Today

1. Welcome

2. Normal Form Games

3. Extensive Form Games

4. Strategies in Extensive Form Games
The second half of Economics 2010a is organized around several types of games of interest in economics, paying particular attention to

(i) relevant solution concepts in the different settings we will consider, and

(ii) some key economic applications belonging to these settings.

The section slides are only for guidance and will be complemented with work on the blackboard.

You are strongly encouraged to work with the section notes for your preparation.
Course Outline through Game Types

GAME TYPE 1: simultaneous move games with complete information
1. theory: Nash equilibrium and its extensions, rationalizability
2. application: Nash implementation

GAME TYPE 2: simultaneous move games with incomplete information
1. theory: Bayesian Nash equilibrium
2. application: Global Games, Auctions
Course Outline through Game Types

GAME TYPE 3: **sequential move** games with **complete information**

1. theory: Subgame perfect equilibrium
2. application: Bargaining games, repeated games

GAME TYPE 4: **sequential move** games with **incomplete information**

1. theory: Perfect Bayesian equilibrium, sequential equilibrium, strategically stable equilibrium, etc
2. application: Signaling games
A two-player Normal Form game

A typical two-player normal form game:

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<tbody>
<tr>
<td>$T$</td>
<td>1,1</td>
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<tr>
<td>$B$</td>
<td>0,0</td>
<td>2,2</td>
</tr>
</tbody>
</table>

Player 1 ($P_1$) chooses a row ($\text{Top or Bottom}$); player 2 ($P_2$) chooses a column ($\text{Left or Right}$).

Two important things to keep in mind:

1. players choose their strategies $\textit{simultaneously}$.
2. The numbers that appear in a payoff matrix are actually Bernoulli $\textit{utilities}$, not monetary payoffs (recall MWG part of the course).

What about other, more complicated games?
General Definition of a Normal Form Game

Definition

A normal form game \( \mathcal{G} = \langle \mathcal{N}, (S_k)_{k \in \mathcal{N}}, (u_k)_{k \in \mathcal{N}} \rangle \) consists of:

- A finite collection of players \( \mathcal{N} = \{1, 2, \ldots, N\} \)
- A set of pure strategies \( S_j \) for each \( j \in \mathcal{N} \)
- A (Bernoulli) utility function \( u_j : \times_{k=1}^{N} S_k \to \mathbb{R} \) for each \( j \in \mathcal{N} \)

**Cournot Game:** \( S_1 = S_2 = [0, \infty) \),

\[
\begin{align*}
u_1(s_1, s_2) &= p(s_1 + s_2) \cdot s_1 - C(s_1) \\
u_2(s_1, s_2) &= p(s_1 + s_2) \cdot s_2 - C(s_2)
\end{align*}
\]

\( p(\cdot) \) inverse demand function and \( C(\cdot) \) cost function.
Extensive form games

Figure 1: Game of assurance in extensive form.
An extensive form game with imperfect information and chance moves

Figure 2. Chance move distributions: $f(a|(l)) = f(b|(l)) = 0.5$. Information partitions: $\mathcal{I}_1 = \{\emptyset, (l, a)\}$, $\mathcal{I}_2 = \{(r), \{(l, b), (m)\}\}$. 
General Definition of an Extensive Form Game

Definition

A finite-horizon extensive form game $\Gamma$ has the following components:

1. A finite-depth tree with vertices $V$ and terminal vertices $Z \subseteq V$.
2. A set of players $\mathcal{N} = \{1, 2, ..., N\}$.
3. A player function $J : V \setminus Z \rightarrow \mathcal{N} \cup \{c\}$.
4. A set of available moves $M_{j,v}$ for each $v \in J^{-1}(j)$, $j \in \mathcal{N}$. Each move in $M_{j,v}$ associated with a unique child of $v$ in the tree.
5. A probability distribution $f(\cdot \mid v)$ over $v$’s children for each $v \in J^{-1}(c)$.
6. A (Bernoulli) utility function $u_j : Z \rightarrow \mathbb{R}$ for each $j \in \mathcal{N}$.
7. An information partition $\mathcal{I}_j$ of $J^{-1}(j)$ for each $j \in \mathcal{N}$, whose elements are information sets $I_j \in \mathcal{I}_j$. Required: that $v, v' \in I_j \Rightarrow M_{j,v} = M_{j,v'}$. 
Properties

Every finite normal form game $G = \left< \mathcal{N}, (S_j)_{j=1}^N, (u_j)_{j=1}^N \right>$ may be converted into an extensive form game of imperfect information with $1 + \sum_{L=1}^{N} \prod_{j=1}^{L} |S_j|$ vertices.

How?
Further Properties

General Definition allows characterizing games of perfect information as a special case.

Definition
An extensive form game is called a **game of perfect information** if all the information sets of all players contain one node.

An example?
Strategies in Extensive Form Games: pure strategies

Think of pure strategy as a program. If j’s turn to play, program input: information set and program output: a feasible move. Program encodes a complete contingency plan for playing the game: returns a legal move at every vertex of the game tree where j might play.

Definition

In an extensive form game, a pure strategy for player j is a function \( s_j : I_j \rightarrow \bigcup_{v \in J^{-1}(j)} M_{j,v} \), so that \( s_j(I_j) \subseteq M_{j,I_j} \) for each \( I_j \in I_j \).

Write \( S_j \) for the set of all pure strategies of player j.
Example: pure strategies in Figure 2?
Let for a set $A$ be $\Delta(A)$ the set of probability distributions over $A$. Two possible ways to randomize between pure strategies in extensive form games:

**Definition**

A **mixed strategy** for player $j$ is an element $\sigma_j \in \Delta(S_j)$.

**Definition**

A **behavioral strategy** for player $j$ is a collection of distributions $\{b_{I_j}\}_{I_j \in \mathcal{I}_j}$, where $b_{I_j} \in \Delta(M_{I_j}, I_j)$.

These are different objects!
Randomization in Extensive Form Games

They can be equivalent for all purposes of analysis:

Definition

A mixed strategy $\sigma_j$ and a behavioral strategy $\{b_{ij}\}$ are equivalent if they generate the same distribution over terminal vertices regardless of the strategies used by opponents, which may be mixed or behavioral.
Randomization in Extensive Form Games

Example in Figure 2.

A behavioral strategy for P1: $b^*_0(l) = 0.5, b^*_0(m) = 0, b^*_0(r) = 0.5, b^*_0(l,a)(t) = 0.7, b^*_0(l,a)(d) = 0.3$.

Consider 4 pure strategies: $s_1^{(1)}(\emptyset) = l, s_1^{(1)}(l,a) = t; s_1^{(2)}(\emptyset) = l, s_1^{(2)}(l,a) = d; s_1^{(3)}(\emptyset) = r, s_1^{(3)}(l,a) = t; s_1^{(4)}(\emptyset) = r, s_1^{(4)}(l,a) = d$ and define mixed strategy $\sigma_j^*$ so that $\sigma_j^*(s_1^{(1)}) = 0.35, \sigma_j^*(s_1^{(2)}) = 0.15, \sigma_j^*(s_1^{(3)}) = 0.35, \sigma_j^*(s_1^{(4)}) = 0.15$.

Then $b^*$ is equivalent to $\sigma_j^*$.

There are games with mixed strategies without an equivalent behavioral strategy and vice versa.
**Randomization in Extensive Form Games: Games of Perfect Recall**

**Definition**

An extensive form game has **perfect recall** if whenever \( v, v' \in I_j \), the two paths leading from the root to \( v \) and \( v' \) pass through the same sequence of information sets and take the same actions at these information sets.

Almost all games in this course will be with perfect recall.
Randomization in Extensive Form Games: Kuhn’s Theorem

Theorem (Kuhn 1957)

In a finite extensive game with perfect recall, (i) every mixed strategy has an equivalent behavioral strategy, and (ii) every behavioral strategy has an equivalent mixed strategy.
Ec 2010a - Game Theory Section 2

Jetlir Duraj

Harvard University
Today

1. Strategies in Normal Form Games

2. Nash Equilibrium and some Properties

3. Solving for Nash Equilibria
Notation

If \( X_1, X_2, \ldots, X_N \) sequence of sets with typical elements
\( x_1 \in X_1, x_2 \in X_2, \ldots: \)

\( X_{-i} \) means \( \times_{1 \leq k \leq N, k \neq i} X_k \)

\( X \) sometimes means \( \times_{k=1}^{N} X_k \).

\((x_i)\) refers to a vector/profile. \((x_1, x_2, \ldots, x_N)\).
So \((x_i)\) is an element in \( \times_{k=1}^{N} X_k \). Parentheses used to distinguish it from \( x_i \), which is an element of \( X_i \).

\( x_{-i} \) is an element in \( X_{-i} \), i.e. \( \times_{1 \leq k \leq N, k \neq i} X_k \).
Mixed Strategies

Definition

Suppose $G = \langle \mathcal{N}, (S_k)_{k \in \mathcal{N}}, (u_k)_{k \in \mathcal{N}} \rangle$ is a normal form game where each $S_k$ is finite. Then a mixed strategy for player $i$ $\sigma_i$ is a probability distribution over $S_i$. We denote the collection of all of these probability distributions by $\Delta(S_i)$.

