IMMERSIONS UP TO COBORDISM

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Contents

1. Introduction

Recall the goal in this seminar:

Theorem 1.1 (Immersion conjecture)**.** *Let $n \geq 2$. Then every compact smooth n-manifold immerses in $\mathbb{R}^{2n-\alpha(n)}$ where $\alpha(n)$ is the number of ones in the binary expansion of n.*

Our goal today is to prove a weaker version of this theorem, which will eventually be used in Cohen's proof of the immersion conjecture.

Theorem 1.2 (R. L. Brown, '71)**.** *Every closed n-manifold is cobordant to one that immerses in $\mathbb{R}^{2n-\alpha(n)}$.*

Our strategy is direct.

(i) First, we classify all manifolds up to cobordism. This calculation was done by Thom [5].
(ii) Second, we find representatives in each cobordism class that immerses in the correct dimension. These manifolds were studied by Dold [3] and Brown [1].

The reference for the material in this talk is the lectures on immersion theory by Cohen-Tillmann [2, §I.2]. Everything not in those notes can be found in either [5] or [1].

2. Review of unoriented cobordism

Note: all homology and cohomology will be taken with $\mathbb{F}_2$-coefficients.

2.1. **Thom complexes.** If E is a vector bundle over a manifold M, then we can form its *Thom complex*

$$\mathrm{Th}(E) := D(E)/S(E),$$

where $D(E)$ and $S(E)$ are the unit disk bundle and the unit sphere bundle of E respectively.

If γ^k is the universal k-plane bundle over $BO(k)$, we define

$$MO(k) := \mathrm{Th}(\gamma^k).$$

We have a pullback square

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$$\begin{tikzcd}
\gamma^k \oplus 1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \gamma^{k+1} \arrow[d] \\
BO(k) \arrow[r] & BO(k+1),
\end{tikzcd}$$

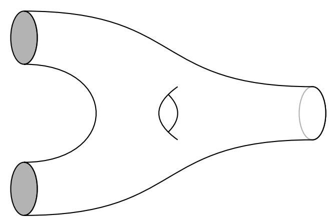
where $BO(k) \to BO(k+1)$ is the canonical inclusion map. Thus, there is an induced map on Thom complexes

$$\Sigma MO(k) \simeq \mathrm{Th}(\gamma^k \oplus 1) \to \mathrm{Th}(\gamma^{k+1}) = MO(k+1).$$

The data of $\{MO(k)\}$ and these bonding maps forms a spectrum called MO.

Furthermore, the map $BO(k) \times BO(k') \to BO(k+k')$ classifying $\gamma^k \boxplus \gamma^{k+1}$ Thomifies to give a map $MO \wedge MO \to MO$, and makes MO into a ring spectrum.

2.2. **Unoriented cobordism.** Let M and M' be two compact n-manifolds. They are *cobordant* if there exists a $(n+1)$-dimensional manifold W such that $\partial W \cong M \sqcup M'$. For example,



is a cobordism from $S^1 \sqcup S^1$ to S^1 (read from left to right).

For $n \in \mathbb{N}$, the *n-dimensional unoriented cobordism group* is

$$\mathfrak{N}_n = \{\text{compact } n\text{-manifolds}\}/\text{cobordism}.$$

Note that $\mathfrak{N}_* = \bigoplus_{n\in\mathbb{N}} \mathfrak{N}_n$ is a graded ring with addition given by disjoint union and multiplication given by cartesian product of manifolds. Moreover, $\mathfrak{N}_*$ has characteristic two, since $\partial(M \times I) = M \sqcup M$ for any M.

Theorem 2.1 (Pontryagin-Thom)**.** $\mathfrak{N}_* \cong \pi_* MO$.

Proof sketch. We're just going to describe the maps implementing this isomorphism; verification of their well-definedness and their required properties is left to the reader.

