Introduction of coordinates

Maintaining choices of \( l_0, l_1, \) and \( 0, 1 \in \mathbb{A} \), choose a line \( l_2 \neq l_1 \) with \( 0 \in l_2 \), and another point \( 0 \neq P \in l_2, 1 \in l_2 \). We assign homogenous coordinates to elements of \( \mathbb{P} \) as follows.

1. If \( a \in l_1 \) and \( a \in l_2 \), then we assign \( a \) the coordinate \( [a, 0, 1] \).

2. If \( 0 \neq P \in l_2 \) and \( P \neq \infty \), there is
a unique dilatation \( \phi \) with \( \phi(0) = 0 \) and \( \phi(1) = p \). Writing \( a = \phi(1) \), we give 

\( p \) the coordinate \( [0, a, 1] \).

(3) We declare that \( l_1 \cap \lambda_\infty = \{ [1, 0, 0] \} \) and \( l_2 \cap \lambda_\infty = \{ [0, 1, 0] \} \).

(4) If \( p \neq \lambda_\infty \), we give \( p \) the coordinate \( [a, b, 1] \), where 

\[
[a, 0, 1] \cap \lambda_\infty = l_1 \cap \lambda_\infty \cap \phi([0, 1, 0])
\]

and 

\[
[0, b, 1] \cap \lambda_\infty = l_2 \cap \lambda_\infty \cap \phi([1, 0, 0])
\]
(5) If \( p \neq l_\infty \) and \( p \neq l_1, ul_2 \), we give \( p \) the coordinates \([1, m, 0]\), where \([1, m, 1]\) is the intersection of \( l_2 \) with the line through \([1, 0, 0]\) and the intersection of \( l(1, [0, 1, 0]) \) and \( l(0, p) \).

Observation: Assigning coordinates in this way gives a bijection \( P \to \mathbb{P}^2 \).
The bijection \( P \to \mathbb{D}P^2 \) is an isomorphism.

This theorem amounts to finding equations describing lines \( \ell \) in terms of the coordinates we've introduced. First, we describe translations and dilations in coordinates.

**Notation** As before, \( T_\alpha \) is the translation taking \( 0 = [0,0,1] \) to \( \alpha = [\alpha,0,1] \), and \( T_{\alpha'} \) is the translation taking the same point to \( [0,0,1] \).
Lemma \( \mathcal{T}_a' = \mathcal{C}_{-\mathcal{T}'; \mathcal{F}_a} \)

Proof The RHS is the unique translation sending 0 to \( \mathcal{F}_a(1) = [0, a, 1] \) (by definition).

It follows from the result of the last class that \( \mathcal{T}_a' \rightarrow \mathcal{T}_a' \) is a homomorphism from \( \text{Trans}_1 \) to \( \text{Trans}_2 \).

Proposition If \( \mathcal{T} \) is a translation with

\[
\mathcal{T}([0,0,1]) = [a,b,1], \text{ then }
\]

\[
\mathcal{T}([x,y,1]) = [x+a, y+b, 1].
\]
Proof We have $Z = Z_a Z_b$ by definition, so

$$Z([x, y, 1]) = Z_x Z_y([0, 0, 1])$$

$$= Z_a Z_b Z_x Z_y([0, 0, 1])$$

$$= Z_a Z_x Z_b' Z_y'([0, 0, 1])$$

$$= Z_a Z_x Z_{b+y}([0, 0, 1])$$

$$= [x + a, y + b, 1]. \ \
\Box$$
If \( \varphi \) is a dilatation fixing \( 0 = (0, 0, 1) \), then \( \varphi = \varphi_a \) for some \( a \in D \), and

\[
\varphi \left( \left[ x, y, 1 \right] \right) = \left[ x_0, y_0, 1 \right].
\]

**Proof:** Define \( a = \varphi(1) \). Then \( \varphi = \varphi_a \), and

\[
\varphi \left( \left[ x, y, 1 \right] \right) = \varphi_a \cdot x \cdot \varphi_y \left( \left[ 0, 0, 1 \right] \right)
\]

\[
= \varphi_a \cdot x \cdot \varphi_y \cdot \varphi_a^{-1} \left( \left[ 0, 0, 1 \right] \right)
\]

\[
= \varphi_a \cdot x \cdot \varphi_a^{-1} \cdot \varphi_y \cdot \varphi_y^{-1} \varphi_a^{-1} \left( \left[ 0, 0, 1 \right] \right)
\]

\[
= T \cdot x \cdot \varphi_y \cdot \varphi_y^{-1} \varphi_a^{-1} \left( \left[ 0, 0, 1 \right] \right).
\]
Thus in coordinates, lines in \((P,L)\) are the solution sets to non-zero equations of the form

\[ ax + by + cz = 0. \]

\textit{Corollary} \((P,L)\) is isomorphic to \(\mathbb{P}^2\).
Proof

(1) By definition, a point lies on \( l \) if and only if its \( z \) coordinate is zero, so \( l = \{ z = 0 \} \).

(2) Also by definition, \( l_1 = \{ y = 0 \} \) and \( l_2 = \{ x = 0 \} \).

(3) Let \( l \) be a line different from \( l_1 \) and \( l_2 \) passing through \( [0,0,1] \). Then \( l \) is determined by its intersection
with \(d\), which is of the form \([1, m, 0]\) for some \(m \in \mathbb{D}\). By definition, \([1, m, 1] \in d\), and, if \([x, y, 1]\) is any other point, there is a unique dilatation \(\phi\) fixing 0 with \(\phi([1, m, 1]) = [x, y, 1]\). By the proposition, \(\phi = \phi_d\) for some \(d \in \mathbb{D}\), and

\[
\phi([1, m, 1]) = [d, md, 1].
\]
Thus, \([x, y, 1] = [d, m d, 1]\), or \(y = mx\), so \(l \subseteq \{y = mx\}\). Conversely, if \(y = mx\), the dilatation \(y \rightarrow y\) sends \([1, m, 1]\) to \([x, y, 1]\), so \(\{y = mx\} \subseteq l\), and equality holds.

(4) If \(l\) does not contain \([0, 0, 1]\), then \(l\) intersects a line \(\{y = mx\}\) on \(l_2\) by the previous case. The intersection of \(l\) with \(l_2\) is necessarily of the
form $[0, b, 1]$, so $T_1$ defines a bijection $\{y = mx^3\} \rightarrow e$. It follows from the proposition that $L = \{y = mx + b\} \cup \{[1, m, 0]\}$, or $L = \{mx - y + bz = 0\}$. \qed