Sometimes the mixed strategy putting probability $p_1$ on action $s_1^{(1)}$ and probability $1 - p_1$ on action $s_1^{(2)}$ — written as $p_1s_1^{(1)} \oplus (1 - p_1)s_1^{(2)}$. 
Remarks

1) When two or more players play mixed strategies, their randomizations are assumed to be independent.

2) Technically, pure strategies also count as mixed strategies. The term “strictly mixed” is usually used for a mixed strategy that puts strictly positive probability on every pure strategy.
Utility from Play

When \((\sigma_k)_{k=1}^{N}\) is played, the assumption on independent mixing, together with previous week’s discussion on payoff matrix entries as Bernoulli utilities in a vNM representation, imply that player \(i\) gets utility:

\[
\sum_{(s_k') \in \times_k S_k} u_i(s'_1, ..., s'_N) \cdot \sigma_1(s'_1) \cdot ... \cdot \sigma_N(s'_N).
\]

Fact

For any fixed \(\sigma_{-i}\), the map \(\sigma_i \mapsto u_i(\sigma_i, \sigma_{-i})\) is affine, in the sense that

\[
u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i) \cdot u_i(s_i, \sigma_{-i}).\]
Nash Equilibrium - Definition

Definition
In a normal form game $\mathcal{G} = \langle \mathcal{N}, (S_k)_{k \in \mathcal{N}}, (u_k)_{k \in \mathcal{N}} \rangle$, a **Nash equilibrium in pure strategies** is a pure strategy profile $(s_k^*)_{k \in \mathcal{N}}$ such that for every player $i$, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*)$ for all $s_i' \in S_i$.

Definition
In a normal form game $\mathcal{G} = \langle \mathcal{N}, (S_k)_{k \in \mathcal{N}}, (u_k)_{k \in \mathcal{N}} \rangle$, a **Nash equilibrium in mixed strategies** is a mixed strategy profile $(\sigma_k^*)_{k \in \mathcal{N}}$ such that for every player $i$, $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*)$ for all $s_i' \in S_i$.$^a$

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$^a$Why is it enough to look at $s_i' \in S_i$ instead of $\sigma_i' \in \Delta(S_i)$?

In words: no player can improve upon her own payoff through a **unilateral** deviation, taking as given the actions of others.
Example: Game of Assurance

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<tr>
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<th>L</th>
<th>R</th>
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<tbody>
<tr>
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<td>0,0</td>
</tr>
<tr>
<td>$B$</td>
<td>0,0</td>
<td>2,2</td>
</tr>
</tbody>
</table>

$(T, L)$ and $(B, R)$ are pure strategy Nash equilibria.

P1 plays $\frac{2}{3}T \oplus \frac{1}{3}B$ and P2 plays $\frac{2}{3}L \oplus \frac{1}{3}R$ is a mixed NE.
Best Response Correspondences

Definition

The **individual pure best response correspondence** for player $i$ is $BR_i : S_{-i} \Rightarrow S_i$ where

$$BR_i(s_{-i}) := \arg \max_{\hat{s}_i \in S_i} u_i(\hat{s}_i, s_{-i})$$

**Pure best response correspondence** is a vector of correspondences: $BR : S \Rightarrow S$ where $BR(s) := (BR_1(s_{-1}) \ldots BR_N(s_{-N}))$.

The **individual mixed best response correspondence** for player $i$ is $\overline{BR}_i : \prod_{k \neq i} \Delta(S_k) \Rightarrow \Delta(S_i)$ where

$$\overline{BR}_i(\sigma_{-i}) := \arg \max_{\hat{\sigma}_i \in \Delta(S_i)} u_i(\hat{\sigma}_i, \sigma_{-i})$$

The **mixed best response correspondence** is a vector of correspondences: $\overline{BR} : \prod_i \Delta(S_i) \Rightarrow \prod_i \Delta(S_i)$ where

$\overline{BR}(\sigma) := (BR_1(\sigma_{-1}) \ldots BR_N(\sigma_{-N}))$. 
Nash equilibrium as a fixed point

Proposition

A pure strategy profile is a pure NE iff it is a fixed point of $BR$. A mixed strategy profile is a mixed NE iff it is a fixed point of $BR$.

Interpretation: Fixed points of the best response correspondences reflect stability of NE strategy profiles.
Some Properties of Nash Equilibrium

Property 1: the indifference condition in mixed NEs.

Proposition

Suppose \((\sigma^*_i)\) is a mixed Nash equilibrium. Then for any \(s_i \in S_i\) such that \(\sigma^*_i(s_i) > 0\), we have \(u_i(s_i, \sigma^*_{-i}) = u_i(\sigma^*_i, \sigma^*_{-i})\).
Some Properties of Nash Equilibrium

Property 2: Nash payoffs are greater or equal than the maxmin values of each player.

Proposition

In each normal form game \( G = (\mathcal{N}, (S_i)_{i \in \mathcal{N}}, (u_i)_{i \in \mathcal{N}}) \) with a finite set of players, for every Nash equilibrium \( \sigma^* = (\sigma_1^*, \ldots, \sigma_n^*) \) and for every player \( i \) it holds

\[
    u_i(\sigma^*) \geq \max_{\sigma_i} \min_{\sigma_{-i}} u_i(\sigma_i, \sigma_{-i}), \quad u_i(\sigma^*) \geq \min_{\sigma_{-i}} \max_{\sigma_i} u_i(\sigma_i, \sigma_{-i}).
\]
Some Properties of Nash Equilibrium

Definition
Call a pure strategy $s_i \in S_i$ of player $i$ strictly dominated if there exists a mixed strategy $\sigma_i \in \Delta(S_i)$ so that for every $\sigma_{-i}$ we have

$$u_i(\sigma_i, \sigma_{-i}) > u_i(s_i, \sigma_{-i}).$$

Property 3:

Proposition
If $\sigma^*$ is a Nash equilibrium of the normal form game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, then no player puts positive probability on a strictly dominated strategy. In other words:

$$s_i \text{ strictly dominated for } i \text{ and } \sigma^* \text{ is Nash} \implies \sigma^*_i(s_i) = 0.$$
Mathematical Properties of Set of NEs

For a finite Game the set of its Nash equilibria is **closed** and **bounded**.

The set \( NE \) of Nash equilibria is in general **not convex**. A counterexample: the Battle of the Sexes game

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<th>Opera</th>
<th>Football</th>
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<tbody>
<tr>
<td><strong>Opera</strong></td>
<td>3,2</td>
<td>0,0</td>
</tr>
<tr>
<td><strong>Football</strong></td>
<td>0,0</td>
<td>2,3</td>
</tr>
</tbody>
</table>
Last Property for today

**Symmetric Games.** A game in normal form \( G = (\mathcal{N}, (S_i)_{i \in \mathcal{N}}, (u_i)_{i \in \mathcal{N}}) \) is called a symmetric game if

1) Each player has the same set of strategies: \( S_i = S \) for all \( i \in \mathcal{N} \).

2) The payoff functions satisfy for each pair of players \( i, j \) with \( i < j \)

\[
u_i(s_1, s_2, \ldots, s_n) = u_j(s_1, \ldots, s_{i-1}, s_j, s_{i+1}, \ldots, s_{j-1}, s_i, s_{j+1}, \ldots, s_n).
\]

Every finite symmetric game has a **symmetric Nash equilibrium in mixed strategies**: a Nash equilibrium \( \sigma = (\sigma_i)_{i \in \mathcal{N}} \) satisfying \( \sigma_i = \sigma_j \) for each \( i, j \in \mathcal{N} \).
‘Recipe’ for solving for NEs in finite 2-player games

1) Use **iterated elimination** of strictly dominated strategies to simplify the problem.

2) Find all the **pure strategy Nash equilibria** by considering all cells in the payoff matrix.

3) Look for a **mixed** Nash equilibrium where one player is playing a pure strategy while the other is strictly mixing.

