Bargaining with Endogenous Learning∗

Jetlir Duraj†

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Abstract

I study a discrete-time dynamic bargaining game in which a buyer can choose to learn privately about her value of the good. Information generation takes time and is endogenous. After learning, the buyer can disclose verifiable evidence of her valuation to the seller. Examples include venture capital negotiations or procurement of new technologies, which sometimes feature significant delay due to endogenous costly learning. The buyer receives informational rents for any period-length only if learning is costly. The high-frequency limits of stationary equilibria result in a folk-theorem type of result about the delay until agreement. Maximal delay is achieved in equilibria with mixed pricing. Near the high-frequency limit, all stationary equilibria feature non-extreme prices and non-extreme payoffs. The analysis allows for closed-form solutions and for comparative statics.

1 Introduction

In bargaining situations often one party can learn privately during the negotiations. For instance, before investing in a start-up, an institutional investor typically conducts market surveys or seeks expert advice about proprietary technology. Learning takes time and resources, but creates an informational advantage. The newly-informed party gains strategic option value, because she can influence the negotiations by disclosing the information.

Appraising the market potential of innovative products is in fact of prime importance in negotiations in the venture capital (VC) industry. To quote the legendary Silicon Valley engineer and venture capitalist Eugene Kleiner: No matter how ground-breaking a new

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†duraj@g.harvard.edu
technology, how large a potential market, make certain customers actually want it.\textsuperscript{1} This investigative process is typically arduous and may lead to significant delays in negotiations. In this spirit, the recent survey study Gompers et al. (2019) finds that closing a deal in the VC industry in the U.S. takes on average 83 days.\textsuperscript{2} They also document significant variance in delay until agreement depending on industry, firm characteristics and location; e.g. average delay in California is 65 days, for late-stage firms it is 106 days.

Considerations of market appraisal also afflict venture capital negotiations in emerging markets where initial uncertainty may be related to regulatory trends or the evolution of cultural tastes (see e.g. the case study ‘Sula Vineyards’ in Zeisberger et al. (2017b) about private equity investment in the nascent wine industry in India).

Endogenous learning pervades negotiations beyond those in the VC industry. A publisher usually prospects the market while bargaining with a new author for the copyrights on her book. A real-estate developer hires lawyers to perform title exams and employs market analysts to forecast the value of a property in a year’s time. In government procurement situations, private companies often conduct studies on the benefits of a particular federal contract while at the same time negotiating the terms of agreement.

Motivated by the examples above, this paper studies an abstract dynamic bargaining game in which a buyer can endogenously generate information over time about her valuation of a unit good offered for sale by a seller.

The analysis sheds light on the following questions.

i. When may it be without loss to assume exogenous information, rather than endogenous acquisition of information?

ii. What is the buyer’s optimal information acquisition strategy? How can one compare information acquisition across different bargaining environments (comparative statics)?

iii. How can one quantify the efficiency loss and delay in agreement when information takes time to arrive, is costly and endogenously acquired?

The model I study is a modified version of the standard one-buyer/one-seller dynamic bargaining game as introduced in Fudenberg et al. (1985) and Gul et al. (1986).\textsuperscript{3} Henceforth the buyer is called Buyer (female pronouns) and the seller is called Seller (male pronouns). Seller has zero value for the good he initially owns (a normalization). The good may be of high or low (non-negative) value to Buyer. Initially the two parties share the same prior belief about Buyer’s value.

An informational asymmetry arises over time as Buyer learns endogenously and privately about her valuation. Buyer’s learning is stochastic and occurs over time: I consider both the case in which she is able to influence the rate of learning and the one in which she is not. Once the opportunity to learn arrives, Buyer chooses the extent to which she wants to exploit this opportunity, i.e. how accurate a signal to acquire. I assume new

\textsuperscript{1}See Kleiner’s laws in \url{http://entrepreneurhalloffame.com/eugene-kleiner/} or \url{www.economist.com/obituary/2003/12/04/eugene-kleiner}.

\textsuperscript{2}Deal selection and closing can take even longer in private equity situations not involving innovative products. See Zeisberger et al. (2017a) and Gompers et al. (2019) for case studies that exemplify this.

\textsuperscript{3}In bargaining theory parlance I consider a seller-offer, (weak) gap case game. See Ausubel et al. (2002) for a survey of the classical dynamic bargaining literature with incomplete information.
information is verifiable and hence cannot be misrepresented. Once new information is acquired, Buyer decides whether and when to disclose it to Seller.

This model departs from traditional bargaining models in two crucial ways. First, players are initially symmetrically informed and one party (Buyer) may become more informed over time. Second, disclosure of new information is feasible and any information disclosed is verifiable. Therefore, Seller learns from two different channels about the current valuation of Buyer: from the information disclosure decision, as well as from the rejection of past prices. The latter is the traditional learning channel in the standard dynamic bargaining game, whereas the former is a new channel.

There are two main takeaways from the analysis. First, learning creates option value for Buyer which is strategic in nature. The possibility to learn typically ensures Buyer a surplus. Second, because learning takes time and is costly, Seller also receives some surplus from the negotiations. This contrasts with the outcomes in the standard game of Fudenberg et al. (1985) and Gul et al. (1986) in which prices and Seller equilibrium payoff are generically extreme because Buyer possesses initial private information for free: Seller typically receives minimal rent in the traditional seller-offer game.

Without the possibility to learn, Seller extracts all surplus immediately. With the possibility to learn, Buyer can wait until she learns about her valuation, disclose if her valuation is low and otherwise keep silent to profit from the private information by pretending she has not learned yet. Thus, the fact that learning is possible, but associated with costs, typically implies non-extreme equilibrium payoffs. Non-extreme payoffs imply trade at non-extreme prices.

One can draw an analogy of the strategic option value from learning with the actions of an investor in a capital market. Without the possibility to learn the payoff structure of Buyer at the moment of trade is akin to the investor acquiring a stock at the fair price. She receives zero profit from the trade. The possibility to learn is akin to the investor owning a call option on the stock at the beginning of time: if there is good news the option is exercised at a profit at the moment of trade, if there is bad news there is no downside at the moment of trade.\(^4\) Acquiring and exercising the strategic option in my model is inefficient because learning leads to delay, as well as other costs associated to it. Nevertheless, learning happens in all equilibria and typically results in positive Buyer payoff.

I focus first on the case in which the exploitation of an information source is costless, a useful benchmark for comparison to the often more realistic case of costly information.\(^5\) The costless learning benchmark may also have independent merit in some applications, e.g. when information may arrive through a social network, say of friends or colleagues, rather than through employing internal or external resources in an advisory capacity.

Next, I consider extensively the more realistic case of costly learning. Some of the results change when compared to the case of costless learning; for example, Buyer may receive zero surplus in the costless learning case, so costly learning is needed to guarantee non-extreme outcomes. Learning costs can arise in two distinct dimensions in my model:

\(^4\)The analogy to a perfect capital market is not complete, because Buyer receives a positive payoff overall from exercising the strategic option value. This is because she is the only party who can learn in my model, which corresponds to a capital market imperfection in the investor analogy.

\(^5\)Strictly speaking, whenever Buyer has to wait for information to arrive there are opportunity costs from waiting, which are reflected in the fact that future utility is discounted. With costless learning I mean that there are no direct, physical costs related to acquiring and exploiting information sources.
exploration/search costs for sources of information and exploitation costs of a newly found learning source. For instance, the manager of a pharmaceutical lab may hire additional scientists to search for new test ideas on the effectiveness of a new drug compared to more traditional ones (exploration/search phase), or once a new test idea is available, she may decide on the scale of the study that implements it (exploitation phase). I endogenize both of these aspects of Buyer learning and show that they behave differently from each other.

The next subsection explains the results of the paper in more detail.

1.1 Preview of results

I focus on perfect Bayesian equilibria (PBE): both players follow optimal contingent plans given their beliefs about Buyer’s value, and their beliefs are derived from equilibrium strategies and Bayes’ rule whenever possible.6 Discrete-time dynamic bargaining games are notoriously difficult to study. Therefore, many (but not all) results hold under the qualification that the period-length is small enough (near the high-frequency limit, henceforth near the HFL), and some hold in the limit as period-length shrinks to zero and time becomes continuous (henceforth in the HFL).7 When period-length shrinks I assume the players discount payoffs within that period less and that the probability that Buyer receives an opportunity to learn diminishes. In particular, the probability of learning within a period vanishes as period-length goes to zero.

The efficient outcome in the game is for trade to happen immediately so that learning does not start.8 This is because learning always leads to delay, apart from any additional costs that may be associated with it. All equilibria with/without costs feature positive delay and thus are inefficient. This is because Buyer accepts the delay in order to capture the strategic option value from learning new information: if she learns bad news about her valuation, she may disclose this to get a lower price, whereas if she learns good news she can keep silent and pretend no new information has arrived to buy the good at a price below its actual value. Nevertheless, approximate efficiency is a feasible equilibrium outcome near the HFL, even if learning is costly. A detailed explanation of the results follows.

Exogenous intensity, costless accuracy. I first consider the case where the opportunity to learn new information arrives privately to Buyer at a positive Poisson rate of λ and Buyer can freely choose how much information to acquire. In this case, it is a weakly dominant strategy for Buyer to learn perfectly whenever she gets the chance. In fact, acquiring full information whenever possible is strictly optimal in a PBE in which Buyer receives a positive equilibrium payoff. Intuitively, free information never hurts, and so is always consumed by an agent without commitment. Thus, whenever information is costless but arrives at a random date, the assumption of endogenous choice of informativeness is equivalent to the assumption that information is exogenous, one-shot and delivers conclusive evidence to Buyer about her value.

6See part IV of Fudenberg and Tirole (1991) for a formal definition.
7Period-length is inversely proportional to the frequency of interaction. Thus, near the HFL equivalently means for a high enough frequency of interaction.
8I consider the case of negative good values as an extension (see below). With negative values learning can be welfare-enhancing.
Second, near the HFL there are generically no PBEs in which Buyer with bad news discloses immediately and Seller screens sequentially the Buyer for the other valuations, as long as there is no disclosure. Thus, the usual logic of Seller ‘screening down the demand curve’ that appears in many standard dynamic bargaining models fails in this setup.\(^9\) The reason is that the disclosure decision leads to an \textit{interim} update of Seller about the private information of Buyer. If Buyer with bad news always discloses, no disclosure is interpreted as indication of higher valuations. This counteracts the belief update after a price rejection, which is interpreted by Seller as an indication of lower values. Overall, the first effect is strong enough to overpower the second.

I also show that for high enough \(\lambda\), but fixed period-length, there are equilibria in which Buyer gets zero payoff, because she cannot successfully pretend to be Buyer who has not received any news. Therefore the possibility of private endogenous learning \textit{per se} does not necessarily ensure informational rents for Buyer. In these equilibria, Seller always charges the highest price that may be accepted by some Buyer type he deems feasible, given the information he has about Buyer’s valuation \((\text{high-price equilibria})\). High-price equilibria do not survive costly learning. Moreover, there cannot be any equilibria with zero Buyer payoff whenever period-length is small enough. This is because of Seller’s lack of commitment: the arrival of information in the first period is very unlikely so that asking for a high price is suboptimal already in the first period. This results in informational rents for Buyer.

I focus then the analysis on stationary equilibria in which Seller mixes between at most two prices after non-disclosure, Buyer who has learned good news trades without delay, whereas Buyer who has learned bad news discloses immediately and accepts the revised price of Seller. Henceforth these are called \textit{strongly stationary equilibria}. These always exist near the HFL and besides strongly stationary equilibria with pure pricing by Seller, there are also strongly stationary equilibria near the HFL that feature mixed pricing upon non-disclosure. In particular, there is equilibrium multiplicity.

In the HFL, any mixing by Seller upon non-disclosure disappears and prices converge to a single number. Thus, Seller asks for a flat price in the HFL, unless he sees evidence of bad news at which point he revises the price down and the game ends. In the HFL of strongly stationary equilibria, both players have positive payoffs. The amount of expected real time delay is indeterminate and varies from zero to \(1/\lambda\): for each value strictly between zero and \(1/\lambda\), there is a corresponding equilibrium with expected real time delay equal to said value. In particular, near the HFL there are strongly stationary equilibria with pure pricing whose outcome is arbitrarily close to efficiency. In these equilibria, Buyer who has not learned yet is made indifferent on path between buying now and continuing and chooses to buy now with high probability. This leads to vanishing delay near the HFL and to low probability of learning on path. Overall, this results in payoffs close to efficiency.

The indeterminacy in expected delay and payoffs occurs because the price charged in the HFL makes Buyer who has not learned yet indifferent between waiting to capture the strategic option value and stopping immediately. It is this possibility to wait that in turn creates the strategic option value from learning. This is because it allows Buyer who has learned good news to ‘pool’ with Buyer who has not learned yet and thus possibly get

\(^9\)See e.g. Fudenberg et al. (1985), Gul et al. (1986) for classical and many recent papers, e.g. Fuchs and Skrzypacz (2010) or Hwang and Li (2017) for results featuring equilibrium dynamics in which seller \textit{screens down the demand curve}.
the high-value good at a bargain. On the other hand, because information is not readily available to Buyer at the start of time, Buyer never captures all surplus in any strongly stationary equilibria.

Overall, the combination of endogenous learning and disclosure of bad news ensures the existence of equilibria which feature neither extreme prices, nor extreme payoffs.

**Exogenous intensity, costly accuracy.** I consider two distinct costs of exploiting an information source, conditional on its arrival. In the first case, costs are deterministic and variable and more accurate information costs more. In the second case, costs are stochastic but lump-sum: whenever the opportunity to learn arrives, a cost is drawn from a distribution. If Buyer pays the cost she can exploit the learning source at no additional marginal cost; otherwise, she may wait for future opportunities to learn and more favorable draws of the lump-sum exploitation cost.

Surprisingly, the same set of results can be proven for both models of accuracy costs. First, for any period-length, all PBEs feature a positive payoff for Buyer. This is in contrast to the costless case. Common knowledge of Buyer’s costs of information acts as an insurance device for Buyer’s payoff. The intuition is the following. Because of the strategic option value of learning, all PBEs feature some non-trivial amount of learning from Buyer. Because learning is private, Seller has to compensate Buyer for the costs of learning on average. This results in positive ex-ante surplus for Buyer because of the discretionary nature of her disclosure decision. Recall that in the case of costless accuracy there exist PBEs with zero Buyer payoff.

Second, just as in the case of costless accuracy, there are generically no PBEs near the HFL in which Buyer with bad news discloses and which feature deterministically falling prices on path (Seller does not ‘screen down a demand curve’). The intuition is the same as before.

Subsequently, I focus again on strongly stationary equilibria. Near the HFL there exist such equilibria featuring pure pricing on path from Seller, for any parameter values of intensity and accuracy costs. Under a condition postulating that the arrival rate of the opportunity to learn λ is not too high compared to the players’ impatience level, there are additional strongly stationary equilibria with mixed Seller pricing near the HFL.

In all strongly stationary equilibria with pure pricing Seller incentivizes the information acquisition of Buyer by offering her the opportunity to buy the good at a low price in case of good news. In contrast to the case of costless learning, the price spread does not disappear in the HFL of mixed pricing equilibria. Near the HFL of equilibria with mixed pricing Seller charges most of the time the reservation price of Buyer with good news as long as there is no disclosure. To incentivize learning in such equilibria Seller occasionally charges a low price so that Buyer has incentives to learn, whenever the opportunity arrives. But since the probability to learn in a single period is very low when period-length is very small, Seller promises the low price within a period with probability declining to zero, as period-length goes to zero. This leads to maximal delay in real time for such equilibria, because Buyer has no choice but to wait for the opportunity to learn to realize the strategic option value.