First, given a cobordism class $[M^n] \in \mathfrak{N}_n$, we embed $M^n \hookrightarrow \mathbb{R}^{n+k} \subset S^{n+k}$ with normal bundle ν_M. The tubular neighborhood theorem says that ν_M embeds in $\mathbb{R}^{n+k}$, and so there is a Pontryagin-Thom collapse map

$$\mathrm{PT}_M : S^{n+k} \twoheadrightarrow S^{n+k}/(S^{n+k} - \operatorname{int} D(\nu_M)) \simeq D(\nu_M)/S(\nu_M) = \mathrm{Th}(\nu_M).$$

Composing this with the map induced on Thom complexes by the map classifying ν_M, we have

$$S^{n+k} \xrightarrow{\mathrm{PT}_M} \mathrm{Th}(\nu_M) \to MO(k),$$

which determines a class in $\pi_n MO$.

Conversely, given $f : S^{n+k} \to MO(k)$, we consider the "zero section" $Z \subset MO(k)$. If f is transverse to Z, then $M := f^{-1}(Z)$ is a codimension k submanifold of S^{n+k}, and represents a class in $\mathfrak{N}_n$. $\square$

So the problem of classifying manifolds up to cobordism is reduced to the calculation of $\pi_* MO$.

Theorem 2.2 (Thom)**.** *$\pi_* MO \cong \mathbb{F}_2[\sigma_i \mid i \neq 2^r - 1]$, where $\deg \sigma_i = i$.*

Proof sketch.

(1) There is a Thom isomorphism

$$H^* MO \cong H^* BO \cong \mathbb{F}_2[w_1, w_2, \ldots]$$

as vector spaces.

(2) The vector space H^*MO is actually a connected, graded coalgebra and a module over the Steenrod algebra $\mathcal{A}$. Additionally, the homomorphism $\mathcal{A} \to H^*MO$, $a \mapsto a(u)$, is a monomorphism.

By the Milnor-Moore theorem, we see that H^*MO is free over $\mathcal{A}$; in fact, we have

$$H^*MO \cong \mathcal{A} \otimes \mathbb{F}_2[\sigma_i \mid i \neq 2^r - 1]$$

by counting dimensions.

(3) For each class $\sigma_I \in H^*MO$, we get a map $MO \to \Sigma^{|I|} H\mathbb{F}_2$. We take the sum of these maps to get

$$MO \to \bigvee_I \Sigma^{|I|} H\mathbb{F}_2.$$

This is an equivalence in mod 2 cohomology by construction, and since both sides are 2-local, this is a homotopy equivalence as desired.

□

3. Generators for $\mathfrak{N}_*$

We now have a pretty good understanding of the abstract structure of the unoriented cobordism ring. Next we want a criterion to determine when a manifold M represents an indecomposable element in $\mathfrak{N}_*$.

The idea is that up to cobordism, everything is determined by characteristic numbers, in this case, by Stiefel-Whitney numbers. For instance, recall that two manifolds are cobordant iff they have the same Stiefel-Whitney numbers. So we would like a condition in terms of Stiefel-Whitney numbers.

We use the Hurewicz map h to relate $\mathfrak{N}_*$ with characteristic classes by embedding π_*MO into H_*MO via h.

Proposition 3.1. *The Hurewicz homomorphism can be identified with the map*

$$\mathfrak{N}_* \cong \pi_*MO \overset{h}{\hookrightarrow} H_*MO \cong H_*BO \cong \mathrm{Hom}(H^*BO, \mathbb{F}_2)$$

that sends a cobordism class $[M]$ to the homomorphism $w \mapsto \langle w(\nu_M), [M] \rangle$.

Proof. Let $[M^n] \in \mathfrak{N}_n$. It corresponds to the element in $\pi_n MO$ given by the composite

$$S^{n+k} \xrightarrow{\mathrm{PT}_M} \mathrm{Th}(\nu_M) \to MO(k)$$

where PT_M is the Pontryagin-Thom collapse map associated to an embedding $M^n \hookrightarrow \mathbb{R}^{n+k}$.

