Problem Set 1.

(4) There are many possible solutions.
   (a) The pair $\{(1,2,3), \emptyset\}$ satisfies $I_2$ and $I_3$ but not $I_1$ (note that, since there are no lines, $I_2$ is satisfied vacuously).

   (b) Adding the empty set to the set of lines of the 3-point plane produces an example satisfying $I_1$ and $I_3$ but not $I_2$.

   (c) The pair $\{(1,2), \{\{1,2\}\}\}$ satisfies $I_1$ and $I_2$ but not $I_3$.

(5) (a) Let $\langle P, L \rangle$ be an incidence geometry such that $P$ has three elements, say $P = \{A, B, C\}$. By $I_3$, $P \notin L$, and $I_2$ implies that no set with fewer than two elements lies in $L$. Thus, the only subsets of $P$ that could possibly lie in $L$ are the subsets with two elements, and all three such subsets must in fact lie in $L$ by $I_1$, so

   $$L = \{\{A, B\}, \{B, C\}, \{A, C\}\}.$$

   The function $A \mapsto 1$, $B \mapsto 2$, $C \mapsto 3$ defines an isomorphism of $\langle P, L \rangle$ with the 3-point plane by inspection.

   (b) Let $\langle P, L \rangle$ be an incidence geometry such that $P$ has four elements, say $P = \{A, B, C, D\}$. The same argument as above shows that $P \notin L$ and that no subset with fewer than two elements lies in $P$. There are now two cases.

   (i) There are no lines containing three points. In this case, $I_1$ implies that $L$ is the set of subsets with two elements, so $\langle P, L \rangle$ is isomorphic to the 4-point plane.

   (ii) There exists a line containing three points. If there is a line containing three points, then the uniqueness clause of $I_1$ implies that there is only one such, which we may take to be $\{A, B, C\}$ without loss of generality, and $I_1$ implies that

   $$L = \{\{A, B, C\}, \{A, D\}, \{B, D\}, \{C, D\}\}.$$

Problem Set 2.

(5) There are many possible approaches.

   (a) The vertical line in question is the set $\{(1,0), (1,1), (1,2), (1,3)\}$, but the first coordinate of every point on the line through $(0,0)$ and $(2,1)$ is even.

   (b) The vertical line through $(0,0)$ is parallel to the first line but not parallel to the second; therefore, parallelism is not transitive.
Problem Set 3.

(2) As a matter of notation, we write $m_\ell$ for the line in $(P_\ell, L_\ell)$ that is $m \in L$ with its point of intersection with $\ell$ removed.

(a) We first show that $(P_\ell, L_\ell)$ is an incidence geometry. Distinct points $p, q \in P_\ell$ lie on $m_\ell$ if and only if they lie on $m$; therefore, 11 holds in $(P_\ell, L_\ell)$, since it holds in $(P, L)$. Each $m \in L$ contains at least three points, so each $m_\ell$ contains at least two, which is 12. For 13, it suffices to show the existence of three non-collinear points in $(P, L)$, none of which lie on $\ell$. Choose $p_1, p_2 \in \ell$ and $p_3 \notin \ell$ (here we use 12 and 13 in $(P, L)$, respectively). By strengthened 12, the lines $\ell(p_1, p_3)$ and $\ell(p_2, p_3)$ each contain further points $q_1$ and $q_2$, respectively. Then $p_3, q_1,$ and $q_2$ are non-collinear, since otherwise 11 implies that $\ell(p_1, p_3) = \ell(p_2, p_3) = \ell$, a contradiction.

To check the Euclidean parallel postulate, fix $m_\ell$ and $p \notin m_\ell$. A line $m_\ell'$ is parallel to $m_\ell$ if and only if the intersection of $m$ and $m_\ell'$ lies on $\ell$. Since there is a unique line through $p$ and the intersection of $m$ with $\ell$, there is a unique such parallel, as required.