4) Look for a **mixed** Nash equilibrium where **both** players are strictly mixing.
Example 1

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<tr>
<th></th>
<th>( L )</th>
<th>( R )</th>
<th>( Y )</th>
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<tbody>
<tr>
<td>( T )</td>
<td>2, 2</td>
<td>-1, 2</td>
<td>0, 0</td>
</tr>
<tr>
<td>( B )</td>
<td>-1, -1</td>
<td>0, 1</td>
<td>1, -2</td>
</tr>
<tr>
<td>( X )</td>
<td>0, 0</td>
<td>-2, 1</td>
<td>0, 2</td>
</tr>
</tbody>
</table>
Example 2

\[
\begin{array}{c|cc|}
\text{A} & \text{L} & \text{R} \\
\hline
\text{T} & 1,1,1 & 0,1,3 \\
\text{B} & 1,3,0 & 1,0,1 \\
\end{array}
\quad
\begin{array}{c|cc|}
\text{C} & \text{L} & \text{R} \\
\hline
\text{T} & 3,0,1 & 1,1,0 \\
\text{B} & 0,1,1 & 0,0,0 \\
\end{array}
\]

We claim, that there is only one mixed Nash equilibrium and it is in pure strategies: \((T, L, A)\).
Ec 2010a - Game Theory Section 3

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Today

1. More Actions can be worse

2. Correlated Equilibrium

3. Rationalizability
More actions can be worse

A single rational agent cannot be made worse off by enlarging the set of possible actions she possesses in a decision problem. BUT

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<tr>
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<td>3, 2</td>
<td>0, 0</td>
</tr>
<tr>
<td>Football</td>
<td>0, 0</td>
<td>2, 3</td>
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It has three Nash equilibria which have strictly positive payoffs for the agents. Consider now

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<th>Football</th>
<th>Stay Home</th>
</tr>
</thead>
<tbody>
<tr>
<td>Opera</td>
<td>3, 2</td>
<td>0, 0</td>
<td>−1, 4</td>
</tr>
<tr>
<td>Football</td>
<td>0, 0</td>
<td>2, 3</td>
<td>−1, 4</td>
</tr>
<tr>
<td>Stay Home</td>
<td>4, −1</td>
<td>4, −1</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

Only Nash is for both players to stay home and it has payoff zero for both players.
Definition of Correlated Equilibrium

Definition

In a normal form game \( G = \langle \mathcal{N}, (S_k)_{k \in \mathcal{N}}, (u_k)_{k \in \mathcal{N}} \rangle \), a **correlated equilibrium** (CE) consists of:

- A **finite set of signals** \( \Omega_i \) for each \( i \in \mathcal{N} \). Write \( \Omega := \times_{k \in \mathcal{N}} \Omega_k \).
- A **joint distribution** \( p \in \Delta(\Omega) \), so that the marginal distributions \( p_i(\omega_i) > 0 \) for each \( \omega_i \in \Omega_i \).
- A **signal-dependent strategy** \( s_i^* : \Omega_i \rightarrow S_i \) for each \( i \in \mathcal{N} \)

such that for every \( i \in \mathcal{N}, \omega_i \in \Omega_i, \hat{s}_i \in S_i \),

\[
\sum_{\omega_i} p(\omega_i | \omega_i) \cdot u_i(s_i^*(\omega_i), s_{-i}^*(\omega_{-i})) \geq \sum_{\omega_i} p(\omega_i | \omega_i) \cdot u_i(\hat{s}_i, s_{-i}^*(\omega_{-i}))
\]
Important Remarks

(1) The signal space and its associated joint distribution, \((\Omega, p)\), are not part of the game \(G\), but part of the equilibrium.

(2) There is no institution compelling player \(i\) to play the action \(s_i^*(\omega_i)\), but \(i\) finds it optimal to do so after seeing the signal \(\omega_i\). Traffic lights analogy.

(3) A Nash equilibrium is always a correlated equilibrium.

(4) The set of correlated equilibria of a finite normal form game is convex.
Example 1

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<tbody>
<tr>
<td>L</td>
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<td>0,0</td>
</tr>
<tr>
<td>R</td>
<td>0,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

The following is a correlated equilibrium: \( \Omega_1 = \Omega_2 = \{ l, r \} \), \( p(l, l) = 0.3 \), \( p(l, r) = 0.1 \), \( p(r, l) = 0.2 \), \( p(r, r) = 0.4 \), \( s_i^*(l) = L \) and \( s_i^*(r) = R \) for each \( i \in \{ 1, 2 \} \).

Here’s another correlated equilibrium: \( \Omega_1 = \Omega_2 = \{ l, r \} \), \( p(l, l) = 0.8 \), \( p(r, r) = 0.2 \), \( p(l, r) = 0 \), \( p(r, l) = 0 \), \( s_i^*(l) = L \) and \( s_i^*(r) = R \) for each \( i \in \{ 1, 2 \} \).
Example 2

**Public randomization device:** Fix any normal form game $G$ and fix $K$ of its pure Nash equilibria, $E_1, \ldots, E_K$, where each $E_k$ abbreviates some pure strategy profile $(\vec{s}_1^{(k)}, \ldots, \vec{s}_N^{(k)})$.

Then, for any $K$ probabilities $p_1, \ldots, p_K$ with $p_k > 0$, $\sum_{k=1}^{K} p_k = 1$, consider the signal structure with $\Omega_i = \{1, \ldots, K\}$, $p(k, \ldots, k) = p_k$ for each $1 \leq k \leq K$, and $p(\omega) = 0$ for any $\omega$ where not all $K$ dimensions match. Consider the strategies $s^*_i(k) = \vec{s}_i^{(k)}$ for each $i \in N$, $1 \leq k \leq K$.

Then $(\Omega, p, s^*)$ is a correlated equilibrium.
Algorithm 1

Iterated elimination of strictly dominated strategies, “IESDS”:

Put $\hat{S}_i^{(0)} := S_i$ for each $i$.

Then, having defined $\hat{S}_i^{(t)}$ for each $i$, define $\hat{S}_i^{(t+1)}$ in the following way:

\[
\hat{S}_i^{(t+1)} := \left\{ s_i \in \hat{S}_i^{(t)} : \forall \sigma_i \in \Delta \left( \hat{S}_i^{(t)} \right) \text{ s.t. } u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \ \forall s_{-i} \in \hat{S}_{-i}^{(t)} \right\} .
\]

Define $\hat{S}_i^\infty := \bigcap_{t \geq 0} \hat{S}_i^{(t)}$. 
Algorithm 2

Iterated elimination of never best responses, “IENBR”:
Put $\tilde{S}_i^{(0)} := S_i$ for each $i$.

Then, having defined $\tilde{S}_i^{(t)}$ for each $i$, we define $\tilde{S}_i^{(t+1)}$ in the following way:

$$\tilde{S}_i^{(t+1)} := \left\{ s_i \in \tilde{S}_i^{(t)} : \exists \sigma_{-i} \in \Delta \left( \tilde{S}_{-i}^{(t)} \right) \text{ s.t. } u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \ \forall s'_i \in \tilde{S}_i^{(t)} \right\}$$

Define $\tilde{S}_i^{\infty} := \bigcap_{t \geq 0} \tilde{S}_i^{(t)}$.

It is important to note that $\Delta \left( \tilde{S}_{-i}^{(t)} \right) \neq \times_{k \neq i} \Delta \left( \tilde{S}_k^{(t)} \right)$. 
Algorithms are equivalent

Proposition

$$\hat{S}_i^{(t)} = \tilde{S}_i^{(t)} \text{ for each } i \in \mathcal{N} \text{ and } t = 0, 1, 2, \ldots \text{ In particular, } \hat{S}_i^\infty = \tilde{S}_i^\infty.$$ 

We call $\tilde{S}_i^\infty$ the “(correlated) rationalizable strategies of player $i$”, but note that it can be computed through either IENBR or IESDS.
Proposition

If \((\Omega, p, s^*)\) is a correlated equilibrium, then \(s_i^*(\omega_i) \in \hat{S}_i^\infty\) for every \(i\), \(\omega_i \in \Omega_i\).

In words: only rationalizable pure strategies can come up in a correlated equilibrium with positive probability
\(\implies\) same is true for NE.

We have shown

\[
\text{Rat}(G) \supseteq \text{CE}(G) \supseteq \text{NE}(G).
\]
Today

1. Mechanism Design and Nash Implementation

2. Bayesian Games
Mechanism Design Problem

Definition

A mechanism design problem (MDP) consists of the following:

- A finite collection of players $\mathcal{N} = \{1, \ldots, N\}$
- A set of states of the world $\Theta$
- A set of outcomes $A$
- A state-dependent utility $u_i : A \times \Theta \rightarrow \mathbb{R}$ for each player $i \in \mathcal{N}$
- A social choice rule $f : \Theta \rightrightarrows A$ (this is a correspondence)
Mechanism

MDP presents an information problem. Designer can choose any outcome $x \in A$. Outcome it wants to pick depends on the state of the world, $f(\theta)$. Goal of Designer is to come up with an incentive scheme, called a mechanism, that induces self-interested players to choose one of the Central Authority’s favorite outcomes.

Definition

Given a MDP, a mechanism is a set of pure strategies $(S_i)_{i \in N}$ for each player and a map $g : S \to A$. 
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In state $\theta$, the mechanism $\langle (S_k)_{k \in \mathcal{N}}, g \rangle$ gives rise to a normal form game, $\mathcal{G}(\theta)$, with set of actions of player $i$ $S_i$ and the payoff of $i$ from strategy profile $(s_1, \ldots, s_N)$ is $u_i(g(s_1, \ldots, s_N), \theta)$.

Let $\text{NE}(\mathcal{G}(\theta))$ the set of Nash equilibria of $\mathcal{G}(\theta)$ in each state of the world.

