This course is concerned with the study of the following spaces.

**Definition** (Configuration spaces). The *configuration space* of $k$ ordered points in the topological space $X$ is

$$\text{Conf}_k(X) := \{(x_1, \ldots, x_k) \in X^k : x_i \neq x_j \text{ if } i \neq j\},$$

endowed with the subspace topology. The *unordered* configuration space is the quotient

$$B_k(X) := \text{Conf}_k(X)/\Sigma_k.$$  

**Remark.** We warn the reader that, in the literature on configuration spaces, there are almost as many traditions of notation as there are references.

Our goal is to attempt to understand the topology of these spaces, in the case that $X$ is a manifold or nearly so, through the lens of homotopy groups and (co)homology. Before beginning our study in earnest, we first mention a few reasons that configuration spaces are particularly interesting objects of study.

**Perspective** (Invariants). The homotopy type of a fixed configuration space is a homeomorphism invariant of the background space, and, in the case of a manifold, it turns out that these invariants remember a rather large amount of information. A simple-minded example is provided by Euclidean spaces of different dimension; indeed, as we will see, there is a homotopy equivalence $B_2(\mathbb{R}^m) \simeq B_2(\mathbb{R}^n)$ if and only if $m = n$. In other words, configuration spaces are sensitive to the dimension of a manifold.

A somewhat more sophisticated example is provided by the fact that $B_2(T^2 \setminus \text{pt}) \not\simeq B_2(\mathbb{R}^2 \setminus S^0)$, which can be shown by a homology calculation. Note that $T^2 \setminus \text{pt}$ and $\mathbb{R}^2 \setminus S^0$ have the same dimension and homotopy type, having $S^1 \vee S^1$ as a common deformation retract. On the other hand, $(T^2 \setminus \text{pt})^+ \cong T^2 \not\simeq S^1 \vee S^1 \vee S^2 \simeq (\mathbb{R}^2 \setminus S^0)^+$, so we might conclude from this example that configuration spaces are sensitive to the proper homotopy type of a manifold.

In order to discuss the most striking illustration of the sensitivity of configuration spaces, we recall that the *Lens spaces* are a family of compact 3-manifolds given by

$$L(p, q) := S^3/C_p,$$

where the cyclic group $C_p$ acts on $S^3 \subseteq \mathbb{C}^2$ by multiplication by $(e^{2\pi i/p}, e^{2\pi i q/p})$. It is a classical theorem of Reidemeister that

$$L(p, q_1) \simeq L(p, q_2) \iff q_1 q_2 \equiv \pm n^2 \mod p,$$

$$L(p, q_1) \cong L(p, q_2) \iff q_1 \equiv q_2^\pm 1 \mod p.$$  

In particular, $L(7, 1)$ and $L(7, 2)$ are homotopy equivalent but not homeomorphic, and, according to a theorem of Longoni-Salvatore [LS05], their configuration spaces distinguish them. Thus, configuration spaces are sensitive at least to the *simple* homotopy type of a manifold.

**Date:** 30 August 2017.
Perspective (Braids). A point moving in $B_k(\mathbb{R}^2)$ traces out $k$ different paths that weave among one another but can never overlap. For this reason, we think of the fundamental group $\pi_1(B_k(\mathbb{R}^2))$ as the group of geometric braids on $k$ strands, with composition given by concatenation of braids. As we shall see, this braid group admits the remarkably simple presentation

$$\pi_1(B_k(\mathbb{R}^2)) \cong \langle \sigma_1, \ldots, \sigma_{k-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1 \rangle,$$

due originally to Artin [Art47]. The combinatorial, algebraic, and geometric properties of these and related braid groups are of fundamental importance to a vast swath of mathematics that encompasses knot theory, mapping class groups [Bir75], quantum groups [Kas95], category theory [JS86], and motion planning [Far17].

Perspective (Embeddings). In the company of manifolds with trivialized tangent bundles, it is possible to speak of a framed embedding, which is to say an embedding respecting the fixed trivialization, possibly up to a homotopy through bundle maps. The tangent bundle of any Euclidean space is canonically trivialized, and evaluation at the origin determines a homotopy equivalence

$$\text{Emb}^\mathcal{F}(\bigvee_k \mathbb{R}^n, \mathbb{R}^n) \xrightarrow{\sim} \text{Conf}_k(\mathbb{R}^n).$$

As a consequence of this observation and the fact that framed embeddings compose, we see that the collection \{Conf_k(\mathbb{R}^n)\}_{k \geq 0} of homotopy types is equipped with hidden algebraic structure. This algebraic structure is that of an operad, which goes by the name of $E_n$. This perspective has many important consequences, three of which we name.