Now we take homology, use the Thom isomorphism, and dualize. The map $h([M])$ is represented by the composite

$$H^n BO \xrightarrow{[\nu_M]^*} H^n M \cong H^{n+k}\,\mathrm{Th}(\nu_M) \xrightarrow{\mathrm{PT}_M^*} H^{n+k} S^{n+k} \cong \mathbb{F}_2$$

given by

$$w \mapsto w(\nu_M) \mapsto u \smile w(\nu_M) \mapsto \mathrm{PT}_M^*(u \smile w(\nu_M)) \mapsto \left\langle \mathrm{PT}_M^*(u \smile w(\nu_M)), [S^{n+k}] \right\rangle,$$

where $u \in H^k\,\mathrm{Th}(\nu_M)$ is the Thom class.

We have

$$\left\langle \mathrm{PT}_M^*(u \smile w(\nu_M)), [S^{n+k}] \right\rangle = \left\langle u \smile w(\nu_M), \mathrm{PT}_{M*}[S^{n+k}] \right\rangle = \left\langle w(\nu_M), u \frown \mathrm{PT}_{M*}[S^{n+k}] \right\rangle.$$

Now, $\mathrm{PT}_{M*}[S^{n+k}] = [\mathrm{Th}(\nu_M)]$ since $\deg \mathrm{PT}_M = \pm 1$. Integrating along a fiber gives $u \frown [\mathrm{Th}(\nu_M)] = [M]$. This completes the proof. □

We are reduced to studying indecomposables in H_*BO, or dually, primitives in H^*BO.

Consider the map $(\mathbb{R}P^\infty)^{\times n} \to BO(n)$ classifying the sum of n line bundles. In cohomology, we have

$$\mathbb{F}_2[t_1, t_2, \ldots, t_n] \cong (H^*\mathbb{R}P^\infty)^{\otimes n} \leftarrow H^*BO(n) \cong \mathbb{F}_2[w_1, w_2, \ldots, w_n].$$

Since taking the sum of bundles is a commutative operation, the image of $H^*BO(n)$ lies in $\mathbb{F}_2[t_1, \ldots, t_n]^{\Sigma_n}$. In fact, the map

$$H^*BO(n) \xrightarrow{\cong} \mathbb{F}_2[t_1, \ldots, t_n]^{\Sigma_n}$$

is an isomorphism. This follows from the calculation

$$w((\gamma^1)^{\boxplus n}) = \prod_{i=1}^{n}(1+t_i) = 1 + e_1 + \cdots + e_n,$$

where the e_k's are the elementary symmetric polynomials in $t_1, \ldots, t_n$, which are known to generate the ring of Σ_n-invariants in $\mathbb{F}_2[t_1, \ldots, t_n]$.

We define

$$s_n = \sum_i t_i^n.$$

This is a symmetric polynomial in the t_i's, so it determines a class $s_n = s_n(w_1, \ldots, w_n)$ in $H^n BO$.

Lemma 3.2.

(a) If E and E' are two bundles, $s_n(E \oplus E') = s_n(E) + s_n(E')$.

(b) s_n is primitive in $H^* BO$.

Proof.

(a) This is immediate from the splitting principle.

(b) The coproduct ψ on H^*BO is induced by the map $\mu : BO \times BO \to BO$ classifying the Whitney sum of universal bundles. Recall that γ^k is the universal k-plane bundle over $BO(k)$, so that $\mu^*\gamma^{k+k'} = \gamma^k \boxplus \gamma^{k'}$.

We want to show that $\psi(s_n) = 1 \otimes s_n \otimes s_n \otimes 1$. We have

$$\psi(s_n(\gamma^{k+k'})) = s_n(\mu^*\gamma^{k+k'}) = s_n(\gamma^k \boxplus \gamma^{k'}) = \mathrm{pr}_1^* s_n(\gamma^k) + \mathrm{pr}_2^* s_n(\gamma^{k'}) = 1 \otimes s_n(\gamma^{k'}) + s_n(\gamma^k) \otimes 1.$$

□

Write $H^*\mathbb{R}P^\infty \cong \mathbb{F}_2[t]$ for t a generator of H^1. Then $H_*\mathbb{R}P^\infty \cong \mathbb{F}_2\{1, b_1, b_2, \ldots\}$ where b_i is dual to t^i. So, $H_*BO \cong \mathrm{Sym}\, \tilde{H}_*\mathbb{R}P^\infty \cong \mathbb{F}_2[b_1, b_2, \ldots]$.