In the HFL of strongly stationary equilibria with pure pricing expected delay is again indeterminate. Moreover, near the HFL there are strongly stationary equilibria that are almost efficient, and all such equilibria exhibit pure pricing. This is despite costly learning. The reasoning for these results is the same as in the costless case.
In the HFL, the payoffs of Buyer and Seller give insight into the inefficiency sources: the deviation from full efficiency is a weighted sum of the ex-ante surplus and the learning costs incurred on path. The inefficiency becomes larger the more impatient the players are or the lower the arrival rate $\lambda$.

**Extension: endogenous intensity.** I extend the model to allow for endogenous costly choice of the learning intensity $\lambda$. In this most general version of the model, both aspects of learning, intensity and accuracy, are endogenous.\(^{10}\) Costs of intensity are deterministic and variable and they are incurred at the beginning of every period, as long as bargaining continues. All general results from the model with costly accuracy but exogenous intensity carry over in this more general framework. In particular, there are again strongly stationary equilibria near the HFL, which are uniquely parametrized by the average price quoted by Seller upon non-disclosure. Compared to the case of exogenous intensity, there is now an additional potential source of inefficiency coming from the intensity costs. This is quantified in the HFL, just as the previous sources of inefficiency.

Once the rate of opportunities to learn is a choice variable for Buyer, comparative statics for information choice are possible. Across all strongly stationary equilibria and in the HFL, the endogenous and stationary level of intensity increases in the ex-ante level of optimism and decreases in the patience level of the players. In contrast, accuracy choice is broadly speaking ‘reverse-U-shaped’ in the level of ex-ante optimism and independent of the impatience level.

**Extension: pre-learning negotiations.** The reasons why learning is inefficient in my main model are two-fold. First, trade is ex-ante efficient and therefore learning does not add to the social welfare. Second, I assume that Buyer starts to learn before Seller can make the first price offer. This is a natural feature of many real-world negotiations: the party who becomes first interested in the trade may naturally start to gather information privately before she actually shows her interest to the other side of the market.

If Buyer could commit to not start learning before she approaches Seller and instead allows him to make a first, pre-learning offer, then the inefficiency would disappear. This is true independently of period-length. I show this by adding an ex-ante stage to the bargaining game. In this stage, Seller can make a first price offer before the learning from Buyer’s side can start. In equilibrium, Buyer accepts the price offer immediately and learning does not happen on path. This extension suggests, that there is scope for a more systematic study of the design of bargaining institutions for markets in which parties typically engage in costly private learning before and during negotiations.

**Extension: possible negative Buyer value.** Suppose that the lowest value of Buyer is negative but that in expectation trade is efficient ex-ante (i.e. positive Buyer value occurs with high enough probability to compensate for the possibility of negative value). Assume in addition that Buyer is free to walk away from the bargaining at any moment. In this situation learning can strictly improve welfare, despite being costly. This is because Buyer can always walk away when she learns bad news that lead to a negative valuation for the good.

\(^{10}\)Results about costless choice of $\lambda$ are contained in the online appendix.
In this set up Buyer’s information acquisition choice is very similar to the case of non-negative values and in strongly stationary equilibria bargaining ends whenever Buyer receives news about the good. When she receives good news she trades without delay. When she receives bad news, she either discloses immediately whenever it leads to a positive valuation or she walks away immediately whenever it leads to a negative valuation.

If learning were impossible, the efficient outcome under imperfect information about the value of the good would again be to trade immediately. This would require trade in both states of the world, i.e. Buyer would incur a loss from trade ex-post, whenever value of good is negative. I illustrate that learning limits this downside by lowering the probability of trade in the case of a negative valuation.

Outline of the rest of the paper. The next subsection discusses related literature. Section 2 introduces the basic model, states auxiliary results which are valid for all model versions and studies the case of costless learning. Section 3 introduces costs for accuracy and studies their implications. Section 4.1 discusses endogenous intensity and contains some comparative statics results. Section 4.2 shows how efficiency can be restored through pre-learning negotiations. Section 4.3 discusses negative values. Section 5 concludes. Formal proofs are contained in the appendix. Results that are not central to the main takeaways of this paper are contained in the online appendix.

1.2 Related literature

The study of bargaining games with informational asymmetries has a long tradition in economic theory. This literature begins with the seminal papers Fudenberg and Tirole (1983), Sobel and Takahashi (1983), Cramton (1984), Fudenberg et al. (1985), Fudenberg et al. (1985) and Gul et al. (1986). These papers focus on the case in which one or both bargaining parties have initial private information about their valuations and study how bargaining parties learn about the private information of their strategic opponent from price offers and rejections. A focal point of the analysis is the validity of the Coase conjecture. This conjecture prescribes that, as bargaining parties interact more and more frequently, delay until agreement vanishes and the informed party’s rent is maximal.

Almost since its beginning, the literature with asymmetric information has focused on understanding the economic forces behind inefficient delay and whether the Coase conjecture survives, in one form or another, more complicated economic environments. To mention a few seminal contributions, Cramton (1984) and Chaterjee and Samuelson (1987) find that two-sided private initial information may lead to costly delay, Rubinstein (1985) that delay is possible whenever a player is uncertain about the time preferences of her bargaining counterpart, whereas Deneckere and Liang (2006) find that the same may happen if the parties have interdependent values.11

Starting from Fudenberg et al. (1987), the literature has studied bargaining under the existence of other potential trading partners, or more generally outside options. Board and Pycia (2014) shows that Seller may get significant surplus (and thus the Coase conjecture fails), even though agreement is immediate, whenever Buyer has an outside option at the beginning of the game. Fuchs and Skrzypacz (2010) as well as Hwang and Li (2017),

\footnote{This list is by far incomplete. Other models of delay include: Abreu and Gul (2000) due to irrational players and reputation building, Feinberg and Skrzypacz (2005) due to higher-order beliefs and Yildiz (2004) due to ex-ante optimism (non-common priors).}
Hwang (2018b) and Lomys (2017) focus on the effects of stochastic arrivals of outside options. Hwang and Li (2017) and Hwang (2018b) in particular look at the case where the exogenous arrival of the outside option leads to a FOSD-shift upwards of the valuation of Buyer. They consider private arrival of the outside option and find similar equilibrium dynamics as the strongly stationary equilibria in this paper, in addition to verifying the Coasian conjecture under some parameter restrictions.

While the existence or arrival of an outside option might also be interpreted as additional private information, none of the above mentioned papers models information acquisition explicitly. Learning new information leads to a mean-preserving-spread (MPS) in the distribution of Buyer valuations and in this paper, this MPS turns into a FOSD-shift upwards in the equilibrium dynamics of strongly stationary equilibria through the equilibrium disclosure choice of Buyer who receives bad news. Thus, the resulting FOSD-shift in valuations is an equilibrium property, rather than assumed in the model primitives. Additionally, results here show that the existence of sequential screening dynamics near the HFL hinges upon the assumption of private initial information, which is absent in the model of this paper. Finally, in the set up considered in this paper Coase conjecture fails and equilibria typically exhibit non-extreme prices and payoffs for all parameter values of the game considered.

This paper also connects to the literature on evolving valuations/roles in dynamic situations. Ortner (2017) and Ortner (2019) consider bargaining situations in which the change is exogenous, whereas Bergemann and Välimaki (2019) offer a review of the recent related literature on dynamic mechanism design, in which the standing assumption is commitment from the part of one of the players. More related to this work, Ravid (2019) studies a seller-offer game, in which Seller has the private initial information about the quality of the good and Buyer is rationally inattentive to past prices and the product’s quality. Similarly to this work, Buyer’s valuation changes endogenously, there may be delay until agreement and Buyer obtains positive surplus in the equilibria characterized. In contrast to this work, there is no disclosure decision for information, because the costs of information are not due to information generation/production but rather to information-processing. Maximal delay in my model is achieved by equilibria with mixed pricing, whereas Ravid (2019) focuses on equilibria with pure pricing. Moreover, positive costs of information are not necessary to ensure positive Buyer payoff in the model of this paper, whereas they are in Ravid (2019). Finally, in the setting of this paper sequential screening dynamics do not play a role, whereas they play an important one in Ravid (2019).

Daley and Green (2019) considers a model of bargaining in which Seller has initial private information about the value of the good and public, exogenous news delivers garbled information about the value of the good. In their setting interdependence of values is necessary but not sufficient for delay. Esö and Wallace (2019) considers a game with interdependent values in which both Buyer and Seller can learn their value of the good exogenously at a random date, privately and independently of each other. They assume each player can disclose verifiably their updated valuation to the other party. The interplay between interdependent values and two-sided exogenous learning implies there is no scope for equilibria with mixed prices and that with infinite horizon there is no delay near the high-frequency limit.

In this paper only Buyer learns endogenously and privately and I model explicitly the

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12 FOSD stands for First-order stochastic dominance.
costly generation process of new information. In terms of results, I characterize equilibria which feature delay in real time, even though values of Buyer and Seller are independent and trade is efficient in every state of the world. Moreover, I show that equilibria with price mixing on path, which are absent from the above mentioned papers, are not knife-edge cases but a robust prediction of endogenous private learning.

Crémer and Khalil (1992) and Crémer et al. (1998) are classical works on information acquisition before the signing of a contract. More recently, Shi (2012) and Li (2019) consider auction settings in which Buyer types can acquire costly signals about their valuation before bidding in the auction.\textsuperscript{13} Kirpalani and Madsen (2019) studies how private information acquisition and social learning through public investments affect investment timing in settings in which investments are non-rival.

In contrast, this paper studies a bilateral dynamic bargaining model and focuses on the delay caused by endogenous learning, besides studying the optimal information acquisition of Buyer in both of its dimensions: intensity and accuracy.

Finally, this paper relates to the classic literature on strategic information transmission of verifiable information without commitment, beginning with Grossman (1981) and Milgrom (1981).\textsuperscript{14} In particular, the strongly stationary equilibria of this paper feature partial unravelling analogous to Dye (1985): Buyer with bad news presents evidence if she has it, while Buyer with good news pools with Buyer without evidence. Esö and Wallace (2014) consider a static bargaining model with verifiable disclosure and two-sided private information and focus on the value of verifiability of private information. Most of the work in the literature on verifiable disclosure has been on static disclosure; notable exceptions are Acharya et al. (2011) and Guttman et al. (2014), which focus on dynamic verifiable disclosure to a market rather than a strategic audience. Most recently, DeMarzo et al. (2019) studies a model in which a seller designs tests she can use to certify product quality to a market. Similar to this paper, it considers the case of costly design and shows the existence of equilibria with partial revelation. However, the audience of the disclosure is non-strategic and the model is static; thus any study of potential inefficient delay due to costly test design is moot.

2 The model

There are two players: Buyer and Seller. Seller owns an indivisible good whose value for Seller is zero (a normalization). Buyer has potential value \( \theta \in \{\bar{v}, v\} \) for the good with \( \bar{v} > v \geq 0 \).

Time is discrete, denoted by \( t = \Delta, 2\Delta, \ldots \) with period-length given by \( \Delta > 0 \). I use the expression near the HFL to mean for all \( \Delta > 0 \) small enough and in the HFL to mean in the limit as \( \Delta \to 0 \).

At the beginning of the game Buyer does not know her value for the good and both Buyer and Seller share a common prior of high value \( \pi_0 \), which is strictly between zero and one. Denote by \( \hat{v} = \pi_0\bar{v} + (1 - \pi_0)v \) the ex-ante common knowledge valuation of Buyer. Players are impatient with common discount factor \( \delta = e^{-r\Delta} \), for some \( r > 0 \).

\textsuperscript{13}Shi (2012), Hwang (2018a) and Esö and Wallace (2019) are to the best of my knowledge the only other papers which consider an incomplete information game between a set of buyers and a seller without the assumption of initial private information.

\textsuperscript{14}See Milgrom (2008) and Dranove and Jin (2010) for surveys.
**Learning.** Every period starting from \( t = 1 \) with probability \( \mu = 1 - e^{-\lambda A}, \lambda > 0 \) opportunities to learn about \( \theta \) arrives for the *Buyer*. For tractability, I make the major simplifying assumption that Buyer can exploit only one opportunity to learn, i.e. she can learn additional information only once. Its arrival is private information of Buyer. Whenever the opportunity to learn arrives, Buyer can pick an experiment of the form\(^{15}\)

\[
\mathcal{E}_a : \{v, \bar{v}\} \to \mathcal{P} \{\{H, L\}\} \text{ with } \mathcal{E}_a(\bar{v})(H) = \mathcal{E}_a(v)(L) = a \in \left[\frac{1}{2}, 1\right].
\]

The accuracy chosen by Buyer is also private. \( a \) is the accuracy and it is a choice variable of Buyer. \( H \) stands for a signal which gives (possibly partial) evidence of high value and \( L \) for a signal which gives (possibly partial) evidence of low value. The informativeness of the experiment \( \mathcal{E}_a \) is denoted by \( I(a) \) and is given by \( I(a) = \frac{a}{1-a} \).

Results would go through with other parametric forms of experiments, as long as they lead to concave value of information. If one allows for general experiments, this requires introducing two informativeness parameters, since there are two states of the world. It turns out that value of information can be non-concave in this model, when general two-parametric experiments are allowed.\(^{16}\)

The parameter \( \lambda \) is called *intensity* throughout, whereas \( a \) is called *accuracy*. Intensity is exogenously given and time-invariant until section 4.1. In analogy to the terminology of the experimentation literature, intensity describes the *exploration/search* rate of the learning process, whereas accuracy choice is akin to the choice of how much to *exploit* an already available learning source.

Observing signal \( L \) or \( H \) after performing experiment \( \mathcal{E}_a \) with accuracy \( a \in [\frac{1}{2}, 1] \) leads to an updated valuation for Buyer (expected value given the signal and the accuracy \( a \) chosen). The possible valuations in \([v, \bar{v}]\) that Buyer may have at each moment in time are called the *Buyer’s type*. These consist of \( \hat{v} \), the valuation of Buyer before she learns, as well as the updated valuations after learning. For all purposes of the analysis, Seller’s belief after a public history about the private information of Buyer may be summarized through a probability distribution over possible valuations in \([v, \bar{v}]\).

**Disclosure choice.** In every period, after opportunity to learn and learning, Buyer can choose to disclose verifiably her updated valuation, or choose to not disclose her updated valuation. Buyer can delay disclosure.

Under this assumption, non-disclosure can be due to two reasons only: Buyer has not learned yet or she has learned and chosen not to disclose until that point in time. The timeline of the game within a period is as in Figure 1.

---

\(^{15}\)Henceforth \( \mathcal{P}(X) \) for a metric space \( X \) denotes the set of Borel probability measures over \( X \).

\(^{16}\)See online appendix for this. The possibility of non-concavity of the value of information in information acquisition models is a well-known phenomenon since the seminal work of Radner and Stiglitz (1984). See Chade and Schlee (2002) for a modern treatment of this issue.
Histories, strategies and equilibrium. This game has two types of histories: private and public. A public history consists of a sequence of disclosure or non-disclosure events, as well as of rejected prices. Only Buyer has access to private histories, which include, in addition to publicly available information, both the occurrence of the arrival of the opportunity to learn as well as the learning outcome. A strategy for Buyer after a private history prescribes her choice of \( a \), if that history ends with the arrival of the opportunity to learn, her choice of a probability of disclosure, if that history prescribes probability of disclosure at the end and there has been an opportunity to learn in the past. Finally, it prescribes an acceptance probability for a price quoted by Seller, if the history ends with that price quoted by Seller. A strategy for Seller prescribes a (possibly mixed) price offer after every public history ending with a disclosure/non-disclosure by Buyer.

