Static and Dynamic Consistency of Preferences in Optimal Stopping Problems*

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Abstract

Suppose an agent with a static risk preference faces prize processes given by regular diffusions and decides when to stop the process and consume the prize. We show that the agent satisfies dynamic consistency of preferences if and only if she is an Expected Utility agent. This extends a classical result from Hammond (1988) and Gul and Lantto (1990) to a continuous-time setup in which the classical proof techniques do not apply, because the ‘decision trees’ the agent faces are not the usual discrete-time, finite-horizon lottery trees.

1 Introduction

Many dynamic choice situations in economics and finance are modeled as optimal stopping problems of a continuous-time (prize) process. Canonical examples of such dynamic choice situations include the decision by a trader when to sell a stock or exercise an option or the decision of a gambler when to leave a casino with the accumulated earnings or losses. A popular technical assumption is that the prize uncertainty that the agent faces derives from a Brownian motion.

The standard model of risk preferences is that of Expected Utility (EU).\(^1\) Besides the theoretical justification for EU contained in the axiomatization from Von Neumann and Morgenstern (see Proposition 6.B.3 in Mas-Collel et al. (1995) and their discussion of the Independence Axiom), the classical model also enjoys a dynamic foundation, first pointed out in Hammond (1988) and Hammond (1989).\(^2\) Suppose the agent faces finite-horizon, discrete-time ‘decision trees’ in which the risk about a final outcome is resolved over time. Suppose she can rank these decision trees according to a weak order, based on the risky lottery they ultimately induce on final outcomes. Suppose that her risk preference does not change with time as she ‘moves along a decision tree’ and in particular, it is not influenced by past experiences of risk or by events which ‘could have occurred but didn’t’. Then her risk preference conforms with EU, only if her observed choices are dynamically consistent. Roughly speaking, this means that an outside party cannot manipulate the agent into accepting ex-post choices she considered ex-ante suboptimal.\(^3\) This gives a

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\(^2\)See chapter 6 in Mas-Collel et al. (1995) for a detailed exposition.

\(^3\)See Green (1987) for a possible formalization of this argument.
dynamic foundation for EU, because dynamic consistency ties choices of the same agent from different moments in time. The classical formal arguments of this result rely on the particular technical set up chosen: the agent is in discrete-time, faces a finite-horizon and the objects of choice are finite decision trees.

A large theoretical literature motivated by compelling experimental evidence against EU has sought to replace the standard model with models of risk behavior which better explain observed behavior in static choices. A recent literature in economics has considered implications of several of these ‘behavioral’ preferences to stopping problems; see e.g. Barberis (2012), Xu and Zhou (2013), Ebert and Strack (2015), Ebert and Strack (2016), Henderson et al. (2017) or He et al. (2016) who focus on various specifications of Cumulative Prospect Theory (Tversky and Kahneman (1992)). The assumption in all of these papers is that the agent’s risk preference remains stable over time and is not influenced by the decision problem or past experiences.

In the settings of continuous-time processes driven by Brownian motion noise, e.g. Xu and Zhou (2013), Ebert and Strack (2015), Ebert and Strack (2016) and Duraj (2019), choices of the agent across time are tied through assumptions like naive or sophisticated behavior. The leading case of optimal stopping with non-Expected Utility preferences considered in this literature is again that of agents with probability weighting; see Huang et al. (2017) and Huang and Nguyen-Huu (2018) for recent studies on both naïve and sophisticated agents. The latter works contain a formal discussion of what it means for an agent to be time consistent. Time Consistency says that an agent’s optimal contingent plan in a stopping problem is such that the agent does not prefer to switch to another plan as time progresses. EU agents are Time Consistent in every decision problem while non-EU agents typically are not. The question remains: what is a condition on preferences over stopping times, that implies that the agent in continuous-time stopping problems of diffusions is EU? I.e. what is the pendant of the results in Hammond (1988) and Hammond (1989) for the continuous time setting of diffusions?

In this note we define for a setting of prize processes driven by Brownian motion what it means precisely for preferences to be dynamically consistent and verify formally that this property is equivalent to the assumption that the static risk preference of the agent conforms with EU. Thus, we establish a continuous-time foundation for EU, a pendant to the results of Hammond (1988) and Hammond (1989).