(1) Iterated loop spaces. For any $X$, there is a collection of maps

$$\text{Emb}^\mathcal{F}(\bigvee_k \mathbb{R}^n, \mathbb{R}^n) \times (\Omega^n X)^k \to \Omega^n X,$$

which arise from a particularly simple variant of the Pontrjagin-Thom collapse construction. The compatibilities among these maps are summarized by saying that $\Omega^n X$ is an $E_n$-algebra; in particular, the homology groups of the configuration spaces of $\mathbb{R}^n$ encode algebraic operations on $H_*(\Omega^n X)$. Remarkably, there is also a partial converse, often called May’s recognition principle [May72], which amounts to an algebraic classification of $n$-fold loop spaces.

(2) Factorization homology. The operad $E_n$ is defined in terms of embeddings among disjoint unions of Euclidean spaces, and so, after choosing a coordinate chart, we think of the structure of an $E_n$-algebra $A$ as being “defined locally” on a manifold $M$. Patching this local structure together across the elements of an atlas produces a manifold invariant, called the factorization homology of $M$ with coefficients in $A$ and denoted $\int_M A$, which can be thought of as a kind of space of configurations in $M$ labeled by elements of the algebra [AF15].

In the example of an $n$-fold loop space discussed above, the non-Abelian Poincaré duality of Salvatore [Sal01] and Lurie [Lur03] supplies the identification

$$\int_M \Omega^n X \xrightarrow{\sim} \text{Map}_c(M, X),$$

as long as $X$ is $(n-1)$-connected. Later in the course, we will encounter a precursor to this result in the configuration space models for mapping spaces introduced by McDuff [McD75, Böd87].

In a sense, factorization homology is a method for using configuration spaces to probe manifolds, but it can also be used to study the configuration spaces themselves, reversing this flow of information [Knu17]. In particular, a theorem of Ayala-Francis shows that
configuration spaces enjoy a kind of Mayer-Vietoris property in the existence of a quasi-isomorphism

\[ C_*(B(M)) \cong C_*(B(M_1)) \otimes_{C_*(B(N \times \mathbb{R}))} C_*(B(M_2)), \]

where \( B(X) := \coprod_{k \geq 0} B_k(X) \). The fundamental fact underlying this quasi-isomorphism is the contractibility of the unordered configuration spaces of \( \mathbb{R} \) (see below).

(3) **Embedding calculus.** The embedding calculus of Goodwillie and Weiss [Wei99, GW99] produces a tower of approximations

\[
\vdots \quad T_2 \text{Emb}(M, N) \quad \text{↓} \quad \text{Emb}(M, N) \quad \text{↓} \quad T_1 \text{Emb}(M, N),
\]

which can be thought of as algebraic approximations, where algebra is construed in the operadic sense. Often these approximations become arbitrarily good—in particular, according to a theorem of Goodwillie-Klein [GK15], this occurs in codimension at least 3—so that one obtains a cofiltration of the space of embeddings. The layers of this cofiltration are described as spaces of sections of certain bundles over configuration spaces, so hard questions about embeddings may be translated, at least in principle, into softer questions about the ordinary topology of configuration spaces.

In order to develop a feel for how configuration spaces behave, we close with a few examples.

**Example** \((\emptyset, \mathbb{R}^0)\). It is usually best to begin with the trivial cases. In the case of the empty manifold, we have

\[
\text{Conf}_k(\emptyset) = \begin{cases} 
\text{pt} & k = 0 \\
\emptyset & \text{else},
\end{cases}
\]

while

\[
\text{Conf}_k(\mathbb{R}^0) = \begin{cases} 
\text{pt} & k = 0, 1 \\
\emptyset & \text{else}.
\end{cases}
\]

Notice that \( \text{Conf}_0(X) \) is a singleton for any space \( X \).

**Example** \((\mathbb{R} \cong (0, 1))\). From the definition \( \text{Conf}_2((0, 1)) = (0, 1)^2 \setminus \{(x, y) : x = y\} \), it is clear that the ordered configuration space two points in \((0, 1)\) is a disjoint union of two open 2-simplices, and that \( \Sigma_2 \) acts by permuting these components. In particular, \( B_2((0, 1)) \cong \Delta^2 \simeq \text{pt} \). This description generalizes readily to higher \( k \). Note that the natural orientation of \((0, 1)\) induces a second ordering on the coordinates of any configuration, which is to say a permutation of \( \{1, \ldots, k\} \), that any such permutation is possible, and that the assignment of configuration to permutation is locally constant by the intermediate value theorem. Thus, we have a \( \Sigma_k \)-equivariant bijection \( \pi_0(\text{Conf}_k((0, 1))) \cong \Sigma_k \), and the unordered configuration space is naturally
identified with the path component of the identity permutation. For this space, we define a map

\[ B_k((0,1)) \rightarrow \tilde{\Delta}^k := \left\{ (t_0, \ldots, t_k) \in \mathbb{R}^{k+1} : t_i > 0, \sum_{i=0}^{k} t_i = 1 \right\} \]

\[ \{x_1, \ldots, x_k\} \mapsto (x_1, x_2 - x_1, \ldots, 1 - x_k), \]

where the set \( \{x_1, \ldots, x_k\} \) is ordered so that \( x_1 < \cdots < x_k \). This map is a homeomorphism; in particular, the unordered configuration spaces of \( \mathbb{R} \) are all contractible.