Define a function $f : \mathcal{P}_\ell \to P$ by setting $f(p) = p$ for $p \in P_\ell$ and defining $f([m_\ell])$ to be the point of intersection of $m$ and $\ell$ for $m_\ell \in L_\ell$. To see that $f$ is well-defined, we must check that the definition of $f$ on ideal points is independent of the choice of representative $m_\ell$ for the pencil $[m_\ell]$ of parallel lines, but this follows from our earlier characterization of lines parallel to $m_\ell$. To see that $f$ is a bijection, we exhibit the inverse function $g$, the definition of which is clear on $P_\ell \subseteq P$, and which sends a point $p \in \ell$ to the ideal point $[m_\ell]$, where $m$ is any line through $p$ different from $\ell$ (all such lie in a common pencil, by our discussion). Now, $f$ sends the line $m_\ell$ to $m$ and the line $\ell_\infty$ to $\ell$, and vice versa for $g$, so $f$ is an isomorphism.

(b) Let $m, m', \text{ and } \ell$ be distinct lines in $(P, L)$. Then $m_\ell$ and $m'_\ell$ are in bijection, since $(P_\ell, L_\ell)$ is an affine plane, and $m$ and $m'$ differ from these lines by the addition of a single point, so they are also in bijection.

(c) Let $(P, L)$ be a finite projective plane. By the previous part, every line has a common number of points, which we call $n + 1$. Then $(P_\ell, L_\ell)$ is an affine plane of order $n$, therefore having $n^2$ points and $n(n + 1)$ lines. Since $(P, L)$ differs by the addition of $n + 1$ additional points and one additional line, the number of points and the number of lines in $(P, L)$ are both $n^2 + n + 1$.

Problem Set 4.

(4)

(a) Yes, this follows from two applications of the statement “the dual of a projective plane is a projective plane.”

(b) The set $P^{**}$ is the set of pencils of lines through points of $P$. A subset of pencils is an element of $L^{**}$ if and only if it is the set of pencils through the points of a line in $L$.

(c) Define a function $f : P \to P^{**}$ by declaring $f(p)$ to be the pencil of lines through $p$. No two points have the same pencil, since assuming otherwise quickly leads to a
Problem Set 5.

(5)

(a) Let \( X \neq X' \) be planes. By S4, the intersection \( X \cap X' \) contains the line \( \ell \). On the other hand, if \( p \in X \cap X' \) does not lie on \( \ell \), then S2 guarantees that \( p \) and any two points on \( \ell \) lie in a unique plane \( X'' \), so \( X = X'' = X' \), a contradiction. Thus, the reverse inclusion \( X \cap X' \subseteq \ell \) also holds, hence equality.

(b) Let \( p_1 \neq p_2 \) be any two points. By S5, there are four non-coplanar points \( q_1, q_2, q_3, \) and \( q_4 \), no three of which are collinear. Some of these points may coincide with \( p_1 \) or \( p_2 \); regardless, no more than two may lie on \( \ell(p_1, p_2) \). We proceed by cases.

If exactly two of the four points lie on \( \ell(p_1, p_2) \), then we may take these points to be \( q_1 \) and \( q_2 \) without loss of generality. It suffices for this case to prove that the set \( \{p_1, p_2, q_1, q_2\} \) is non-coplanar. Assuming otherwise, part (a) guarantees that the plane \( X \) containing it intersects the plane through \( q_1, q_2, \) and \( q_3 \) in a line (we use that these three points are non-collinear in order to guarantee the existence of this plane, and we use that the four points are non-coplanar to guarantee its distinctness from \( X \)). Since this line is necessarily \( \ell(p_1, p_2) = \ell(q_1, q_2) \), it follows that \( q_1, q_2 \in X \), contradicting the assumption that \( \{q_1, \ldots, q_4\} \) is non-coplanar.

If exactly one of the four points lies on \( \ell(p_1, p_2) \), then we may take this point to be \( q_1 \) without loss of generality. It suffices in this case to prove that the set \( \{p_1, p_2, q_1, q_i\} \) is non-coplanar for some \( i \in \{3, 4\} \). Assuming otherwise, part (a) guarantees, if the two resulting planes \( X_3 \) and \( X_4 \) are distinct, that they intersect in a line, which is necessarily \( \ell(p_1, p_2) \), contradicting the assumption that \( q_2 \notin \ell(p_1, p_2) \). On the other hand, if \( X_3 = X_4 \), then it follows that \( q_1 \in X_3 \), and the four are coplanar, another contradiction; indeed, the plane containing \( q_1, q_2, \) and \( p_1 \) (which are non-collinear since \( q_2 \notin \ell(p_1, p_2) \)) intersects \( X_3 \) in a line, which is necessarily \( \ell(p_1, p_2) \), containing \( q_1 \).