On a technical level the proof of the dynamic foundations of EU is distinct from the discrete-time, finite-horizon version, because the decision trees are now hard to visualize and have to be identified with scaling functions, i.e. mappings from the space of prizes to real numbers. Nevertheless, a functional separation argument allows to construct ‘continuous-time decision trees’ tailored to the ones needed for the verification of the Independence Axiom in the setting of uncertainty driven by Brownian noise.

The fact that dynamic consistency implies again a EU preference for risk in the continuous-time setup with Brownian noise has the same implications as the classical result in discrete-time. It follows that to be able to pin down uniquely the dynamic

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4Duraj (2019) studies optimal stopping behavior without making functional form assumptions on the risk preference of the agent, besides continuity and monotonicity with respect to first-order-stochastic-dominance.

5See e.g. Definition 2.1 in resp. Huang et al. (2017) and Huang and Nguyen-Huu (2018).

6See section 6 in Hammond (1989) or the argument in pp. 166-167 of Gul and Lantto (1990). Here, the separation argument for diffusions is contained in Claim 1. of the proof of Lemma 1.
behavior of the agent, when the agent is not an Expected Utility-maximizer, additional assumptions on dynamic behavior are needed. The additional rules of naiveté and sophistication considered in the works above correspond to natural classical relaxations of dynamic consistency in discrete-time dynamic choice problems.\footnote{These rules were (to the best of knowledge) first introduced in Strotz (1956).}

**Organization of the rest of the paper.** The next subsection contains the model primitives, whereas section 2 defines in terms of preferences over stopping times what it means to be dynamically consistent in the setup of this paper and states the result of this note. The last section contains the formal proof of the result.

### 1.1 Model

**Preferences.** The agent possesses a weak order $\succeq$ (i.e. complete and transitive order) over the space of Borel probability distributions over a prize space $[w, b]$, which in turn is denoted by $\Delta([w, b])$. This is the agent’s risk preference.\footnote{In the following the notation $\Delta(A)$ where $A$ is a metric space is to be understood as the space of Borel probability distributions over the metric space $A$.} We denote by $\succ$ the irreflexive part of $\succeq$. In text we use interchangeably the name *lottery* for probability distributions over prizes and denote by $\delta_x$ for $x \in [w, b]$ the degenerate probability distribution giving $x$ with probability one. Throughout we assume the following for $\succeq$.

**Assumption 1 - Continuity:** For all $G \in \Delta([w, b])$ the sets $\{ F \in \Delta([w, b]) : F \succeq G \}$ and $\{ F \in \Delta([w, b]) : F \preceq G \}$ are closed in the topology of convergence in distribution.

**Assumption 2 - FOSD-monotonicity** If $F$ strictly FOSD-dominates $G$, then $F \succ G$.

Here $F$ *FOSD-dominates* $G$ if for all $x \in [w, b]$ we have $F(x) \leq G(x)$ and ‘strictly FOSD-dominates’ is the irreflexive part of the ‘FOSD-dominates’ relation.

It is well-known, that under Assumption 1 the agent has a utility function $V : \Delta([w, b]) \to \mathbb{R}$ which is continuous in the topology of convergence in distribution (see e.g. Debreu and Hildenbrand (1983)). Under Assumption 2 this utility function is increasing with respect to the FOSD-order.

**Technology of the prize process.** We assume the agent faces a sequence of lotteries in continuous time $t \in \mathbb{R}_+$, which are generated by a diffusion

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = y_0.$$ \hspace{1cm} (1)

We call a pair $(X, y_0)$ consisting of a diffusion process and a starting point $y_0 \in [w, b]$ a stopping problem.

Here $(W_t)_{t \in \mathbb{R}_+}$ is a Brownian motion and the drift $\mu : [w, b] \to \mathbb{R}$ together with the volatility $\sigma : [w, b] \to (0, +\infty)$ are assumed Lipschitz continuous. The process $X_t, t \geq 0$ lives in the Wiener space $C([0, \infty), [w, b])$ of continuous functions with image in $[w, b]$ and adapted to the (completed) filtration of the Brownian motion, which we denote by $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$. Lipschitz continuity can be relaxed without changing the results of
We also assume that the diffusion is stopped, once it leaves \([w, b]\): the lower bound is a limited liability constraint of the agent, while the upper bound excludes gambles with arbitrarily large prizes. The assumption that the variance coefficient \(\sigma\) is bounded away from 0 means that the uncertainty the agent faces at each moment in time is non-negligible, independently of the current state of the diffusion. We call diffusions satisfying these properties regular diffusions. One can think of them as modeling the value of an asset or the wealth process of a gambler in a casino.