Proposition 3.3. *For $x \in H_n BO$,*

$$\langle s_n, x\rangle = \begin{cases} 1 & x \textit{ indecomposable},\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. Suppose $x = x'x''$ where x' and x'' are in positive degrees. Then

$$\langle s_n, x'x''\rangle = \langle \psi(s_n), x' \otimes x''\rangle = \langle 1 \otimes s_n + s_n \otimes 1, x' \otimes x''\rangle = \langle s_n, x'\rangle + \langle s_n, x''\rangle .$$

This is zero, since $\deg x'$ and $\deg x''$ are less than n, and so x' and x'' cannot pair nontrivially with s_n.

On the other hand, if x is indecomposable, then $x \equiv b_n$ mod decomposables, and we have

$$\langle s_n, x\rangle = \langle s_n, b_n\rangle = \langle t^n, b_n\rangle = 1.$$

□

We can finally state and prove our criterion for detecting indecomposables in $\mathfrak{N}_*$.

Theorem 3.4 (Thom)**.** *Let M be a closed n-manifold. Then $[M]$ is indecomposable in $\mathfrak{N}_*$ iff $\langle s_n(\nu_M), [M]\rangle = 1$.*

Proof. If $[M]$ is decomposable in $\mathfrak{N}_*$, then $h([M])$ is decomposable in H_*BO, and so

$$\langle s_n(\nu_M), [M]\rangle = \langle s_n, h([M])\rangle = 0.$$

The other direction follows once we produce manifolds M^n such that $\langle s_n(\nu_M), [M]\rangle = 1$ for every $n \neq 2^r - 1$. □

Remark 3.5. Let M be an n-manifold. Since $TM \oplus \nu_M$ is trivial, we have $s_n(TM) + s_n(\nu_M) = 0$, so $s_n(\nu_M) = s_n(TM)$ and we may use $s_n(TM)$ interchangeably with $s_n(\nu_M)$ in our criterion above.

Example 3.6. Our first example is $M = \mathbb{R}P^n$. We have

$$T\mathbb{R}P^n \oplus 1 \cong (n+1)\gamma,$$

where γ is the tautological line on $\mathbb{R}P^n$. So we have

$$s_n(T\mathbb{R}P^n) = (n+1)s_n(\gamma) = (n+1)t^n.$$

This shows that if n were even, then $[\mathbb{R}P^n]$ is a polynomial generator for $\mathcal{N}_*$.

Example 3.7 (Milnor)**.** For the odd-dimensional generators, we need to do something different. Let $M_{a,b}$ be a hypersurface of bidegree $(1,1)$ in $\mathbb{R}P^a \times \mathbb{R}P^b$, i.e., the projectivization of the zero locus of a bilinear form on $\mathbb{R}^{a+1} \times \mathbb{R}^{b+1}$. We have $n := \dim M_{a,b} = a + b - 1$. Let $i : M_{a,b} \hookrightarrow \mathbb{R}P^a \times \mathbb{R}P^b$ be the inclusion, so that $\nu_{M_{a,b}} \cong i^*(\gamma_1 \otimes \gamma_2)$, where γ_1 (respectively γ_2) is the tautological line on $\mathbb{R}P^a$ (respectively $\mathbb{R}P^b$). We write $H^*(\mathbb{R}P^a \times \mathbb{R}P^b) \cong \mathbb{F}_2[t_1, t_2]/(t_1^{a+1}, t_2^{b+1})$ where $w(\gamma_1) = 1 + t_1$ and $w(\gamma_2) = 1 + t_2$.