**Definition**

The mechanism $\langle (S_k)_{k \in \mathcal{N}}, g \rangle$ Nash-implements social choice rule $f$ if $g(\text{NE}(\mathcal{G}(\theta))) = f(\theta)$ for every $\theta \in \Theta$.

Note: this is an equality of sets.
Maskin Monotonicity and No Veto Power

Definition

A social choice rule $f$ satisfies **Maskin monotonicity** (MM) if for all $\theta, \theta' \in \Theta$, whenever (1) $x \in f(\theta)$, and (2) 
\[
\{ y : u_i(y, \theta) \leq u_i(x, \theta) \} \subseteq \{ y : u_i(y, \theta') \leq u_i(x, \theta') \}
\] for every $i$, then $x \in f(\theta')$ too.

Definition

A social choice rule $f$ satisfies **no veto power** if for any $i \in \mathcal{N}$ and any $x \in A$, $u_j(x, \theta) \geq u_j(y, \theta)$ for all $j \neq i$ and all $y \in A$ implies $x \in f(\theta)$. 
Nash Implementability Theorem

Theorem

If \( f \) is Nash-implementable, then it satisfies MM. If \( N \geq 3 \) and \( f \) satisfies MM and no veto power, then \( f \) is Nash-implementable.

A full characterization for \( N = 2 \) agents much more complicated.
Example: social choice rule satisfying NVP but not MM

Suppose $N \geq 3$ and individuals have strict preferences over outcomes $A$ in any state of the world. Consider the social choice rule “top-ranked rule” $f^{\text{top}}$, where $x \in f^{\text{top}}(\theta)$ iff for all $z \in A$,

$$\#\{i : u_i(x, \theta) > u_i(y, \theta) \text{ for all } y \neq x\} \geq \#\{i : u_i(z, \theta) > u_i(y, \theta) \text{ for all } y \neq z\}.$$

That is, $f^{\text{top}}$ chooses the outcome(s) top-ranked by the largest number of individuals.
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Suppose individuals have strict preferences over outcomes \( A \) in any state of the world. Consider the social choice rule “dictator’s rule” \( f^D \), where \( f^D \) simply chooses the top-ranked outcome of player 1, a dictator.

\( f^D \) satisfies MM.

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(Consider \( A = \{x, y\} \), \( N = 3 \), and a state of the world with \( u_1(x, \theta) = 1, u_1(y, \theta) = 0, u_2(x, \theta) = 0, u_2(y, \theta) = 1, u_3(x, \theta) = 0, u_3(y, \theta) = 1 \))
Definition of a Bayesian Game

Definition

A Bayesian game with common prior assumption (CPA) is

\[
\mathcal{B} = \langle \mathcal{N}, (S_k)_{k \in \mathcal{N}}, (\Theta_k)_{k \in \mathcal{N}}, \mu, (u_k)_{k \in \mathcal{N}} \rangle
\]

consisting of:

A finite collection of **players** \( \mathcal{N} = \{1, 2, ..., N\} \)

A set of **actions** \( S_i \) for each \( i \in \mathcal{N} \)

A set of **states of the world** \( \Theta = \times_{k=1}^{N} \Theta_k \).

A **common prior** \( \mu \in \Delta(\Theta) \)

A **Bernoulli utility function** \( u_i : \times_{k=1}^{N} S_k \times \Theta \rightarrow \mathbb{R} \) for each \( i \in \mathcal{N} \)
Strategies in Bayesian Games

Definition

A pure strategy of player $i$ in a Bayesian game is a function $s_i : \Theta_i \rightarrow S_i$. A mixed strategy of player $i$ in a Bayesian game is a function $\sigma_i : \Theta_i \rightarrow \Delta(S_i)$.

When mixed strategies are used, we assume again that players, for each realization of the state of the world, randomize independently of each other.
In words...

A (CPA) Bayesian game proceeds as follows.

State of the world $\theta$ is drawn according to $\mu$

Player $i$ learns the $i$-th dimension, $\theta_i$

Player $i$ of type $\theta_i$ then takes a pure action from her strategy set $S_i$ or a mixed action from $\Delta(S_i)$

Outcome is realized.

The utility of player $i$ depends on the profile of actions as well as the state of the world $\theta$.

In particular it might depend on the dimensions of $\theta$ that $i$ does not observe.

The subset of Bayesian games where the Bernoulli utility $u_i$ does not depend on $\theta_{-i}$ are called private values games.
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Model this situation as a Bayesian game.
Bayesian Nash Equilibrium: Definition

Definition

A pure Bayesian Nash equilibrium (pBNE) is a strategy profile \((s^*_i)\) in a Bayesian game, such that for each player \(i \in \mathcal{N}\), each type \(\theta_i \in \Theta_i\),

\[
s^*_i(\theta_i) \in \arg\max_{\hat{s}_i \in S_i} \left\{ \sum_{\theta_{-i}} u_i(\hat{s}_i, s^*_{-i}(\theta_{-i}), (\theta_i, \theta_{-i})) \cdot \mu(\theta_{-i}|\theta_i) \right\}
\]

A mixed Bayesian Nash equilibrium (mBNE) is a mixed strategy profile \((\sigma^*_i)\) in a Bayesian game, such that for each player \(i \in \mathcal{N}\), each type \(\theta_i \in \Theta_i\),

\[
\sigma^*_i(\theta_i) \in \arg\max_{\hat{\sigma}_{-i} \in \Delta(S_{-i})} \left\{ \sum_{\theta_{-i}} \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}, (\theta_i, \theta_{-i})) \cdot \hat{\sigma}_{-i}(s_{-i}) \cdot \sigma^*_{-i}(\theta_{-i})(s_{-i}) \cdot \mu(\theta_{-i}|\theta_i) \right\}
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What are all the BNEs in Example 1?

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**Pure BNE:** only one $s_1^*(l) = M$, $s_1^*(r) = B$, $s_2^*(0) = R$.
No BNEs in **mixed** strategies.
Today

1. Mechanism Design and Nash Implementation

2. Bayesian Games
Mechanism Design Problem

Definition

A mechanism design problem (MDP) consists of the following:

- A finite collection of players $\mathcal{N} = \{1, \ldots, N\}$
- A set of states of the world $\Theta$
- A set of outcomes $\mathcal{A}$
- A state-dependent utility $u_i : A \times \Theta \rightarrow \mathbb{R}$ for each player $i \in \mathcal{N}$
- A social choice rule $f : \Theta \rightrightarrows \mathcal{A}$ (this is a correspondence)
Mechanism

MDP presents an information problem. Designer can choose any outcome \( x \in A \). Outcome it wants to pick depends on the state of the world, \( f(\theta) \).

Goal of Designer is to come up with an incentive scheme, called a mechanism, that induces self-interested players to choose one of the Central Authority’s favorite outcomes.

**Definition**

Given a MDP, a mechanism is a set of pure strategies \((S_i)_{i \in N}\) for each player and a map \( g : S \to A \).
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In state $\theta$, the mechanism $\langle (S_k)_{k \in \mathcal{N}}, g \rangle$ gives rise to a normal form game, $G(\theta)$, with set of actions of player $i S_i$ and the payoff of $i$ from strategy profile $(s_1, \ldots, s_N)$ is $u_i(g(s_1, \ldots, s_N), \theta)$.

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Model this situation as a Bayesian game.
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\]
Example 1 continued

What are all the BNEs in Example 1?

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**Pure BNE:** only one \( s^*_1(l) = M, s^*_1(r) = B, s^*_2(0) = R \).

No BNEs in **mixed** strategies.
Today: Auctions

1. Auction Model

2. Solving for BNEs in Auctions

3. Revenue Equivalence Theorem with Applications
In this course:
(1) auctions are private values;
(2) the type distribution is symmetric and independent across players;
(3) there is one seller who sells one indivisible item;
(4) players are risk neutral,

This implies:

getting the item with probability $H$ and having to pay $P$ in expectation gives a player $i$ with type $\theta_i$ a utility of

$$\theta_i \cdot H - P.$$
Auctions: Formal Definition

Definition

An auction $\mathcal{A} = \langle \mathcal{N}, F, [0, \bar{\theta}], (H_k)_{k \in \mathcal{N}}, (P_k)_{k \in \mathcal{N}} \rangle$ consists of:

- A finite set of bidders $\mathcal{N} = \{1, ..., N\}$
- A type distribution $F$ over $[0, \bar{\theta}] \subseteq \mathbb{R}$, which admits a continuous density $f$ with $f(\theta_i) > 0$ for all $\theta_i \in [0, \bar{\theta}]$.
- For each $i \in \mathcal{N}$, an allocation rule $H_i : \mathbb{R}_+^N \rightarrow [0, 1]$ that specifies the probability that player $i$ gets the item for every profile of $N$ bids.
- For each $i \in \mathcal{N}$, a payment rule $P_i : \mathbb{R}_+^N \rightarrow \mathbb{R}$ that specifies the expected payment of player $i$ for every profile of $N$ bids.
Auction timeline

At the start of the auction, each player $i$ learns her own valuation $\theta_i$. The valuations of different players are drawn i.i.d. from $F$, which is supported on the interval $[0, \bar{\theta}]$.

