Def A projective plane is an incidence geometry in which every line contains at least three points, and every pair of lines has a point in common.

Recall This idea came from imagining that parallel lines in an affine plane meet "at the horizon."

Def The completion of the affine plane \((P, L)\) has points and lines as follows.
A point in the completion is an element of \( P \) or a pencil of parallel lines. A line is either one of the sets \( \overline{L} = L \cup \{e\} \) for \( \lambda \in \Lambda \) or the set \( L_\infty = \{[\lambda] \mid \lambda \in \Lambda^{0}\} \). This last line is the line at infinity, and the point \([e]\) is the point at infinity in the direction of \( e \), also called an ideal point.
Prop The completion of an affine plane is a projective plane.

Proof We first verify (I1). Let $P_1$ and $P_2$ be points in the completion.

Case 1 If $P_1, P_2 \in P$, then they lie on a unique line $\ell \equiv (P_1P_2)$, hence $P_1, P_2 \in \ell$, and this line is the unique such line in the completion, since neither $P_1$ nor $P_2$ is an ideal point.

Case 2 If $P_1 \in P$ and $P_2 = [\ell]$, then $P_1, P_2 \in \overline{m}$, where $m$ is the parallel to $\ell$
through \( P_1 \). Uniqueness follows from the Euclidean parallel postulate.

(Case 3) If \( P_1 \) and \( P_2 \) are ideal, then \( l_\infty \) is the unique line containing them.

Next, we verify the strengthened version of (I2). Since each \( L \) contains at least two points, \( L \) contains at least three. As for \( l_\infty \), we must verify that \( (P,L) \) has at least three distinct pencils of parallel lines, which follows from the fact that every point is contained
in at least three lines (homework).

For (I3), any three non-collinear points in \((P, L)\) remain non-collinear in the completion. Thus, it remains to check that any two lines intersect.

Case 1. Any \(I\) intersects \(L_\infty\) at \([l]\).

Case 2. If \(l_1, l_2 \in L\) intersect at \(P\), then so do \(\overline{l_1}\) and \(\overline{l_2}\).

Case 3. If \(l_1, l_2 \in L\) are parallel, then \(\overline{l_1}\) and \(\overline{l_2}\) intersect at \([l_1]\] = \([l_2]\). \(\square\)
Upshot: Any example of an affine plane gives an example of a projective plane.

Example (Fano plane)

Completing the 4-point plane:

Example (Real projective plane $\mathbb{RP}^2$)

A point in $\mathbb{RP}^2$ is a line through the origin in $\mathbb{R}^3$. A line in $\mathbb{RP}^2$ is a set of lines lying in a plane.
Prop: The completion of the Euclidean plane $\mathbb{R}^2$ is isomorphic to $\mathbb{RP}^2$.

Homogeneous coordinates

A line through the origin in $\mathbb{R}^3$ is determined by any nonzero vector therein, and two such vectors determine the same line if and only if they are scalar multiples of one another.

$\ell = \{t\bar{v} | t \in \mathbb{R}\}$
More formally, there is a bijection

\[ \{ \text{lines through } \vec{0} \in \mathbb{R}^3 \} \leftrightarrow \mathbb{R}^3 \setminus \{ \vec{0} \} \]

\[ \{ t \vec{v} | t \in \mathbb{R} \} \]

where we declare that \( \vec{v} \sim \vec{w} \)

if \( \vec{v} = c \vec{w} \) for \( c \in \mathbb{R} \).

Now, a plane in \( \mathbb{R}^3 \) is determined by an equation of the form

\[ ax + by + cz = 0, \]

\( a, b, c \in \mathbb{R} \) not all zero.
So we can describe $\mathbb{RP}^2$ analytically as

- points equivalence classes of nonzero vectors under scaling, $[x,y,z]$
- lines solution sets to nonzero linear equations $ax + by + cz = 0$.

Picture preview of proof:

By-plane

$z=1$