Then we have

$$\begin{aligned}
\left\langle s_n(\nu_{M_{a,b}}), [M_{a,b}]\right\rangle &= \langle i^*(t_1+t_2)^n, [M_{a,b}]\rangle \\
&= \left\langle (t_1+t_2)^n, [\mathbb{R}P^a \times \mathbb{R}P^b] \frown w_1(\nu_{M_{a,b}})\right\rangle \\
&= \left\langle (t_1+t_2)^{n+1}, [\mathbb{R}P^a \times \mathbb{R}P^b]\right\rangle \\
&= \left\langle \binom{n+1}{a} t_1^a t_2^b, [\mathbb{R}P^a \times \mathbb{R}P^b]\right\rangle \\
&= \binom{n+1}{a}.
\end{aligned}$$

For n odd, $n \neq 2^r - 1$, we write $n = 2^p(2q+1) - 1$. Then $\binom{2^p(2q+1)}{2^p}$ is odd (use Lucas' theorem!), so $[M_{2^{p+1}q, 2^p}]$ is a polynomial generator for $\mathcal{N}_*$.

4. Twisted products

In order to prove R. L. Brown's theorem, it remains to find generators of the cobordism ring that immerse in the Euclidean space of the dimension predicted by the immersion conjecture. The generators we found above are not quite good enough, so we describe a different set of generators in this section.

Given a n-manifold M, consider the product $S^k \times M \times M$. There is a C_2-action on this space given by the antipodal action on S^k and the swap action on $M \times M$. This involution is free, so its orbit space is again a manifold.

Definition 4.1. We define the twisted product $P(k, M) := S^k \times_{C_2} (M \times M)$.

Roughly, we want to show that $P(k, -)$ turns generators of $\mathfrak{N}_*$ into other generators.

Since the method we use to check whether a manifold is a generator is via characteristic numbers, we first compute the cohomology ring of $P(k, M)$. There is a fiber bundle

$$\begin{array}{ccc}
M \times M & \longrightarrow & P(k, M) \\
 & & \downarrow \\
 & & S^k \times_{C_2} * \cong \mathbb{R}P^k.
\end{array}$$

Note that there is a section $s : \mathbb{R}P^k \to P(k, M)$ that sends u to (u, x, x) where x is a chosen point of M.

Anyway, this fiber sequence makes the following proposition reasonable.

Proposition 4.2. *$H^*P(k, M) \cong H^*(\mathbb{R}P^k, (H^*M)^{\otimes 2})$.*

Proof. Either study the Serre spectral sequence for the fiber bundle, or write down a C_2-equivariant cell decomposition of $S^k \times M \times M$ using a cell decomposition of M and the cell decomposition of S^k with two i-dimensional cells for each $0 \leq i \leq k$. $\square$

Anyway, this means that an additive basis for $H^*P(k,M)$ is given by elements of the form

- $1 \otimes (x \otimes y + y \otimes x)$,
- $g \otimes (x \otimes y + y \otimes x)$, and
- $t^i \otimes x \otimes x$,

where $H^*\mathbb{R}P^k \cong \mathbb{F}_2[t]/t^{k+1}$, g is the generator of $H^k S^k$, and x, y are cohomology classes of M.

For convenience, for $\alpha \in H^*M$, we will write

$$\begin{aligned} e(\alpha) &= 1 \otimes \alpha + \alpha \otimes 1, \\ d(\alpha) &= \alpha \otimes \alpha. \end{aligned}$$

Now, observe that $P(k,-)$ is functorial, and that if ξ is a rank r bundle on M, then $P(k,\xi)$ is a rank $2r$ bundle on $P(k,M)$.