Each player simultaneously submits a nonnegative real number as her bid.

When the profile of bids $(s_1, ..., s_N)$ is submitted, player $i$ gets the item with probability

$$H_i(s_1, ..., s_N)$$

and pays

$$P_i(s_1, ..., s_N).$$
Auction as Bayesian Game

1) \( \mathcal{N} \) is the set of players.
2) Player \( i \)’s action set \( S_i = \mathbb{R}_+ \), interpreted as bids.
3) States of the world \( \Theta = [0, \bar{\theta}]^N \), where the \( i \)-th dimension is the valuation of player \( i \).
4) The common prior \( \mu \) on \( \Theta \) is the product distribution on \( [0, \bar{\theta}]^N \) derived from \( F \).
5) Utility function \( u_i \) given by

\[
    u_i(s_1, \ldots, s_N, \theta) = \theta_i \cdot H_i(s_1, \ldots, s_N) - P_i(s_1, \ldots, s_N)
\]

A pure BNE is a strategy profile \( (s^*_k)_{k \in \mathcal{N}} \) s.t. for each player \( i \) and valuation \( \theta_i \in \Theta_i \),

\[
s^*_i(\theta_i) \in \arg\max_{\hat{s}_i \in \mathbb{R}_+} \mathbb{E}_{\theta_{-i}} \left[ \theta_i \cdot H_i(\hat{s}_i, s^*_i(\theta_{-i})) - P_i(\hat{s}_i, s^*_i(\theta_{-i})) \right]
\]
Solving for Auction BNEs: the case of weakly dom. BNEs

**Definition**
In a private-value Bayesian game, a strategy $s_i : \Theta_i \rightarrow S_i$ is **weakly dominant** for $i$ if for all $s_{-i} \in S_{-i}$ and all $\theta_i \in \Theta_i$,

$$s_i(\theta_i) \in \arg \max_{\hat{s}_i \in S_i} \{ u_i(\hat{s}_i, s_{-i}, \theta_i) \}$$

**Proposition**

In a private-value Bayesian game, consider a strategy profile $(s_k^*)_{k \in \mathcal{N}}$ where for each $i \in \mathcal{N}$, $s_i^*$ is weakly dominant for $i$. Then $(s_k^*)_{k \in \mathcal{N}}$ is a BNE.

Application on blackboard: second-price auction **with a reserve price**.
Solving for Auction BNEs: the FOC approach

In the BNE of an auction, fixing $i$ and an interior valuation $\theta_i \in (0, \bar{\theta})$ we have:

$$s_i^*(\theta_i) \in \arg \max_{\hat{s}_i \in \mathbb{R}_+} \mathbb{E}_{\theta_i} \left[ \theta_i \cdot H_i(\hat{s}_i, s_{-i}^*(\theta_{-i})) - P_i(\hat{s}_i, s_{-i}^*(\theta_{-i})) \right]$$

so in particular,

$$\theta_i \in \arg \max_{\hat{\theta}_i \in \Theta_i} \mathbb{E}_{\theta_i} \left[ \theta_i \cdot H_i(s_i^*(\hat{\theta}_i), s_{-i}^*(\theta_{-i})) - P_i(s_i^*(\hat{\theta}_i), s_{-i}^*(\theta_{-i})) \right] \quad (1)$$
Solving for Auction BNEs: the FOC approach

Consider now the objective function of (1),

\[ \hat{\theta}_i \rightarrow \mathbb{E}_{\theta_{-i}} \left[ \theta_i \cdot H_i(s^*_i(\hat{\theta}_i), s^*_{-i}(\theta_{-i})) - P_i(s^*_i(\hat{\theta}_i), s^*_{-i}(\theta_{-i})) \right] \tag{2} \]

If it is differentiable (which can be assured to hold) and \( \theta_i \in (0, \bar{\theta}) \), then FOC of optimization implies

\[ \frac{d}{d\hat{\theta}_i} \left\{ \mathbb{E}_{\theta_{-i}} \left[ \theta_i \cdot H_i(s^*_i(\hat{\theta}_i), s^*_{-i}(\theta_{-i})) - P_i(s^*_i(\hat{\theta}_i), s^*_{-i}(\theta_{-i})) \right] \right\} (\theta_i) = 0 \]

Sometimes this can be solved for \( s^*_i(\theta_i) \) after manipulation.
Revenue Equivalence Theorem

Let a BNE \((s^*_k)_{k \in \mathcal{N}}\) of auction game \(\mathcal{A}\) be given.

Define two functions \(G_i, R_i : \Theta_i \to \mathbb{R}\) for each player \(i\), so that \(G_i(\hat{\theta}_i)\) and \(R_i(\hat{\theta}_i)\) give the \textbf{expected probability of winning} and \textbf{expected payment} when bidding as though valuation is \(\hat{\theta}_i\).

\[
G_i(\hat{\theta}_i) := \mathbb{E}_{\theta_{-i}} \left[ H_i(s^*_i(\hat{\theta}_i), s^*_{-i}(\theta_{-i})) \right]
\]

\[
R_i(\hat{\theta}_i) := \mathbb{E}_{\theta_{-i}} \left[ P_i(s^*_i(\hat{\theta}_i), s^*_{-i}(\theta_{-i})) \right]
\]

The expectations are taken over opponents’ types.

**Importantly**, \(G_i\) and \(R_i\) are dependent on the BNE \((s^*_k)_{k \in \mathcal{N}}\).
The nameless result and RET

Proposition

(Nameless Result) Fix a BNE \((s^*_k)_{k \in \mathcal{N}}\) of the auction game. Under regularity conditions, \(R_i(\theta_i) = \int_0^{\theta_i} xG_i'(x)dx + R_i(0)\) for all \(i\) and \(\theta_i\).

And finally...

Corollary

(Revenue equivalence theorem) Under regularity conditions, for two BNEs of two auctions such that \(G_i(\theta_i) = G_i^0(\theta_i)\) for all \(i, \theta_i\) and \(R_i(0) = R_i^0(0)\) for all \(i\), we have \(R_i(\theta_i) = R_i^0(\theta_i)\) for all \(i, \theta_i\).
RET is an equilibrium property

We stress again: RET is not a statement comparing two potentially different auction formats, but a statement comparing two equilibria of two auction formats.

“Expected Revenue” is an equilibrium property and an auction game might admit multiple BNEs with different expected revenues.
Example

Consider a first-price auction with two bidders. The two bidders’ types are distributed i.i.d. with $\theta_i \sim \text{Uniform}[0, 1]$. Each bidder submits a nonnegative bid and whoever bids higher wins the item and pays her own bid. If there is a tie, then each bidder gets to buy the item at her bid with equal probability. It is known that this auction has a symmetric BNE where (i) $s_i^*(\theta_i)$ is differentiable, strictly increasing in $\theta_i$; (ii) the associated Equation (2) is differentiable. Find a closed-form expression for $s_i^*(\theta_i)$.

We solve this on the blackboard using two different methods: FOC approach and RET.
Ec 2010a - Game Theory Section 6

Jetlir Duraj

Harvard University
Today

1. Subgame-Perfect Equilibrium

2. Infinite-Horizon Games and One-Shot Deviation

3. Rubinstein-Stahl Bargaining

4. Introduction to Repeated Games

5. Folk Theorem, infinitely repeated
Extensive Form Games

Recall the Definition of Extensive Form games

Definition

A finite-horizon extensive-form game $\Gamma$ has the following components:

1) A finite-depth tree with vertices $V$ and terminal vertices $Z \subseteq V$.
2) A set of players $N = \{1, 2, ..., N\}$.
3) A player function $J : V \setminus Z \to N \cup \{c\}$.
4) A set of available moves $M_{j,v}$ for each $v \in J^{-1}(j)$, $j \in N$. Each move in $M_{j,v}$ is associated with a unique child of $v$ in the tree.
5) A probability distribution $f(\cdot|v)$ over $v$’s children for each $v \in J^{-1}(c)$.
6) A (Bernoulli) utility function $u_j : Z \to \mathbb{R}$ for each $j \in N$.
7) An information partition $\mathcal{I}_j$ of $J^{-1}(j)$ for each $j \in N$, whose elements are information sets $l_j \in \mathcal{I}_j$. It is required that $v, v' \in l_j \Rightarrow M_{j,v} = M_{j,v'}$. 
Strategies in an Extensive Form Game

Definition

In an extensive-form game, a **pure strategy** for player $j$ is a function $s_j : \mathcal{I}_j \rightarrow \bigcup_{v \in J^{-1}(j)} M_{j,v}$, so that $s_j(l_j) \in M_{j,l_j}$ for each $l_j \in \mathcal{I}_j$. Write $S_j$ for the set of all pure strategies of player $j$.

