Toward the immersion conjecture

Thm (Cohen) M^n immerses in $\mathbb{R}^{2n-\alpha(n)}$.

By Hirsch-Smale, equivalent to the lifting problem

$$\begin{tikzcd}
BO/I_n \arrow[r, dashed, "?"] & BO(n-\alpha(n)) \arrow[d] \\
M \arrow[u, dashed] \arrow[r, "\nu"] & BO
\end{tikzcd}
\qquad
\begin{tikzcd}
MO/I_n \arrow[r] & MO(n-\alpha(n)) \arrow[d] \\
T\nu \arrow[u, dashed] \arrow[r] & MO
\end{tikzcd}$$

Reminders on BO/I_n

(0) There is no BO/I_n (very non-canonical)

(1) BO/I_n is a cell complex of dim. n

(2) $T(BO/I_n \to BO) = MO/I_n = \bigvee_{|w| \le n} S^{|w|} \wedge J_{\mathcal{J}_{n-|w|}}$,

$$\mathcal{J}_k = B(B_k \to \Sigma_k \to O(k)).$$

(3) BO/I_n is constructed by step by step de-Thomification of a specific Adams resolution

$$MO/I_n \to \cdots \to J_{j+1} \to J_j \to \cdots \to J_0 = MO,$$

and taking the n-skeleton of the inverse limit. This tower is a wedge of towers for Brown-Gitler spectra.

(4) BO/I_n has the following lifting property. A stable vector bundle over a CW complex of dimension n lifts to BO/I_n provided its Thom spectrum admits a cohomology injection from a spectrum E satisfying the following duality condition (Lannes):

(D_n) The n-dual of E is (2-locally) a finite CW complex Y of dimension n, i.e.,

$$\exists\, S^n \longrightarrow E \wedge Y$$

inducing a perfect pairing on mod 2 H^*.

<u>Consequences of (4)</u>

(A) v_M lifts to BO/I_n

(B) ∂_k lifts to BO/I_k

(C) $BO/I_k \times BO/I_{n-k} \to BO$ lifts to BO/I_n

Much of the difficulty of Cohen's proof is in controlling ambiguity arising from BO/I_n and the use of (B) and (C).

<u>Problem</u> Construct a specific instance of BO/I_n that is conceptual, explicit, and/or structured. OR,

<u>Problem</u> Package the ambiguity and carry it through the argument.

<u>Problem</u> Understand (B) in terms of braids.

<u>Goal</u> De-Thomify $MO/I_n \to MO(n-\alpha(n))$

<u>Idea</u> MO/I_n is a <u>summand</u> of a spectrum TX_n such that $TX_n \twoheadrightarrow MO/I_n \to MO(n-\alpha(n))$ <u>obviously</u> de-Thomifies.

Writing $MO/I_n = \bigvee_w S^{|w|} \wedge T\sigma_{n-w}$, we choose a representative M_w for the wth generator of MO_* such that the immersion conjecture holds for M_w (this is possible by R.L. Brown's theorem as explained by Jim Hou).

We then have the commuting diagram

$$\begin{tikzcd}
M_w \times B_{n-|w|}(\mathbb{C}) \arrow[r, dashed, "① \times ②"] \arrow[d, "v_w \times \gamma"'] & BO/I_{|w|} \times BO/I_{n-|w|} \arrow[dl] \arrow[d, dashed, "③"] \\
BO \times BO \arrow[r] & BO & BO/I_n \arrow[l]
\end{tikzcd}$$

The dashed maps exist by the versorl property of $BO/I_\bullet$. Beware non-canonicity!

Thus, setting $X_n = \coprod_w M_w \times B_{n-|w|}(\mathbb{C})$, we have the maps $X_n \xrightarrow{g} BO/I_n$ and

$$\sigma = \bigvee_w \left(S^{|w|} \xrightarrow{PT} T\nu_w \right) \wedge T\gamma_{n-w}.$$

<u>Claim</u> $Tg \circ \sigma = id_{MO/I_n}$

Now, since the immersion conjecture holds for M_w, and using the same fact about the braid bundles from before, we have

$$\begin{tikzcd}
M_W \times B_{n-|W|}(\mathbb{C}) \arrow[r, dashed] \arrow[d, "\nu_W \times \gamma"'] & BO(|W|-\alpha(|W|)) \times BO(n-|W|-\alpha(n-|W|)) \arrow[d] \arrow[dl] \\
 & BO(n-\alpha(|W|)-\alpha(n-|W|)) \arrow[d] \\
BO \times BO \arrow[r] & BO & BO(n-\alpha(n)) \arrow[l]
\end{tikzcd}$$