Proposition 4.3. *Let ξ be a line bundle on M such that $w_1(\gamma) = \alpha$. Then*

$$w(P(k,\xi)) = 1 + t + e(\alpha) + d(\alpha).$$

Proof. The idea is to pull the bundle back to other spaces for which the entire cohomology of $H^*P(k,M)$ is detected. First, we pull $P(k,\xi)$ back along the identification map $q : S^k \times M \times M \to P(k,M)$. We have

$$\begin{tikzcd} \xi \boxplus \xi \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P(k,\xi) \arrow[d] \\ S^k \times M \times M \arrow[r, "q"'] & P(k,M). \end{tikzcd}$$

So,

$$q^* w(P(k,\xi)) = w(\xi \boxplus \xi) = (1+\alpha) \otimes (1+\alpha) = 1 + e(\alpha) + d(\alpha).$$

Next, pull $P(k,\xi)$ back along the section $s : \mathbb{R}P^k \to P(k,M)$.

$$\begin{tikzcd} S^k \times_{C_2} (\mathbb{R} \oplus \mathbb{R}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P(k,\xi) \arrow[d] \\ \mathbb{R}P^k \arrow[r, "s"'] & P(k,M). \end{tikzcd}$$

The pullback bundle is the twisted product construction applied to the fiber of ξ above a point in M. We have

$$S^k \times_{C_2} (\mathbb{R} \oplus \mathbb{R}) \cong (S^k \times_{C_2} \mathrm{Sym}^2(\mathbb{R}^2)) \oplus (S^k \times_{C_2} \mathrm{Alt}^2(\mathbb{R}^2)) \cong (1 \downarrow \mathbb{R}P^k) \oplus (\gamma \downarrow \mathbb{R}P^k).$$

So

$$s^* w(P(k,\xi)) = w(1 \oplus \gamma) = 1 + t.$$

Comparing the results from these two pullbacks, we deduce the desired formula for $w(P(k,\xi))$. $\square$

We can now prove the following theorem.

Theorem 4.4 (Dold, Brown)**.** *Let M be an n-manifold. Then $[P(k,M)]$ is indecomposable iff $[M]$ is indecomposable and $\binom{n+k-1}{n}$ is odd.*

Proof. We assume $n > 0$; the proof in that case is simpler. We use our criterion for indecomposability and compute the Stiefel-Whitney number $\langle s_{k+2n}(TP(k,M)), [P(k,M)] \rangle$. First, since $P(k,M)$ is a fiber bundle, we can decompose its tangent bundle as the sum of the tangent bundle of the base and the bundle along the fiber, i.e.,

$$TP(k,M) \cong T\mathbb{R}P^k \oplus P(k,TM).$$

The total Stiefel-Whitney class of $T\mathbb{R}P^k$ is $(1+t)^{k+1}$.

For $P(k, TM)$, we may assume without loss of generality by the splitting principle that TM is a sum of line bundles $\bigoplus_{i=1}^{n} L_i$ with $w_1(L_i) = \alpha_i$. Then by the previous proposition, we have

$$w(P(k, TM)) = \prod_{i=1}^{n}(1 + t + e(\alpha_i) + d(\alpha_i)).$$

By pulling back further if necessary, we may assume that $P(k, L_i)$ themselves split as sums of two line bundles, and write

$$1 + t + \epsilon(\alpha_i) + d(\alpha_i) = (1 + u_i)(1 + v_i)$$

for some u_i, v_i such that $u_i + v_i = t + \epsilon(\alpha_i)$ and $u_i v_i = d(\alpha_i)$.

Therefore, we have

$$w(TP(k, M)) = (1 + t)^{k+1} \prod_{i=1}^{n}(1 + u_i)(1 + v_i).$$

The "Stiefel-Whitney roots" of $TP(k, M)$ consist of $(k + 1)$ copies of t, and the u_i's and v_i's. So we have

$$s_{k+2n}(TP(k, M)) = (k + 1)t^{k+2n} + \sum_{i=1}^{n} u_i^{k+2n} + v_i^{k+2n}.$$

The term t^{k+2n} vanishes since $t^{k+1} = 0$. Next, we rewrite $u_i^{k+2n} + v_i^{k+2n}$ in terms of elementary symmetric polynomials. Using Newton's identity, we have

$$u_i^{k+2n} + v_i^{k+2n} = \sum_{p+2q=k+2n} \binom{p+q-1}{q}(u_i + v_i)^p(u_i v_i)^q = \sum_{p+2q=k+2n} \binom{p+q-1}{q}(t + e(\alpha_i))^p d(\alpha_i)^q.$$

