Neglected concept: Distance

Ex. Could have defined the incidence $\mathbb{R}^2$ structure of $\mathbb{R}^2$ by saying a line minimizes the distance between any two points on it ("geodesic")

This tactic generalizes to lots of examples and leads eventually to Riemannian geometry.
Ex The upper hemisphere

\[ S^2_+ = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, z > 0 \}. \]

It turns out that the distance minimizing paths on the sphere are great circles (intersection of planes through the origin) by calculus.
These are parametrized as \( \mathbf{u} \cos \theta + \mathbf{v} \sin \theta \),
and the distance is \( \cos^{-1}(\mathbf{u} \cdot \mathbf{v}) \).

\[ H^2_+ = \left\{ (x, y, z) \in \mathbb{R}^3 \mid -x^2 - y^2 + z^2 = 1, \quad z > 0 \right\} \]

Ex. The hyperboloid (upper sheet)

- The distance minimizers here are parametrized as \( \mathbf{u} \cosh \theta + \mathbf{v} \sinh \theta \), and
- the distance is \( \cosh^{-1}(\mathbf{u} \cdot \mathbf{v}) \).
It's a little hard to "eyeball" how parallelism works in these geometries, so let's do as people have been doing for centuries and make a map.

Ex "Gnomonic" projection (from the center of the sphere onto the plane \( z = 1 \)) is a bijection

\[ S^2_+ \xrightarrow{\sim} \mathbb{R}^2, \]

which ends the intersection of a
plane with $S^2$ to its intersection with the xy-plane, which is a line. In other words, this projection respects lines and is therefore an isomorphism.
So the incidence geometry $S^2$ is the familiar Euclidean geometry, which satisfies the Euclidean parallel postulate.
Ex: Gnomonic projection

For the same reason, "lines" are sent to lines, but this projection isn't surjective. The slope of the line of projection is always greater than 1, and it tends to 1 in the limit, so the image of the projection is the unit disk $D^2$ in the $xy$-plane.
Upshot: The hyperboloid $H^2_+$ is isomorphic to the incidence geometry with:

- points $\{ (x, y) \mid x^2 + y^2 < 1 \}$
- lines intersections of lines in $\mathbb{R}^2$

This incidence geometry is called "the Klein dish," and it satisfies the hyperbolic parallel postulate.
So the HPP appears in nature. What about other examples?

Ex (Pomcevich disk) Take points to be points of $D^2$ again and lines to be intersections of circles orthogonal to boundary, and diameters of the unit circle.
Ex (Upper half plane model) Take points to be \( \{ (x,y) \in \mathbb{R}^2 \mid y > 0 \} \) and lines either half-circles centered on the x-axis or vertical rays.
Amazingly, all four models of hyperbolic geometry (hyperboloid, Klein disk, Poincaré disk, upper half-plane) are isomorphic. This is good news, since each model has its own advantages.

(1) From hyperboloid to Klein: "gnomonic" projection.

(2) From hyperboloid to Poincaré: "stereographic" projection.
(3) From half-plane to Poincaré:

\[ z \mapsto \frac{z-i}{z+i} \]
But why are they the same? The explanation lies "at infinity", in the boundary of the disk or half-plane (which is not part of the model). Thinking of this line as the real projective line, linear fractional transformations act, and the action extends to "Möbius transformations" of the entire disk/half-plane. As we've learned, automorphisms carry
much of the information of a geometry, so it's not unreasonable to hope to proceed as we did with projective planes: given an abstract hyperbolic plane (suitably axiomatized), use automorphisms to build a "boundary", which will be a projective line, and proceed from there to coordinatize the whole plane. We sketch this process next time.