In this lecture, we give easy proofs of the invariants and additivity theorems. The key observation is that a more convenient basis appears naturally after passing from cohomology to homology. This passage is justified by the tensor/hom adjunction; indeed, for a finite group $G$ and $G$-module $M$, we have

\[ H^* (G; M) \cong \operatorname{Ext}^*_F [G] (F, M) \cong \operatorname{Ext}^*_F (F, \operatorname{Ext}^*_F (M, F)) \cong \operatorname{Ext}^*_F (\operatorname{Tor}^*_F (F, M), F) \cong H_* (G; M). \]

In particular, in the case of interest, we have the isomorphisms

\[ H^* (\operatorname{Conf}_p (R^n)) \cong (H^* (\operatorname{Conf}_p (R^n)))^\Sigma_p \cong H_* (C_p; H^* (\operatorname{Conf}_p (R^n))) \cong H_* (C_p; H_* (\operatorname{Conf}_p (R^n))). \]

Recall that $H_* (\operatorname{Conf}_p (R^n))$ is spanned by classes indexed by $p$-forests modulo the Jacobi identity and graded antisymmetry, with a preferred basis given by the tall forests. The advantage this perspective is that $\Sigma_p$ acts on a forest by permuting the leaves, which are independent of one another, whereas the indices of a monomial such as $\alpha_{ab} \alpha_{bc} \alpha_{ac}$ are far from independent.

The first of the theorems in question is now almost immediate; indeed, we gave the argument previously when computing $H^* (B_k (R^n); \mathbb{Q})$.

**Proof of Invariants Theorem.** Let $\alpha$ be the class of a tall forest with at least three leaves; thus, $|\alpha| \geq 2(n-1)$. By the Jacobi identity, $3 |\alpha| = 0$ in $H_* (\operatorname{Conf}_p (R^n)) \Sigma_p$; therefore, since $p > 3$, the map from $H_* (\operatorname{Conf}_p (R^n))$ to the module of coinvariants is zero, since it annihilates a basis. Since this map is also surjective, the claim follows in degree $2(n-1)$ and higher. The argument in degree 0 is trivial, and the argument in degree $n-1$ has already been given. \(\square\)

We turn now to the vanishing theorem, which we first reformulate in homological terms.

**Vanishing Theorem.** If $s > 0$ and $0 < t < (n-1)(p-1)$, then

\[ H_* (C_p; H^* (\operatorname{Conf}_p (R^n))) = 0. \]

**Remark.** We emphasize that this theorem requires no restriction on $p$ (note that the case $p = 2$ is vacuous).

We will use the following vanishing criterion.

**Proposition.** Let $V$ be a $C_p$-module over $\mathbb{F}$ and $\sigma \in C_p$ a fixed generator. If $V$ admits a decomposition of the form

\[ V \cong \bigoplus_{i=1}^{p} V_{\sigma^i}, \]

such that $\sigma (V_{\sigma^i}) \subseteq V_{\sigma^{i+1}}$, then $H_* (C_p; V) = 0$ for $s > 0$. 

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Proof. The trivial $G_p$-module $\mathcal{F}$ admits the so-called periodic resolution
\[ \cdots \to \mathcal{F}[C_p] \xrightarrow{N} \mathcal{F}[C_p] \xrightarrow{\sigma^{-1}} \mathcal{F}[C_p] \xrightarrow{\epsilon} \mathcal{F}, \]
where $N = 1 + \sigma + \cdots + \sigma^{p-1}$ and $\epsilon$ denotes the augmentation. Thus, the group homology of $V$ is computed by the complex
\[ \cdots \to V \xrightarrow{N} V \xrightarrow{\sigma^{-1}} V, \]
so it suffices to show that the inclusion $\text{im}(N) \subseteq \ker(\sigma - 1)$ is an equality.

Suppose that $\sigma(v) = v$. Our assumption on $V$ provides the unique decomposition
\[ \sum_{i=1}^{p} v_{\sigma^i} = v = \sigma(v) = \sum_{i=1}^{p} \sigma(v_{\sigma^i}). \]
Since $\sigma(v_{\sigma^i}) \in V_{\sigma^{i+1}}$, it follows by induction that $v_{\sigma^i} = v$ for all $1 \leq i \leq p$, so $v = Nv$. □

In order to apply this observation to our situation, we recall that a $p$-forest is simply an ordered partition of $\{1, \ldots, p\}$ together with a binary parenthesization of each block of the partition, and that changing the order of the partition introduces an overall sign. Since the Jacobi identity and antisymmetry do not change the partition of a forest, we have the direct sum decomposition
\[ H_s(\text{Conf}_p(R^n)) \cong \bigoplus_{1 \leq \ell \leq p} \bigoplus_{[\pi] \in \text{Surj}(p, \ell)_{\Sigma_\ell}} F_{[\pi]}, \]
where $F_{[\pi]}$ denotes the subspace spanned by the forests with underlying unordered partition $[\pi]$. We now make three observations.

1. The degree 0 subspace is exactly the $\ell = p$ summand. Thus, we disregard this summand.
2. The degree $(n-1)(p-1)$ subspace is exactly the $\ell = 1$ summand. Thus we, disregard this summand.
3. The symmetric group acts via the action on $\text{Surj}(p, \ell)_{\Sigma_\ell}$ given by pre-composition. In particular, the $\ell$th summand above is closed under the action of $\Sigma_p$.

We conclude that the Vanishing Theorem is equivalent to the claim that
\[ H_s \left( C_p; \bigoplus_{\text{Surj}(p, \ell)_{\Sigma_\ell}} F_{[\pi]} \right) = 0 \]
for $s > 0$ and $1 < \ell < p$. The essential observation in establishing this claim is the following.

Lemma. For any $1 < \ell < p$ and $[\pi] \in \text{Surj}(p, \ell)_{\Sigma_\ell}$,
\[ [\pi \circ \sigma] \neq [\pi]. \]

Proof. There are numbers $1 \leq i, j \leq \ell$ such that $|\pi^{-1}(i)| \neq |\pi^{-1}(j)|$; indeed, assuming otherwise implies that $\ell \mid p$, which contradicts our assumption that $\ell \notin \{1, p\}$. Now, if $\rho$ is a cyclic permutation taking any element of $\pi^{-1}(i)$ to $\pi^{-1}(j)$, then $[\pi] \neq [\pi \circ \rho] = [\pi \circ \sigma^i]$, which leads to a contradiction under the assumption that $[\pi \circ \sigma] = [\pi]$. □

Proof of Vanishing Theorem. Since $F_{[\pi]} \cap F_{[\rho]} = 0$ for $[\pi] \neq [\rho]$, the lemma implies that $F_{[\pi]} \cap F_{[\pi \circ \sigma]} = 0$ for any $1 < \ell < p$ and $[\pi] \in \text{Surj}(p, \ell)_{\Sigma_\ell}$. Thus, for fixed $[\pi]$, the submodule $\bigoplus_{\ell} F_{[\pi \circ \sigma]}$ satisfies the hypotheses of the proposition. By induction on $\text{Surj}(p, \ell)_{\Sigma_\ell}$, $F_{[\pi]}$ decomposes as a direct sum of submodules of this form. The proposition now implies the claim. □