Throughout, an equilibrium is a perfect Bayesian equilibrium (PBE).\(^{17}\) Buyer’s strategy prescribes an optimal move after every private history, given Seller’s strategy and Bayes updating about her value of the good. Seller’s strategy prescribes an optimal mixing over prices after every public history in which he is called upon to quote a price, given Buyer strategy and Bayes’ updating about the evolution of Buyer’s valuation (whenever possible using Buyer’s strategy).

Introduction of the assumptions on costly learning is deferred to sections 3 and 4.1.

Before delving into the analysis of the main results, I state two auxiliary results, which are very helpful in simplifying the analysis of PBEs throughout.\(^{18}\) The first one mirrors similar results in the classical works Fudenberg et al. (1985) and Gul et al. (1986) on the seller-offer game with initial private information. It is valid for all model versions considered in this work. Before stating it, I define the concept of reservation prices.

**Definition 1.** Fix a PBE, a Buyer type \( w \) and a private history \( h \) which ends just before Seller has the possibility to quote a price. The reservation price of type \( w \) after \( h \) is the highest price that type \( w \) is willing to pay after \( h \) given continuation play in the PBE.

**Lemma 1.** In any PBE the following hold true.

1) Fix a public history \( h \) after which Seller is asked to quote a price and let \( w \) be the lowest possible Buyer valuation according to Seller’s belief distribution over Buyer-types after \( h \). Seller asks for at least \( w \) after \( h \).

---

\(^{17}\)See part IV of Fudenberg and Tirole (1991) for a formal definition.

\(^{18}\)Both Lemma 1 and Lemma 2 here are also valid word-for-word for the models with costs. Thus, I do not restate them in sections 3 and 4.1.
2) In any PBE, after every public history in which it is Seller’s turn to move, Seller asks for prices among all reservation prices (given continuation play) of Buyer types she thinks are feasible right after that history.

3) After every private history, the Buyer type with the highest reservation price that has positive probability after that history, accepts an offer equal to that reservation price with positive probability.

4) After any disclosure event, Seller quotes a price equal to the disclosed valuation with probability one.

The strategies of a player in a PBE may be history-dependent and look back at more than just the preceding periods. Therefore, there is the theoretical chance that Buyer can reward Seller for prices lower than \( v \) by using history-dependent continuation play. Part 1) shows that this can never happen in a PBE.

The proof of part 1) also establishes the skimming property. It says that, after every public history, if Buyer of type \( w \) accepts a price \( p \) then so does every type with a strictly higher valuation \( w' > w \).

Part 2) follows immediately from part 1) and the skimming property: if Seller wouldn’t charge reservation prices, he would be leaving surplus on the table at no future benefit. Part 3) holds necessarily in every PBE to ensure that best responses of Seller are well-defined. Finally, part 4) is a direct implication of the assumption that learning is one-shot, disclosure is verifiable and that this is common knowledge: if Buyer discloses, Seller knows her valuation will not change in the future and so asks for the full surplus. Parts 1) and 4) imply that Buyer with bad news has zero surplus in any PBE.

Lemma 1 has several important economic implications whose proofs are contained in the appendix. First, it implies that in every PBE the reservation prices are strictly increasing in the type of Buyer. Second, it implies that there are no quiet periods in any PBE, i.e. after any history in which Seller is called upon to play, the probability of agreement is positive.

The second auxiliary result concerns the disclosure decision.

**Lemma 2.** It holds true in all PBEs:

- Buyer has strict incentives not to disclose good news on path, whenever the PBE features a positive Buyer payoff
- Buyer is indifferent between disclosing or not disclosing bad news on path,
- there are no strict incentives to delay disclosure of bad news.

Intuitively, if Buyer has received good news she cannot have strict incentives to disclose because she hopes to get a price lower than her reservation price. If she receives bad news, she knows her continuation payoff is zero and thus is indifferent between disclosing and not disclosing.

### 2.0.1 Equilibrium refinements

I close this subsection by introducing several equilibrium refinements which are used in the rest of the paper.

**Refinement with respect to the disclosure decision.** Equilibria in which Buyer with bad news does not disclose with positive probability are not ‘robust’ to the introduction of some slightly more realistic features into the model. For instance, Seller may
have very small inventory costs for the good. Alternatively, he or Buyer may have small but positive overhead costs for continuing the bargaining, e.g. paperwork costs or costs for intermediaries for the communication or other meeting costs. In all of these cases the game ends once Buyer receives bad news. If Seller has small overhead costs for bargaining or inventory costs for the good, she ‘bribes’ Buyer with bad news into disclosing immediately by offering her a negligible surplus. If Buyer has small overhead costs for continuing the bargaining, she discloses bad news immediately to avoid future overhead costs.

In the following, I call an equilibrium a *disclosure equilibrium*, if it prescribes that Buyer who receives bad news on path discloses immediately and accepts with probability one the price offered subsequently by Seller.

**Stationarity.** As is usual for many dynamic bargaining games, I often focus the analysis on *stationary equilibria.*

Stationary equilibria satisfy the following properties.

i. Buyer’s on-path actions depend only on her current type and Seller’s current belief over Buyer types,

ii. Seller’s on-path actions depend only on his belief distribution over Buyer types,

iii. If off-path play leads to a Seller-belief that happens with positive probability on path, ensuing play of Seller follows his on-path strategies.

Under a very mild technical requirement, the online appendix shows that for every sequence of disclosure equilibria as $\Delta \to 0$ the beliefs at the start of each period converge to the degenerate distribution on $\hat{v}$, as long as bargaining goes on. This motivates requiring this property near the HFL as well.

A stationary equilibrium is called **strongly stationary**, if as long as bargaining goes on, Seller starts each period on path with belief concentrated on Buyer type $\hat{v}$. 19 Strongly stationary equilibria are analytically tractable and have intuitive closed-form solutions.

Note that Buyer with good news in a strongly stationary equilibrium never rejects her reservation price on path. In particular, under strong stationarity the game never continues past the period of the arrival of information. This bounds the delay across all strongly stationary equilibria with disclosure because it takes in expectation $\frac{1}{\lambda}$ in real time for the opportunity to learn new information to arrive.

Finally, in the rest of the paper I call an equilibrium a **strongly stationary equilibrium with mixed pricing** if it satisfies

i. Seller mixes after on-path histories,

ii. it is a disclosure equilibrium,

iii. it is strongly stationary.

I call an equilibrium a **strongly stationary equilibrium with pure pricing** it it satisfies ii. and iii. above and i. is replaced with

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19 Proposition 3 from subsection 2.1 below shows that there exist stationary equilibria which are not strongly stationary. Theorem 1, Propositions 6 and 7 show that strongly stationary equilibria exist near the HFL for both costless and costly learning.
Seller does not mix after on-path histories.

The main economic property of the strongly stationary equilibria characterized in the rest of the paper is the strategic option value of Buyer from learning. This results in non-extreme payoffs for both Buyer and Seller and non-extreme prices.

2.1 A benchmark: costless learning

I now consider the benchmark case of costless learning. Thus, intensity is fixed \( \lambda > 0 \) throughout and Buyer can acquire any level of accuracy \( a \) for free, whenever the opportunity arises. The first major implication of costless choice of accuracy is that in every PBE it is a (weakly) best response for Buyer to learn conclusively, whenever the opportunity to learn arises. Learning conclusively is necessarily a strict best response in any PBE in which Buyer payoff is positive. The proof of the following Proposition is in the online appendix.

Proposition 1. There does not exist any PBE with a private history \( h \) (either on- or off-path), such that all of the following conditions are fulfilled

i. Buyer is uncertain of the value of good,

ii. at the end of \( h \) Buyer has an opportunity to learn,

iii. with positive probability after \( h \) and after opportunity to learn, Buyer picks \( a < 1 \),

iv. Buyer has positive continuation payoff after picking \( a < 1 \).

This result gives a micro-foundation for the assumption of exogenous arrival of a one-shot and conclusive learning opportunity, thus answering Question 1 in the introduction. If the learning is costless, then this assumption is without loss of generality for any PBE in which Buyer has a positive payoff. Moreover, this result simplifies the analysis of the costless case in that, there are only two possibilities: either Buyer receives zero equilibrium payoff or her information acquisition decision on the equilibrium path is trivial, because she chooses to learn perfectly whenever she can.

The next result formalizes the idea, that the usual logic of sequential screening and the related intuition of the optimality of ‘screening down the demand curve’ (so-called Coasian dynamics) fail in this model. First, I define the concept of sequential screening of valuations in this set up.

Definition 2. Say that a PBE features sequential screening of valuations if and only if on path

- as long as there is no disclosure, Seller quotes a decreasing sequence of deterministic prices \( \{r_l, l \leq K\} \) \( (K \leq \infty) \) with \( r_1 \) a reservation price of Buyer with good news

- the sequence of Seller-beliefs \( \gamma_l \in \mathcal{P}([v, \bar{v}]), 2 \leq l \leq K \) that Seller entertains at the beginning of every period starting from the second, is strictly decreasing over time in the FOSD-sense.\(^{20}\)

\(^{20}\)At the start of the game there is no initial private information, i.e. \( \gamma_1(\hat{v}) = 1 \).
I first show that the logic of Seller pricing is enough to exclude disclosure PBEs when \( v > 0 \) if the period-length is small enough. The case \( v = 0 \) is more difficult to treat and I introduce additional assumptions.\(^{21}\) It is treated in detail in the appendix.\(^{22}\)

**Proposition 2.** Suppose accuracy is costless and that \( v > 0 \). Then there are no disclosure equilibria near the HFL in which Seller screens the valuations sequentially.

Recall that the economic rationale for equilibria in which Seller ‘screens down the demand curve’ is that of screening for the initial private information that Buyer might possess at the start of the game. When there is common knowledge of an initial distribution of valuations and Buyer cannot learn, the fact that bargaining continues can only be interpreted as indication of lower valuations.\(^{23}\) When the demand curve is endogenous and there is a disclosure decision, under the assumption that Buyer with bad news discloses immediately, Seller updates twice within a period as long as bargaining continues. He updates once from non-disclosure (an indication of higher valuations) and once from the rejection of a price (an indication of lower valuations). The first movement in beliefs is large enough as to neutralize the effect of the second, so that overall the classical sequential screening result fails.

### 2.1.1 Positive Buyer payoff only near the HFL

This subsection shows that when accuracy is costly, the mere possibility to learn does not ensure informational rents for Buyer, unless the frequency of interaction with Seller is high enough.

I first look for stationary equilibria in which Seller quotes \( p_H = \bar{v} \) with probability one every period on path, as long as there is no disclosure. Let this type of equilibrium be called a *stationary high-price equilibrium*. Buyer of type \( \bar{v} \) accepts \( p_t = \bar{v} \) with some probability \( q_1 \in (0, 1) \) at \( t = 1 \). This results upon rejection of \( p_H \) in positive probability of type \( \bar{v} \) at the beginning of period \( t = 2 \). Let this probability be \( \gamma \) and let \( q(\gamma) \) be the probability with which the type \( \bar{v} \) rejects \( p_H \) in \( t \geq 2 \).

Upon non-disclosure of \( v \) within a period, Seller updates her belief of Buyer with good news from \( \gamma \) to

\[
U(\gamma) = \frac{\gamma + (1 - \gamma)\mu\pi_0}{1 - (1 - \gamma)\mu(1 - \pi_0)}.
\]

This *interim* update is higher than \( \gamma \): there is a *positive selection effect*, because no disclosure is stronger indication that Buyer may have learned good news.

If Buyer of type \( \theta = \bar{v} \) accepts \( p_H \) with probability \( q \), Seller updates the belief of \( \bar{v} \) from \( U(\gamma) \) to

\[
B(U(\gamma), q) = \frac{U(\gamma)(1 - q)}{U(\gamma)(1 - q) + 1 - U(\gamma)}.
\]

\(^{21}\)In particular, I restrict to PBEs which are stationary and satisfy an equilibrium refinement called ‘divinity in bargaining’. See the appendix for more.

\(^{22}\)In the online appendix Proposition 2 is generalized to its ‘real-time’ counterpart: the sequential screening dynamics are allowed to start at some date \( T(\Delta) \geq 1 \) and so that \( T(\Delta)\Delta \to 0 \) as \( \Delta \to 0 \).

\(^{23}\)This belief updating logic underlies the traditional ‘Coasian dynamics’ result. See chapter 10 of *Fudenberg and Tirole (1991)* for more on Coasian dynamics.
This is also the belief with which Seller starts the new period. It is strictly lower than
the interim update $U(\gamma)$, because of a negative selection effect: rejection of a price is
indication of lower valuations.

The condition for stationary beliefs on path from $t = 2$ on is given by

$$B(U(\gamma), q(\gamma)) = \gamma.$$  \hfill (3)

Thus, the positive and negative selection effects balance out at $\gamma$, whenever the type $\bar{v}$ rejects the price $p_H = \bar{v}$ with probability $1 - q(\gamma)$.

In the following let $W(\gamma)$ denote the stationary payoff of Seller from $t = 2$ onwards in the stationary high-price equilibrium. This aggregates over time the profit Seller makes from the arrival of the type $\bar{v}$, conditional on her accepting the price and the profit she makes from the type $v$ who discloses immediately.

Given that type $\bar{v}$ discloses, the only possibly viable deviation for Seller is to ask for
the reservation price of Buyer who has not learned yet. The following necessary conditions
need to be satisfied for the stationary equilibrium.

$$U(0) \cdot q_1 \cdot \bar{v} + (1 - U(0)q_1) \delta W(\gamma) \geq \hat{v}, \quad \text{Seller-optimality at } t = 1,$$  \hfill (4)

and

$$U(\gamma)q(\gamma) \cdot \bar{v} + (1 - U(\gamma)q(\gamma))\delta W(\gamma) \geq \hat{v}, \quad \text{Seller-optimality at, } t \geq 2$$  \hfill (5)

with

$$\gamma = B(U(0), q_1).$$

$\gamma$ here gives the stationary belief at the beginning of periods $t \geq 2$. In the appendix I show the following result.

**Proposition 3.** Let $U(0) = \frac{\mu \pi_0}{1 - \mu + \mu \pi_0}$ with $\mu = 1 - e^{-\lambda \Delta}$, be the probability on the type $\bar{v}$ in the first period after no disclosure and $W = \frac{\mu}{1 - \delta + \delta \mu}(U(0)\bar{v} + (1 - U(0))\hat{v})$. Then whenever the parameters satisfy

$$(C - \text{high}) \quad U(0)\bar{v} + (1 - U(0))\delta W > \hat{v},$$

there exists a stationary high-price equilibrium. In this equilibrium Buyer payoff is zero
and Seller asks with probability one for $\bar{v}$ as long as the bargaining continues and there is
no disclosure.

For fixed other parameters of the game, $(C - \text{high})$ is always satisfied when $\lambda$ is high
enough.

The intuition for the existence of the stationary high-price equilibria is simple. When $\lambda$
is large Seller knows that with high probability Buyer will know the value of the good very
soon after bargaining starts. Non-disclosure is a strong indicator of good news whenever
$\lambda$ is large. Thus, Buyer with good news cannot successfully pool with Buyer who has not
learned yet. But the strategic option value from learning comes precisely from being able
to pool with type $\hat{v}$!

Stationary high-price equilibria from Proposition 3 do not survive near the HFL. The
intuition is that as $\Delta \to 0$ the probability that Buyer has learned before any fixed period
$K$ goes to zero as well, at a speed of $\Delta$. Thus, Seller cannot ask for the highest price
already at $t = 1$. The price should be lower than $\bar{v}$ with positive probability in the
first period, whenever $\Delta$ is small enough. But this implies that Buyer receives a positive information rent with positive probability already in the first period. This intuition does not depend on the assumption of stationarity. Therefore, more generally Buyer always ensures a positive payoff near the HFL.

