-connected whenever each \( X_i \) is \( k \)-connected, then \( \psi \) induces a weak equivalence after multilinearization.
Proof. The hypothesis implies that $\Omega^n \varphi(\Sigma X_1, \ldots, \Sigma X_n)$ is $((n+1)k - (n - 1 + c))$-connected. Viewing this map as a natural transformation

$$\psi : \Omega^n F (\Sigma(-), \ldots, \Sigma(-)) \to \Omega^n G (\Sigma(-), \ldots, \Sigma(-))$$

Lemma 15 implies that $\psi$ induces a weak equivalence on multilinearizations. Since the multilinearizations of these functors coincide with the respective multilinearizations of $F$ and $G$, the claim follows. 

Now, we have the equivalences

$$\Omega \cr_n (\text{id}) (\Sigma X_1, \ldots, \Sigma X_n) \simeq \Omega \text{tfib} \left( S \mapsto \bigvee_{i \in S \subseteq [n-1]} \Sigma X_i \right)$$

$$\simeq \Omega \text{tfib} \left( S \mapsto \Sigma \left( \bigvee_{i \in S \subseteq [n-1]} X_i \right) \right)$$

$$\simeq \text{tfib} \left( S \mapsto \Omega \Sigma \left( \bigvee_{i \in S \subseteq [n-1]} X_i \right) \right)$$

$$\simeq \text{cr}_n (\Sigma \Sigma)(X_1, \ldots, X_n).$$

Thus, in order to obtain the kind of connectivity estimate required by Corollary 16, we may instead study the cross effects of the functor $\Omega \Sigma$. We will be aided in this task by the following classical result.

**Theorem 17** (Hilton-Milnor). Let $\{X_i\}_{i=1}^n$ be pointed, connected spaces. There is a canonical natural weak equivalence

$$\prod_{w \in L_n} \Omega \Sigma (w(X_1, \ldots, X_n)) \sim \Omega \Sigma (X_1 \vee \cdots \vee X_n).$$

We pause to explain some of the terms of this theorem.

(1) The symbol $\prod'$ denotes the weak infinite product, which is defined as the colimit of products over all finite subsets of the indexing set.

(2) The set $L_n$ is an additive basis for the free Lie algebra on generators $\{x_1, \ldots, x_n\}$. These basis elements are often called basic products.

(3) The space $w(X_1, \ldots, X_n)$ is obtained by substituting $X_i$ for $x_i$ and $\wedge$ for $[-,-]$ in the expression for a basic product. For example, $[x_1, [x_2, x_3]] = X_1 \wedge X_2 \wedge X_3$.

(4) The map is given by taking products of nested Samelson brackets patterned after the basic products $w$.

See [Whi78] for a proof of this theorem. It should be emphasized that the Hilton-Milnor map is not $\Sigma_n$-equivariant, since the set of basic products is not closed under the action of $\Sigma_n$.

**Corollary 18.** If $X_i$ is $k$-connected for every $1 \leq i \leq n$, then

$$\pi_m (\text{cr}_n (\Sigma \Sigma)(X_1, \ldots, X_n)) \cong \pi_m \left( \bigwedge_{i=1}^n X_i \right)^{(n-1)!}$$

for $0 \leq m \leq (n+1)(k+1) - 1$. 

Proof. We calculate that
\[
\text{cr}_n(\Omega \Sigma)(X_1, \ldots, X_n) = \text{tfib} \left( S \mapsto \Omega \Sigma \left( \bigvee_{i \in S \subseteq [n]} X_i \right) \right) \\
\simeq \text{tfib} \left( S \mapsto \prod_{L_{n-|S|}} \Omega \Sigma(w(X_1, \ldots, X_n)) \right) \\
\simeq \prod_{L_n^0} \Omega \Sigma(w(X_1, \ldots, X_n)),
\]
where \(L_n^0\) is the set of basic products involving each \(x_i\) at least once. The claim now follows from the fact that there are \((n-1)!\) basic products involving each \(x_i\) exactly once, together with the fact that the factor pertaining to any longer basic product is \(((n+1)(k+1) - 1)\)-connected. \(\square\)

On the other hand, by the Freudenthal suspension theorem, we compute that, in the range of interest,
\[
\pi_m \left( \bigwedge_{i=1}^n X_i \right)^{(n-1)!} \simeq \pi_{m+n} \left( \bigwedge_{i=1}^n \Sigma X_i \right)^{(n-1)!} \\
\simeq \pi_m \left( \Omega \text{Map}_+ \left( \bigvee_{(n-1)!} S^{n-1}, \bigwedge_{i=1}^n \Sigma X_i \right) \right).
\]
This calculation, together with the criterion of Corollary 16, leads us to hope for the following result.

**Proposition 19.** There is a canonical weak equivalence
\[
\bigvee_{(n-1)!} S^{n-1} \xrightarrow{\sim} \Delta_n
\]

The proof proceeds through the intermediary complex \(\tilde{\Delta}_n := \{ t \in \Delta_n : t_{ij} = 0 \text{ for } j > 1 \}\). Concerning this space, we have the following.