If none of the four points lie on \( \ell(p_1, p_2) \), we claim that the set \( \{p_1, p_2, q_1, q_i\} \) is non-coplanar for some \( i \in \{2, 3, 4\} \). Otherwise, we obtain three planes \( X_2 \), and \( X_3 \), and \( X_4 \). Not all three can be the same, since \( \{q_1, \ldots, q_4\} \) is non-coplanar; however, if (without loss of generality) \( X_2 \neq X_3 \), then \( X_2 \cap X_3 \) is a line that is necessarily equal to \( \ell(p_1, p_2) \). It follows that this line contains \( q_1 \), contrary to assumption.

(c) Let \( \ell \) be a line and \( p_1 \) and \( p_2 \) two points thereon. If \( p_3 \) and \( p_4 \) are the two points guaranteed by part (b), then planes containing \( \{p_1, p_2, p_3\} \) and \( \{p_1, p_2, p_4\} \) exist by the non-collinearity clause and are distinct by the non-planarity clause. By part (a), their intersection is a line, which is necessarily \( \ell = \ell(p_1, p_2) \).

(6) Suppose that \( (P, L) \) satisfies P6. Then \( (P, L) \) satisfies P5 as well and is therefore isomorphic to the projective plane over a division ring, which is a field by P6. It follows that \( (P, L) \) is self-dual, so the dual of P6, which holds in the dual of \( (P, L) \), must also
hold in \((P,L)\).

**Problem Set 6.**

(1) Depending on which variant of the definition of “isomorphism” one works with, the solution looks slightly different. This argument uses the phrasing “\(f\) is an isomorphism if \(f\) is a bijection and \(S \subseteq P_1\) is a line if and only if \(f(S) \subseteq P_2\) is a line.”

Supposing that \(f\) is an isomorphism, let \(S \subseteq P_1\) be a collinear set, so \(S \subseteq \ell\) for some line \(\ell \in L_1\). Then \(f(S) \subseteq f(\ell)\), and the latter is a line, so \(f\) preserves collinearity. On the other hand, if \(T \subseteq P_2\) is a collinear set, so that \(T \subseteq m\) for some \(m \in L_2\), then \(f^{-1}(T) \subseteq f^{-1}(m)\); therefore, to conclude that \(f^{-1}\) preserves collinearity, it suffices to show that \(f^{-1}(m)\) is a line. But \(f(f^{-1}(m)) = m\) is a line, and \(f\) is an isomorphism, so the claim follows.

Conversely, suppose that \(f\) and \(f^{-1}\) both preserve collinearity, and let \(\ell \in L_1\) be a line. Then \(f(\ell) \subseteq m\) for some line \(m \in L_2\), since \(f\) preserves collinearity, and \(f^{-1}(m) \subseteq \ell\), since \(f^{-1}\) does so. It follows that \(\ell \subseteq \ell'\), so the two lines are equal by \(\text{II}\); therefore, \(f(\ell) = m\), a line. On the other hand, suppose that \(S \subseteq L_1\) is a subset such that \(f(S) \subseteq L_2\) is a line. Since \(f^{-1}\) preserves collinearity, it follows that \(S\) is collinear, so \(S \subseteq \ell\) for some \(\ell \in L_1\). Since \(f\) preserves collinearity, \(f(\ell) \subseteq m\) for some line \(m \in L_2\). As before, it follows that \(f(S) = f(\ell) = m\), so \(S = \ell\) is a line. Thus, \(f\) is an isomorphism.

(b) Consider the 4-point plane and the incidence geometry with four points \(P = \{1, 2, 3, 4\}\) and \(L = \{\{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}\}\). The identity function on \(\{1, 2, 3, 4\}\) preservers collinearity when viewed as function from the set of points of the 4-point plane to the set of points of \((P,L)\), but its inverse (also the identity function) does not.