**Proposition 4.** Fix all parameters of the game except for the period-length $\Delta$. There are no equilibria with zero Buyer payoff if $\Delta$ is small enough.

If an equilibrium has zero Buyer payoff it is necessarily a high-price equilibrium (albeit maybe not stationary): Seller quotes $\bar{v}$ as long as there is no disclosure and bargaining goes on. Otherwise Buyer waits until the first period that Seller quotes with positive probability a price lower than $\bar{v}$ to realize informational rents with positive probability. The same argument as above for stationary equilibria shows that Seller would do better by offering some positive information rent already in the first period, as period-length shrinks to zero.

Intuitively, Seller would like to commit to quoting prices less often as period-length shrinks, so that he can become relatively certain that Buyer has learned in the meanwhile and her willingness to pay has increased. When there is no commitment across periods Seller quotes a price every period and thus allows Buyer to achieve positive information rent already in the first period with positive probability.\footnote{Incidentally, Proposition 4 also shows that costs of information are not necessary for Buyer to ensure a positive equilibrium payoff, unless there are rational-inattention costs of processing information, as Ravid (2019) shows.}

### 2.1.2 Strongly stationary equilibria.

Next, I focus on strongly stationary equilibria in which Seller may or may not mix between prices on path upon non-disclosure. If he mixes on path, then he puts positive probability on two prices $p_H > p_L$. $p_H$ is the reservation price of Buyer who has learned good news, whereas $p_L$ of Buyer who has not learned yet. If Seller does not mix on path after non-disclosure, he necessarily quotes the price $p_L$. This follows from the requirements of strong stationarity and the refinement with respect to the disclosure decision.

Let $p \in [0, 1)$ be the probability with which $p_H$ is quoted upon non-disclosure. Suppose in the following, that in equilibrium Buyer of type $\hat{\nu}$, who has the option value, accepts her reservation price $p_L$ with some stationary probability $q \in [0, 1]$.

I look extensively at the HFL of sequences of strongly stationary equilibria.

**Definition 3.** Say that a sequence of strongly stationary equilibria indexed by period-length $\Delta > 0$ with $\Delta \to 0$ converges in the HFL, if as $\Delta \to 0$

1. the average price quoted by Seller upon non-disclosure $\hat{p}(\Delta)$ converges,
2. the sequence $q(\Delta)$ of acceptance probabilities of Buyer type $\hat{\nu}$ satisfies

$$\frac{q(\Delta)}{\Delta} \to \kappa,$$

for some $\kappa \in [0, \infty]$.

Say that a HFL corresponds to some $\kappa$ if there exists a sequence of strongly stationary equilibria such that $\frac{q(\Delta)}{\Delta}$ converges to $\kappa$ as $\Delta \to 0$.\footnote{Incidentally, Proposition 4 also shows that costs of information are not necessary for Buyer to ensure a positive equilibrium payoff, unless there are rational-inattention costs of processing information, as Ravid (2019) shows.}
Let $U(0) = \frac{\mu \pi_0}{1 - \mu + \mu \pi_0}$. This is the (stationary) on-path probability that Seller puts on the type $\bar{v}$ after non-disclosure. Denote by $V_\Delta(q, p)$ the stationary Seller-payoff in the equilibrium, if the period-length is $\Delta$.

Seller optimality upon non-disclosure in strongly stationary equilibria with pure pricing is ensured if

$$
(U(0) + (1 - U(0))q)p_L + (1 - U(0))(1 - q)\delta V_\Delta(q, 0) \geq U(0)p_H + (1 - U(0))\delta V_\Delta(q, 0) \quad (6)
$$

(6) ensures that deviating to $p_H$ is not profitable for Seller. Lemma 1 ensures that these are the only ‘relevant’ deviations for Seller on path.

Seller indifference upon non-disclosure in strongly stationary equilibria with mixed pricing is equivalent to

$$
U(0)p_H + (1 - U(0))\delta V_\Delta(q, p) = p_L(U(0) + (1 - U(0))q) + (1 - U(0))(1 - q)\delta V_\Delta(q, p) \quad (7)
$$

The left-hand side is the payoff from charging $p_H$ upon non-disclosure, whereas the right-hand side is the payoff from charging $p_L$ upon non-disclosure.

Part 2) of Lemma 1 implies the following relations for the reservation pricing of types $\bar{v}, \hat{v}$ in strongly stationary equilibria with mixing probability of Seller given by $p \in [0, 1)$.

$$
\hat{v} - p_H = \delta(p(\bar{v} - p_H) + (1 - p)(\bar{v} - p_L)) \quad (8)
$$

and

$$
\hat{v} - p_L = \frac{\mu \pi_0(\bar{v} - p_H) + \delta(1 - \mu)(1 - p)(\hat{v} - p_L)}{1 - \delta p(1 - \mu)} \quad (9)
$$

On the left-hand side of (8) and (9) is the payoff of the respective type if she decides to buy now when facing her reservation price, and on the right-hand side is the payoff if she decides to continue. Denote $\hat{p} = pp_H + (1 - p)p_L$ the average price quoted on path upon non-disclosure. (9) can be re-written with use of (8) as

$$
\hat{v} - p_L = \frac{\delta \mu}{1 - \delta + \delta \mu} \pi_0(\bar{v} - \hat{p})
$$

This depicts the option value of the type $\hat{v}$. Buyer has the option to wait for the opportunity to learn at which case she gets a payoff of $\bar{v} - \hat{p}$, if she learns good news and of zero, if she learns bad news. The payoff from exercising the option value is discounted by $\frac{\delta \mu}{1 - \delta + \delta \mu}$. This is the ‘effective discount rate’ for the option value, because the type $\hat{v}$ sometimes stops at price $p_L$ (payoff has weight $1 - \delta$) and otherwise continues next period in the hopes of getting the chance to exercise the option value (payoff has weight $\delta \mu$).

The next result characterizes strongly stationary equilibria near the HFL and gives a complete characterization of their convergence in HFL.

**Theorem 1.** 1) [Existence near the HFL] Strongly stationary equilibria with both pure and mixed pricing exist near the HFL.

2) [Uniqueness near the HFL] Near the HFL, every strongly stationary equilibrium with mixed pricing is unique up to the mixing probability $p \in (0, 1)$ of Seller.

Near the HFL, every strongly stationary equilibrium with pure pricing is unique up to the acceptance probability $q$ of Buyer of type $\hat{v}$.
3) [Delay in the HFL] There exists HFL of strongly stationary equilibria corresponding to any $\kappa \in [0, \infty]$. In any HFL of strongly stationary equilibria with mixed pricing $\kappa$ is 0 and positive expected delay is $\frac{1}{\lambda+\kappa}$.
In any HFL of strongly stationary equilibria with pure pricing $\kappa$ is in $(0, \infty)$ and expected delay is given by
$$
\begin{cases} 
\frac{1}{\lambda+\kappa}, & \text{if } \kappa \in (0, \infty), \\
0, & \text{if } \kappa = \infty.
\end{cases}
$$

4) [Pricing in the HFL] In any HFL of strongly stationary equilibria prices converge to
$$
\psi = \frac{rv + \lambda(1-\pi_0)v}{r + \lambda(1-\pi_0)}.
$$
In particular, there is no price spread in the HFL of equilibria with mixed pricing.

5) [Payoff and efficiency properties in the HFL] Buyer and Seller payoffs in any converging sequence of strongly stationary equilibria with $q(\Delta) \rightarrow \kappa$ are unique. Buyer and Seller payoffs lie in $(v, \hat{v})$ for all $\kappa \in [0, \infty]$.

The efficiency loss in the HFL of equilibrium sequences with $\kappa \in [0, \infty)$ is given by
$$
\frac{r}{r + \lambda + \kappa} \hat{v}.
$$
There is no efficiency loss in the HFL of sequences with $\kappa = \infty$.

Near the HFL, the mixing probability $p$ of Seller upon non-disclosure is a sufficient statistic for the construction of the equilibria with mixed pricing: whenever two strongly stationary equilibria with mixed pricing share the same mixing probability of Seller, they prescribe identical play on path.\(^{25}\) The Seller indifference condition pins down a unique $q$. Strongly stationary equilibria with pure pricing allow for a unique Seller mixing probability $p = 0$, but the acceptance probability for the reservation price of type $\hat{v}$ is determined only up to a lower bound.

The pricing in the HFL has a simple structure: Seller asks for a flat price $\psi$, unless he sees evidence that $\theta = v$ and subsequently revises price down to $v$. Buyer waits with some probability until she gets the information to end the game with either of the two prices. This is optimal due to two reasons. First, because $\psi$ is lower than $\hat{v}$, but not too low as to compensate for the option value from learning. Second, as $\Delta \rightarrow 0$ the loss due to impatience from waiting an additional period is small, whereas the option value from learning remains strictly positive as $\Delta \rightarrow 0$.

The price spread $p_H - p_L$ in equilibria with mixed pricing, which is due to Seller attempting to screen the types $\{\hat{v}, \bar{v}\}$, disappears as $\Delta$ becomes smaller and smaller. This shows that the inefficient delay near the HFL of such equilibria originates mostly from Buyer of type $\hat{v}$ waiting to realize her option value from learning, rather than Seller trying to screen the types $\{\hat{v}, \bar{v}\}$ upon non-disclosure.

\(^{25}\)Whenever discussing strongly stationary equilibria, I use the words a collection of variables are a sufficient statistic for the equilibrium in the sense they are used here, i.e. the equilibrium is unique within the class of strongly stationary equilibria, once the value of the variables in consideration is fixed.
ψ corresponds to the HFL of the reservation pricing of type \( \hat{v} \). This leads to the indeterminacy in expected delay in the HFL and implies that equilibrium multiplicity survives HFL. Depending on the acceptance probability, expected delay in the HFL can be any number in \([0, \frac{1}{\lambda}]\). The degree of inefficiency in the HFL of a strongly stationary equilibrium is characterized by the difference between the ex-ante surplus \( \hat{v} \) and the sum of Buyer and Seller payoff in the HFL. The last part of Theorem 1 shows that near the HFL there are strongly stationary equilibria with (necessarily) pure pricing that are arbitrarily close to efficiency. These correspond to cases in which \( \kappa \) is ‘very large’, i.e. the sequence of acceptance probabilities \( q(\Delta) \) falls relatively slowly on the scale of \( \Delta \).

In the HFL Seller pricing, and Buyer and Seller payoffs are non-extreme. In particular, even after accounting for losses from delay due to learning, Seller’s payoff is not minimized as in the classical seller-offer game of Fudenberg et al. (1985) and Gul et al. (1986).

Finally, the potential inefficiency from learning is ceteris paribus decreasing in the patience level of the players. This is because delay hurts more, the more impatient players are.

### 2.2 Discussion of alternative assumptions

To get a better sense of the economic factors driving the results in the costless case and beyond, it is instructive to consider variations in the model assumptions with the classical game from Fudenberg et al. (1985) and Gul et al. (1986) in mind.

Suppose for a moment that Buyer knows her true valuation before she approaches Seller and that this would be common knowledge. Thus, Buyer has initial private information. Suppose that Buyer can disclose her valuation verifiably. Similar to the results in the model of this paper, type \( g \) is indifferent in her disclosure decision. In equilibria in which she discloses immediately, the average prices will be approximately \( \bar{v} \) near the HFL unless there is disclosure. In equilibria in which she never discloses, the average prices are near \( \bar{v} \) near the HFL. In either case, one bargaining party receives all the surplus and prices are extreme near the HFL, just as in the traditional game. Therefore, in my model it is the assumption of private arrival of information that ensures non-extreme equilibrium payoffs and prices, whenever the assumption of verifiable disclosure is maintained.

Suppose alternatively, that types are endogenous as in the model of this paper, but that communication between Buyer and Seller is impossible. I conjecture that in this case the equilibria are again extreme near the HFL, in that they either have extreme payoffs or equilibrium play ultimately exhibits extreme prices. In contrast, the combination of verifiable disclosure and stochastic evolution of types in my model enables equilibria near the HFL which exhibit neither extremeness of payoffs nor extremeness of prices.

### 3 Costly learning

This section introduces costs for accuracy. I consider two distinct models of accuracy costs and always assume that parametric forms on costs are common knowledge. In the first case, costs are deterministic and marginal costs of picking a higher accuracy are positive. I assume that acquiring an experiment of very low accuracy costs very little. In the second case, the costs are stochastic and independent of accuracy. Thus, the second model is one of fixed costs of learning. I assume in this second case, that arbitrarily low
costs have positive probability. Formally, the assumptions are as follows.

**Deterministic variable costs of accuracy.** The experiment $E_a$ with informativeness $I(a) = \frac{a}{1-a}$ costs $c(I(a))$ with $c : [1, \infty) \rightarrow \mathbb{R}_+$ satisfying

- $c(1) = 0$ and $c'(1) = 0$
- $c$ is strictly convex and increasing
- $\lim_{I \rightarrow \infty} c'(I) = +\infty$.

It is easy to see that in the case of deterministic variable costs, in any PBE, Buyer always learns whenever she gets the chance, be it even by a bit, provided the option value from learning is strictly positive. This follows from the assumption that $c'(1) = 0$, i.e. an experiment close to uninformative costs almost nothing. It follows that the learning rate and the intensity $\lambda$ are the same for deterministic variable costs of accuracy, in any equilibrium in which Buyer has positive option value from learning.

**Stochastic fixed costs of accuracy.** Conditional on an opportunity to learn having arrived and independent of everything else, a fixed cost $c \in (0, \infty)$ is drawn, which is distributed according to a distribution $F$. If Buyer pays $c$, she can verify the state (equivalently: can pick $a = 1$) at no additional cost. Otherwise she can wait for future draws. $F$ satisfies the following requirements.

- $F$ is continuous and has finite first moment
- (Possibility of arbitrarily low costs) $F$ puts positive probability to a neighborhood of zero.

Because of the lump-sum nature of the stochastic costs, and the fact that Buyer can always wait for lower cost draws, whenever the current cost draw is too high, the rate at which the agent learns becomes endogenous and distinct from the rate of arrival of opportunities to learn, given by the intensity parameter $\lambda$. To avoid confusion in notation, I use the definition $\mu_0 = 1 - e^{-\lambda \Delta}$ for the case of stochastic fixed costs only, for the probability of the arrival of the opportunity to learn within a period (intensity) and keep the notation $\mu$ for the probability with which Buyer actually learns within a period in a stationary equilibrium. The latter is now endogenous.

Despite their significant differences, broadly speaking the same set of results turns out to be true for both of the models of costly learning. The first result establishes that costs ensure a positive Buyer payoff in every PBE. This is true for any $\Delta$, in contrast to the case of costless learning, in which, for every $\Delta > 0$ one can construct PBEs with zero Buyer payoff.

**Theorem 2.** Pick any $\Delta > 0$. If learning is costly, every PBE has a positive Buyer payoff.

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26 An alternative and for-all-purposes-equivalent assumption is that $F$ is continuous, has bounded support contained in $[0, +\infty)$.

27 Recall that in the case of deterministic variable costs these are the same.

28 It is easily established that this object is time-stationary in a stationary equilibrium.
**Proof-sketch.** Fix a $\Delta > 0$.

A PBE with zero Buyer payoff can only happen if on path Seller erases the option value from learning. He can only do this by quoting, after every public history when it is his turn to move, a price equal to the reservation price of the highest Buyer type she deems feasible at that moment in the game. But if there is no option value from learning Buyer strictly prefers not learn on path, because learning is costly. If Buyer does not to learn on path, then the best response of Seller is to ask for $\hat{v}$, after every public history, as long as there is no agreement.