Fix a diffusion \(X\) as in (1) and a starting point \(y_0 \in (w, b)\). Denote the elements of the filtration generated by the diffusion \(X\) started at \(y_0\) by \(F_T^X(y_0), T \in [0, \infty)\). \(F_T^X(y_0)\) encodes the prize uncertainty resolved till time \(T\) for the diffusion \(X\) started at time zero at the prize level \(y_0\).

Denote by \(F_X(y_0)\) the union of \(F_T^X(y_0)\) as \(T\) ranges across all positive time periods and by \(F(y_0)\) the union of all \(F_X(y_0)\) as \(X\) ranges across all regular diffusions. The former encodes all possible histories of prize path realizations when fixing a particular regular diffusion and the latter when considering all regular diffusions started at \(y_0\). \(F_X(y_0)\) is a well-defined object because all diffusions \(X\) are adapted to the filtration of the underlying Brownian motion \(W\) of the diffusion.

The diffusion model in (1) can accommodate costs of continuation or costs arising from impatience into the drift of the diffusion as long as they are time independent: for a given cost parameter \(c\), the net drift including costs of continuation/impatience would be modified to \((\mu(X_t) - c)dt\). There does not seem to exist a general theory of impatience for an arbitrary static risk preference as discussed in this paper so that any way of modeling impatience in our model is bound to have an ad-hoc flavor. If the model is interpreted as that of a trader or gambler active only within a period, like a day or week, the horizon can be interpreted as too short for discounting to be an important economic force.

We finish the statement of the model assumptions with a comment on the requirement of time-homogeneity of coefficients in (1). Suppose instead that one allows for prize processes which correspond to time-inhomogeneous diffusions. Then the coefficients of the prize process depend on time explicitly: \(\mu(t, X_t), \sigma(t, X_t)\). As the proof of Theorem 1 below shows, to characterize when the static risk preference of the agent is Expected Utility, it is sufficient to focus on stopping time choices in the case of homogeneous diffusions as in (1). Therefore, the assumption of time-homogeneity of the prize processes is without loss of generality in this work.

## 2 A behavioral model of optimal stopping.

In the following we explain the behavioral foundations of the optimal stopping model stated in the previous section. First, we introduce some helpful notation.

Unless otherwise stated, in the following the index \(X\) runs over regular diffusions as defined in (1). Fix such a diffusion \(X\) and a starting point \(y_0 \in (w, b)\). Recall, that \(F_T^X(y_0)\) encodes the prize uncertainty resolved till time \(T\) if \(X\) is started at \(y_0\). An arbitrary element \(n\) from \(F_T^X(y_0)\) can be interpreted as a ‘node’ of depth \(T\) of the ‘uncertainty tree’ defined by the diffusion \(X\) started at \(y_0\). We identify the singleton set \(F_0^X(y_0)\) with the trivial sigma-algebra generated by \(\{y_0\}\). We say that \(n' \in F_T^X(y_0)\) is a continuation.
of \( n \in \mathcal{F}_X^T(y_0) \) if \( T' > T \) and the occurrence of \( n' \) implies that of \( n \). Formally, the latter requirement means \( n' \subseteq n \).

The choice objects of the agent at each moment in time are given by the set \( \mathcal{S} \) of (uniformly integrable) stopping times. All stopping times are assumed to be adapted to the filtration of the underlying Brownian motion which drives the diffusions in (1). For each \( n \in \mathcal{F}_X(y_0) \) we assume that the agent has a complete and transitive preference relation \( \succeq_n \) over \( \mathcal{S} \). An agent can thus be identified with the collection of preference relations \( \mathcal{A} = \left( (\succeq_n)_{n \in \mathcal{F}_X(y_0)} \right)_{X,y_0 \in \{w,b\}} \) : for each diffusion \( X \) and starting point \( y_0 \) a collection of preference relations for each node that can be reached by the diffusion.