A **mixed strategy** of player $j$, $\sigma_j$, is a probability distribution over $S_j$, i.e. an element from $\Delta(S_j)$. 
Payoff from a strategy

A pure strategy profile \((s_k)_{k \in \mathcal{N}}\) induces a distribution over terminal vertices \(Z\). Randomness only from moves of Nature;

A mixed strategy \((\sigma_k)_{k \in \mathcal{N}}\) profile does the same. Randomness now from moves of Nature and the (independent) randomization of the players. Write \(p(\cdot|(\sigma_k)) \in \Delta(Z)\) for the implied distribution over terminal vertices in both cases.

Define \(U_i : \times_{k \in \mathcal{N}} \Delta(S_k) \rightarrow \mathbb{R}\) where

\[
U_i(\sigma_i, \sigma_{-i}) := \mathbb{E}_{z \sim p(\cdot|(\sigma_k))} [u_i(z)]
\]

In all, the extensive-game payoff to player \(i\) is her expected utility from terminal vertices, according to Bernoulli utility \(u_i\) and the distribution over terminal vertices induced by the strategy profile and moves of Nature.
Nash Equilibrium and its problematic

Definition

A **Nash equilibrium** in finite-horizon extensive-form game is a strategy profile \((\sigma^*_k)_{k \in \mathcal{N}}\) where \(U_i(\sigma^*_i, \sigma^*_{-i}) \geq U_i(\hat{\sigma}_i, \sigma^*_{-i})\) for all \(\hat{\sigma}_i \in \Delta(S_i)\).

Problems with NE in dynamic games:
sometimes a NE in a dynamic game is sustained by a **non-credible threat** on decision vertexes never reached on the equilibrium path.
Subgames

Definition

In a finite-horizon extensive-form game $\Gamma$, any $x \in V \setminus Z$ such that every information set is either entirely contained in the subtree starting at $x$ or entirely outside of it defines a subgame, $\Gamma(x)$.

This subgame is an extensive-form game and inherits the payoffs, moves, and information structure of the original game $\Gamma$ in the natural way.
**Subgame-Perfect Equilibrium**

**Definition**

A (possibly mixed) strategy profile \((\sigma^*_k)_{k \in N}\) of \(\Gamma\) is called a pure **subgame-perfect equilibrium** (SPE) if for every subgame \(\Gamma(x)\), \((\sigma^*_k)_{k \in N}\) restricted to \(\Gamma(x)\) forms a Nash equilibrium in \(\Gamma(x)\).

Recall that we can rewrite mixed strategies as behavioral strategies. Then, for each player, the restriction of the strategies to a subgame is just the collection of the behavioral strategies corresponding/relevant to info sets in the subgame.

Every SPE is an NE (the whole game is always a subgame), but **not** conversely.
Solving for SPE: Backwards Induction

**Idea:** successively replace subgames with terminal vertices corresponding to SPE payoffs of the deleted subgames.

Start with a non-terminal vertex **furthest away** from the root of the game, say $v$.

Choose one of $J(v)$’s potentially mixed strategies, $m^*$, that maximizes her payoff in $\Gamma(v)$, then replace the subgame $\Gamma(v)$ with the terminal vertex corresponding to payoff profile induced by $m^*$.

Repeat this procedure, working backwards from the vertices further away from the root of the game.

Eventually, the game tree is reduced to a single terminal vertex, whose payoff is an SPE payoff of the extensive-form game, while the (mixed) actions chosen throughout the deletion process form a SPE strategy profile.
Backwards Induction Uniqueness Question

If $u_i(z) \neq u_i(z')$ for every $i$ and $z, z' \in Z$ with $z \neq z'$, then backwards induction finds the unique SPE of the extensive-form game.

Otherwise, the game may have multiple SPEs and backwards induction may involve choosing between several indifferent moves.

Depending on the moves chosen, during the tie-breaking, backwards induction may lead to different SPEs.
Possibly Infinite-Horizon Games

We generalize the Definition of Extensive Form games to include Infinite-Horizon Games

Definition

An extensive-form game with perfect information and no chance moves has the following components:

A possibly infinite-depth tree with vertices \( V \) and terminal vertices \( Z \subseteq V \).

A set of players \( \mathcal{N} = \{1, 2, \ldots, N\} \).

A player function \( J : V \setminus Z \to \mathcal{N} \).

A set of available moves \( M_{j,v} \) for each \( v \in J^{-1}(j), j \in \mathcal{N} \). Each move in \( M_{j,v} \) is associated with a unique child of \( v \) in the tree.

A (Bernoulli) utility function \( u_j : Z \cup H^\infty \to \mathbb{R} \) for each \( j \in \mathcal{N} \), where \( H^\infty \) refers to the set of all infinite length paths.
Preferences

An infinite-horizon game might end at a terminal vertex or it might never reach a terminal vertex.

It follows: each player must have a preference not only over the set of terminal vertices, but also over the set of infinite histories.

In the bargaining game, for instance, it is specified that $u_j(h) = 0$ for any $h \in H^\infty$, $j = 1, 2$, i.e. every infinite history in the game tree (i.e. never reaching an agreement) gives 0 utility to each player.
One-Shot Deviation Principle

Note that in truly infinite-horizon games backwards induction has no bite. Thus checking for/finding SPEs can be daunting. Fortunately

Theorem

*(One-shot deviation principle)*

If $\Gamma$ fulfills certain regularity requirements\(^a\), then a strategy profile $(s^*_k)_{k \in \mathcal{N}}$ is an SPE of $\Gamma$ iff for every player $i$, every subgame $\Gamma(x)$ with $J(x) = i$, and every strategy $\hat{s}_i$ of $\Gamma(x)$ such that $s^*_i(v) = \hat{s}_i(v)$ at every $v \in J^{-1}(i)$ except possibly at $v = x$,

$$U_i^{\Gamma(x)}(s^*_i, s^*_{-i}) \geq U_i^{\Gamma(x)}(\hat{s}_i, s^*_{-i})$$

where $U_i^{\Gamma(x)}$ denotes the payoff of $i$ in subgame $\Gamma(x)$.

\(^a\)These are always fulfilled in this course.
In words

The one-shot deviation principle says for extensive-form games to check whether a profile of strategies \((s_i^*)_{i \in \mathcal{N}}\) is SPE, we need only check for each \(i\) and each subgame \(\Gamma(x)\):

\(i \) does not have a profitable deviation amongst strategies that differ from \(s_i^*\) only at \(x\).

This is a much restricted set of alternative strategies to check!

One-Shot Deviation is related to the dynamic programming technique you learn and apply in the Macro sequence.
The Rubinstein-Stahl bargaining game, or simply “bargaining game” is comparable to the static ultimatum game, but with two important differences:

(i) the game is infinite-horizon, so that first rejection does not end the game. Instead, players alternate in making offers;  
(ii) the good that players bargain over is assumed infinitely divisible, so that any allocation of the form \((x, 1-x)\) for \(x \in [0, 1]\) is feasible.
Bargaining Game

Part of the bargaining game tree, showing only some of the branches in the first 2 periods. The root $\emptyset$ has (uncountably) infinitely many children of the form $(x^1, 1 - x^1)$ for $x^1 \in [0, 1]$. At each such child, P2 may play R or A. Playing A leads to a terminal node with payoffs $(x^1, 1 - x^1)$; playing R continues the game with P2 to make the next offer.
Example: Asymmetric bargaining power

Please read at home Example 87...
It is a good application of One-Shot Deviation Principle.
Repeated Game

A repeated game is an extensive form game where in each period of time the players play the same stage game.¹

Definition

For a normal-form game $G = \langle \mathcal{N}, (A_k)_{k \in \mathcal{N}}, (u_k)_{k \in \mathcal{N}} \rangle$ and a positive integer $T$, denote by $G(T)$ the extensive-form game where $G$ is played in every period for $T$ periods and players observe the action profiles from all previous periods. $G$ is called the stage game and $G(T)$ the ($T$-times) repeated game.

Terminal vertices of $G(T)$: $(a^1, a^2, ..., a^T) \in (\times_{k=1}^{\mathcal{N}} A_k)^T =: A^T$ and payoff to player $i$ at such a terminal vertex is $\sum_{t=1}^{T} u_i(a^t)$. A pure strategy for player $i$ maps each non-terminal history of action profiles to a stage game action, $s_i : \bigcup_{k=0}^{T-1} A^k \rightarrow A_i$.

¹Note that the stage game can itself be an extensive form game. This will mostly not be the case in this course though.
Repeated Games

Definition

For a normal form game \( G = \langle N, (A_k)_{k \in N}, (u_k)_{k \in N} \rangle \) and a \( \delta \in [0, 1) \), denote by \( G^\delta(\infty) \) the extensive-form game where \( G \) is played in every period for infinitely many periods and players act like exponential discounters with discount rate \( \delta \). This is called the (infinitely) repeated game with discount rate \( \delta \).

Note that \( G^\delta(\infty) \) doesn’t have terminal vertices.