Suppose this is the case and consider first the model with deterministic variable costs. After every private history which ends with the arrival of an opportunity to learn, Buyer does want to learn. This is because learning very little costs very little (recall $c'(1) = 0$), whereas the benefit from learning is an order of magnitude larger than the increase in marginal costs. Since it can happen with positive probability that Buyer receives the opportunity to learn in every period that the game goes on and she has not learned before (in particular, also in the first period), such a strategy would give Buyer positive payoff with positive probability. This is a contradiction.

Consider next the model with stochastic fixed costs. In this case, the option value from learning is strictly positive (at least as large as $\pi_0(\hat{v} - \hat{v})$). Again, since the opportunity to learn arrives with positive probability every period that the bargaining goes on, and the probability that the cost draw is below $\pi_0(\hat{v} - \hat{v})$ is strictly positive (due to the assumption of the possibility of arbitrarily low costs), the same argument as in the case of deterministic variable costs leads to a contradiction.

The intuition for this result is surprisingly simple. The surplus may change only through Buyer-learning. Because the learning is private information, Seller can only gives incentives to Buyer to learn on average, and not conditional on every realized learning outcome. Because of the discretionary nature of the information disclosure decision, this creates informational rents for Buyer. That the proof works for any $\Delta > 0$, depends crucially on the assumption of zero marginal costs for experiments close to uninformative for the case of deterministic variable costs and the assumption of the possibility of arbitrarily low costs in the case of stochastic fixed costs.

Theorem 2 complements Proposition 4 in Ravid (2019), because it exhibits another situation in which costs of information (in this case of production, rather than information processing costs) ensure a positive payoff across PBEs. In both models learning creates surplus because it creates private information for Buyer. Positive Buyer payoff does not come from initial private information as in the classical setting of Fudenberg et al. (1985) and Gul et al. (1986), because Coasian forces are absent. The positive buyer payoff comes instead from the fact that learning is private and costly.

Finally, another implication of the proof of Theorem 2 is that there are no PBEs with costly learning in which Buyer chooses not to learn with probability one, whenever the opportunity to learn comes.\(^{29}\) This implies that the no-sequential-screening-of-valuations result from the costless case extends to the case of deterministic variable costs. This is true without additional assumptions even when $v = 0$, because any learning event in the case of deterministic variable costs leads to an updated valuation $w > 0$ (bad news is never conclusive). Moreover, by adapting the proof of Proposition 2 the same result can

\(^{29}\)See Corollary 3 in the appendix.
be shown to hold for the case of stochastic fixed costs with \( v > 0 \). Summarizing, one has the following extension of Proposition 2.

**Proposition 5.** There are no disclosure equilibria near the HFL in which Seller screens the valuations sequentially, in the case of
- deterministic variable costs on accuracy
- stochastic fixed costs of accuracy with \( v > 0 \).

This extension is not surprising, because the proof of Proposition 2 only relies on the logic of Seller-pricing: as the period-length shrinks, the probability that Buyer has learned before a fixed finite date vanishes. Thus, as the length of the period shrinks, there is no common knowledge of a date in which the private information of Buyer is present and Seller can start the sequential screening.

### 3.1 Strongly stationary equilibria with accuracy costs

I construct the same type of strongly stationary equilibria as in the costless case. For strongly stationary equilibria with mixed pricing and costly learning, the sufficient statistic for the construction of the equilibria is the average price upon non-disclosure \( \hat{p} = pp_H + (1 - p)p_L \).

**The case of deterministic variable costs of accuracy.** Suppose the stationary valuation of Buyer with good news is given by \( \bar{w} \), whereas the stationary valuation of Buyer with bad news is given by \( \underline{w} \). It holds

\[
\bar{w} < \underline{w} < \hat{v} < \bar{v}.
\]

The option value from information acquisition is a function of the pair \((a, \hat{p})\):

\[
V_A(a, \hat{p}) = \pi_0 a \bar{v} + (1 - \pi_0)(1 - a)\underline{v} - (\pi_0 a + (1 - \pi_0)(1 - a))\hat{p} = (\pi_0 a + (1 - \pi_0)(1 - a))(\bar{w} - \hat{p}).
\]

When the opportunity to learn arrives, Buyer learns and ends the bargaining in the same period. She discloses bad news to get the lower price \( \underline{w} \), and does not disclose good news in which case she pays in expectation \( \hat{p} \) to Seller.

Let the accuracy chosen on path be \( a \). The valuations \( \bar{w}, \underline{w} \) are given by

\[
\bar{w}(a) = \frac{a \pi_0 \bar{v} + (1 - a)(1 - \pi_0)\underline{v}}{a \pi_0 + (1 - a)(1 - \pi_0)}, \quad \underline{w}(a) = \frac{a \pi_0 \bar{v} + a(1 - \pi_0)\underline{v}}{(1 - a)\pi_0 + a(1 - \pi_0)}.
\]

Whenever the opportunity to learn arrives, optimal learning results in the following two incentive constraints.

\[
(OL - intensive) \quad a \in \arg \max_a \{V_A(\bar{a}, \hat{p}) - c(I(\bar{a}))\}, \quad (10)
\]

and

\[
(OL - extensive) \quad V_A(a, \hat{p}) - c(I(a)) \geq \hat{v} - p_L. \quad (11)
\]
OL stands for ‘optimal learning’. Incentive constraint (10) refers to the intensive margin of the learning decision (i.e. how accurate a signal to acquire), whereas (11) to the extensive margin of the learning decision (i.e. whether to acquire a costly signal).

The reservation prices of Buyer with good news \( \bar{w} \) and of Buyer who has not learned yet are given by

\[
\bar{w}(a) - p_H = \delta(\bar{w}(a) - \hat{p}),
\]

and

\[
\hat{v} - p_L = \frac{\delta \mu}{1 - \delta + \delta \mu} \left( V_A(a, \hat{p}) - c(I(a)) \right).
\]

I denote by

\[
BL(\hat{p}) = V_A(a(\hat{p}), \hat{p}) - c(I(a(\hat{p}))),
\]

the endogenous benefit from learning in the stationary equilibrium with sufficient statistic \( \hat{p} \).

Seller indifference condition remains the same as in (7). Note that because \( \hat{v} \geq p_L \), OL-intensive and the reservation pricing for Buyer of type \( \hat{v} \) imply immediately that OL-extensive is satisfied. Therefore, this constraint can be dropped in the following w.l.o.g. OL-intensive leads to the first-order condition

\[
\pi_0(\bar{v} - \hat{p}) + (1 - \pi_0)(\hat{p} - \bar{v}) = c' \left( \frac{a}{1 - a} \right) \frac{1}{(1 - a)^2}.
\]

This determines uniquely the optimal accuracy \( a(\hat{p}) \) and with it, also the rest of the variables of the equilibrium, except for the acceptance probability \( q \) of the type \( \hat{v} \) for the price \( p_L \). \( q \) is determined uniquely by Seller’s indifference condition in equilibria with mixed pricing and is determined only up to a lower bound in equilibria with pure pricing.

The case of stochastic fixed costs of accuracy. When costs of accuracy are lump-sum but stochastic, there is no intensive margin for the learning decision: Buyer learns perfectly whenever she pays the costs. The extensive margin decision is explained as follows.

Let \( V_N \) be the continuation utility for Buyer of type \( \hat{v} \). Upon non-disclosure, with probability \( p \) she faces a price of \( p_H \) which she rejects with probability one and gets the continuation payoff \( \delta \hat{V} \), where \( \hat{V} \) is the continuation payoff of starting a period in the stationary equilibrium as type \( \hat{v} \). Due to reservation pricing and Seller’s belief dynamics on path, it holds

\[
\delta \hat{V} = \hat{v} - p_L.
\]

On the other hand, with probability \( 1 - p \) type \( \hat{v} \) faces price \( p_L \) and has continuation payoff \( \hat{v} - p_L \). Overall, it follows \( V_N = \hat{v} - p_L \). Let \( V_A \) be the continuation utility if Buyer

\[\text{This follows from some algebra, starting with the reservation price relation for type } \hat{v}:\]

\[
\hat{v} - p_L = \delta(\mu \pi_0 a + (1 - \pi_0)(1 - a))(\bar{w} - pp_H - (1 - p)p_L) - \mu c(I(a)) + (1 - \mu)\delta \hat{V_L} + (1 - \mu)(1 - p)(\hat{v} - p_L),
\]

where \( V_L \) is the continuation payoff of the type \( \hat{v} \) when she starts a new period in the stationary equilibrium. Due to reservation pricing it holds \( \delta \hat{V_L} = \hat{v} - p_L \).

\[\text{Another way to see that } OL\text{-extensive is redundant is to combine Corollary 3 from the appendix and use the assumed stationarity of the PBE.}\]
learns, with the learning costs not yet subtracted. With probability \( \pi_0 \) Buyer becomes the high type \( \bar{v} \) and so receives continuation utility \( p(\bar{v} - p_H) + (1 - p)(\bar{v} - p_L) = \bar{v} - \hat{p} \). With probability \( 1 - \pi_0 \) Buyer becomes type \( \underline{v} \), discloses immediately, receives a payoff of zero and the game ends. It follows \( V_A = \pi_0(\bar{v} - \hat{p}) \).

The costs \( c \) are worth paying if and only if
\[
c \leq V_A - V_N,
\]
that is, if and only if they are low enough. In particular, the stationary probability \( \mu \) that Buyer of type \( \hat{v} \) learns within a period is given by
\[
\mu = \mu_0 F(\pi_0(\bar{v} - \hat{p}) - \hat{v} + p_L).
\] (14)

In the following let \( \bar{\mu}(\hat{p}, \Delta) = F(\pi_0(\bar{v} - \hat{p}) - (\hat{v} - p_L(\hat{p}, \Delta))) \) be the probability of incurring the costs, conditional on the opportunity to learn having arrived. Denote also \( \bar{\mu}(\hat{p}) = F(\pi_0(\bar{v} - \hat{p}) - (\hat{v} - \bar{p}_L(\hat{p}))) \) for any HFL of \( \bar{\mu}(\hat{p}, \Delta) \) as \( \Delta \to 0 \). In difference to the case of deterministic variable costs, \( \bar{\mu} \) is equilibrium-dependent and different from the probability of learning \( \mu \).

Seller indifference condition and the reservation price relation for type \( \bar{v} \) are the same as in the case of costless learning (namely, formally the same as in (7) and (8)), with the major difference that now \( \mu \) is endogenously determined in equilibrium. Reservation pricing for the type \( \hat{v} \) leads to
\[
\hat{v} - p_L = \frac{\delta \mu}{1 - \delta + \delta \mu} BL(\hat{p}),
\]
with the endogenous benefit of learning \( BL(\hat{p}) \) given by
\[
BL(\hat{p}) = \pi_0(\bar{v} - \hat{p}) - \mathbb{E}[c|c \leq V_A - V_N].
\] (15)

The option value from the costless case, given by \( \pi_0(\bar{v} - \hat{p}) \) is reduced in (15) by the expected costs of learning, conditional on the event that learning occurs.

The following Proposition establishes existence of strongly stationary equilibria near the HFL.

**Proposition 6.** Pick any \( \pi_0, \underline{v}, \bar{v} \) and \( \lambda, r \). In both cases of accuracy costs the following holds.

1) [Existence near the HFL] Strongly stationary equilibria with pure pricing always exist near the HFL.

There exists an open neighborhood \( \mathcal{N} \) of \( \hat{v} \) such that strongly stationary equilibria with mixed pricing and average price upon non-disclosure \( \hat{p} \in \mathcal{N} \) exist near the HFL, whenever the following condition is satisfied.

\[
(P) \quad r > \lambda \text{ if } \pi_0 \leq \frac{1}{2} \text{ or } r > \sqrt{2} \lambda \text{ if } \pi_0 > \frac{1}{2}.
\]

2) [Uniqueness near the HFL] For any fixed average price \( \hat{p} \in \mathcal{N} \) the quantities \( p_L(\hat{p}, \Delta), p_H(\hat{p}, \Delta), p(\hat{p}, \Delta), q(\Delta, \hat{p}) \) are uniquely determined in every strongly stationary equilibrium with mixed pricing.

Pure pricing equilibria are unique up to the acceptance probability \( q \) of Buyer of type \( \hat{v} \).
In the case of deterministic variable costs, \( \alpha(\hat{p}) \) is unique and decreasing in \( \hat{p} \) if \( \pi_0 > \frac{1}{2} \), increasing in \( \hat{p} \) if \( \pi_0 < \frac{1}{2} \) and independent of \( \hat{p} \) if \( \pi_0 = \frac{1}{2} \).

In the case of mixed pricing the acceptance probability \( q(\Delta, \hat{p}) \) is unique.

The condition (P) \( r > \lambda \) if \( \pi_0 \leq \frac{1}{2} \) or \( \sqrt{2} < \frac{\pi_0}{\lambda} \) if \( \pi_0 > \frac{1}{2} \) for the existence of mixed pricing equilibria are used in the proof to show existence of the mixing probability \( q(\hat{p}, \Delta) \) of the type \( \hat{v} \), whenever \( \Delta \) is small enough. (P) is not minimal (see the proof of Proposition 6 in the appendix for more on this), but it does not depend on the precise parametric specification of costs. Namely, it ensures existence of mixed pricing equilibria for any cost function \( c \) in the deterministic variable case and for any \( F \) in the stochastic fixed case, as long as these satisfy the original assumptions at the beginning of this section. (P) requires that the intensity is not too high compared to the impatience level of the players.\(^{32} \)

The sufficient statistic for the construction of strongly stationary equilibria with mixed pricing is the average price \( \hat{p} \). In the case of pure pricing there is one additional degree of freedom: the acceptance probability \( q \) for type \( \hat{v} \).

The next result gives the HFL characterization of the strongly stationary equilibria with accuracy costs.

**Theorem 3.** Pick any \( \pi_0, v, \bar{v} \) and \( r, \lambda \).

1) [Existence in the HFL] There exists HFL of strongly stationary equilibria with pure pricing corresponding to any \( \kappa \in [0, \infty) \).

Let condition (P) from Proposition 6 be satisfied and \( N \) as in Proposition 6. For every \( \hat{p} \in N \) with \( N \) as in Proposition 6 there exists a sequence of strongly stationary equilibria with mixed pricing such that the sequence of average prices \( \hat{p}(\Delta) \) along the sequence converges to \( \hat{p} \).

2) [Delay in the HFL] In any HFL of strongly stationary equilibria expected delay is given by

\[
\begin{cases} 
\frac{1}{\lambda + \kappa}, & \text{if } \kappa \in [0, \infty) \text{ and accuracy costs are deterministic and variable}, \\
\frac{1}{\lambda \bar{\mu}(\hat{p}) + \kappa}, & \text{if } \kappa \in [0, \infty) \text{ and accuracy costs are stochastic and fixed}, \\
0, & \text{if } \kappa = \infty.
\end{cases}
\]

3) [Pricing in the HFL] In both cases of accuracy costs the price spread in the HFL of a sequence of strongly stationary equilibria with mixed pricing is bounded away from zero, and the low price is charged with vanishingly small probability.

4) [Payoff and efficiency properties in the HFL] Buyer and Seller payoffs in any converging sequence of strongly stationary equilibria with \( \frac{\alpha(\Delta)}{\Delta} \rightarrow \kappa \) are unique. Buyer and Seller payoffs lie in \( (\bar{v}, \hat{v}) \) for all \( \kappa \in [0, \infty) \).

