Reminders on terminology can be found at https://scholar.harvard.edu/files/knudsen/files/terms.pdf. This problem set has two pages.

(1) **Complete quadrangles and Fano’s axiom.** First, a definition: a **complete quadrangle** in a projective plane \((P,L)\) is a set of four points, no three of which are collinear. Such a set of points determines six lines called **sides**, and opposite pairs of sides intersect at **diagonal points**. We think of all of these points and lines as being part of the figure that is the quadrangle.

(a) Draw a complete quadrangle, together with its sides and diagonal points. If you’d like visual inspiration, try wikipedia, but beware of spoilers.

(b) Does it look like the three diagonal points in your figure could be collinear? Formulate a conjecture based on your observation.

(c) Find a complete quadrangle in the Fano plane whose diagonal points are collinear. If necessary, revise your conjecture (hint: your piece of paper and the Fano plane pertain to two specific fields).

Fano’s axiom **P7** is the following statement: the diagonal points of any complete quadrangle are non-collinear. Like **P5** and **P6**, this axiom may or may not hold for any particular projective plane.

(d) Show that Fano’s axiom does not hold in \(\mathbb{F}P^2\) if the characteristic of \(\mathbb{F}\) is 2 (hint: find a complete quadrangle with diagonal points \([0,1,1],[1,1,0],[1,0,1]\)).

In fact, the converse of the last statement is also true: Fano’s axiom holds in \(\mathbb{F}P^2\) as long as the characteristic of \(\mathbb{F}\) is different from 2. You may use this fact in the rest of the problem set.

(2) **Harmonic points.** Another definition: a **harmonic quadruple** is an ordered set of collinear points \((A,B,C,D)\) such that \(A\) and \(B\) are diagonal points of some complete quadrangle, and \(C\) and \(D\) lie on the remaining two sides.

(a) Draw a harmonic quadruple together with a complete quadrangle witnessing it as such. If you’d like visual inspiration, try wikipedia, but beware of spoilers.

Assuming **P5** and **P7**, it is a fact that, given an ordered triple of distinct points on a line, there is a **unique** fourth point such that the four points form a harmonic quadruple. You may use this fact from now on.

(b) Let \((A,B,C,D)\) be a harmonic quadruple. Show that a perspectivity \(\varphi: \ell \overset{O}{\sim} \ell'\) such that \(A\in\ell\cap\ell'\) sends this harmonic quadruple to another harmonic quadruple (hint: consider the complete quadrangle \(\{O,X,B,D\}\), where \(X\) is the point of intersection of \(\ell(B,\varphi(C))\) with \(\ell(O,A))\).

(c) Let \(\psi: m \overset{O}{\sim} m'\) be an arbitrary perspectivity, and set \(m'' = \ell(A,\psi(B))\). Show that \(\psi\) is equal to the projectivity

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\frac{P}{m \overset{O}{\sim} m'' \overset{P}{\sim} m'}. 
\]
(d) Conclude from (2c) that perspectivities preserve harmonic quadruples (hint: use (2b) twice).

(e) Conclude from (2d) that projectivities preserve harmonic quadruples.

(f) We proved that, given three points on a line \( \ell \) in any projective plane, there is a projectivity sending those three points to any three other points on \( \ell \). Conclude from (2e) that the corresponding statement for four points is false.

(3) Transformations of the cross-ratio. Work Exercises 5.8.1-5.8.6 in Stillwell. In Problem 5.8.4, a function obtained by “combining \( 1/y \) and \( 1 - y \)” refers to the result of performing a finite number of substitutions of the form \( y \mapsto 1/y \) or \( y \mapsto 1 - y \) starting with \( y \) itself.

(4) Cross-ratio and harmonic quadruples. Working in the ordinary Euclidean plane \( y \neq 0 \) inside \( \mathbb{RP}^2 \), show that a set of points \( (p, q, r, s) \) on the \( t = x/y \)-axis is a harmonic quadruple if and only if \([p, q; r, s] = -1\).

(5) Cross-ratio and projectivities. Let \( \mathbb{F} \) be a field. The cross-ratio of four points \( \{p, q, r, s\} \subseteq \mathbb{F} \cup \{\infty\} \) is defined in exactly the same way as in the case \( \mathbb{F} = \mathbb{R} \). In this problem, you will show that the cross-ratio exactly characterizes projectivities among all functions.

(a) Show that the cross-ratio is preserved by every fractional linear transformation (hint: you only need to check three cases). Be careful not to omit cases in which one of the points is 0 or \( \infty \).

(b) Conversely, show that any function preserving the cross-ratio is a linear fractional transformation (hint: compare \([0, 1; \infty, t]\) with \([a, b; c, f(t)]\), where \( a = f(0) \) and so on).