To simplify notation in the following we denote for a real-valued random variable \( Y \) which is a measurable function of the path of \( W_t \) and \( n \) an event from some sub-sigma-algebra \( \sigma \) of the filtration of the Brownian motion \( W \), by \( F_{Y|n} \) the distribution of the random variable conditional on the event \( n \) occurring.

In our model, the preference relations \( \succeq_n \) are related to each other by the existence of a fixed, static risk preference, which evaluates prize lotteries in a history-independent way. The following definition is an adaptation of the similar Definition of Consistency in Gul and Lantto (1990), who consider choice in discrete time problems modeled as finite lottery trees.

**Definition 1.** The agent exhibits Static Consistency of Preferences (SCP) if there exists a risk preference functional \( V : \Delta([w,b]) \to \mathbb{R} \) such that for all stopping problems \((X, y_0)\) and all events \( n \in \mathcal{F}_X(y_0)\), it holds
\[
\succeq_n \text{ is represented by } V(F_{X|n}).
\]

Thus, the value of stopping time \( \tau \) if event \( n \) has occurred is given by the static utility of the distribution of \( X_\tau \), evaluated conditional on \( n \) having occurred. We maintain SCP in the following.

The next definition gives a formal statement of the well-known Dynamic Consistency axiom in our setting.

**Definition 2.** An agent satisfies Dynamic Consistency of Preferences (DCP) if for all stopping problems \((X, y_0)\) and three different \( n, n_1, n_2 \in \mathcal{F}_X(y_0) \) such that

\( i \) \( n_1, n_2 \) are continuations of \( n \) and they are disjoint events,

\( ii \) the probability that either \( n_1 \) or \( n_2 \) happens conditional on \( n \) happening, is one,

the following implication is true for any four stopping times \( \tau_1, \tau'_1, \tau_2, \tau'_2 \) from \( \mathcal{S} \):

\[
\text{if } \tau_1 \succeq_n \tau'_1 \text{ and } \tau_2 \succeq_n \tau'_2 \text{ then } \tau = \begin{cases} 
\tau_1, & \text{if } n_1 \\
\tau_2, & \text{if } n_2
\end{cases} \quad \tau' = \begin{cases} 
\tau'_1, & \text{if } n_1 \\
\tau'_2, & \text{if } n_2
\end{cases}.
\]

Dynamic Consistency of Preferences says, that if for two mutually exclusive and exhaustive continuation events \( n_1, n_2 \) of \( n \), stopping time \( \tau_1 \) is preferred to \( \tau'_1 \) at event \( n_1 \) and \( \tau_2 \) is preferred to \( \tau'_2 \) at event \( n_2 \), it should also hold that the combined stopping time \( \tau \) is preferred to the stopping time \( \tau' \) after event \( n \). DCP is violated when there is an event \( n \) and two stopping times \( \tau, \tau' \) from \( \mathcal{S} \) such that at \( n \) it holds \( \tau' \succ_n \tau \) even though
the stopping time \( \tau \) leads to (weakly) better prospects than \( \tau' \) in all future continuations of event \( n \).

DCP puts very strong restrictions on the preference \( \succeq \) which the agent uses to evaluate lotteries from \( \Delta([w,b]) \). The following Theorem is the result of this paper.

**Theorem 1.** Suppose an agent satisfies SCP with static risk preference \( V \). The static risk preference functional \( V \) of an agent has an Expected Utility representation with a strictly increasing, bounded and continuous Bernoulli utility function \( u : [w,b] \to \mathbb{R} \) if and only if the following requirements are met:

A. \( V \) satisfies FOSD-monotonicity and is continuous in the topology of convergence in distribution.

B. The agent satisfies Dynamic Consistency of Preferences.

There are two leading cases of failure of Dynamic Consistency of Preferences considered in the literature. In the case of a naive agent at an event \( n \), she projects the ‘current’ preference \( \succeq_n \) into all preferences of continuation events: she decides on the stopping time at event \( n \) by assuming that \( \succeq_{n'} = \succeq_n \) for all continuation events \( n' \) of \( n \). In the case of the sophisticated agent on the other hand, the agent restricts her choice set of stopping times \( S \) at event \( n \) to the set \( S(n) \) of stopping times she knows will not lead to preference reversals, no matter the continuation of the process.