The efficiency loss in the HFL of equilibrium sequences with \( \kappa \in [0, \infty) \) is given by

\[
\frac{r}{r + \lambda + \kappa} \hat{v} + \frac{\lambda}{r + \lambda + \kappa} c(I(a(\hat{p}))),
\]

in the case of deterministic variable costs of accuracy. It is given by

\[
\frac{r}{r + \lambda \bar{\mu}(\hat{p}) + \kappa} \hat{v} + \frac{\lambda \bar{\mu}(\hat{p})}{r + \lambda \bar{\mu}(\hat{p}) + \kappa} \mathbb{E}_F[c | c \leq \pi_0(\bar{v} - \hat{p}) - (\hat{v} - \bar{p}_L(\hat{p}))],
\]

\(^{32}\)In section 4.1 the choice of \( \lambda \) is endogenous and costly, so that these parametric assumptions can be transferred to corresponding assumptions on the cost of choosing the intensity \( \lambda \).
in the case of stochastic fixed costs of accuracy.

There is no efficiency loss in the HFL of sequences with $\kappa = \infty$.

Theorem 3 showcases the differences as well as the commonalities between the cases of costless and costly learning.

First, the expected delay in the HFL of all $\kappa$ but $\kappa = \infty$ is positive in both cases. It equals that of the case of costless information in the case of deterministic variable accuracy costs. It becomes equilibrium-dependent in the case of stochastic fixed accuracy costs, because the rate of learning diverges from the rate of arrival of opportunities to learn.

Second, in the case of costly learning the price spread does not disappear for equilibria with mixed pricing. The reason for this is that it is necessary even in the limit to subsidize the information costs incurred with positive probability for any $\Delta > 0$. For $\Delta$ positive but small, Seller promises to occasionally charge a low price so that Buyer has incentives to learn whenever the opportunity arrives (gets the high value good at a bargain). Since the probability to learn in a single period is very low when $\Delta$ is very small, Seller promises the low price within a period less and less often as $\Delta$ vanishes. Overall, in the HFL of mixed pricing equilibria the reason for the delay is the same as in the costless case: Buyer waits to realize the option value associated with learning new information.

Third, the price distribution of Seller in mixed pricing equilibria converges in the HFL to the reservation price of Buyer with good news, so that in the HFL of mixed pricing equilibria Buyer who has not learned yet waits until she can acquire new information. This leads to maximal expected delay in the HFL. In the HFL of pure pricing equilibria Seller incentivizes learning by offering the reservation price of Buyer who has not learned yet. Just as in the case of costless learning this this allows for multiplicity in expected delay in the HFL. In particular, near the HFL there are strongly stationary equilibria with (necessarily) pure pricing that are arbitrarily close to efficiency. This holds true despite the fact that in every PBE with costly learning Buyer learns with positive probability on path! The reason is that along a sequence of strongly stationary equilibria which converge to efficiency as $\Delta$ shrinks to zero, learning happens less and less often and Buyer of type $\hat{v}$ accepts her reservation price with higher and higher probability.

Fourth, the first part of both (16) and (17) quantifies how much of the ex-ante surplus is wasted whenever there is delay in the HFL. The second term of the sum in (16) and (17) quantifies the loss due to costly learning. The share of the ‘pie’ lost due to delay does not depend on Seller pricing in the model with deterministic variable costs, but it does so in the model with stochastic fixed costs. In the costless case, less patient players waste more of the overall gains from trade because waiting hurts more. If learning is costly, there are two effects whenever players become more impatient and equilibrium features inefficient delay. First, delay until agreement hurts more and so leads to higher inefficiency. This is the same effect as in the costless case. Second, a more impatient Buyer discounts learning costs more, which ceteris paribus should lower inefficiency. Since the costs of learning incurred in any equilibrium are less than the ex-ante surplus $\hat{v}$, the first effect dominates, so that inefficiency falls with the patience level even when learning is costly.

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33 The proof shows that in general, the HFL of mixed pricing equilibria cannot correspond to HFL of pure pricing equilibria. Therefore, the two cases must be treated separately also in the HFL.

34 To see that costs of learning on path are necessarily lower than $\hat{v}$, note that they are lower than the option value from learning, which in turn is always less than the ex-ante surplus $\hat{v}$. This is straightforward

28
Fifth, the inefficiency in the HFL is always decreasing in $\kappa > 0$. This is intuitive: a larger and positive $\kappa$ means that Buyer of type $\hat{v}$ waits longer in real time to realize her strategic option, even though she may not be worse off by accepting the current price offer of Seller. In the case of mixed pricing it holds $\kappa = 0$, because Seller quotes the reservation price of Buyer with good news as long as there is no disclosure.

Overall, just as in the costless case in the HFL Seller pricing, and Buyer and Seller payoffs are non-extreme.

4 Extensions

4.1 Costly choice of intensity

I now introduce costly choice of intensity and thus endogenize the intensity choice of Buyer. Exploration/search and exploitation are very different aspects of endogenous learning. Typically, an agent decides first how actively she is going to search for new information sources and only after, how extensively she is going to exploit the ones she has found. This distinction bears out in the results of this section because these two aspects of learning behave differently in the HFL of strongly stationary equilibria. Here I focus on the case of costly choice of intensity. In the online appendix I also consider the case of costless choice of intensity.\(^{35}\) Formally, costs on intensity are modeled as follows.

**Costs on intensity.** At the beginning of every period $t \geq 1$ Buyer picks the probability $\mu$ that a learning opportunity arrives within that period at the cost $C(\Delta, \mu)$. The following is satisfied for $C$:

- the function $C : (0, \infty) \times [0, 1) \rightarrow \mathbb{R}_+$ is differentiable
- it satisfies
  \[
  \lim_{\Delta \to 0} \frac{C(\Delta, \lambda \Delta)}{\Delta} = f(\lambda), \quad \lim_{\Delta \to 0} \frac{\partial}{\partial \mu} C(\Delta, \Delta \lambda) = f'(\lambda) \quad (18)
  \]
  uniformly on $\lambda > 0$, with $f : [0, \infty) \rightarrow \mathbb{R}_+$ differentiable, strictly increasing and convex with
  - $f(0) = f'(0) = 0$ and
  - $\lim_{\lambda \to \infty} f'(\lambda) = +\infty$.

The second requirement states that $C$ scales appropriately with time: as period-length goes to zero, both absolute and marginal costs of picking intensity go to zero on the same scale as the period-length. $f$ is therefore the cost of intensity in real time. An example that satisfies the conditions is $C(\Delta, \mu) = \Delta \cdot f(\mu)$ for $f$ satisfying the properties above.

\(^{35}\)The results on costless choice of intensity have some implications for the classical result of generic uniqueness of sequential equilibria in the traditional seller-offer, weak gap game from Fudenberg et al. (1985) and Gul et al. (1986). Namely, the generic uniqueness result for Coasian dynamics established in Fudenberg et al. (1985) and Gul et al. (1986) depends crucially on the assumption that Buyer *can commit* to having private initial information, before she approaches Seller.
All general insights from section 3 apply to the set up with costs on intensity. In particular, all equilibria feature positive Buyer payoff for every $\Delta > 0$ and there exist strongly stationary equilibria near the HFL. In the HFL of strongly stationary equilibria with mixed pricing the price spread does not disappear, but the low price quoted upon non-disclosure is quoted only with vanishing probability. Moreover, despite costly learning, there are near the HFL strongly stationary equilibria (necessarily with pure pricing) that are arbitrarily close to efficiency. The intuitions are the same as in the case of exogenous intensity.

The condition (P) for the existence of strongly stationary equilibria with mixed pricing in the statement of Proposition 6 can now be transferred to assumptions that involve the real-time cost of intensity $f$. The new existence conditions look as follows.

\[(P') \quad \text{if } \pi_0 \leq \frac{1}{2}, \text{ then } f'(r) > \frac{1}{2} \pi_0 \bar{v}, \quad \text{if } \pi_0 > \frac{1}{2}, \text{ then } f' \left( \frac{1}{\sqrt{2}} r \right) > \frac{\sqrt{2}}{\sqrt{2} + 1} \pi_0 \bar{v}. \]  

\[(P') \text{ assumes that acquiring intensity is not too cheap.}^{36} \text{ Just as in the case of exogenous intensity, the average price upon non-disclosure } \hat{p} \text{ is a sufficient statistic for the construction of the equilibria. Analogously, in the case of pure pricing, equilibria are unique up to the price quoted by seller upon non-disclosure and the acceptance probability of Buyer of type } \hat{v}.\]

At the beginning of every period in a strongly stationary equilibrium, Buyer picks the intensity $\lambda$. She trades off the physical costs with the benefit from learning. Because the benefit of learning is stationary and not influenced by the acceptance probability $q$ of type $\hat{v}$, so is the intensity choice in any strongly stationary equilibrium. This implies that the average price upon non-disclosure is a sufficient statistic for the information acquisition choice of Buyer in strongly stationary equilibria.

The first-order condition for the choice of intensity in the HFL of deterministic variable costs is given by

\[f'(\hat{\lambda}) \frac{r + \lambda}{r} = BL(\hat{p}), \]  

where $BL(\hat{p})$ is the stationary benefit from learning defined in (13). For the case of stochastic fixed costs the respective intensity choice is characterized by the first-order condition

\[f'(\hat{\lambda}) \frac{r + \lambda \bar{\mu}(\hat{p})}{r \bar{\mu}(\hat{p})} = BL(\hat{p}), \]  

where $BL(\hat{p})$ is the stationary benefit from learning defined in (15) and $\bar{\mu}(\hat{p}) = F(\pi_0(\bar{v} - \hat{p}) - (\hat{v} - \bar{p}_L(\hat{p})))$ is the HFL of the stationary probability that the state is verified by Buyer, conditional on the opportunity to learn having arrived. Here, $\bar{p}_L(\hat{p}) = \lim_{\Delta \to 0} p_L(\hat{p}, \Delta)$ is the HFL of the reservation price of the type $\hat{v}$.

In both cases of costs on accuracy the endogenous intensity $\lambda(\hat{p})$ is strictly decreasing in $\hat{p}$. In the HFL of the case of deterministic variable costs, the marginal costs of picking the intensity $f'(\lambda) \frac{r + \lambda}{r}$ do not depend on the pricing of Seller, whereas in the model with

\[^{36}\text{The proof of existence is contained in section (C) of the appendix. Here I focus only on the analysis in the HFL.}\]
stochastic fixed costs the marginal costs of picking the intensity \( f'(\lambda) \frac{r + \lambda \hat{\mu}(\hat{p})}{\hat{\mu}(\hat{p})} \) depend on the average price \( \hat{p} \). This is because the rate of arrival of opportunities to learn and the rate of learning coincide in the model with deterministic variable costs, but they diverge in the model with stochastic fixed costs. Recall that in the latter case, the probability of learning conditional on having the opportunity to learn is an endogenous object.

The following Proposition summarizes some of the interesting results from the analysis in the HFL. I focus only on delay and payoff properties in the HFL for brevity’s sake.

**Proposition 7.** Let \( \lambda(\hat{p}) \) satisfy (20) in the case of deterministic variable costs and (21) in the case of stochastic fixed costs.

1) [Delay in the HFL] In any HFL of strongly stationary equilibria expected delay is given by

\[
\begin{cases}
\frac{1}{\lambda(\hat{p})} + \kappa, & \text{if } \kappa \in [0, \infty), \hat{p} \in \mathcal{N} \text{ and accuracy costs are deterministic and variable}, \\
\frac{1}{\lambda(\hat{p})\hat{\mu}(\hat{p})} + \kappa, & \text{if } \kappa \in [0, \infty), \hat{p} \in \mathcal{N} \text{ and accuracy costs are stochastic and fixed}, \\
0, & \text{if } \kappa = \infty.
\end{cases}
\]

2) [Payoff and efficiency properties in the HFL] Buyer and Seller payoffs lie in \((v, \hat{v})\) for all \( \kappa \in [0, \infty] \).

The efficiency loss in the HFL of equilibrium sequences with \( \kappa \in [0, \infty) \) is given by

\[
\frac{r}{r + \lambda(\hat{p})} \hat{v} + \frac{\lambda(\hat{p})}{r + \lambda(\hat{p}) + \kappa} c(I(a(\hat{p}))) + \frac{f(\lambda(\hat{p}))}{r + \lambda(\hat{p}) + \kappa},
\]

in the case of deterministic variable costs of accuracy. It is given by

\[
\frac{r}{r + \lambda(\hat{p})\hat{\mu}(\hat{p})} \hat{v} + \frac{\lambda(\hat{p})\mu(\hat{p})}{r + \lambda(\hat{p})\hat{\mu}(\hat{p}) + \kappa} \mathbb{E}_F[c \leq \pi_0(\hat{v} - \hat{p}) - (\hat{v} - \bar{p}_L(\hat{p}))] + \frac{f(\lambda(\hat{p}))}{r + \lambda(\hat{p}) + \kappa},
\]

in the case of stochastic fixed costs of accuracy.

There is no efficiency loss in the HFL of sequences with \( \kappa = \infty \).

There are two main differences between (22) and (16), as well as (23) and (17). First, for fixed \( \kappa < \infty \) the share of the ex-ante surplus that is lost due to the delay until agreement is endogenous in the case of deterministic variable accuracy costs. It also depends on Seller-pricing. This is because the rate of learning is now a strategic variable for Buyer in the case of deterministic variable costs. Second, there is a third term in the sum of inefficiencies that quantifies the additional welfare loss due to the fact that intensity is costly for Buyer. In the HFL, the decision on accuracy \( a \) in the case of deterministic variable costs and the decision whether to verify the state in the case of stochastic fixed costs do not depend directly on the arrival rate \( \lambda \). Therefore, for fixed deterministic variable cost \( c \) or distribution of stochastic lump-sum cost \( F \), as costs of intensity \( f \) converge to zero uniformly the inefficiency due to delay and due to costly choice of intensity disappears, because \( \lambda(\hat{p}) \) becomes arbitrarily large. But the inefficiency due to costly accuracy persists.
4.1.1 Comparative statics for information acquisition

In a learning model in which intensity and accuracy are endogenous it is natural to ask the question of how these two distinct aspects of learning behave across ex-ante different environments. The Proposition in this subsection delivers an answer to this question for the quantities $a, \lambda$ in the HFL. Despite the equilibrium multiplicity the comparative static comparisons are clean, if one compares stationary equilibria with the same average price $\hat{p}$ upon non-disclosure. Comparing equilibria with the same average price upon non-disclosure is natural. $\hat{p}$ is potentially empirically observable and at the same time it is a sufficient statistic for the information acquisition choice of Buyer in strongly stationary equilibria.\(^{37}\)

The result of this subsection is as follows.\(^{38}\)

**Proposition 8.** [Comparative statics in the HFL.]

1) Suppose there are two strongly stationary equilibria in the HFL with the same average price $\hat{p}$ and all parameters the same, except for the discount rates $r_1 > r_2$. Then the equilibrium intensity is higher for $r_1$ than $r_2$.

2) Suppose there are two strongly stationary equilibria with the same average price $\hat{p}$ and all parameters the same, except for prior of high value $\pi_1^0 > \pi_2^0$. Then the equilibrium intensity is higher for $\pi_1^0$.

3) (deterministic variable accuracy costs). Equilibrium accuracy is independent of the discount rate. Suppose there are two strongly stationary equilibria with the same average price $\hat{p}$ and all parameters the same except for $\pi_0^1 > \pi_0^2$. Equilibrium accuracy is higher for $\pi_0^1$ if $\bar{v} + \frac{v}{2} > \hat{p}$, and it is higher for $\pi_0^2$ if $\frac{\bar{v} + \hat{p}}{2} < \hat{p}$.

4) (stochastic fixed accuracy costs). Suppose in the case of stochastic fixed costs there are two strongly stationary equilibria with the same average price $\hat{p}$ and all parameters the same, except for the distribution of lump-sum costs $F_1 >_{FOSD} F_2$.\(^{39}\) Then $\lambda_1$ is lower than $\lambda_2$.

