Toward the immersion conjecture

Theorem (Cohen) $M^n$ immerses in $\mathbb{R}^{n-2(q)}$.

By Hirsch-Smale, equivalent to the lifting problem

\[ M \xrightarrow{?} B\Omega \xrightarrow{\nu} B\Omega(n-2q) \]

\[ M \xrightarrow{\nu} B\Omega \xleftarrow{T_y} M_0 \]

\[ M_0/\mathcal{I}_n \rightarrow M_0(n-2q) \]

\[ \text{MO} \]
Reminders on $BO/In$

(0) There is no $BO/In$ (very non-canonical)

(1) $BO/In$ is a cell complex of dim. $n$

(2) $T(BO/In \to BO) = MO/In = \bigvee_{\text{dim } n} S^{(n)} \wedge \Omega_{n-1, n}$

\[ T = B(\Sigma_k \to \Omega_k \to O(k)) \]

(3) $BO/In$ is constructed by step by step de- Thomification of a specific Adams resolution

\[ MO/In \to \cdots \to T_{i+1} \to T_i \to \cdots \to T_0 = MO, \]
and taking the n-skeleton of the inverse limit. This tower is a wedge of towers for Brown-Gitler spectra.

(4) $BO/In$ has the following lifting property. A stable vector bundle over a CW complex of dimension $n$ lifts to $BO/In$ provided its Thom spectrum admits a cohomology injection from a spectrum $E$ satisfying the following duality condition (Lannes):
(Dn) The n-dual of $E$ is (2-locally) a finite CW complex $Y$ of dimension $n$, i.e.,

$$\exists S^n \to E \times Y$$

inducing a perfect pairing on $\text{mod } 2 \tilde{H}^\ast$.

**Consequences of (4)**

(A) $v_m$ lifts to $BO/I_n$
(B) $\tau_m$ lifts to $BO/I_k$
(C) $BO/I_k \times BO/I_{n-k} \to BO$ lifts to $BO/I_n$
Much of the difficulty of Cohen's proof is in controlling ambiguity arising from $B_0/I_n$ and the use of $(B)$ and $(C)$.

Problem: Construct a specific instance of $B_0/I_n$ that is conceptual, explicit, and/or structured.

OR,

Problem: Package the ambiguity and carry it through the argument.

Problem: Understand $(B)$ in terms of braids.
Goal: De-Thomify $\text{MO}/\text{In}_n \rightarrow \text{MO}(n-\alpha(n))$

Idea: $\text{MO}/\text{In}_n$ is a summand of a spectrum $\text{TX}_n$ such that $\text{TX}_n \rightarrow \text{MO}/\text{In}_n \rightarrow \text{MO}(n-\alpha(n))$, obviously de-Thomify is free.

Writing $\text{MO}/\text{In}_n = V S^{\infty} \wedge T \alpha_n w$, we choose a representative $M^w$ for the $w^\text{th}$ generator of $\text{MO}$, such that the immersion conjecture holds for $M^w$ (this is possible by R.L. Brown's theorem as explained by Jan A. Høn).
We then have the commutivity diagram

\[
\begin{array}{ccc}
M_w \times B_{n-1,1} & \xrightarrow{1 \times 2} & B_0/I_{n,1} \times B_0/I_{n-1,1} \\
\downarrow \mu \times \sigma & & \downarrow (3) \\
B_0 \times B_0 & \rightarrow & B_0 & \leftarrow & B_0/I_n
\end{array}
\]

The dashed maps exist by the versal property of $B_0/I_n$. Beware non-commonicity!
Thus, setting $X_n = \frac{1}{11} M_w \times B_{n-1w1}(C)$, we have the maps $X_n \xrightarrow{g} B_0/\text{In}$ and

$$\Gamma = \bigvee_w \left( S_{1w1w} \xrightarrow{\text{pt}} T_{1w} \right) \cap T_2 T_{n-w}.$$ 

