You may consult your personal notes, the homework, the posted lecture notes, and the textbook at will. Please do not try to find answers online or discuss the problems with your classmates. This exam has five pages.

(1) **Pencils in** $F_4^2$. In this problem, we work in the affine plane over the field $F_4$ with four elements. Here are the addition and multiplication tables for $F_4$:

$$
\begin{array}{cccc}
+ & 0 & 1 & \omega \\
0 & 0 & 1 & \omega \\
1 & 1 & 0 & \omega \\
\omega & \omega & \omega & 0 \\
\bar{\omega} & \bar{\omega} & \omega & 1 \\
\end{array}
\quad
\begin{array}{cccc}
\times & 0 & 1 & \omega \\
0 & 0 & 0 & 0 \\
1 & 0 & 1 & \omega \\
\omega & \omega & \omega & 1 \\
\bar{\omega} & \bar{\omega} & 1 & \omega \\
\end{array}
$$

As a reminder, points in $F_4^2$ are ordered pairs $(x, y)$ of elements of $F_4$, and lines are solution sets to equations $ax + by + c = 0$ with coefficients in $F_4$, where $a$ and $b$ are not both zero. The slope of such a line is the element $m = ab^{-1} \in F_4$ if $b \neq 0$, and otherwise we declare that $m = \infty$. As discussed in class, lines are parallel if and only if they have the same slope (the proof is the same as for $\mathbb{R}^2$).

(a) What is the order of the affine plane $F_4^2$?

(b) What does your answer to (1a) tell you about the number of pencils of parallel lines in this affine plane? How does this accord with the number of options for the slope of a line?

(c) What does your answer to (1a) tell you about the number of lines in each pencil of parallel lines? How does this accord with the number of equations $ax + by + c = 0$ with a given slope?

In the rest of this problem, we will represent the points of $F_4^2$ as a $4 \times 4$ grid, using the ordering $0, 1, \omega, \bar{\omega}$ from left to right and bottom to top. Thus, for example, the top row of points from left to right is $(0, \bar{\omega}), (1, \bar{\omega}), (\omega, \bar{\omega}),$ and $(\bar{\omega}, \bar{\omega})$. **Do not confuse this grid with any portion of the usual plane.**

(d) Draw the pencil of lines of slope 1 on the grid below. Mark each line with its equation. Here are some “sanity checks” to consider (these are just to help you and don’t require a response).

(i) How many lines should you be drawing?

(ii) How many points should be contained in each line?

(iii) How many points should be contained in the entire pencil?

(iv) If all of your lines “look like” they have slope 1, something has gone awry!
(e) Repeat this process with the pencil of slope $\omega$.

(f) Repeat this process with the pencil of slope $\overline{\omega}$.
(g) There are two pencils left. Describe them in words.
(h) Compare and contrast your depiction of $F_2^4$ with the $4 \times 4$ Pac-Man plane from the second homework.
(i) Based on your pictures, do you think the function that swaps $\omega$ and $\overline{\omega}$ is an automorphism of $F_2^4$? Why or why not?

(2) **Pappus implies Desargues.** In this problem, you will prove that a projective plane satisfying $P_6$ automatically satisfies $P_5$. We refer throughout to the diagram on the last page. Your first task is to construct this diagram from the axioms. Begin with triangles $\Delta ABC$ and $\Delta A'B'C'$ in point perspective from $O$. Assume that the vertices and the sides are all distinct (otherwise there is nothing to prove).

(a) Define the points $P$, $Q$, and $R$ in terms of the two triangles.
(b) Define $S$ in terms of the two triangles.
(c) Show that $\ell(O, S)$ and $\ell(B, C)$ are distinct. Define $T$ in terms of these lines.
(d) Show that $\ell(O, A)$ and $\ell(B, C')$ are distinct. Define $U$ in terms of these lines.
(e) Show that $\ell(O, S)$ and $\ell(B', C')$ are distinct. Define $V$ in terms of these lines.

Now that the diagram has been constructed, your task is to show that the conclusion of Desargues’ theorem holds for the two triangles.

(f) The conclusion of Desargues’ theorem for the triangles $\Delta ABC$ and $\Delta A'B'C'$ is a statement about the collinearity of certain points. Which points are they?
(g) Why are $B$, $S$, and $A$ collinear? Why are $O$, $C$, and $C'$ collinear?
(h) Use $P_6$ to show that $T$, $U$, and $Q$ are collinear.
(i) Why are $O$, $B$, and $B'$ collinear? Why are $C'$, $A'$, and $S$ collinear?
(j) Use $P_6$ to show that $U$, $V$, and $P$ are collinear.
(k) Why are $B$, $C'$, and $U$ collinear? Why are $V$, $T$, and $S$ collinear?
(l) Use $P_6$ and what you have shown so far to draw the desired conclusion.

(3) **When is a projectivity a perspectivity?** We work in a projective plane satisfying $P_6$. Consider the projectivity given by the composite

$$
O \xrightarrow{\ell \not=} m \xrightarrow{P} n
$$

of perspectivities, where $\ell \neq n$. Call this projectivity $\varphi : \ell \rightarrow n$, and denote the point of intersection of $\ell$ and $n$ by $X$. In this problem, you will characterize when $\varphi$ itself is a perspectivity. Consider the following conditions:

(A) $\varphi$ is a perspectivity;
(B) $\varphi(X) = X$;
(C) either $\ell$, $m$, and $n$ intersect or $O$, $P$, and $X$ are collinear.

You will show that these conditions are equivalent (assuming $P_6$).

(a) Show that (A) implies (B) (you’ve essentially already done this as a homework problem).
(b) Show that (B) implies (C) (hint: why is it enough to show that $O$, $P$, and $X$ are collinear assuming (B) and that $X \notin m$?).
(c) Show that, if \( \ell, m, \) and \( n \) intersect, then there exists \( Q \in \ell(O,P) \) such that \( \varphi \) is equal to the perspectivity from \( \ell \) to \( n \) through \( Q \) (hint: use \textbf{P5}. Why is this allowed?).

(d) Suppose that \( \ell, m, \) and \( n \) do not intersect but \( O, P, \) and \( X \) are collinear. Let \( Y \) be the intersection point of \( \ell \) with \( m \) and \( Z \) the intersection point of \( m \) with \( n \).

(i) Show that \( \ell(O,Z) \) and \( \ell(P,Y) \) are distinct, so their point of intersection \( Q \) is defined.

(ii) Show that \( \varphi \) is equal to the perspectivity from \( \ell \) to \( n \) through \( Q \) (hint: use \textbf{P6}).

(e) Explain why it follows that (A), (B), and (C) are equivalent.
Pappus $\Rightarrow$ Desargues.