Incidence geometry

Definition: An incidence geometry is a set $P$, whose elements are called points, together with a set $L$ of subsets of $P$, whose elements are called lines, subject to the following axioms:

(I1) every pair of distinct points is contained in a unique line;

(I2) every line contains at least two distinct points;

(I3) there exist three points not contained in the same line.
Termmology  Let \((P,L)\) be an incidence geometry.

- If \(A \in P\), \(l \in L\), and \(A \in l\), say \(A\) lies on \(l\).
- If \(A, B \in P\), write \(AB\) for the unique element of \(L\) containing \(A + B\).
- Say \(A, B \in P\) are collinear if \(A, B \in l\) for some \(l \in L\).
- Say \(l_1, l_2 \in L\) intersect if \(l_1 \cap l_2 \neq \emptyset\).
- etc.

A few basic results are accessible.
Prop. 1  Lines intersect in at most one point.

Proof  If \( A \neq B \) lie on \( l_1 + l_2 \), and if \( A \neq B \), then \( l_1 \parallel AB = l_2 \) by (II).

Prop 2  Given a line \( l \) and a point \( A \) on \( l \), there exists a point on \( l \) distinct from \( A \).

Proof  By (II) \( l \) contains two points \( P \neq Q \).

If \( P \neq A \neq Q \), either \( P \) or \( Q \) suffices.

If \( A = P \), \( Q \) suffices, and vice versa.

Prop 3  Given a line \( l \), there exists a point not on \( l \).
Proof By (I3), there are non-collinear points $A, B, C$. If two, one, or none lie on $l$, we're done, but not all three lie on $l$ by non-collinearity.

Interpretations + Models

Ex (3-point plane)

$P = \{A, B, C\}$

$L = \{\{A, B\}, \{B, C\}, \{A, C\}\}$
Ex (4-point plane)

\[ P = \{1, 2, 3, 4\} \]
\[ L = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\} \]

Ex (5-point plane)

\[ P = \{A, E, I, O, V\} \]
\[ L \text{ the set of all subsets with two elements} \]
**Ex (Wend 5-point plane)**

\[ P = \{1, 2, 3, 4, 5\} \]

\[ L = \{\{1,2,3\}, \{2,3,4\}, \{3,4,5\}, \{1,4,5\} \} \]

**Ex (Fano plane)**

\[ P = \{1, \ldots , 7\} \]

\[ L = \{\{1,4,6\}, \{4,5,3\}, \{5,2,3\}, \{1,7,5\}, \{4,7,3\}, \{6,7,1\} \} \]
Ex (Euclidean plane)

\[ P = \mathbb{R}^2 \]

L the set of solution sets of all equations of the form \[ x = c \]

\[ y = mx + b, \]

where \( c, m, b \) are real numbers.

Prop The Euclidean plane is an incidence geometry.

Proof Homework.
**Non-example**

\[ P = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \} \]

\[ L \text{ the set of great circles} \]

**Parallel postulates**

**Definition** Let \((P, L)\) be an incidence geometry. We say \(L_1, L_2 \in L\) are parallel if they do not intersect (or if \(L_1 = L_2\)).

- **No parallels**
- **Unique parallels**
- **Multiple parallels**
Elliptic parallel postulate: There are no parallel lines.

Euclidean parallel postulate: For any line \( l \) and point \( A \) not on \( l \), there is a unique line through \( A \) parallel to \( l \).

Hyperbolic parallel postulate: For any line \( l \) and point \( A \) not on \( l \), there are at least two distinct lines through \( A \) parallel to \( l \).

Note: There are other options (weird 5-point plane).

Note: These are independent of the axioms.