An infinite history of the form \((a^1, a^2, \ldots, ) \in A^\infty \) gives player \( i \) the payoff \( \sum_{t=1}^{\infty} \delta^{t-1} u_i(a^t) \).

A pure strategy for player \( i \) maps each finite history of action profiles to a stage game action, \( s_i : \bigcup_{k=0}^{\infty} A^k \rightarrow A_i \).
A strategy for player $i$ in $G(T)$ or $G^\delta(\infty)$ specifies an action of the stage game $G$ after any non-terminal history $(a^1, ..., a^k) \in A^k$, including histories that would never be reached under $i$’s strategy.

Example: if P1’s strategy in repeated prisoner’s dilemma is to always play defect, she still needs to specify $s_1(((C, C), (C, C)))$. 

Strategies
In $G(T)$, the **average payoff** to $i$ at a terminal vertex $a = (a^1, a^2, \ldots, a^T) \in A^T$ is

$$U(a) = \frac{1}{T} \sum_{t=1}^{T} u_i(a^t).$$

In $G^\delta(\infty)$, the **discounted average payoff** to $i$ at the infinite history $a = (a^1, a^2, \ldots) \in A^\infty$ is

$$U(a) = (1 - \delta) \cdot \sum_{t=1}^{\infty} \delta^{t-1} u_i(a^t).$$
Feasible Payoffs and Public Randomization

Definition

The set of **feasible payoffs** is defined as $\text{co}(\{u(a) : a \in A\}) \subseteq \mathbb{R}^N$, where $\text{co}$ is the convex hull operator.

**Set of feasible payoffs** can be obtained if players use a **public randomization device** to correlate their actions:

given $v \in \text{co}(\{u(a) : a \in A\})$ write as weighted average $v = \sum_{k=1}^r p_k \cdot u(a^k)$ where $p_k \geq 0$, $\sum_{k=1}^r p_k = 1$ and $a^k \in A$ for each $k$.

Construct a correlated strategy profile where all players observe a public random variable that realizes to $k$ with probab. $p_k$, then player $i$ plays $a^k_i$ upon observing $k$.

The expected payoff profile under this correlated strategy profile is $v$. 
Public Randomization in Repeated Games

Public Randomization will be used in two ways in this course:

1) to realize specific feasible payoffs in a SPE in certain periods as described above (on equilibrium path, but not only). That agents will follow the prescriptions of the public randomization device, follows from the optimization property of the (overall) equilibrium.

2) to construct SPEs, which have as payoffs a mixture of the payoffs from two (or more) SPEs. Here, that agents will follow the prescriptions of the public randomization device, follows because no matter the realization of the public randomization device a SPE strategy profile will be picked.
Minimax Payoff

Definition
In a normal-form game $G$, player $i$’s minimax payoff is defined as

$$v_i := \min_{\alpha_{-i} \in \Delta(A_{-i})} \max_{a_i \in A_i} u_i(a_i, \alpha_{-i})$$

Call a payoff profile $v \in \mathbb{R}^N$ individually rational if $v_i \geq v_i$ for every $i \in N$.
Call $v$ strictly individually rational if $v_i > v_i$ for every $i \in N$.

An immediate consequence:

Proposition

Suppose $(s^*_k)_{k \in N}$ is an NE for $G(T)$ or $G^\delta(\infty)$. Then the average payoff profile associated with $(s^*_k)_{k \in N}$ is feasible and individually rational for the stage game $G$. 
First Result about SPEs

**Proposition**

*If $G$ has a unique NE, then for any finite $T$ the repeated game $G(T)$ has a unique SPE. In this SPE, players play the unique stage game NE after every non-terminal history.*

Lucky case... too lucky to be worthy of testing in exams.
Fudenberg and Maskin’s Folk Theorem

Theorem

(Fudenberg and Maskin 1986)
Write $V^*$ for the set of feasible, strictly individually rational payoff profiles of $G$. Assume $G$ has full dimensionality. For any $v^* \in V^*$, there corresponds a $\bar{\delta} \in (0, 1)$ so that $v^* \in \mathcal{E}(G^{\delta}(\infty))$ for all $\delta \in (\bar{\delta}, 1)$.

Proof constructs SPE strategies with average payoff $v^*$. To ensure $-i$ minimax $i$ in the event that $i$ deviates one promises a reward of $\epsilon > 0$ in all future periods to players who carry out their roles as minimaxers (stick and carrot strategy).

On Interpretation: Note that this is an anything goes result!

\[^{2}\text{This might be needed as it can happen that some } j \neq i \text{ might even prefer to be minimaxed rather than minimax } i.\]
Ec 2010a - Game Theory Section 7

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1. Folk Theorem with two players

2. Folk Theorem, finitely repeated

3. Refinements of NE in extensive and normal-form games

4. Signaling Games
Folk Theorem with two players

The case of two players doesn’t need the full dimensionality assumption (don’t need to reward minimaxers).

Theorem

(Fudenberg and Maskin 1986) Suppose \( N = 2 \). Write \( V^* \) for the set of feasible, strictly individually rational payoff profiles of \( G \). For any \( v^* \in V^* \), there corresponds a \( \tilde{\delta} \in (0, 1) \) so that \( v^* \in \mathcal{E}(G^\delta(\infty)) \) for all \( \delta \in (\tilde{\delta}, 1) \).
Folk-Theorem Strategies with two players

Two phases. In **normal phase**, players publicly randomize over vertices \(\{u(a) : a \in A\}\) to get \(\nu^*\) as an expected payoff profile.

In the **mutual minimax phase**, P1 plays the minimax strategy against P2 while P2 also plays the minimax strategy against P1, for \(M\) periods.

If any player deviates in the normal phase, go to the mutual minimax phase.

If any player deviates in the mutual minimax phase, restart the mutual minimax phase.

If the mutual minimax phase completes without deviations, go to normal phase.

This works for \(M\) large enough for all \(\delta\) near 1 (see blackboard).
Folk Theorem, finitely repeated

We know that for $G(T)$ to have multiple SPE $G$ needs to have more than one Nash eq...

**Proposition**

Suppose $G$ has a pair of NEs $\bar{a}^{(i)}$ and $\underline{a}^{(i)}$ for each player $i \in \mathcal{N}$ such that $u_i(\bar{a}^{(i)}) > u_i(\underline{a}^{(i)})$.

Write $d_i := \max_{a_i,\hat{a}_i \in A_i, a_{-i} \in A_{-i}} u_i(\hat{a}_i, a_{-i}) - u_i(a_i, a_{-i})$ for an upperbound on deviation utility to $i$ in the stage game and set $M_i \in \mathbb{N}$ with $M_i \cdot (u_i(\bar{a}^{(i)}) - u_i(\underline{a}^{(i)})) \geq d_i$.

For any feasible payoff profile $v^*$ where $v^*_i \geq u_i(\underline{a}^{(i)})$ for each $i$ and $T \geq \sum_{k \in \mathcal{N}} M_k$, there exists an SPE of $G(T)$ where the average payoff for all except the last $\sum_{k \in \mathcal{N}} M_k$ periods is $v^*$.

Why so complicated? The game unravels at the last period: only Nash equilibrium can be played at time $T$ so less room for incentives as the game nears the end.
Example

Suppose the following game is repeated $T$ times and each player maximizes the sum of her payoffs in these $T$ plays. Show that, for every $\epsilon > 0$, we can choose $T$ big enough so that there exists an SPE of the repeated game in which each player’s average payoff is within $\epsilon$ of 2.

\[
\begin{array}{ccc}
A & B & C \\
A & 2,2 & -1,3 & 0,0 \\
B & 3,-1 & 1,1 & 0,0 \\
C & 0,0 & 0,0 & 0,0 \\
\end{array}
\]

See blackboard...
Refinements of NE

We recall the Definitions from the lecture one-by-one.

Definition

A perfect Bayesian equilibrium (PBE) is a behavioral strategy profile \((p_j)_{j \in N}\) together with a belief system \((\pi_j(\cdot | l_j))_{j \in N, l_j \in I_j}\) so that:

1) \(p_j(l_j)\) maximizes expected payoffs starting from information set \(l_j\) according to belief \(\pi_j(\cdot | l_j)\), for each \(j \in N, l_j \in I_j\)

2) beliefs are derived from Bayes’ rule at all on-path information sets (an information set is called on-path if it is reached with strictly positive probability under \((p_j)_{j \in N}\). Else, it is called off-path.)

Main novelty: equilibrium concept now includes beliefs for every information set! Given beliefs play should be rational on-and-off path and beliefs should be derived from Bayes rule and play, whenever possible.
Refinements of NE

If $l_j$ is reached with strictly positive probability under $(p_j)_{j \in N}$, then the conditional probability of having reached each $v \in l_j$ given that $l_j$ is reached, is well-defined: $\pi_j(v|l_j)$.