The comparative statics of $\lambda$ are all intuitive: the more impatient Buyer and/or the more optimistic at the outset of the bargaining she is, the higher the incentives to explore for information sources. The higher the accuracy costs, the lower the incentives to explore in the first place.

Equilibrium accuracy in the case of deterministic variable costs is independent of the impatience level. Whenever Buyer has the chance to exploit an opportunity to learn, the option value from learning is independent of the discount rate and so are the incurred accuracy costs. This is because, as long as there is no agreement, Seller uses the same stationary pricing strategy upon non-disclosure. To get an intuition for the rest of the result about accuracy, suppose $\frac{\bar{v} + \hat{p}}{2} > \hat{p}$ and $\hat{p}$ is close to $\hat{v}$. Then Buyer is relatively pessimistic ex-ante about the value of the good. As long as she becomes more optimistic

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\(^{37}\)For small changes of parameters like ex-ante prior of high value $\pi_0$ and discount rate $r$ the existence neighborhoods $\mathcal{N}$ in Proposition 7 for mixed pricing equilibria overlap and so comparative statics for stationary equilibria with the same $\hat{p}$ are possible even if they are of the mixed pricing sort, at least for marginal changes in $\pi_0, r$ or of the cost distribution $F$.

\(^{38}\)More complete results can be found in subsection C.2.2 of the appendix. I focus here on $a, \lambda$ for brevity’s sake.

\(^{39}\)This means that $F_1$ dominates $F_2$ in the sense of first-order stochastic dominance. Formally, $F_1(x) \leq F_2(x)$ for all $x > 0$ with the inequality strict for some $x > 0$. 

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(\frac{1}{2} > \pi_0^1 > \pi_0^2), she also becomes more uncertain about the value of the good and as a consequence has stronger incentives to exploit the learning opportunity. If instead \( \frac{\bar{v} + \bar{v}}{2} < \hat{\rho} \) and \( \hat{\rho} \) is close to \( \hat{\nu} \), then Buyer is relatively optimistic ex-ante about the value of the good. The marginal benefit of additional learning when she becomes even more optimistic (\( \pi_0^1 > \pi_0^2 > \frac{1}{2} \)) is low in this case, so that the equilibrium accuracy \( a(\hat{\rho}) \) is lower for the higher prior \( \pi_0^1 \).

Other comparative results can be proven, and their proofs are skipped here for the sake of length. E.g. for the case of deterministic variable costs, one can easily show that, whenever accuracy costs fall pointwise (\( c_1(a) \leq c_2(a) \) for all \( a \in [\frac{1}{2}, 1] \)) equilibrium intensity \( \lambda(\hat{\rho}) \) weakly increases. A slightly more convoluted argument is needed to show that the probability of learning conditional on an opportunity to learn \( \bar{\mu}(\hat{\rho}) = F(\pi_0(\bar{v} - \hat{\rho}) - (\bar{v} - \bar{p}_L(\hat{\rho}))) \) falls, if \( F \) increases in the FOSD-sense.\(^{40} \)

### 4.2 Pre-learning negotiations

In the game introduced in section 2 learning is a socially wasteful activity for every \( \Delta > 0 \), because it does not raise the ex-ante surplus, nor does it eliminate any pre-game informational asymmetry. Therefore, a third party who has social welfare in mind would prohibit learning altogether, if that were feasible. Another way to avoid the inefficiency from learning is to give Buyer the opportunity to commit to pre-learning negotiations. In fact, such commitment would lead to efficient outcomes.

To see this formally, consider an extensive-form expansion of the basic game from section 2 in which there is a first stage at a period \( t = 0 \) in which Seller can make an offer to Buyer, before learning can start. If the offer at \( t = 0 \) is accepted the game ends with agreement, whereas if the offer is rejected by Buyer the bargaining game from section 2 is played.

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\(^{40}\)There is a direct effect, as can be seen from the definition of \( \bar{\mu}(\hat{\rho}) \), but also an indirect one counteracting it, because the HFL of \( p_L(\hat{\rho}) \) given by \( \bar{p}_L(\hat{\rho}) \), increases when \( F \) increases in the FOSD-sense. The direct effect is stronger.
The following Proposition is straightforward.

**Proposition 9.** Pick any $\Delta > 0$. All perfect Bayesian equilibria of the game with pre-learning negotiations are efficient. In particular, all perfect Bayesian equilibria feature agreement at time $t = 0$.

Note that Buyer’s payoff in the game with pre-learning negotiations quantifies the value of learning from an ex-ante perspective. Suppose that the PBE has a price $p$ accepted with probability one in $t = 0$. Then it holds $\hat{v} - p = \delta V_B(p)$ with $V_B(p)$ the payoff of Buyer in the continuation bargaining game, if she rejects $p$. Then the opportunity to learn is worth precisely $\hat{v} - p$. Proposition 3 implies that the value of learning can be zero whenever learning is costless but not immediate, so that Buyer may not get an informational rent from the opportunity to learn. In contrast, Theorem 2 shows that Buyer always receives an informational rent whenever learning is costly.

Proposition 9 gives a rationale for the intervention of a third party with social welfare in mind or for the introduction of commitment devices that allow Buyer to commit to pre-learning negotiations. This intervention is the more valuable, the larger the period-length $\Delta > 0$ is, or equivalently the lower the frequency of interaction between the bargaining parties.

The crucial assumption that allows efficiency to be restored for any $\Delta > 0$ is that trade is ex-ante efficient. In other situations, e.g. Buyer may also experience disutility from acquiring the good, learning is not necessarily wasteful and delay might be efficiency-enhancing. This is illustrated in the next subsection.

### 4.3 Negative lowest Buyer value

Consider a variation of the game introduced in section 2 with two differences. First, assume that $\underline{v} < 0 < \hat{v} = \pi_0 \hat{v} + (1 - \pi_0)\underline{v}$. Thus, trade is not efficient in every state of the world, even though it is efficient to trade immediately, if learning is impossible. Second, assume that Buyer can walk away from negotiations whenever she chooses. Many negotiation settings share these features. E.g. a significant share of VC, or more generally, private equity negotiations break down after the investors learn new information and conclude that the transaction is not worthwhile.

The analysis of behavior for this variation of the model follows closely the results of the main model.\footnote{The changes are minor. Details are available upon request.} This holds throughout except for payoff and efficiency considerations.

For brevity’s sake, here I illustrate only that learning can be welfare-enhancing, even in strongly stationary equilibria with $\kappa = 0$ (recall that these are the strongly stationary equilibria with the largest expected delay in the baseline model). I look at the case of costly learning to give inefficiency its ‘best shot’. For simplicity of exposition I assume intensity $\lambda > 0$ is exogenously given and focus on the mixed pricing equilibria.

**Proposition 10.** Suppose that $\underline{v} < 0 < \hat{v}$.

1) If learning is impossible the efficient outcome with respect to the available information is to trade with probability one.

2) [Deterministic variable costs of accuracy] Consider the HFL of strongly stationary equilibria with mixed pricing and $\hat{p} = \hat{v}$ and the following parameter restrictions:
\[ \pi_0 > \frac{1}{2}, \sqrt{2}\lambda + 2\epsilon > r > \sqrt{2}\lambda + \epsilon \text{ with some } \epsilon > 0 \text{ and} \]
\[
\frac{\sqrt{2}}{\sqrt{2} + 1} \frac{\pi_0}{1 - \pi_0} < -\frac{v}{\hat{v}} < \frac{\pi_0}{1 - \pi_0}.
\]

Then learning is welfare enhancing whenever the accuracy costs \( c \) are near enough to zero with respect to the topology of uniform convergence on compact sets and \( \epsilon \) is small enough.

3) [Stochastic variable costs of accuracy] Consider the HFL of strongly stationary equilibria with mixed pricing and \( \hat{p} = \hat{v} \) and the following parameter restrictions:
\[
\frac{r}{r + \lambda} \frac{\pi_0}{1 - \pi_0} < -\frac{v}{\hat{v}} < \frac{\pi_0}{1 - \pi_0}.
\]

Then learning is welfare enhancing whenever the distribution of fixed accuracy costs \( F \) is near enough to zero with respect to the topology of weak convergence of probability distributions.

Learning is not always welfare enhancing when \( v < 0 \). Intuitively, this holds true in case the costs \( F \) or \( c \) are ‘high enough’ with respect to the other parameters of the game.

5 Conclusions

In most real-life bargaining situations a bargaining party has the chance to learn privately about the terms of trade during the negotiation process. She will typically take up this opportunity, even if it is costly and leads to delay, because additional information may give her a strategic advantage in negotiations. This paper considers such a situation explicitly and shows that this may lead to delay, even in situations in which ex-ante trade is efficient.

Moreover, this paper models the learning process explicitly in a bargaining environment. Endogenous information acquisition and disclosure of new information with the aim of influencing bargaining positions is a realistic feature of many real-world market interactions, from merger and acquisition negotiations to government leases of natural resources. This paper is a first attempt to introduce this realistic feature into the dynamic bargaining theory literature.

Many extensions and variations to this work are natural topics for future research. First, many real-world bargaining situations feature initial information asymmetries in addition to the possibility of endogenous learning as negotiations progress. Examples include mergers and acquisitions involving industry leaders or management buyouts. It is natural to expect that sequential screening dynamics reappear in situations in which, besides the possibility of endogenous learning and selective disclosure of information, initial private information is a prominent feature.

Second, in my model learning is wasteful because, besides being costly, it creates information asymmetry between the two bargaining parties. In situations in which there is initial private information and the values of Buyer and Seller are interdependent, learning and the possibility to communicate learning outcomes may lead to improvements in efficiency.\(^{42}\) A careful study of the effects of endogenous learning in settings with interdependent values is left for future research.

\(^{42}\) These are typically situations in which the lemons problem is prevalent (Akerlof (1970)).
Third, a serious study of the interaction of competitive pressure and endogenous learning is missing from this paper, even though competitive pressure is a significant factor for negotiations in many business situations, e.g. in private equity or procurement of investment goods.\footnote{One \emph{ad hoc} way to incorporate ideas of competition into the model of this paper is to assume that the discount factor of the players reflects, besides impatience, the possibility that negotiations break down due to exogenous reasons, one of them being the arrival of a superior bargaining partner who ‘steals away’ the chance for a deal between the original players. A more realistic model could have discount factors of the players depend on their respective estimate of the current Buyer valuation.} A better understanding of the interaction between initial information asymmetries, competitive pressure and endogenous information acquisition in dynamic bargaining environments is an exciting topic left for future research.

Finally, my model implies that one-sided endogenous private learning in bargaining situations is compatible with a wide variety of welfare outcomes, from approximate efficiency to significant inefficiency. This suggests studying the role of limited commitment and other institutional forms with the aim of designing bargaining outcomes. One hopes that deepening our understanding of endogenous costly learning in bargaining situations will lead to valuable insights on how to better devise institutions that facilitate negotiations in the real world.

References


## Appendices

A remark on notation in the appendix and online appendix:

For limit statements with respect to $\Delta \to 0$, I often use the Landau notation $o(\Delta), O(\Delta)$.\(^{44}\)

### A Proofs for section 2

#### A.1 Proofs of general properties of PBEs

**A.1.1 Proof of Lemma 1 and its corollaries.**

The proof of Lemma 1 relies heavily on similar arguments in the proofs of the Lemmas 1 and 2 in Fudenberg et al. (1985). The arguments need to be adapted to account for Buyer valuation changing over time due to learning.

Before giving the proof of Lemma 1, I remark an important fact used in the proof of the Lemma.

**Remark 1.** Suppose a Bayesian learner has a prior $F$ over $\mathbb{R}$ for the variable $\theta$ and uses an unbiased experiment $\mathcal{E} : \text{supp}(F) \to \Delta(S)$, with $S$ finite, to learn about $\theta$ where $S$ is a signal space. Let $E_F[\theta|s]$ be the posterior mean after signal $s$ when prior for $\theta$ is $F$. If $F'$ is another prior so that $F' \overset{FOSD}{\succ} F$ then for every $s \in S$ it holds

$$E_{F'}[\theta|s] \geq E_F[\theta|s],$$

i.e. the estimates increase pointwise when the prior increases in the FOSD sense.\(^{45}\)

**Proof of Remark 1.** One can realize all random variables needed in one large enough probability space where $\theta, \theta'$ are such that $\theta \sim F$, $\theta' \sim F'$ and $\theta' = \theta + y$ in distribution with $y \geq 0$ a random variable. This larger probability space has as sample space the collection of pairs $(\theta, y)$. Note here the signal space $S$ as well as the experiment $\mathcal{E}$, which

\(^{44}\)See e.g. chapter 5, section 4 of Lang (1997) for formal definitions.

\(^{45}\)Note, it is in general not true that the distribution of posteriors increases in the FOSD-sense.
is a random variable from $\Theta \supseteq \text{supp}(F) \cup \text{supp}(F')$ to $\Delta(S)$ is being kept fixed. In this larger probability space it holds conditional on the realization of a signal $s$

$$\mathbb{E}[\theta'|s] = \mathbb{E}[\theta + y|s] = \mathbb{E}[\theta|s] + \mathbb{E}[y|s] \geq \mathbb{E}[\theta|s]. \quad (24)$$

Because $\mathbb{E}_F[\theta|s], \mathbb{E}_F[\theta|s]$ depend only on the joint distributions of $(\theta, s)$ (in the case of $F$ given by $F(\theta) \cdot \mathcal{E}(\theta)(s)$ and in the case of $F'$ given by $F'(\theta) \cdot \mathcal{E}(\theta)(s)$), the conclusion follows from (24).

\[ \square \]

**Proof of Lemma 1.** First, I show some auxiliary claims. These imply that in any PBE any price $p < v$ is accepted immediately by all Buyer types $w$. For the auxiliary claims let $h$ be a public history which ends with a period $t$ and the rejection of the price quoted at the end of history $h$ by Seller. Let $V(w, h)$ be the expected equilibrium payoff of Buyer type $w$ after public history $h$ (starting from period $t + 1$).

**Claim 1.** $w \mapsto V(w, h)$ is strictly increasing and Lipschitz with constant one.

**Proof of Claim 1.** Define $\tau(h, w)$ to be the stopping time that gives the agreement date in an equilibrium for Buyer of type $w$ after history $h$.