Claim \quad $T_g \circ \Gamma = \text{id}_{B_0/\text{In}}$

Now, since the immersion conjecture holds for $M_w$, and using the same fact about the braid bundles from before, we have
$M_w \times B_{n-lw1}(C) \longrightarrow BO(\mathbb{R}^w_1 - \alpha(\mathbb{R}^w_1)) \times BO(\mathbb{R}^{n-1w1}_1 - \alpha(\mathbb{R}^{n-1w1}_1))$

$\nu \times \sigma$

$BO \times BO \longrightarrow BO$ \leftarrow $BO(\mathbb{R}^{n-\alpha(n)})$

These maps assemble into $X_n \xrightarrow{f} BO(\mathbb{R}^{n-\alpha(n)})$, and we have the diagram:

$X_n \xrightarrow{f} BO(\mathbb{R}^{n-\alpha(n)})$

$\downarrow \quad \downarrow$

$BO/\mathbb{R}^n_0 \quad \longrightarrow \quad BO$
Apply Thom spectra:

\[ T \xrightarrow{\text{If}} M\Omega(n-\Delta(n)) \]

\[ Tg \downarrow \text{If} \circ \Gamma \]

\[ M\Omega/\text{In} \xrightarrow{} M\Omega \]

Remark: We have no preferred lift, and the ambiguity is the same as before, namely the choices for \( B_k(C) \).

Problem: Construct preferred classifying maps:

\[ B_k(C) \rightarrow BO(k-\Delta(k)) \]

Problem: Construct \( BO/\text{In} \) in terms of \( B_k(C) \).
Using this tactic, we can almost prove the conjecture for \( n - \alpha(n) \) odd (Lannes).

In this case, we may consider the partial relative Postnikov tower

\[
\begin{align*}
K(\mathbb{Z}/\mathbb{Z}, v_j) \\
\downarrow \\
B_0(n - \alpha(n)) \to B_r \to \cdots \to B_j \to B_{j-1} \to \cdots \to B_0 = B_0
\end{align*}
\]
and attempt to lift \( BO/I_n \rightarrow BO \) inductively:

\[
\begin{array}{c}
\text{BO}(n-a(n)) \rightarrow \cdots \rightarrow B_j \rightarrow B_{j-1} \rightarrow \cdots \rightarrow BO \\
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{ 
X_n \ar[r]^{f} \ar[dr]_{f_j} & B_j \ar[r]^{g} & B_{j-1} \ar[r] & \cdots \ar[u]_{h_{j-1}} & BO/I_n \\
& & & & \}
\end{array}
\]

The obstruction to the existence of the filler is a class in \( H^*(BO/I_n, X_n) \) pulled back from \( (B_{j-1}, B_j) \), so it suffices that \( (MO/I_n, TX_n) \rightarrow (TB_{j-1}, TB_j) \) is zero in mod 2 cohomology.
This is guaranteed if the diagram

\[ T X_n \xrightarrow{Tf} MO(n - \alpha(n)) \]

\[ \sigma \xrightarrow{\uparrow} \xrightarrow{\uparrow} Tf \]

\[ MO/In \xleftarrow{Tg} TX_n \]

commutes, since then there is the map \( Tf \circ \sigma \) fits into the following commuting diagram, which factors the map of pairs in question through a pair with trivial cohomology.
Bad news: No component of $(\star)$ is canonical, so it has no reason to commute.

Good news: Failure is controlled by deformation obstruction theory, and we can change our choices as we go to make obstructions vanish.

Corollary of good news: ~100 pages of induction.
Extra flexibility in making and modifying all these choices will be offloaded (apparently) by working over the homotopy pullback

\[
P \longrightarrow \text{BD}(n-d(n))
\]

\[
\downarrow \quad \downarrow \quad \downarrow
\]

\[
\text{BD}/\Sigma_n \quad \quad \text{BD}
\]

What Cohen proves is the following:
Lemma B. There is a space $X_n \xrightarrow{h} P$ such that

1. $T \left( X_n \xrightarrow{g} B_0/I_n \right)$ admits a section $p$.

