Today and for a while: hidden algebraic structure in projective planes, by analogy with

\[ \mathbb{R} \xrightarrow{\text{algebra}} \mathbb{R}^2 \xrightarrow{\text{completion}} \mathbb{R}P^2. \]

Affine picture of addition:

slope \(-\frac{1}{a}\)
<table>
<thead>
<tr>
<th>Ingredient</th>
<th>In $\mathbb{RP}^2$</th>
<th>In $(P,L)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Origin</td>
<td>$[0,0,1]$</td>
<td>Any point $O$</td>
</tr>
<tr>
<td>Axes</td>
<td>$[x,0,1]$, $[0,y,1]$</td>
<td>Two lines $l_x$, $l_y$ through $O$</td>
</tr>
<tr>
<td>Points $a$, $b$</td>
<td>$[a,0,1]$, $[b,0,1]$</td>
<td>Two points $a$, $b$ on $l_x$</td>
</tr>
<tr>
<td>Line $y=1$</td>
<td>$[x,1,1]$</td>
<td>$??$</td>
</tr>
<tr>
<td>Connecting lines</td>
<td>connects through the same ideal point</td>
<td>Incidence axioms</td>
</tr>
<tr>
<td>Parallel lines</td>
<td>$??$</td>
<td>$??$</td>
</tr>
</tbody>
</table>

To fill these in, choose any line $l$ not containing $O$ to play the role of $l_\infty$, and choose any reference line $r$ intersecting $l_\infty$ and $l_x$ at the same point to play the role of $y=1$. 

\[ \Rightarrow \]
Projective setup:
Projective addition:

\[ y = -\frac{1}{2}x + 1 \]

analogous to \( x = \frac{1}{2} \) for line through \((0,1)\)
Independence of reference line:

Why does this coincidence occur in the picture?
The two triangles are in point perspective, hence in line perspective in our picture. Since Desargues is true in \( \mathbb{R}P^2 \).
So if we assume that (P,L) satisfies (P5) then we may conclude that "addition" is well-defined, i.e., independent of r.

By reversing the construction, we also have well-defined "subtraction"—more precisely, we know that \( a + b = c \) can be solved for \( b \) — and it is easy to show that 0 is an additive unit, i.e., \( a + 0 = a \) for every \( a \). We don't yet know that "addition" is associative (homework) or commutative (deferred).
Affine picture of multiplication (as in Lecture 1):

Projective picture:
We can (and will) show that this “multiplication” is associative — this will be the most substantive consequence of (P5) — and reversing the construction gives division (with some care about 0). The question of commutativity comes down to this diagram, which is an instance of (P6):
Upshot: We believe a projective plane satisfying (P5) has addition that behaves as expected and an associative multiplicative with division that is commutative if (P6) is also satisfied.
Definition A division ring or skew field is a set $D$ equipped with "addition" and "multiplication" operations

$$(a, b) \rightarrow a + b$$

$$(a, b) \rightarrow ab$$

such that

1. both are associative
2. addition is commutative
3. there is an additive unit $0$ and a multiplicative unit $1$, and these are distinct
4. additive inverses exist
5. multiplicative inverses exist except for $0$
6. multiplicative distributes over addition
A field is a division ring with commutative multiplication (usually written \( \mathbb{F} \)).

Ex: \( \mathbb{R} \) is a field

Ex: \( \mathbb{Q} \) is a field

Ex: \( \mathbb{F}_2 = \{0, 1\} \) is a field

\[
+ \begin{array}{c|cc}
  & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 0 \\
\end{array}
\]

\[
\cdot \begin{array}{c|ccc}
  & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 0 & 1 \\
\end{array}
\]