### 3 Proof of Theorem 1

Before giving the full proof of Theorem 1, which is technically involved, we sketch the main idea behind the hardest part of the proof: sufficiency of DCP for the risk preference to have an Expected Utility representation, i.e. to satisfy Independence. Suppose the agent prefers distribution \( F \) to \( G \) and assume that she faces the choice between the mixed lotteries \( \lambda F + (1 - \lambda)H \) and \( \lambda G + (1 - \lambda)H \) for some other arbitrary lottery \( H \) and \( \lambda \in (0, 1) \). Suppose agent satisfies DCP and there is a stopping problem where the agent has two stopping strategies \( \tau_{F,H} \) and \( \tau_{G,H} \) which, if implemented, lead to a history \( h_1 \) where \( H \) is realized with probability \( 1 - \lambda \) and otherwise to a history \( h_2 \) where respectively \( F \) or \( G \) is realized with probability \( \lambda \). Under this situation DCP will imply that the agent prefers \( \tau_{F,H} \) to \( \tau_{G,H} \) at the current moment of time, because conditional on either history \( h_1 \) or \( h_2 \) the agent prefers \( \tau_{F,H} \) to \( \tau_{G,H} \). But the distribution induced by, respectively \( \tau_{F,H} \) or \( \tau_{G,H} \), is \( \lambda F + (1 - \lambda)H \) or \( \lambda G + (1 - \lambda)H \)! Now SCP will imply that the agent prefers \( \lambda F + (1 - \lambda)H \) to \( \lambda G + (1 - \lambda)H \) in a static problem as well. This implies that the risk preference of the agent satisfies Independence. The latter fact and the other assumed technical conditions imply due to classical results, that the risk preference has an Expected Utility representation.\(^{10}\)

**Proof of Theorem 1. Necessity:** Let \( V \) be given by \( V(F) = \mathbb{E}_F[u] \) for some \( u : [w,b] \to \mathbb{R} \) strictly increasing, bounded and continuous. Checking requirement A. is then standard.

\(^{10}\)See Theorem 3 in Grandmont (1972).
Regarding Requirement B: take \( \tau_1, \tau_1', \tau_2, \tau_2', n, n_1, n_2 \) as in Definition 2. It follows for \( \tau \) either \( \tau \) or \( \tau' \) that
\[
\mathbb{E}[u(X_\tau)|n] = \mathbb{E}[u(X_\tau), n_1|n] + \mathbb{E}[u(X_\tau), n_2|n] \\
= \mathbb{E}[\mathbb{E}[u(X_\tau)|n_1]|n] + \mathbb{E}[\mathbb{E}[u(X_\tau)|n_2]|n],
\]
where the last equality follows from the Markov property of diffusion processes. Both summands at the end of the calculation above are weakly higher for \( \bar{\tau} = \tau \) than \( \bar{\tau} = \tau' \) by hypothesis. This shows necessity of the requirements.

**Sufficiency:** The proof consists of two steps. First, we establish the following important Lemma which states that under the three Requirements SCP, A and B, \( V \) satisfies the Independence axiom of Expected Utility.

**Lemma 1.** Under the Requirements SCP, A and B, \( V \) satisfies Independence, i.e. for all \( H, G_1, G_2 \in \Delta([w, b]) \) and \( \alpha \in [0, 1] \) we have
\[
V(G_1) \geq V(G_2) \text{ implies } V(\alpha H + (1 - \alpha)G_1) \geq V(\alpha H + (1 - \alpha)G_2).
\]

**Proof.** Take \( H \) first as in the statement and assume throughout the proof w.l.o.g. that \( \alpha \in (0, 1) \). We assume for now additionally that \( G_i, i = 1, 2 \) are step functions, i.e. correspond to lotteries of finite support. We relax this assumption at the very end of the proof of Lemma 1. We divide the proof for this case in several steps.

**Step 1.** Assume first, that \( \{G_1, G_2\} \) is ordered by FOSD-monotonicity. In particular, it holds \( G_1 >_{FOSD} G_2 \), the other case being excluded by \( V(G_1) \geq V(G_2) \). It then follows that
\[
\alpha H + (1 - \alpha)G_1 >_{FOSD} \alpha H + (1 - \alpha)G_2,
\]
and by Requirement A that \( V(\alpha H + (1 - \alpha)G_1) > V(\alpha H + (1 - \alpha)G_2) \).

