PROBLEM SET 3  
MATH 130, SPRING 2019  
DUE 2/21 IN CLASS

Proofs should be written in complete sentences and include justifications of each step. The word show is synonymous with prove. An “explanation” need not be a proof.

Reminders on terminology can be found at https://scholar.harvard.edu/files/knudsen/files/terms.pdf. You may find Chapter 5 of Stillwell useful as supplementary reading.

This problem set has two pages.

(1) Hexagons revisited. Make a perspective drawing of a tiling of the plane by hexagons (if you’d like a hint, this problem is broken into two pieces as 5.2.1 and 5.2.2 in Stillwell).

(2) Uncompletion. Let \((P, L)\) be a projective plane. Given a line \(\ell \in L\), set \(P_\ell = P \setminus \ell\) and \(L_\ell = \{m \setminus m \cap \ell : \ell \neq m \in L\}\). In words, \((P_\ell, L_\ell)\) is the result of removing the line \(\ell\) and all of its points from \((P, L)\).
   (a) Show that \((P_\ell, L_\ell)\) is an affine plane whose completion is isomorphic to \((P, L)\).
   (b) Show that any two lines in a projective plane are in bijection. Don’t work too hard!
   (c) What can you say about the number of points in a finite projective plane? What about the number of lines?
   (d) Do you notice a coincidence in your answer to (2c)? Think about how you might explain this coincidence (write down your speculations if you want to, it’s up to you).

(3) The Fano plane.
   (a) Show that the Fano plane is isomorphic to the completion of the 4-point plane.
   (b) Show that the Fano plane is the smallest projective plane in the sense that no projective plane has fewer than 7 points, and any projective plane with exactly 7 points is isomorphic to the Fano plane. Don’t work too hard!

(4) Navigating \(\mathbb{RP}^2\).
   (a) Work problems 5.4.1 and 5.4.2 in Stillwell.
   (b) Interpret (4a) in terms of incidence in \(\mathbb{RP}^2\).
   (c) Do you notice a coincidence in your answer to (4a)? Does it seem at all related to the coincidence you described in (2d)? Think about how you might explain these coincidences (write down your speculations if you want to, it’s up to you). We’ll get to a full answer soon in class.
   (d) Show that the unique line containing the points \([x_1, y_1, z_1] \neq [x_2, y_2, z_2]\) in \(\mathbb{RP}^2\) is the set of points of the form

\[ [sx_1 + tx_2, sy_1 + ty_2, sz_1 + tz_2], \]

where \(s, t \in \mathbb{R}\) are not both zero.
(e) Find the point of intersection of the line through \([p, q, 1]\) and \([0, 0, 1]\) with the line determined by the equation \(z = 0\).

(5) Getting started on a tiling. Begin by refreshing your memory on the perspective drawing of a square tiling that we made in class (or see Section 5.2 in Stillwell). In this problem, we explore the possibility of creating such a drawing in an arbitrary projective plane.

For the purposes of this problem, let’s call a tile in a projective plane any collection \(\{p_1, p_2, p_3, p_4\}\) of points such that no three of the lines \(\ell_{12}, \ell_{23}, \ell_{34}, \text{ and } \ell_{41}\) have a point in common (here, \(\ell_{12}\) is the line containing \(p_1\) and \(p_2\), and so on).

(a) Draw a schematic picture of a tile.
(b) Show that \(\{p_1, p_2, p_3, p_4\}\) is a tile provided no three points are collinear (if you need a hint, this problem is broken into two pieces as 5.3.3 and 5.3.4 in Stillwell).
(c) Using (5b), show that every projective plane contains a tile.
(d) When we made our perspective drawing in class, what did we need to get going in addition to the first tile (hint: it was the very first thing we drew)? Show that, once you have a tile in a projective plane, this other part comes for free.
(e) Think carefully about the steps used in our perspective drawing. Do any of them seem to work by coincidence (hint: consider how many points we can guarantee to coincide on a line)? Eventually, this coincidence will reappear in the form of the so-called “little” Desargues theorem.
(f) Based on your observations, do you think our perspective drawing can be carried out in any projective plane?

(6) The rational plane: congruence. Let \((P, L)\) be an incidence geometry equipped with a betweenness relation satisfying the axioms B1-B5 from the second problem set.

(a) Using the betweenness relation, formulate a definition of the term segment. Specifically, for points \(A, B \in P\), define the segment \(\overline{AB}\).
(b) Using the betweenness relation, formulate a definition of the term ray. Specifically, for points \(A, B \in P\), define the ray \(\overrightarrow{AB}\).

Suppose now that \((P, L)\) is equipped with an equivalence relation on the set of segments. This relation is written \(\overline{AB} \cong \overline{CD}\), and we say that \(\overline{AB}\) is congruent to \(\overline{CD}\). Here are two axioms that congruence could satisfy.

- C1. Given a segment \(\overline{AB}\) and a ray \(\overrightarrow{CD}\), there is a unique point \(E\) on \(\overrightarrow{CD}\) such that \(\overline{AB} \cong \overline{CE}\).
- C2. If \(A * B * C\) and \(D * E * F\), and if \(\overline{AB} \cong \overline{DE}\) and \(\overline{BC} \cong \overline{EF}\), then \(\overline{AC} \cong \overline{DF}\).

Before you do anything with these axioms, be sure you think they’re reasonable.

(c) Using the betweenness relation you defined on the previous problem set, define a congruence relation on segments in the rational plane. Pick one of the congruence axioms, and show that it is satisfied.
(d) Does your definition and argument extend to the Euclidean plane? Why or why not?