**Lemma 20.** There is a canonical homeomorphism
\[
\tilde{\Delta}_n \cong \bigvee_{(n-1)!} S^{n-1}
\]

**Proof.** The space in question is obtained as a quotient of an \((n-1)\)-dimensional cube by the “fat diagonal,” which is the subspace where any two coordinates agree, together with the subspace where any coordinate is either 0 or 1. Thus,
\[
\tilde{\Delta}_n \cong \text{Conf}_{n-1}((0,1))^+ \\
\cong \left( \prod_{\sigma \in \Sigma_{n-1}} \{ (s_1, \ldots, s_{n-1}) : 0 < s_{\sigma(1)} < \cdots < s_{\sigma(n-1)} < 1 \} \right)^+ \\
\cong \bigvee_{(n-1)!} (\Delta^{n-1})^+.
\]
Using the standard identification \(S^{n-1} \cong \Delta^{n-1}/\partial \Delta^{n-1}\), the proof is complete. \(\square\)
Thus, in order to prove Proposition 19, it suffices to check that the inclusion \( \tilde{\Delta}_n \to \Delta_n \) is a weak equivalence. Setting \( W = \bigcup_{1 \leq i < j \leq n} W_{ij} \), the (homotopy) pushout squares

\[
\begin{array}{ccc}
\tilde{Z} \cup \tilde{W} & \longrightarrow & I^{n-1} \\
\downarrow & & \downarrow \\
\text{pt} & \longrightarrow & \tilde{\Delta}_n
\end{array}
\quad
\begin{array}{ccc}
Z \cup W & \longrightarrow & I^{n(n-1)} \\
\downarrow & & \downarrow \\
\text{pt} & \longrightarrow & \Delta_n
\end{array}
\]

show that it suffices to check that the inclusion \( \tilde{Z} \cup \tilde{W} \to Z \cup W \) is a weak equivalence, and the (homotopy) pushout squares

\[
\begin{array}{ccc}
\tilde{Z} \cap \tilde{W} & \longrightarrow & \tilde{Z} \\
\downarrow & & \downarrow \\
\tilde{W} & \longrightarrow & \tilde{Z} \cup \tilde{W}
\end{array}
\quad
\begin{array}{ccc}
Z \cap W & \longrightarrow & Z \\
\downarrow & & \downarrow \\
W & \longrightarrow & Z \cup W
\end{array}
\]

show that it suffices to check that the inclusion \( \tilde{Z} \cap \tilde{W} \to Z \cap W \) is a weak equivalence, since \( W \) and \( Z \) are contractible. In order to verify this equivalence, we make use of the following result, which is called the “Nerve Theorem.”

**Theorem 21** (Borsuk). If \( X \) is covered by subcomplexes \( K_i \) such that every nonempty finite intersection \( K_{i_1} \cap \cdots \cap K_{i_r} \) is contractible, then \( X \) is weakly equivalent to the nerve of the partially ordered set of finite intersections of elements of \( \{K_i\} \).

**Proof of Proposition 19.** We will show that both \( \tilde{Z} \cap \tilde{W} \) and \( Z \cap W \) admit a cover by subcomplexes whose associated poset is isomorphic to the poset of nontrivial partitions \( \lambda \) of the set \( \{1, \ldots, n\} \) (recall that a partition is simply an equivalence relation, and a partition is trivial if either \( i \sim \lambda j \) for all \( i \) and \( j \) or \( i \neq \lambda j \) unless \( i = j \)).

The cover is by the subcomplexes \( \{Z \cap W_{ij}\}_{1 \leq i < j \leq n} \) (resp. \( \tilde{W}_{ij} \)). The finite intersections corresponding to \( \lambda \) is the subspace of matrices with the following properties:

1. all diagonal entries vanish;
2. some entry is 1; and
3. the \( i \)th row and the \( j \)th row coincide if \( i \sim \lambda j \).

(in the “tilde” case, there is the further condition that all columns but the first vanish). Each of these is contractible by a coordinatewise straight line homotopy sending \( t_{ij} \) to 0 if \( i \sim \lambda j \) and sending \( t_{ij} \) to 1 otherwise (in the “tilde” case we perform the homotopy only on the first column).

\( \square \)

**Remark 22.** From this calculation and Theorem 13, it follows that \( \partial_n(\text{id}) = D(\Sigma^\infty+2N(P_n)) \), where \( P_n \) is the category of nontrivial partitions of \( \{1, \ldots, n\} \) under refinement. This reformulation due to Arone-Mahowald [AM99] has borne much subsequent fruit—see [Chi05], for example.

To summarize the results of our investigation so far, the domain and codomain of the map

\[
\varphi : \text{cr}_n(\text{id})(X_1, \ldots, X_n) \to \text{Map}_\ast \left( \Delta_n, \bigwedge_{i=1}^n X_i \right)
\]

constructed above have homotopy groups that are abstractly isomorphic in the desired range. Thus, all that remains is to verify that \( \varphi \) induces this isomorphism. The key fact in this verification is the construction of a family of maps with the following properties.
Proposition 23. For each $\sigma \in \Sigma_{n-1}$ there is a map $C_\sigma$ fitting into a commuting diagram

$$\begin{array}{ccc}
\prod_{i=1}^{n} X_i & \xrightarrow{C_\sigma} & \Omega \text{cr}_n(id)(\Sigma X_1, \ldots, \Sigma X_n) \\
\downarrow q & & \downarrow \Omega \text{Map}_n \left( \Delta_n, \bigwedge_{i=1}^{n} \Sigma X_i \right) \\
\bigwedge_{i=1}^{n} X_i & \xrightarrow{\Gamma_\sigma \wedge (-)} & \Omega^n \Sigma^n \left( \bigwedge_{i=1}^{n} X_i \right)
\end{array}$$

for each $\tau \in \Sigma_{n-1}$, where $\lambda_\tau : S^{n-1} \to \bigvee_{\Sigma n-1} S^{n-1} \to \Delta_n$ is the inclusion of the $\tau$ factor. Moreover, $\deg(\Gamma_\sigma \tau) = \delta_\sigma \tau$.

Given these maps, a diagram chase in reduced homology implies that $\Omega \varphi$ induces the desired isomorphism. The maps $C_\sigma$ are constructed by modifying the iterated commutators appearing in the Hilton-Milnor map with explicit homotopies in order to map into the total fiber.

References