**Step 2.** Assume now that \( \{G_1, G_2\} \) is not ordered by FOSD-monotonicity and that \( H \) has support contained in \( (w, b) \).

Recall from the statement and proof of Lemma 2 in Duraj (2019) the set \( C^{2,L}_{inc} \). Recall, that these correspond to scaling functions for regular diffusions started at a point \( y_0 \in [w, b] \).

In more detail, for every \( S \in C^{2,L}_{inc} \) there is a \( y_0 \in [w, b] \) such that \( S(y_0) = 0 \) and
\[
S(x) = \int_{y_0}^{x} \exp \left( -2 \int_{y_0}^{z} \frac{\mu(t)}{\sigma^2(t)} dt \right) dz, \quad x \in [w, b], \tag{2}
\]
with \( \mu : [w, b] \to \mathbb{R} \) and \( \sigma : [w, b] \to (0, +\infty) \) Lipschitz continuous functions.

Equip it with the metric given by the maximum norm. Consider the map \( \psi : C^{2,L}_{inc} \to \mathbb{R} \) given by
\[
\psi(S) = \mathbb{E}_{G_1}[S] - \mathbb{E}_{G_2}[S].
\]
This map is continuous and since \( C^{2,L}_{inc}([w, b]) \) is a convex metric space (in particular it is connected), it follows that the image of \( \psi \) in \( \mathbb{R} \) is connected. In particular, it is an interval. We need the following auxiliary Claim.

**Claim 1:** There exists a \( S \in C^{2,L}_{inc}([w, b]) \) such that \( \psi(S) = 0 \).

**Proof of Claim 1.** Assume this is not the case. It then follows that the whole image of \( \psi \), it being connected, consists of either only negative or only positive numbers.
It then follows that either

Case 1: \( \mathbb{E}_{G_1}[S] > \mathbb{E}_{G_2}[S] \) for all \( S \in C_{1c}^{2,L}([w, b]) \)

or

Case 2: \( \mathbb{E}_{G_1}[S] > \mathbb{E}_{G_2}[S] \) for all \( S \in C_{1c}^{2,L}([w, b]) \)

We close the proof of the Claim by showing that this implies that \( \{G_1, G_2\} \) is ordered by FOSD. Focus on Case 1, the other one being analogous. Pick a finite sequence \( x_1 < x_2 < \cdots < x_n \) in \([w, b]\) which consists of the union of the support of the lotteries corresponding to \( G_1 \) and \( G_2 \). For a \( k \in \{2, \ldots, n\} \) pick \( S_k \in C_{1c}^{2,L}([w, b]) \) with \( S_k(x_j) = 1 - \epsilon(n - j) \) for \( k \leq j \leq n \) and \( S_k(x_j) = \epsilon j \) for all \( 1 \leq j < k \). This is possible for \( \epsilon > 0 \) small enough by combining linear interpolations and mollification arguments.\(^{11}\) It follows from Case 1 that

\[
\sum_{j=k}^{n} (1 - \epsilon(n - j))(G_1(x_j) - G_1(x_{j-1})) + \epsilon \sum_{j=1}^{k-1} G_1(x_j) - G_1(x_{j-1})
\]

\[
> \sum_{j=k}^{n} (1 - \epsilon(n - j))(G_2(x_j) - G_2(x_{j-1})) + \epsilon \sum_{j=1}^{k-1} G_2(x_j) - G_2(x_{j-1})
\]

for all \( \epsilon > 0 \) small enough. Letting now \( \epsilon \) go to zero we recover for all \( k \in \{2, \ldots, n\} \) that

\( G_1(x_j) \leq G_2(x_j), j = 1, \ldots, n. \)

But this implies that \( G_1 \) FOSD-dominates \( G_2.\(^{12}\)

End of Proof of Claim 1.