**Example** \((\mathbb{R}^n)\). In the configuration space of two points in \( \mathbb{R}^n \), there are exactly three pieces of data—the direction, the center of mass, and the distance—and only one of these is not a contractible choice. Precisely, the map

\[ \text{Conf}_2(\mathbb{R}^n) \rightarrow S^{n-1} \times \mathbb{R}_{>0} \times \mathbb{R}^n \]

\[ (x_1, x_2) \mapsto \left( \frac{x_2 - x_1}{\|x_2 - x_1\|}, \frac{x_2 - x_1}{\|x_2 - x_1\|}, \frac{x_1 + x_2}{2} \right) \]

is a homeomorphism, and, in particular, the Gauss map

\[ \text{Conf}_2(\mathbb{R}^n) \rightarrow S^{n-1} \]

given by the first coordinate of this homeomorphism is a homotopy equivalence. Since this map is also \( \Sigma_2 \)-equivariant for the antipodal action on the target, we conclude that \( B_2(\mathbb{R}^n) \simeq \mathbb{RP}^{n-1} \). The Gauss map will play a fundamental role in our later study; in particular, by pulling back a standard volume form, this map furnishes us with our first example of a nonzero higher dimensional class in the cohomology of configuration spaces.

We cannot be as explicit about \( \text{Conf}_k(\mathbb{R}^n) \) for higher \( k \), but it should be noted that this space is an example of the complement of a hyperplane arrangement, since

\[ \text{Conf}_k(\mathbb{R}^n) = \mathbb{R}^{nk} \setminus \bigcup_{1 \leq i < j \leq k} \{(x_1, \ldots, x_k) \in \mathbb{R}^{nk} : x_i = x_j\}, \]

which is an interesting and classical type of mathematical object in its own right with its own literature [OT92].

**Example** \((S^1)\). By definition \( \text{Conf}_2(S^1) \) is the 2-torus with its diagonal removed, which is homeomorphic to a cylinder, and, by cutting and pasting, one can see directly that the corresponding unordered configuration space is the open Möbius band [Tu02]. In the general case [CJ98, p. 292], make the identification \( S^1 = \mathbb{R}/\mathbb{Z} \), and consider first the subspace \( A \subseteq \text{Conf}_k(S^1) \) of configurations \( (x_1, \ldots, x_k) \) whose ordering coincides with the cyclic ordering induced by the standard orientation of \( S^1 \). For \( 1 \leq i \leq k \), the difference \( t_i = x_{i+1} - x_i \in (0, 1) \) is well-defined, where we set \( t_{k+1} = t_1 + 1 \), and \( \sum_{i=1}^{k} t_i = 1 \). Recording the normalized first coordinate and the \( t_i \) defines a \( C_k \)-equivariant homeomorphism \( A \cong S^1 \times \tilde{\Delta}^{k-1} \), which induces a \( \Sigma_k \)-equivariant homeomorphism

\[ (S^1 \times \tilde{\Delta}^{k-1}) \times_{C_k} \Sigma_k \xrightarrow{\approx} \text{Conf}_k(S^1). \]

In particular, \( B_k(S^1) \cong S^1/C_k \cong S^1 \) for \( k > 0 \).

**Example** \((S^n)\). In the case of two points in the \( n \)-sphere, as with Euclidean space, the choice of the direction in \( S^n \) between the two points is the fundamental parameter. Precisely, the projection onto the first coordinate defines a map \( \text{Conf}_2(S^n) \rightarrow \text{Conf}_1(S^n) \), and recording the
unit tangent vector in the direction of the second coordinate produces a commuting diagram

\[
\begin{array}{ccc}
\text{Conf}_2(S^n) & \longrightarrow & \text{Sph}(TS^n) \\
\downarrow & & \downarrow \\
\text{Conf}_1(S^n) & \longrightarrow & S^n
\end{array}
\]

in which the top map is a homotopy equivalence. The projection map is a special case of a family of maps, the Fadell-Neuwirth fibrations [FN62], which relate different cardinalities of configuration space in any background manifold. The existence of this family of maps, which will be one of our most important tools in what follows, places very strong constraints on configuration spaces; for example, it ultimately implies that the $i$th Betti number of $B_k(M)$ is constant for large $k$ [Chu12, RW13]. This phenomenon of homological stability, and the corresponding representation stability in the ordered case, has since become an active area of study in its own right [Far].

**References**