**Problem Set 7.**

(4) Since \(H\) is non-empty, there is an element \(h \in H\), hence \(h^{-1} \in H\), since \(H\) is closed under the formation of inverses. By closure under multiplication \(hh^{-1} = e \in H\).

(b) We choose a bijection of \(g: P \xrightarrow{\approx} \{1, \ldots, n\}\). Define a function \(\overline{g}: \text{Aut}(P,L) \to S_n\) by \(\overline{g}(f) = g \circ f \circ g^{-1}\), which is well-defined since the composite of bijections is again a bijection. Since \(g \circ f_1 \circ g^{-1} \circ g \circ f_2 \circ g^{-1} = g \circ f_1 \circ f_2 \circ g^{-1}\), \(\overline{g}\) is a group homomorphism, and it follows that the set \(H_g\) of permutations \(\sigma \in S_n\) such that \(\sigma = \overline{g}(f)\) for some automorphism \(f\) is in fact a subgroup of \(S_n\), and \(\overline{g}\) restricts to a group homomorphism from \(\text{Aut}(P,L)\) to \(H_g\), which is surjective by definition and also injective, since \(g \circ f_1 \circ g^{-1} = g \circ f_2 \circ g^{-1}\) implies that \(f_1 = f_2\) after composing on the left with \(g^{-1}\) and on the right with \(g\). Therefore, \(\text{Aut}(P,L)\) is isomorphic to this subgroup \(H_g\).

(c) The subgroup \(H_g\) and the isomorphism depend on the choice of bijection \(g\). There is no canonical ordering of the points of an incidence geometry, so this choice cannot be eliminated.
(d) If $\sigma \in S_3$ is any permutation, then $\{i,j\}$ is a set of distinct elements of $\{1,2,3\}$ if and only if $\{\sigma(i),\sigma(j)\}$ is a set of distinct elements. Since the lines of the 3-point plane are precisely the sets of two distinct elements, we conclude that every permutation is an automorphism, i.e., that the subgroup in question is $S_3$ itself.

(e) Replacing $\{1,2,3\}$ with $\{1,2,3,4\}$ in the previous argument shows that the automorphism group of the 4-point plane is isomorphic to $S_4$.

(f) The same argument shows that the automorphism group of the $n$-point plane is isomorphic to $S_n$.

(g) Suppose that $\sigma_1(4) = \sigma_2(4) = 4$. Then $\sigma_1(\sigma_2(4)) = \sigma_1(4) = 4$, and $\sigma_1^{-1}(4) = \sigma_1^{-1}(\sigma_1(4)) = 4$, so the set of elements with this property is closed under multiplication and the formation of inverses. It is also non-empty; for example, it contains the permutation that switches 1 and 2 and leaves 3 and 4 fixed. Therefore, this subset is a subgroup. This subgroup is isomorphic to $S_3$, since a permutation fixing 4 is determined by its values on $\{1,2,3\} \subseteq \{1,2,3,4\}$, and since any permutation of 1, 2, and 3 determines an element of $S_4$ by declaring its value on 4 to be 4.

(h) Let $P = \{1,2,3,4\}$ and $L = \{\{1,2,3\}, \{1,4\}, \{2,4\}, \{3,4\}\}$. Let $H \leq S_4$ denote the stabilizer subgroup of 4, as in the previous problem. We claim that $\text{Aut}(P,L) = H$ (we may write an equals sign rather than an isomorphism here, since the set of points in this specific example is $\{1,2,3,4\}$, rather than an abstract set with four elements). Both $\text{Aut}(P,L)$ and $H$ are subgroups of $S_4$, so it suffices to show that each contains the other. Any permutation fixing 4 sends lines to lines by inspection, so $H \subseteq \text{Aut}(P,L)$. On the other hand, suppose that $f$ is a bijection that does not fix 4, so $f(4) = 1$ without loss of generality. Then $f(\{1,2,3\}) = \{4,2,3\}$, which is not a line; therefore, $f$ is not an automorphism. It follows that $\text{Aut}(P,L) \subseteq H$, as required.

(i) The automorphism groups of the 3-point plane and the weird 4-point plane are both isomorphic to $S_3$, but the two are certainly not isomorphic, having different numbers of points.