It follows, that there exists some \( S \in C_{1c}^{2,L}([w, b]) \) with

\[ \mathbb{E}_{G_1}[S] = \mathbb{E}_{G_2}[S]. \]

As in the proof of Lemma 2 of the Appendix of Duraj (2019), it follows that there exists some \( y_2 \) and some diffusion \( X \), started at \( y_2 \in [w, b] \) with scaling function \( S : [w, b] \times [w, b] \to \mathbb{R} \) of the form as in (2), so that \( S(y_2, y_2) = 0 \) and

\[ \mathbb{E}_{x \sim G_1}[S(x, y_2)] = \mathbb{E}_{x \sim G_2}[S(x, y_2)] = 0. \]

Consider now the function \( \rho : [w, b] \to \mathbb{R} \) given by

\[ \rho(z) = \mathbb{E}_{x \sim H}[S(x, z)]. \]

This is a continuous function with \( \rho(b) < 0 \) and \( \rho(w) > 0 \). In particular, it follows, there exists \( y_1 \in (w, b) \) with \( \rho(y_2) = 0 \) (that \( y_1 \) can be chosen different from \( w, b \) follows from the intermediate assumption made in Step 2 above, that \( \text{supp}(H) \subset (w, b) \)). Due to Proposition 1 in Duraj (2019) there exists \( \tau_H, \tau_{G_1} \) and \( \tau_{G_2} \) such that if the diffusion \( X \) is started at \( y_1 \) then \( F_{X_{\tau_H}} = H \) and if \( X \) is started at \( y_2 \) then \( F_{X_{\tau_{G_i}}} = G_i, i = 1, 2. \)

\(^{11}\)Mollification arguments allow to approximate arbitrarily well locally, and uniformly, a function which is continuous and piecewise differentiable by functions which are infinitely differentiable. See chapter 5 and in particular section 5.3 of Evans (2010) for details on mollification arguments.

Step 2a. Assume first, that $y_1 > y_2$ and consider the stopping time $\tau_{y_2, y_1}$. We show that Independence holds for $H$ as assumed in Step 2. The case of $y_2 > y_1$ is similar.

We know from the proof of Theorem 2 in Duraj (2019) and the discussion preceding it, that for $y \in (y_1, y_2)$ the probability $p(y)$ that $X$ started at $y$ reaches $y_1$ before it reaches $y_2$ is a strictly increasing continuous function $p : [y_1, y_2]$ with $p(y_1) = 1, p(y_2) = 0$. There exists thus a $y_0 \in (y_1, y_2)$ with $p(y_0) = \alpha$. Consider now the stopping times $\tau_i$ for $i = 1, 2$ given by\textsuperscript{13}

$$\tau_i = \begin{cases} 
\tau_H, & \text{if } X_{\tau_{y_1, y_2}} = y_1 \\
\tau_{G_i}, & \text{if } X_{\tau_{y_1, y_2}} = y_2. 
\end{cases}$$

This is again a uniformly integrable stopping time, i.e. an element of $S$.\textsuperscript{14} The Markov property of $X$ shows that $X_{\tau}$ has the distribution $\alpha H + (1 - \alpha) G_i$.

Take the ‘root’ event $\{y_0\} \in \mathcal{F}_X(y_0)$. It follows that on the event $n = \{X_{\tau_{y_1, y_2}} = y_1\}$

$$V(F_{X_{\tau_{y_1, y_2}}}n) = V(F_{X_{\tau_{y_1, y_2}}}n) = V(H),$$

while on the event $m = \{X_{\tau_{y_1, y_2}} = y_2\}$

$$V(F_{X_{\tau_{y_1, y_2}}}m) = V(F_{X_{\tau_{y_1, y_2}}}m) = V(G_i).$$

Here we have used the fact that diffusions satisfy the strong Markov property which implies that the distribution of $X_{\tau}$ conditional on an event $A$ in the sigma-algebra of another stopping time $\hat{\tau}$, which is finite with probability one, depends on its sigma-algebra $\sigma(\hat{\tau})$ only through $X_{\tau}$.

Note that the union of $m$ and $n$ is the whole sample space and thus has probability of occurring equal to one. This uses continuity of the diffusion process $X$.

Because $V(G_1) \geq V(G_2)$, DCP yields that $\tau_1 \preceq y_0 \tau_2$. In all, due to the Definition 1 (SCP) and DCP, it follows $V(\alpha H + (1 - \alpha) G_1) \geq V(\alpha H + (1 - \alpha) G_2)$.