These maps assemble into $X_n \xrightarrow{f} BO(n-\alpha(n))$, and we have the diagram

$$\begin{tikzcd}
X_n \arrow[r, "f"] \arrow[d, "g"'] & BO(n-\alpha(n)) \arrow[d] \\
BO/J_n \arrow[r] & BO
\end{tikzcd}$$

Apply Thom spectra:

$$\begin{tikzcd}
TX_n \arrow[r, "Tf"] \arrow[d, "Tg"'] & MO(n-\alpha(n)) \arrow[d] \\
MO/I_n \arrow[r] \arrow[ur, dashed, "Tf\circ\sigma"] & MO
\end{tikzcd}$$

<u>Remark</u> We have no preferred lift, and the ambiguity is the same as before, namely the choices for $B_k(\mathbb{C})$.

<u>Problem</u> Construct preferred classifying maps

$$B_k(\mathbb{C}) \longrightarrow BO(k-\alpha(k)).$$

<u>Problem</u> Construct BO/I_n in terms of $B_k(\mathbb{C})$.

Using this tactic, we can almost prove the conjecture for $n-\alpha(n)$ odd (Lannes).

In this case, we may consider the partial relative Postnikov tower

$$\begin{tikzcd}
 & & K(\mathbb{Z}/2\mathbb{Z}, q_j) \arrow[d] & & \\
BO(n-\alpha(n)) \arrow[r] & B_r \arrow[r] & \cdots \arrow[r] & B_j \arrow[r] \arrow[d] & B_{j-1} \arrow[r] & \cdots \arrow[r] & B_0 = BO \\
 & & & K(\mathbb{Z}/2\mathbb{Z}, q_j+1) & & &
\end{tikzcd}$$

$2(n-\alpha(n))+1$-connected

and attempt to lift $BO/I_n \to BO$ inductively:

$$\begin{tikzcd}
BO(n-\alpha(n)) \arrow[r] & \cdots \arrow[r] & B_j \arrow[r] & B_{j-1} \arrow[r] & \cdots \arrow[r] & BO \\
X_n \arrow[u, "f"] \arrow[urr, "f_j"] \arrow[rrr, "g"] \arrow[rrrr, bend right] & & & & BO/I_n \arrow[ull, dashed, "\exists ?\, h_j"'] \arrow[ul, "\exists\, h_{j-1}"'] \arrow[u, "h_0"'] &
\end{tikzcd}$$

The obstruction to the existence of the filler is a class in $H^*(BO/I_n, X_n)$ pulled back from (B_{j-1}, B_j), so it suffices that $(MO/I_n, TX_n) \longrightarrow (TB_{j-1}, TB_j)$ is zero in mod 2 cohomology.

This is guaranteed if the diagram

$$\begin{tikzcd}
TX_n \arrow[r, "Tf"] & MO(n-\alpha(n)) \\
MO/_{I_n} \arrow[u, "\sigma"] & TX_n \arrow[l, "Tg"'] \arrow[u, "Tf"']
\end{tikzcd} \qquad (\star)$$

commutes, since then there is the map $Tf_j \circ \sigma$ fits into the following commuting diagram, which factors the map of pairs in question through a pair with trivial cohomology.

$$
\begin{tikzcd}
 & TX_n \arrow[r, "Tg"] \arrow[dl, "Tf"'] \arrow[dd, "Tf_j"] & MO/I_n \arrow[dr, "Tf_j \circ \sigma"] \arrow[ddl, bend left=60, "Th_{j-1} \circ Tg \circ \sigma"] & \\
MO(n-\alpha(n)) \arrow[dr] & (\star) & & MO(n-\alpha(n)) \arrow[dl] \\
 & TB_j \arrow[r, equal] \arrow[d, equal] & TB_j \arrow[d] & \\
 & TB_j \arrow[r] & TB_{j-1} &
\end{tikzcd}
$$

$$Th_{j-1} \circ Tg \circ \sigma = Th_{j-1}$$

<u>Bad news</u> No component of (☆) is canonical, so it has no reason to commute.