But: we cannot use Bayes’ rule in an off-path information set $\implies$ PBE places no restrictions on these off-equilibrium beliefs (additional ’degree of freedom’ which can be used to support specific PBEs).
Refinements of NE

Definition

A **sequential equilibrium (SE)** is a behavioral strategy profile \((p_j)_{j \in N}\) together with a belief system \((\pi_j(\cdot | l_j))_{j \in N, l_j \in I_j}\) so that:

1) \(p_j(l_j)\) maximizes expected payoffs starting from information set \(l_j\) according to belief \(\pi_j(\cdot | l_j)\), for each \(j \in N, l_j \in I_j\).

2) **there exists** a sequence of strictly mixed behavioral strategies \((p_j^{(m)})_j\) so that \(\lim_{m \to \infty} (p_j^{(m)})_j = (p_j)_j\), and furthermore \(\lim_{m \to \infty} (\pi_j^{(m)})_j = (\pi_j)_j\), where \((\pi_j^{(m)})_j\) is the unique belief system consistent with \((p_j^{(m)})_j\).

Note: with strictly mixed behavioral strategies every info-set is on path so Bayes rule is always applicable. So along the sequence beliefs are pinned down uniquely.
Refinements of NE

Compared to PBE, SE places **additional restrictions on off-equilibrium beliefs:** off-path beliefs must be attainable as the limiting beliefs of a sequence of strictly mixed strategy profiles that converge to \((p_j)_{j \in \mathcal{N}}\). Given a strictly mixed \((p_j^{(m)})_{j \in \mathcal{N}}\), every information set is reached with strictly positive probability. Therefore, \((\pi_j^{(m)})_{j \in \mathcal{N}}\) is uniquely defined.

**Note:** there are **no assumptions of rationality/optimization** on the sequence of strategies \((p_j^{(m)})_j\). In particular, there is no requirement that \((p_j^{(m)})_j\) forms any kind of equilibrium under beliefs \((\pi_j^{(m)})_j\).
Refinements of NE

There is one special case where a PBE is automatically a SE.

**Proposition**

*If a PBE is so, that all non-singleton information sets of all players are on the equilibrium path of the PBE, then that PBE is a SE.*

The two remaining refinements use **trembles**: a small, positive probability is associated to all moves in each information set, interpreted as the minimum weight that any strategy must assign to all valid moves.

The strategy profile \((p_j, p_{-j})\) is said to be an \(\epsilon\)-**equilibrium** if at each information set \(l_j\) and with respect to belief \(\pi_j(\cdot|l_j)\):

- \(p_j\) maximizes \(j\)’s expected utility subject to the constraint of minimum weights from the **vector** of trembles \(\epsilon\).\(^1\)

\(^1\)In principle we allow for each info-set its own tremble.
Refinements of NE

Definition

A **trembling-hand perfect equilibrium (THPE)** is a behavioral strategy profile \((p_j)_{j \in N}\) so that there exists a sequence of trembles \(\epsilon^{(m)}\) converging to 0 and a sequence of strictly mixed behavioral strategies \((p^{(m)}_j)_{j}\), such that \((p^{(m)}_j)_{j}\) is an \(\epsilon^{(m)}\)-equilibrium and \(\lim_{m \to \infty} (p^{(m)}_j)_{j} = (p_j)_j\).

Definition

A **strategically stable equilibrium (SSE)** is a behavioral strategy profile \((p_j)_{j \in N}\) so that for every sequence of trembles \(\epsilon^{(m)}\) converging to 0, there exists a sequence of strictly mixed behavioral strategies \((p^{(m)}_j)_{j}\) such that \((p^{(m)}_j)_{j}\) is an \(\epsilon^{(m)}\)-equilibrium and \(\lim_{m \to \infty} (p^{(m)}_j)_{j} = (p_j)_j\).
Refinements of NE

Remarks:

1) THPE and SSE are also defined for normal-form games. The tremble $\epsilon$ specifies minimum weights for the different actions of various players.

2) For extensive-form game $\Gamma$, the following inclusions\(^2\) hold:

$$
\text{NE}(\Gamma) \supseteq \text{PBE}(\Gamma) \supseteq \text{SPE}(\Gamma) \supseteq \text{SE}(\Gamma) \supseteq \text{THPE}(\Gamma) \supseteq \text{SSE}(\Gamma)
$$

3) For a finite extensive-form game $\Gamma$, THPE($\Gamma$) $\neq \emptyset$, though it is possible that SSE($\Gamma$) $= \emptyset$.

\(^2\)Inclusion in terms of strategy profiles. Technically, NE is not in the same universe as PBE, SE, as these later objects require a belief system in addition to a strategy profile.
Example

Find the pure NE, PBE, SE of this game. See blackboard...
Signaling Games

Nature picks state of the world, $\theta \in \{\theta_1, \theta_2\}$, according to a common prior. P1 is informed of this state and sends to P2 a message from a (possibly infinite) message set $\mathcal{M}$. P2 does not observe the state of the world, observes P1’s message: P2 has one information set for every message in $A_1$. 

![Signaling Game Diagram](image-url)
PBEs in Signaling Games

A PBE in the signaling game has the following components:

1) A strategy for P1: what message to send in state $\theta_1$, what message to send in state $\theta_2$.

2) A strategy for P2: how to respond to every $a_1 \in A_1$ that P1 could send (even for off-path messages not sent by P1’s strategy).

3) A belief system $\{\pi_2(\cdot | a_1)\}_{a_1 \in A_1}$ over states of the world (one such $\pi_2(\cdot | a_1) \in \Delta(\{\theta_1, \theta_2\})$ for each message $a_1$).

4) P2’s belief system derived from Bayes’ rule whenever possible.

5) P2’s strategy after every message $a_1$ optimal given $\pi_2(\cdot | a_1)$.

6) P1’s strategy optimal in every state of the world.
PBEs in Signaling Games

When two states of the world (i.e. two “types” of P1), pure PBEs can be classified into two families.

1) In a **separating PBE**, the two types of P1 send different messages, say \( a'_1 \neq a''_1 \). By Bayes’ rule, each of these two messages perfectly reveals the type of the sender in the PBE.

2) In a **pooling PBE**, the two types of P1 send the same message, say \( a''' \). By Bayes’ rule, P2 should keep his prior about the state of the world after seeing \( a''' \) in such a PBE.

**Note:** In a PBE from either family, most of P2’s information sets are off-path. PBE allows P2 to hold arbitrary beliefs on these off-path information sets.
Example: Lawsuit Game

Consider a plaintiff (P1) and a defendant (P2) in a civil lawsuit. Plaintiff knows whether she has a strong case ($\theta_H$) or weak case ($\theta_L$), but defendant does not. Defendant has prior belief that $\pi(\theta_H) = \frac{1}{3}$, $\pi(\theta_L) = \frac{2}{3}$. The plaintiff can ask for a low settlement or a high settlement ($A_1 = \{1, 2\}$). The defendant accepts or refuses, $A_2 = \{y, n\}$. If the defendant accepts a settlement offer of $x$, the two players settle out-of-court with payoffs $(x, -x)$. If defendant refuses, the case goes to trial. If the case is strong ($\theta = \theta_H$), plaintiff wins for sure and the payoffs are $(3, -4)$. If the case is weak ($\theta = \theta_L$), the plaintiff loses for sure and the payoffs are $(-1, 0)$.

See blackboard..
Intuitive Criterion

**Definition**

Say a PBE \((a_1(\theta), a_2(a_1)), \pi_2(\cdot|a_1)\) satisfies Intuitive Criterion if there do not exist \((\hat{a}_1, \hat{a}_2, \hat{\theta})\) such that:

1. \(\hat{a}_1 \notin (a_1(\theta_1), a_1(\theta_2))\) (C1)
2. \(u_1(\hat{a}_1, \hat{a}_2, \hat{\theta}) > u_1(a^*_1(\theta), a^*_2(a_1(\theta)), \hat{\theta})\) (C2)
3. \(u_1(\hat{a}_1, a_2, \theta) < u_1(a^*_1(\theta), a^*_2(a_1(\theta)), \theta)\) for all \(a_2 \in A_2, \theta \neq \hat{\theta}\) (C3)
4. \(\hat{a}_2 \notin \arg\max_{a_2 \in A_2} u_2(\hat{a}_1, a_2, \hat{\theta})\). (C4)

In words: there is no deviation for any type \(\hat{\theta}\) of \(P_1\), s.t. given the best response of \(P_2\) to the deviation, type \(\hat{\theta}\) strictly prefers the deviation and no other type \(\theta \neq \hat{\theta}\) wants to (weakly) imitate \(\hat{\theta}\)'s deviation, no matter the action played by \(P_2\).
Intuitive Criterion

Intuitively, if type $\hat{\theta}$ of P1 proposes to P2 the action profile $(\hat{a}_1, \hat{a}_2)$, this breaks the equilibrium, because type $\hat{\theta}$ of P1 is better off than in the equilibrium and P2 is truly optimizing by playing $\hat{a}_2$, since he knows, that only type $\hat{\theta}$ of P1 would have a strict incentive to propose to break the equilibrium.

What happens in the lawsuit game?