Step 2b. We now assume that $H$ with $\text{supp}(H) \subset (w, b)$ as above delivers $y_1 = y_2$ instead. In this case, one can find a sequence of distributions $H_n, n \in \mathbb{N}$ with

1. $\text{supp}(H_n) \subset (w, b)$,
2. $H_n \Rightarrow H, n \to \infty$ in distribution and
3. so that the $y_{1n}$ in $(w, b)$ defined by

$$E_{x \sim H_n}[S(x, y_{1n})] = 0$$

have $y_{1n} > y_2$ (since $S(\cdot, \cdot)$ is increasing in the first argument and decreasing in the second, it suffices to perturb $H$ into $H_n$ by shifting some probability on higher values within the support of $H$). The case above of distinct $y_1, y_2$ gives

$$V(\alpha H_n + (1 - \alpha) G_1) \geq V(\alpha H_n + (1 - \alpha) G_2)$$

and Continuity of $V$ establishes that $V(\alpha H + (1 - \alpha) G_1) \geq V(\alpha H + (1 - \alpha) G_2)$.

Step 3. Finally, it remains to consider the case of general $H \in \Delta([w, b])$ with support

\textsuperscript{13}Intuitively, we paste together the two distributions $G_i$ and $H$ in such a way so that from the perspective of the agent at time 0, when diffusion is started at $y_0$ the probability that $\tau_H$ is realized is precisely $\alpha$.

\textsuperscript{14}One checks easily that the events $\{\tau_i > t\}$ for $t \geq 0$ depend only on the evolution of the process till time $t$. Uniform integrability follows from the fact that the diffusions ‘live’ in $[w, b]$. 

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possibly including $w$ or $b$. In this case, we can again find a sequence $H_n, n \in \mathbb{N}$ with
(1) $\text{supp}(H_n) \subset (w, b)$,
(2) $H_n \rightarrow H, n \rightarrow \infty$ in distribution.
The arguments above then give
\[
V(\alpha H_n + (1 - \alpha)G_1) \geq V(\alpha H_n + (1 - \alpha)G_2)
\]
and Continuity of $V$ establishes again that $V(\alpha H + (1 - \alpha)G_1) \geq V(\alpha H + (1 - \alpha)G_2)$. This establishes the proof for the case of step distributions $G_1, G_2$. We now use the following Claim and continuity to close the proof for the case when $G_1, G_2$ are not necessarily step distributions (i.e. their respective lotteries don’t have finite support).

Claim 2 For $G$ a cdf, there exists sequences of cdf-s $G_{i,n}, n \in \mathbb{N}, i = 1, 2$ which are step functions, so that $G_{i,n} \rightarrow G_i$ in distribution with $G_{i,n} <_{\text{FOSD}} G <_{\text{FOSD}} G_{2,n}$. 

Proof of Claim 2. $G$ is increasing, right-continuous with at most countably many discontinuities. Moreover, each open subinterval of $[w, b]$ contains a continuity point of $G$. One can use this easily to construct the needed sequences.

End of Proof of Claim 2.

Given $V(G_1) \geq V(G_2)$ and $H \in \Delta([w, b])$ arbitrary, pick sequences $G_{1,n}, G_{2,n}, n \in \mathbb{N}$ so that $G_{1,n} >_{\text{FOSD}} G_1, G_{2,n} <_{\text{FOSD}} G_2$ (using Claim 2), so that
\[
V(G_{1,n}) > V(G_1) \geq V(G_2) < V(G_{2,n}).
\]
Moreover note that $G_{i,n} \rightarrow G_i$, weakly for $n \rightarrow \infty$.

Now note that it follows $\alpha H + (1 - \alpha)G_{1,n} >_{\text{FOSD}} \alpha H + (1 - \alpha)G_1, \alpha H + (1 - \alpha)G_{2,n} <_{\text{FOSD}} \alpha H + (1 - \alpha)G_2$. This, Continuity of $V$ and the fact that Independence holds for the triplets $G_{1,n}, G_{2,n}, H$ finishes the proof of the Lemma.

Given the result of Lemma 1 and the Continuity assumption on $\succeq$, Theorem 3 of Grandmont (1972) yields a representation $V(F) = \mathbb{E}_F[u]$ with a bounded and continuous $u : [w, b] \rightarrow \mathbb{R}$. Finally, requirement A establishes that $u$ is also strictly increasing. This finishes the proof of sufficiency and therefore also of Theorem 1.

References