Recall that in the ring $H^*P(k, M)$, we have $t^{k+1} = 0$, $e(\alpha_i)^p d(\alpha_i)^q = 0$ if $p + 2q > 2n$, and $te(\alpha_i) = 0$. So in fact the sum above is very simple:

$$u_i^{k+2n} + v_i^{k+2n} = \binom{n+k-1}{n} t^k d(\alpha_i^n).$$

Therefore, we have

$$s_{k+2n}(TP(k, M)) = \binom{n+k-1}{n} t^k d(\sum_{i=1}^{n} \alpha_i^n)$$

(the function d is not linear, but all the cross terms are annihilated by t). But now note that $\sum_{i=1}^{n} \alpha_i^n = s_n(TM)$. So we have

$$\langle s_{k+2n}(TP(k, M)), [P(k, M)]\rangle = \binom{n+k-1}{n} \left\langle t^k, [\mathbb{R}P^k]\right\rangle \langle s_n(TM), [M]\rangle^2 = \binom{n+k-1}{n} \langle s_n(TM), [M]\rangle ,$$

which is nonzero iff $\binom{n+k-1}{n}$ is odd and $\langle s_n(TM), [M]\rangle \neq 0$. $\square$

5. Proof of the main theorem

We are now ready to prove R. L. Brown's theorem. Inductively define manifolds B^n of dimension n by

- $B^0 = *$ (which is indecomposable),
- if B^q is defined for $q < n$, $q \neq 2^r - 1$, write $n = 2^i + 2m$ where 2^i is the least significant bit of n, and define
$$B^n = P(2^i, B^m).$$

Since $\binom{2^i+m-1}{m}$ is odd by our choice of 2^i, we see that B^n is indecomposable if B^m is. Let $b_n = [B_n]$. We have shown

Corollary 5.1. $\mathfrak{N}_* \cong \mathbb{F}_2[b_n \mid n \neq 2^r - 1]$.

Example 5.2. If $n = 2^r$, then $B^n = P(n, B^0) = \mathbb{R}P^n$. In general, B^n is a twisted product of real projective spaces.

Proposition 5.3. *Every B^n immerses in $\mathbb{R}^{2n-\alpha(n)}$.*

Proof. If $\alpha(n) = 1$, then $B^n = \mathbb{R}P^n$. The (strong) Whitney immersion theorem says that there is an immersion $\mathbb{R}P^n \looparrowright \mathbb{R}^{2n-1} = \mathbb{R}^{2n-\alpha(n)}$.

So assume $\alpha(n) \geq 2$ and induct on n. We have $B^n = P(2^i, B^m)$. By the induction hypothesis, there is an immersion $B^m \looparrowright \mathbb{R}^{2m-\alpha(m)}$, so we get an immersion $B^n \looparrowright P(2^i, \mathbb{R}^{2m-\alpha(m)})$.

The twisted product $P(2^i, \mathbb{R}^{2m-\alpha(m)})$ is the total space of the bundle $(2m-\alpha(m))(\gamma \oplus 1)$ over $\mathbb{R}P^{2^i}$, and thus immerses in codimension 2^i (see the next lemma). So we have

$$B^n \looparrowright P(2^i, \mathbb{R}^{2m-\alpha(m)}) \looparrowright \mathbb{R}^{2^{i+1}+4m-2\alpha(m)}.$$

We are done, because

$$2^{i+1} + 4m - 2\alpha(m) = 2n - 2\alpha(n) + 2 \leq 2n - \alpha(n)$$

since $\alpha(n) \geq 2$. □

In the course of the previous proof, we needed a result about immersing vector bundles.

Lemma 5.4. *If M is an n-manifold and $E \to M$ is a rank r bundle, then the total space E immerses in $\mathbb{R}^{2n+r}$.*

The Cohen-Tillmann notes seem to suggest there is a simple proof, but here is the argument I came up with.

