Special (P5) If \( \triangle ABC \) and \( \triangle ADEF \) are in point perspective from \( O \in \infty \), and if \( l(A,B) \cap l(DE) \) and \( l(B,C) \cap l(E,F) \) are both contained in \( \infty \), then so is \( l(A,C) \cap l(D,F) \).

Affine picture:
Thm Special (P5) holds iff, for any \( p, p' \neq \infty \), there is a unique translation \( \tau \) with \( \tau(p) = p' \).

Def A parallelogram in (PIL) is an ordered set of distinct points \( (p, p', q, q') \) not contained in \( \infty \) such that \( \ell(p, p') \cap \ell(q, q') \leq \ell \infty \) and \( \ell(p, q) \cap \ell(p', q') \leq \ell \infty \).
Lemma 1: Given non-collinear points \( P, P', Q \neq \infty \), there is a unique \( Q' \) such that \((P, P', Q, Q')\) is a parallelogram.

Lemma 2: Assume special (P5). If two of the three quadrangles in the following diagram are parallelograms, then so is the third.
Lemma 3 If \( \tau \) is a translation and \( p,q \in \mathbb{R}^2 \) are distinct points, then \( (p,q,\tau(p),\tau(q)) \) is a parallelogram.
Proof of Theorem Assume Special (P5), and choose $p, p' \neq \infty$. If $p = p'$, then we may take $\tau$ to be the identity, which is the unique translation with a fixed point away from $\infty$; so assume $p \neq p'$.

Uniqueness of $\tau$ is immediate, since $\tau' \tau^{-1}$ would be a translation fixing $p$, hence the identity, so it suffices to construct such a $\tau$. Given $q$, we define $\tau(q)$ by cases.
Case 0: If $q \in \infty$, $\tau(q) = q$.

Case 1: If $q \not\in l(p, p')$, we define $\tau(q)$ by ensuring that $(p, p', q, \tau(q))$ be a parallelogram (invoking Lemma 1).
Case 2. If \( q \in l(p, p') \), choose any \( r \) not in \( l(p, p') \) or \( l(\infty) \). Then \( \tau(r) \) is defined by case 1, and we define \( \tau(q) \) by requiring that 

\[ (q, \tau(q), r, \tau(r)) \]

be a parallelogram. We must check that \( \tau(q) \) is independent of the choice of \( r \), so let \( r' \in l(p, p') \) be another point. By Lemma 2 and special (P5),
$(r, r', \tau(r), \tau(r'))$ is a parallelogram. Replacing $p, p'$ with $q, q'$ in this diagram shows that $\tau(q)$ as defined by $r$ is the same as $\tau(q)$ as defined by $r'$.
Since \( T(p) = p' \) by construction, it remains to check that \( T \) is a translation. First, \( T \) is a bijection, since \( T^{-1} \) may be obtained by performing the same construction, reversing the roles of \( p, p' \). Since \( T \) fixes \( \infty \) pointwise and fixes no other points, it suffices to show that \( T \) preserves collinearity (the same argument will show that \( T^{-1} \) does so as well). Let \( \ell \neq \infty \)}
be a line intersecting $l_\infty$ at $s$, and pick $s \neq q$ and $l$. Another appeal to Lemma 2 shows that $T(q') \in \ell(T(s), s)$ for any other point $q \in l$ with $q' \neq q, s$, i.e., $T(l) \subseteq \ell(T(s), s)$, so $T$ preserves collinearity.

Conversely, suppose that such translations exist, and consider triangles $\triangle ABC$ and $\triangle ADEF$ satisfying the assumptions of special (P5).
There is a unique translation $\tau$ with $\tau(D) = A$, and it suffices to show that
\( \pi(E) = B \) and \( \pi(F) = C \), since then it follows that \( \pi(e(D,F)) = \pi(A, C) \), so \( e(D,F) \cap e(A, C) = \emptyset \) since \( \pi \) is a dilatation. But, by Lemma 3, \( (D, E, A, \pi(E)) \) is a parallelogram, so \( \pi(E) = B \) by Lemma 1 and our assumption that \( (D, E, A, B) \) is so. The same argument shows that \( \pi(F) = C \). \[ \square \]