To do list

3. (P6) $\Rightarrow$ 3 point uniqueness of projectivities

1. Lemma on matrices and projectivities

2. If $D$ is a field, (P6) holds.
   - $\text{PGL}_2(D)=\text{PGL}_2(\mathbb{R})$
   - 3-point uniqueness for $\text{PGL}_2$

4. (P5) $\Rightarrow$ isomorphic to $\mathbb{RP}^2$

Recall that $\text{PGL}_2(D)$ is the group of equivalence classes of $2 \times 2$ matrices up to central scalar multiplication.
Matrices act on $X = \{ [x,y,z] \mid z = 0 \} \subseteq \mathbb{P}^2$ by

$$T_m([x,y,0]) = [ax + by, cx + dy, 0],$$

where $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Lemma A (1) $T_m$ is a projectivity.

(2) $T_m = T_{m'}$ iff $M = dM'$ with $d \in \mathbb{D}$ central.

Proof of (2) Suppose $T_m = T_{m'}$. Without loss of generality, $M' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Evaluating at $[1,0,0]$ and $[0,1,0]$ gives
\[ (a, c, 0) = (1, 0, 0) \text{ and } (b, d, 0) = (0, 1, 0), \]
so \( c = b = 0. \) Evaluating at \((1, 1, 0)\) gives \((a, d, 0) = (1, 1, 0), \) so \( a = d. \)
Evaluating at \((x, 1, 0)\) for \( x \in \text{ID} \) gives
\[ (ax, a, 0) = (x, 1, 0) \implies axa^{-1} = x, \]
so \( a \) is central, and \( M = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \)
Linear fractional transformations

Assume \( D = \mathbb{F} \) is a field for simplicity. On \( \mathbb{P} \),

\([x, y, 0] = [x/y, 1, 0]\) as long as \( y \neq 0 \),

where \( x/y = x y^{-1} = y^{-1} x \). Introducing the
coordinate

\[
t = \begin{cases} 
  x/y & \text{if } y \neq 0 \\
  \infty & \text{if } y = 0
\end{cases}
\]

the matrix \([a \ b] \ c d\) acts on \( t \) by

\[
t \rightarrow \frac{at + b}{ct + d} \quad \text{“Linear fractional”}
\]
Because $F$ is a field, the condition for invertibility is $ad - bc = 0$.

**Lemma B** Any LFT is a composite of transformations of the form

(i) $t \mapsto t + b$, $b \in F$

(ii) $t \mapsto at$, $0 \neq a \in F$

(iii) $t \mapsto \frac{1}{t}$,

where we interpret $\infty + b = a \cdot \infty = \frac{1}{0} = \infty$

and $\frac{1}{\infty} = 0$. 
Proof consider \( f(t) = \frac{at+b}{ct+d} \), \( ad-bc \neq 0 \).

If \( c=0 \), then \( g/d \neq 0 \), so \( f(t) = \frac{a}{d} + \frac{b}{d} \) is a composite of types (i) and (iii).

If \( c \neq 0 \), then (exercise)

\[
f(t) = \frac{b - \frac{ad}{c}}{g(t)} + \frac{a}{c},
\]

where \( g(t) = ct+d \). Since \( c \neq 0 \) and \( ad-bc \neq 0 \), this is a composite of the three types. \( \square \)
Lemma C: A linear fractional transformation fixes 0, 1, and \( \infty \) is the identity.

Proof: Let \( f(t) = \frac{at+b}{ct+d} \) fix 0, 1, and \( \infty \).

Then
\[
\frac{b}{d} = 0 \implies b = 0
\]
\[
\frac{a+b}{c+d} = 1 \implies a+b = c+d
\]
\[
\frac{a}{c} = \infty \implies c = 0
\]

\[
\implies a = d
\]

So \( f(t) = t \). \( \square \)