Proof. Using the Whitney immersion theorem, we immerse M in $\mathbb{R}^{2n}$, and let ν_M be the normal bundle. It suffices to show that there is a bundle monomorphism $E \hookrightarrow \nu_M \oplus r$, since by the tubular neighborhood theorem, ν_M embeds in $\mathbb{R}^{2n}$ and the total space $\nu_M \oplus r$ then embeds in $\mathbb{R}^{2n} \times \mathbb{R}^r \cong \mathbb{R}^{2n+r}$.

We claim that the following stronger statement is true. Let E be a rank r bundle on M^n, and let G be a rank $r+n$ bundle on M. Then there exists a bundle F of rank n such that $E \oplus F \cong G$. The proof of this claim is by obstruction theory. The given pair of bundles E and G are classified by a map $M \to BO(r) \times BO(r+n)$. The existence of F is equivalent to finding a lift

$$\begin{tikzcd} & BO(r) \times BO(n) \arrow[d, "1\times\mu"] \\ M \arrow[ur, dashed, "\exists?"] \arrow[r, "{([E],[G])}"'] & BO(r) \times BO(r+n). \end{tikzcd}$$

Let C be the mapping cone $(BO(r) \times BO(r+n))/(BO(r) \times BO(n))$. It has no cells of positive dimension $\leq n$. Since M is homotopic to a CW complex of dimension n, any map $M \to C$ is nullhomotopic by cellular approximation, and thus the desired lift exists. □

Remark 5.5. The immersion dimension of the vector bundle over projective space is not optimal in general. For example, it asserts that we can immerse the trivial line bundle over a circle into $\mathbb{R}^3$, but in fact we can do it in just $\mathbb{R}^2$ as an open annulus. Better bounds for immersing bundles over projective spaces has been found by Mahowald-Milgram [4].

We return to the proof of the main theorem. We just saw that we can immerse the generators B^n into $\mathbb{R}^{2n-\alpha(n)}$. Using the product immersion and the fact that $\alpha(n+n') \leq \alpha(n) + \alpha(n')$, we see that products of B^n's immerse in the correct dimension. Hence, disjoint unions of products of B^n's immerse in the correct dimension. Since every manifold is cobordant to a disjoint union of products of these B^n's, this proves that every n-manifold is cobordant to one that immerses in $\mathbb{R}^{2n-\alpha(n)}$. □

Remark 5.6. This bound is optimal provided $n \neq 3$. Recall from last time that immersions of certain products of real projective spaces into smaller dimensional Euclidean spaces are obstructed by Stiefel-Whitney numbers, which are cobordism-invariant. So there's no way to reduce this bound, even up to cobordism.

Remark 5.7. With a bit more work, one can in fact show that every n-manifold *embeds* in $\mathbb{R}^{2n-\alpha(n)+1}$ up to cobordism.

Finally, we have a corollary.

Corollary 5.8. *The map* $\pi_{2n-\alpha(n)}MO(n-\alpha(n)) \to \pi_n MO$ *is onto.*

Proof. Let $\beta : S^n \to MO$ be a map representing an element of $\pi_n MO$. It corresponds to an n-manifold $[M^n] \in \mathfrak{N}_n$. By the theorem, we may assume that M^n immerses in $\mathbb{R}^{2n-\alpha(n)}$ with normal bundle ν_M. The Pontryagin-Thom construction then gives

$$\beta : S^{2n-\alpha(n)} \to \mathrm{Th}(\nu_M) \to MO(n-\alpha(n)).$$

□

This corollary says that there is a lift

$$\begin{tikzcd} & \Omega^{n-\alpha(n)} MO(n-\alpha(n)) \arrow[d] \\ S^n \arrow[ur, dashed] \arrow[r] & MO. \end{tikzcd}$$

Compare this to a lift in the following diagram:

$$\begin{tikzcd} & BO(n-\alpha(n)) \arrow[d] \\ M^n] \arrow[ur, dashed] \arrow[r, "{[\nu_M]}"'] & BO, \end{tikzcd}$$

which is a reformulation of the immersion conjecture. Thus what we have proved today was a weak, Thomified version of the conjecture.

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