MATH 130: CLASSICAL GEOMETRY
SELECTED HOMEWORK SOLUTIONS

Problem Set 1.

(4) There are many possible solutions.
   (a) The pair \((\{1, 2, 3\}, \emptyset)\) satisfies \(I_2\) and \(I_3\) but not \(I_1\) (note that, since there are no lines, \(I_2\) is satisfied vacuously).

   (b) Adding the empty set to the set of lines of the 3-point plane produces an example satisfying \(I_1\) and \(I_3\) but not \(I_2\).

   (c) The pair \((\{1, 2\}, \{\{1, 2\}\})\) satisfies \(I_1\) and \(I_2\) but not \(I_3\).

(5)
   (a) Let \((P, L)\) be an incidence geometry such that \(P\) has three elements, say \(P = \{A, B, C\}\). By \(I_3\), \(P \notin L\), and \(I_2\) implies that no set with fewer than two elements lies in \(L\). Thus, the only subsets of \(P\) that could possibly lie in \(L\) are the subsets with two elements, and all three such subsets must in fact lie in \(L\) by \(I_1\), so

   \[ L = \{\{A, B\}, \{B, C\}, \{A, C\}\} .\]

   The function \(A \mapsto 1, B \mapsto 2, C \mapsto 3\) defines an isomorphism of \((P, L)\) with the 3-point plane by inspection.

   (b) Let \((P, L)\) be an incidence geometry such that \(P\) has four elements, say \(P = \{A, B, C, D\}\). The same argument as above shows that \(P \notin L\) and that no subset with fewer than two elements lies in \(P\). There are now two cases.

      (i) \textbf{There are no lines containing three points.} In this case, \(I_1\) implies that \(L\) is the set of subsets with two elements, so \((P, L)\) is isomorphic to the 4-point plane.

      (ii) \textbf{There exists a line containing three points.} If there is a line containing three points, then the uniqueness clause of \(I_1\) implies that there is only one such, which we may take to be \(\{A, B, C\}\) without loss of generality, and \(I_1\) implies that

   \[ L = \{\{A, B, C\}, \{A, D\}, \{B, D\}, \{C, D\}\} .\]

Problem Set 2.

(5) There are many possible approaches.

   (a) The vertical line in question is the set \(\{(1, 0), (1, 1), (1, 2), (1, 3)\}\), but the first coordinate of every point on the line through \((0, 0)\) and \((2, 1)\) is even.

   (b) The vertical line through \((0, 0)\) is parallel to the first line but not parallel to the second; therefore, parallelism is not transitive.
Problem Set 3.

(2) As a matter of notation, we write $m_\ell$ for the line in $(P_\ell, L_\ell)$ that is $m \in L$ with its point of intersection with $\ell$ removed.

(a) We first show that $(P_\ell, L_\ell)$ is an incidence geometry. Distinct points $p, q \in P_\ell$ lie on $m_\ell$ if and only if they lie on $m$; therefore, I1 holds in $(P_\ell, L_\ell)$, since it holds in $(P, L)$. Each $m \in L$ contains at least three points, so each $m_\ell$ contains at least two, which is I2. For I3, it suffices to show the existence of three non-collinear points in $(P, L)$, none of which lie on $\ell$. Choose $p_1, p_2 \in \ell$ and $p_3 \notin \ell$ (here we use I2 and I3 in $(P, L)$, respectively). By strengthened I2, the lines $\ell(p_1, p_3)$ and $\ell(p_2, p_3)$ each contain further points $q_1$ and $q_2$, respectively. Then $p_3, q_1,$ and $q_2$ are non-collinear, since otherwise I1 implies that $\ell(p_1, p_3) = \ell(p_2, p_3) = \ell$, a contradiction.

To check the Euclidean parallel postulate, fix $m_\ell$ and $p \notin m_\ell$. A line $m'_\ell$ is parallel to $m_\ell$ if and only if the intersection of $m$ and $m'$ lies on $\ell$. Since there is a unique line through $p$ and the intersection of $m$ with $\ell$, there is a unique such parallel, as required.

Define a function $f : \mathcal{P}_\ell \to P$ by setting $f(p) = p$ for $p \in P_\ell$ and defining $f(\lfloor m_\ell \rfloor)$ to be the point of intersection of $m$ and $\ell$ for $m_\ell \in L_\ell$. To see that $f$ is well-defined, we must check that the definition of $f$ on ideal points is independent of the choice of representative $m_\ell$ for the pencil $\lfloor m_\ell \rfloor$ of parallel lines, but this follows from our earlier characterization of lines parallel to $m_\ell$. To see that $f$ is a bijection, we exhibit the inverse function $g$, the definition of which is clear on $P_\ell \subseteq P$, and which sends a point $p \in \ell$ to the ideal point $\lfloor m_\ell \rfloor$, where $m$ is any line through $p$ different from $\ell$ (all such lie in a common pencil, by our discussion). Now, $f$ sends the line $\overline{m}$ to $m$ and the line $\ell_\infty$ to $\ell$, and vice versa for $g$, so $f$ is an isomorphism.

(b) Let $m, m', \text{ and } \ell$ be distinct lines in $(P, L)$. Then $m_\ell$ and $m'_\ell$ are in bijection, since $(P_\ell, L_\ell)$ is an affine plane, and $m$ and $m'$ differ from these lines by the addition of a single point, so they are also in bijection.

(c) Let $(P, L)$ be a finite projective plane. By the previous part, every line has a common number of points, which we call $n + 1$. Then $(P_\ell, L_\ell)$ is an affine plane of order $n$, therefore having $n^2$ points and $n(n + 1)$ lines. Since $(P, L)$ differs by the addition of $n + 1$ additional points and one additional line, the number of points and the number of lines in $(P, L)$ are both $n^2 + n + 1$.

Problem Set 4.

(4) 

(a) Yes, this follows from two applications of the statement “the dual of a projective plane is a projective plane.”

(b) The set $P^{**}$ is the set of pencils of lines through points of $P$. A subset of pencils is an element of $L^{**}$ if and only if it is the set of pencils through the points of a line in $L$.

(c) Define a function $f : P \to P^{**}$ by declaring $f(p)$ to be the pencil of lines through $p$. No two points have the same pencil, since assuming otherwise quickly leads to a
contradiction of \textbf{13}, so $f$ is injective, and $f$ is obviously surjective, since every pencil of lines is the pencil of lines through some point. Our “if and only if” description of $L^{**}$ above immediately implies that $f$ is an isomorphism.