<u>Good news</u> Failure is controlled by de-Thomification obstruction theory, and we can change our choices as we go to make obstructions vanish.

<u>Corollary of good news</u> ~100 pages of induction.

Extra flexibility in making and modifying all these choices will be afforded (apparently) by working over the homotopy pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\, h"] & BO(n-\alpha(n)) \arrow[d] \\ BO/I_n \arrow[r] & BO \end{tikzcd}$$

What Cohen proves is the following:

<u>Lemma B</u> There is a space $X_n \xrightarrow{h} P$ such that

(1) $T(X_n \to P \to BO/I_n)$ admits a section σ

(2) the following diagram commutes:

$$\begin{tikzcd}
TX_n \arrow[r, "h"] & P \\
MO/I_n \arrow[u, "\sigma"] & TX_n \arrow[l, "g"'] \arrow[u, "h"']
\end{tikzcd} \qquad (\star\star)$$

Constructing X_n amounts to choosing manifolds M_w, which is done inductively on n.

<u>Case 1</u> $n = 2^i - 1$, so that every element of $\pi_n MO$ is decomposable. In this case, we can define the new manifolds to be products of the old manifolds. To get the maps, we need the following:

<u>Lemma 2.10</u> The maps $BO/I_\bullet^2 \longrightarrow BO/I_\bullet$ may be chosen so that the following diagram commutes:

$$\begin{tikzcd} X_\bullet^2 \arrow[r] \arrow[d, "g_\bullet^2"] & X_\bullet \arrow[d, "g_\bullet"] \\ BO/I_\bullet^2 \arrow[r] & BO/I_\bullet \end{tikzcd} \qquad (\star\star\star)$$

This is the kind of thing that "obviously must be true," but nothing in the diagram

is canonical except the top map, so the possibility of making bad choices necessitates another induction argument.

<u>Problem</u> Endow $BO/I_\bullet$ with structure making (☆☆☆) obviously commute.

<u>Case 2</u> $n \neq 2^j - 1$, so that $\pi_n MO$ has one new generator. In this case, we know how to build X_n apart from a single manifold.

Cohen's approach is to show that the splitting $MO/I_n = \bigvee_{w \neq (n)} S^{|w|} \wedge J\mathcal{T}_{n-|w|} \vee S^n$ comes from a decomposition

$$BO/I_n = \widetilde{BO/I_n} \cup_{S^{n-1}} D^n,$$

to repeat the <u>entire</u> argument for $\widetilde{BO/I_n}$ using $\tilde{X}_n$ constructed as in the case $n = 2^i - 1$, then to use obstruction theory to produce the lift

$$\begin{tikzcd}
\widetilde{BO/I_n} \arrow[r] \arrow[d] & BO(n-\alpha(n)) \arrow[d] \\
BO/I_n \arrow[r] \arrow[ur, dashed] & BO .
\end{tikzcd}$$

Only then does he double back to show that Lemma B holds with $M_{(n)}$ <u>any</u> generator.

There is one seemingly important element in the proof that Lemma B implies the immersion conjecture that was not apparent in our heuristic argument, namely the role of the Snaith splitting from Sanath's talk.

I quote: "the central idea is to play off the Thom spectrum level information given in the hypotheses of Lemma B against the Snaith splitting." Specifically, it implies:

<u>Thm 1.6 / Lem. 1.7</u> There is an equivalence

$$QBO \simeq QBO(m) \times BO/BO(m)$$

through dimension $2m$, and $BO/BO(m)$ is a product of mod 2 Eilenberg-MacLane spaces through dimension $2m+1$.

This allows Cohen to work over $QBO(n-\alpha(n))$:

<u>Thm A''</u> There is a homotopy commutative diagram

$$\begin{tikzcd}
 & BO(n-\alpha(n)) \arrow[r] \arrow[dr, phantom, "\Downarrow"] & QBO(n-\alpha(n)) \\
BO/I_n \arrow[ur] \arrow[r] & BO \arrow[r] & QBO \arrow[u, "s"']
\end{tikzcd}$$

with an extra condition on Thom spectra.

Aside from this subtlety, our sketch is a reasonable proxy for the argument. Most of the many pages consist of

(1) endless replacements of maps and spaces up to homotopy

(2) showing maps of pairs are zero after Thomification

(3) obstruction theory (based on QBO rather than BO).

<u>Problem</u> Formulate the argument in a context where replacement is unnecessary.