2. The following diagram commutes:

\[
\begin{array}{ccc}
TX_n & \xrightarrow{h} & P \\
\uparrow & & \uparrow h \\
B_0/I_n & \xleftarrow{p} & TX_n
\end{array}
\]

(\star \star \star)
Constructing $X_n$ amounts to choosing manifolds $M_w$, which is done inductively on $n$.

Case 1 $n = 2^i - 1$, so that every element of $\Pi_n$ is decomposable. In this case, we can define the new manifolds to be products of the old manifolds. To get the maps, we need the following:
Lemma 2.10 The maps $\mathbb{B}^2_0/I_2 \rightarrow \mathbb{B}^2_0/I_1$. may be chosen so that the following diagram commutes:

$$
\begin{array}{cccc}
X^2 & \rightarrow & X_0 \\
\downarrow g^2 & & \downarrow g_0 & \\
\mathbb{B}^2_0/I_2 & \rightarrow & \mathbb{B}^2_0/I_1 & \\
\end{array}
$$

This is the kind of thing that "obviously must be true," but nothing in the diagram
is canonical except the top map, so the possibility of making bad choices necessitates another induction argument.

Problem: Endow $B \mathcal{D} S I$ with structure making (\(*\)\(*\)\(*\)\(*\)\) obviously commute.
Case 2 \( n \neq 2^j - 1 \), so that \( \Pi_n \mathbb{M} \mathbb{O} \) has one new generator. In this case, we know how to build \( X_n \) apart from a single manifold.

Cohen's approach is to show that the splitting \( \mathbb{M} \mathbb{O}/I_n = \bigvee_{W \neq (n)} S^{l_w} \wedge T_{\sigma_{n-1,w}} \wedge S^n \) comes from a decomposition.
\[ \tilde{B}_0/\mathbb{I}_n = \tilde{B}_0/\mathbb{I}_n \cup D^n \]

to repeat the entire argument for \( \tilde{\mathbb{I}} \) using \( X_n \) constructed as in the case \( n = 2^n - 1 \), then to use obstruction theory to produce the lift

\[
\begin{array}{ccc}
\sim \\
\tilde{B}_0/\mathbb{I}_n & \longrightarrow & \tilde{B}_0(n - \xi(n)) \\
\downarrow & & \downarrow \\
\tilde{B}_0/\mathbb{I}_n & \longrightarrow & \tilde{B}_0(n - \xi(n)) \\
& & \\
\tilde{B}_0/\mathbb{I}_n & \longrightarrow & \tilde{B}_0 \\
\end{array}
\]
Only then does he double back to show that Lemma B holds with $M_{cn}$ any generator.

There is one seemingly important element in the proof that Lemma B implies the inversion conjecture that was not apparent in our heuristic argument, namely the role of the Snaith splitting from Sanath's talk.
I quote: "the central idea is to play off the Thom spectrum level information given in the hypotheses of Lemma B against the Snaith splitting." Specifically, it implies:

Thm 1.6/Lem. 1.7 There is an equivalence

\[ \mathbb{Q}BO \simeq \mathbb{Q}BO(m) \times BO/BO(m) \]

through dimension 2m, and \( BO/BO(m) \) is a product of mod 2 Eilenberg–Mac Lane spaces through dimension 2m+1.
This allows Cohen to work over $\text{QBO}(n-\alpha(n))$:

**Thm A** There is a homotopy commutative diagram

$$
\begin{array}{ccc}
\text{BO}(n-\alpha(n)) & \to & \text{QBO}(n-\alpha(n)) \\
\downarrow & \searrow & \uparrow \\
\text{BO/In} & \to & \text{BO} \\
\end{array}
$$

with an extra condition on Thom spectra.
Aside from this subtlety, our sketch is a reasonable proxy for the argument. Most of the many pages consist of (1) endless replacements of maps and spaces up to homotopy (2) showing maps of pairs are zero after Thomification (3) obstruction theory (based on Q3O rather than B0).

Problem Formulate the argument in a context where replacement is unnecessary.