The equilibrium payoff of type $w$ after history $h$ can be written as

$$V(w, h) = \sum_{u \geq 0} \delta^t \alpha_{t+1+u}(w, h)(\mathbb{E}_w[v_{\tau(h, w)}]|\tau(h, w) = t + 1 + u] - \mathbb{E}_w[p_{t+1+u}(w, h)],$$

where $\alpha_{t+1+u}(w, h)$ is the probability of agreement at time $t + 1 + u$ on path if $w$ follows her strategy, $\mathbb{E}_w[v_{\tau(h, w)}]|\tau(h, w) = t + 1 + u]$ is the expected value for Buyer of type $w$ conditional on agreement at time $t + 1 + u$ and $\mathbb{E}_w[p_{t+1+u}]$ is the expected price Buyer pays conditional on agreement at time $t + 1 + u$.\textsuperscript{46} Given an (adapted) learning strategy for Buyer, the expected valuation of the good for Buyer is a Martingale. Thus, it holds

$$\mathbb{E}_w[v_{\tau(h, w)}] = w. \quad (25)$$

Let $w' > w$ be another possible type after history $h$. If Buyer of type $w$ instead uses the optimal stopping strategy of the higher type $w'$, $\tau(h, w')$, it holds again by Martingale property.

$$\mathbb{E}_w[v_{\tau(h, w')} ] = w. \quad (26)$$

Similar to the classical proof, I use a no-imitation argument. From the equilibrium property it follows

$$V(w, h) \geq \sum_{u \geq 0} \delta^t \alpha_{t+1+u}(w', h)(\mathbb{E}_w[v_{\tau(h, w')}]|\tau(h, w') = t + 1 + u] - \mathbb{E}_w[p_{t+1+u}(w', h)].$$

Replacing the formula for $V(w', h)$ from above yields

$$V(w', h) - V(w, h) \leq \mathbb{E}_{w'}[\delta^{\tau(h, w')}v_{\tau(h, w)}] - \mathbb{E}_w[\delta^{\tau(h, w)}v_{\tau(h, w)}].$$

\textsuperscript{46}Note that time has been shifted accordingly.
Next, I show the following relation.

\[ \mathbb{E}_{w'}[\delta^{\tau(h,w)}v_{\tau(h,w)}] - \mathbb{E}_w[\delta^{\tau(h,w)}v_{\tau(h,w)}] \leq w' - w. \tag{27} \]

**Proof of (27).** Fix a stopping time \( \tau \) and consider two bounded, non-negative stochastic processes \((v_t)\) and \((v'_t)\) such that for all \( t \) it holds \( v_t \leq v'_t \). Then it follows

\[ \mathbb{E}[v_{\tau}(1 - \delta^{\tau})] \geq \mathbb{E}[v'_{\tau}(1 - \delta^{\tau})] \iff \mathbb{E}[v_{\tau}] - \mathbb{E}[\delta^{\tau}v_{\tau}] \geq \mathbb{E}[v'_{\tau}] - \mathbb{E}[\delta^{\tau}v'_{\tau}]. \tag{28} \]

When Buyer starts with type \( w' > w \) and follows the same strategy as if she started from \( w \), after every learning opportunity she receives a pointwise weakly higher estimate of the value of the good than under \( w \) (see Remark 1). Since this holds pointwise, one can use the inequality (28) together with (25) and (26) to show (27).

**End of proof of (27).**

This establishes the Lipschitz continuity of \( w \mapsto V(w,h) \). One writes

\[ V(w,h) = \max_{\text{learning},\tau,h}-\text{measurable} \mathbb{E}[\delta^{\tau}v_{\tau} - p_{\tau}]. \]

For every fixed learning and stopping strategy, it holds that \( v_{\tau} \) increases pointwise with \( w \). Therefore monotonicity of \( w \mapsto V(w,h) \) follows from a simple optimization/envelope theorem argument.

**End of proof of Claim 1.**

As a next step, one establishes the following **skimming property**.

**Skimming:** \( w - p \geq \delta V(w,h) \implies w' - p > \delta V(w',h), \) whenever \( w' > w \).

To see this, one uses Lipschitz continuity of \( w \mapsto V(w,h) \) as follows.

\[ w - p \geq \delta(V(w',h) + w - w') \iff w + \delta(w' - w) - p \geq \delta V(w',h) \]
\[ \implies w' - p > \delta V(w',h), \]

where the last step used the fact that \( w' > 0 \).

**Proof of part 1) of the Lemma.** Suppose that after history \( h \) the highest possible on-path Buyer valuation is \( \bar{w} \) and the lowest possible is \( w \). Note that any Buyer type can always ensure 0 by always rejecting after any history. Same holds for Seller; she can always ensure zero after any history by always offering prices which are too high (say prices above \( \bar{v} \)). This implies that after any history, the expected valuation of any Buyer type whenever there is agreement is below the highest valuation \( \bar{w} \). Because the continuation payoff \( V(w,h) \) is increasing in \( w \) it holds then that \( V(w,h) \leq \bar{w} \).

It follows through Lipschitz continuity that

\[ V(w,h) \leq w + \bar{w} - w. \]

In particular, it follows that all Buyer types accept any subsidy larger than \( -w + \bar{w} \) immediately. Given this, it is never optimal for Seller to charge prices below \( w - \bar{w} \). Knowing this, Buyer of type \( w \) accepts any \( p \) satisfying \( w - p \geq \delta(w - (w - \bar{w})) \iff p \leq w - \delta \bar{w} \). Iterating this argument just as in the classical proof, one finds that Seller requires prices strictly below \( w - \delta^n \bar{w} \). Send \( n \to \infty \) to finish the proof.
Proof of part 2) of the Lemma. Suppose Seller after some public history \( h \) asks with positive probability for a price \( p \) smaller than the reservation price of a Buyer of type \( w \). Denote the reservation price of this type by \( r(w, h) \) in the PBE in question. It satisfies \( w - r(w, h) = \delta V(w, h) \). It holds for any \( w' > w \) that \( w' - r(w, h) > \delta V(w', h) \). Because of the skimming property, increasing \( p \) to \( p + \epsilon \) with \( p + \epsilon < r(w, h) \) and keeping the same probability on it as previously on \( p \) leads to higher profits.

Next, assume towards a contradiction that Seller asks for a price above the highest reservation price that is accepted with positive probability after the history \( h \) is optimal. Let \( F_h \) be the distribution of types conditional on history \( h \) and set \( \bar{w} \) for the highest type in the support of \( F_h \). Denote by \( V_S(h) \) the continuation payoff of Seller from the equilibrium strategy. Charging more than the highest reservation price \( r(\bar{w}, h) \) results in a payoff \( \delta V_S(h) \). Suppose that instead Seller charges with positive probability \( p \) also \( \bar{w} - \epsilon \) with some \( \epsilon > 0 \) small. The payoff from this deviation is at least

\[
p \cdot F_h(\bar{w})(\bar{w} - \epsilon) + (1 - p) \cdot \delta V_S(h) > \delta V_S(h),
\]

for all \( \epsilon > 0 \) small enough. The inequality above follows from the fact that in the assumed strategy, game slides into next period with starting distribution \( F_h \) (Seller does not learn anything by charging a price which no Buyer type can afford). In particular it holds

\[
V_S(h) \leq \sup \{ r(w, h') : w \in \text{supp}(F_{h'}), h' \text{ a continuation history of } h \} \leq \bar{w}.
\]

Here, the second inequality uses the fact \( r(w, h) \leq w \) for all \( w \) and all \( h \).

Proof of part 3) of the Lemma. If this probability were zero, then no equilibrium would exist where this price is quoted with positive probability because Seller, whenever the equilibrium would prescribe quoting that price with positive probability, would want to deviate to arbitrarily smaller price offers. In particular, there would not be a well-defined best response of Seller after such a history. This contradicts the existence of the equilibrium.

Proof of part 4) of the Lemma. Upon disclosure of current valuation Seller can calculate the valuation of Buyer and knows that this valuation will not change going forward. The proof of part 4) is finished by an argument that follows the logic of the uniqueness of payoffs for the usual Rubinstein-Stahl bargaining model. See Example 9.A.A.2 in Mas-Colell et al. (1995).\(^{47}\)

The next Remark shows that Lemma 1 remains valid in the environment with costs.

Remark 2 (Lemma 1 in the case of costs). All the statements of Lemma 1 remain valid in the environments with costs from sections 3 and 4.1.

\(^{47}\) The details are as follows. Suppose for simplicity and normalization that valuation to Buyer after this history of disclosure is \( w \). Because this is common knowledge after the verifiable disclosure, the subgame turns into a classical share-the-pie game. Let \( v_S \) be the lowest PBE payoff in the continuation game of Seller, \( \bar{v}_S \) the highest PBE payoff in the continuation game of Seller and define in addition accordingly \( v_B, \bar{v}_B \) to be respectively the lowest and the highest PBE payoffs for Buyer. Note that, because of learning once, the subgame after a rejection is isomorphic to the game started after the history of disclosure. Suppose Seller offers more than \( \delta \bar{v}_B \). Then this is accepted and so \( v_S \geq w - \delta \bar{v}_B \). Now note that \( \bar{v}_B \leq w - \text{PBE payoff of Seller} \leq w - v_S \). Combining with the previous inequality, delivers overall that \( v_S \geq w \). This implies Seller leaves zero surplus to Buyer.
Proof. To see this, note first that all arguments in the Claim 1 of the proof of part 1) of the Lemma remain true if one interprets the price \( p_u(w, h) \) in the proof arguments as the expected costs incurred if agreement is reached at period \( u \) for Buyer of type \( w \) at history \( h \). These costs contain the price costs of the purchase at time \( u \) as well as the information acquisition costs incurred in the period between \( u \) and history \( h \).\(^{48}\) Note also that the probability of agreement \( \alpha_u(w, h) \) used in the proof is history-dependent and thus does not depend on whether the arrival rate of information \( \mu \) is history-independent. This, and the fact that the proof of part 1) of Lemma 1 involves only Buyer payoffs, yield that part 1) remains true in the case of costs.

It is trivial to see that the proof of parts 2) and 4) of Lemma 1 do not depend on whether information is costly or not, because they involve the pricing decision of Seller.

To see that part 3) of the Lemma remains true in the environment with costs just note that any information costs incurred in the past are sunk at the moment of Buyer’s decision of whether to accept an offered price.

Corollary 1. After every private history \( h \) the reservation prices of Buyer types \( r(w, h) \) are strictly increasing in \( w \).

Proof. From the definition of reservation prices \( w - r(w, h) = \delta V(w, h) \) and the skimming property it holds that \( w' > w \) implies \( w' - r(w, h) > \delta V(w', h) \). It follows \( r(w', h) > r(w, h) \).

Corollary 2 (No quiet periods). There is no PBE where after a history in which Seller is called upon to play, the probability of trade is zero.

Proof. This is very similar to the second part of the proof of part 2) of Lemma 1.

Proof of Lemma 2. Disclosure never happens when learning good news and Buyer receives positive payoff in the PBE after receiving good news, since learning is only once and disclosure would lead to zero continuation payoff.

Delaying disclosure of bad news never increases payoffs. Since the learning happens only once, Buyer with bad news knows that she receives zero payoff in equilibrium, no matter disclosure decision. This is true after every private history.

A.2 No sequential screening of valuations near the HFL

First, I define a refinement for a PBE.

Refinement for off-path beliefs: ‘divinity in bargaining’. After an off-path history resulting from the rejection of a price, if the pool of Buyer types contains only one type who was indifferent between accepting and rejecting and all other types, for which Seller’s belief had positive probability, had a strict incentive to accept the quoted price, then Seller starts new period with a belief that puts probability one on the type who was indifferent between accepting and rejecting in the last period.

This equilibrium selection is motivated by the ‘divinity’ criterion for signalling games, see Banks and Sobel (1987). The strongly stationary equilibria constructed in this paper satisfy divinity in bargaining.

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\(^{48}\)The crucial assumption that allows this interpretation is the fact that information acquisition costs enter the overall payoff of Buyer linearly.
Proposition 11. For all $\Delta$ small enough, there are no equilibria which satisfy the following properties.

A. $\nu > 0$ and Buyer of type $\nu$ discloses immediately,

B. sequential screening of valuations.

or

A. $\nu = 0$ and Buyer of type $\nu$ discloses immediately

B. stationarity,

C. sequential screening of valuations,

D. ‘divinity in bargaining’.

Proof. I show that for all $\Delta$ small enough, generically, all PBEs do not satisfy all of the properties in the statement of the Proposition. The proof is split into several claims. First, focus on the case $\nu > 0$.

Claim 1. a) Fix $\Delta > 0$. There is no PBE which satisfies A and has sequential screening dynamics of arbitrary length (i.e. $K = \infty$ is impossible for any $\Delta$).

b) Under the requirement A the number of periods needed in any PBE with sequential screening dynamics does not grow faster than $\frac{1}{\Delta}$, i.e. $K(\Delta)\Delta = O(1)$.

To see a), suppose there is such a PBE. Then the price dynamics, given by the reservation price relation of type $\bar{\nu}$, satisfies

$$r_k = (1 - \delta)\bar{\nu} + \delta r_{k+1}, \quad k \geq 1,$$

with $r_2 < \bar{\nu}$. Then one can solve for the dynamics to get

$$r_{k+1} = \frac{1}{\delta k - 1} r_2 + \left(1 - \frac{1}{\delta k - 1}\right)\bar{\nu}.$$

This dynamics leads to $r_k \to -\infty$ due to $\delta < 1$ and this is a contradiction to results of Lemma 1.

To see b) consider a sequence of equilibria which satisfy A. and have sequential screening dynamics. The reservation price relation for the high type $\bar{\nu}$ given by $\bar{\nu} - r_i = \delta(\bar{\nu} - r_{i+1})$ delivers $r_i - r_{i+1} = (1 - \delta)(\bar{\nu} - r_{i+1})$. A telescopic sum argument gives

$$r_1 - r_K = \sum_{j=1}^{K(\Delta)-1} (r_i - r_{i+1}) \geq (1 - \delta)(K(\Delta) - 1)(\bar{\nu} - r_1),$$

where the last step uses that prices $r_i, i \leq K(\Delta)$ are decreasing. Recall that $1 - \delta = 1 - e^{-r\Delta}$ and from iterating the reservation price relation for the type $\bar{\nu}$ one arrives at $\bar{\nu} - r_1 = \delta^{K(\Delta)-1}(\bar{\nu} - r_{K(\Delta)})$. Overall it follows

$$1 \geq \frac{r_1 - r_{K(\Delta)}}{\bar{\nu} - r_{K(\Delta)}} \geq (1 - \delta)\delta^{K(\Delta)-1}(K(\Delta) - 1).$$

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This delivers through simple estimates
\[ K(\Delta) \Delta \leq Ce^{-rK(\Delta)\Delta}, \]
for some \( C > 0 \) which is independent of \( \Delta \). From here it is easy to see that \( K(\Delta) \Delta \) remains bounded away from infinity.\(^{49}\)

Claim 2. In the last period \( K \) on path, a price \( 0 < r_K \leq \hat{v} \) is quoted.

By the skimming property the sequence of decreasing prices \( r_t \) corresponds, up to the last period \( K(\Delta) < \infty \), to the reservation prices of the high-type \( \bar{v} \). In period \( K(\Delta) \) \( r_{K(\Delta)} \) corresponds to the reservation price of type \( \hat{v} \). If it corresponds to the reservation price of type \( \bar{v} \), then this implies that type \( \hat{v} \) is never screened before period \( K(\Delta) \) and thus, with positive probability, bargaining goes on into period \( K(\Delta) \). This is a contradiction to the definition of \( K(\Delta) \) and to Claim 1.

If \( r_K > \hat{v} \), with positive probability the game does not end at or before \( K \) because Buyer type \( \hat{v} \) who is present in period \( K(\Delta) \) after non-disclosure with positive probability rejects \( r_{K(\Delta)} \). Given that \( r_{K(\Delta)} \) is the reservation price of \( \hat{v} \), Lemma 1 delivers \( r_{K(\Delta)} > 0 \) (recall that type \( \bar{v} \) discloses immediately and has zero payoff after every on-path history). For ease of notation, in the following I sometimes suppress the dependence on \( \Delta \) in \( K(\Delta) \), whenever the argument does not rely on the precise magnitude of \( \Delta \).

Claim 3. If \( \bar{v} > 0 \) then \( K \leq 2 \). In particular, the game never continues past third period, whenever \( \Delta \) is small enough.

To see this, assume that \( K \geq 4 \) in the equilibrium and look for a contradiction. The price at time \( t = K - 1, r_{K-1} \) necessarily satisfies
\[ r_K \leq U(\gamma_{K-1})(r_{K-1} - \delta r_K) + \delta r_K, \]
which can be rewritten with the help of (29) from the proof of Claim 1 as \( U(\gamma_{K-1}) \geq \frac{r_K}{\bar{v}} \). But if \( K \geq 4 \) then it holds \( \gamma_{K-1}(\bar{v}) \leq U(0) \) if there are sequential screening dynamics, because it holds \( \gamma_2(\bar{v}) \leq U(0) \) by virtue of the assumed updating of Seller in Definition 2. \( \gamma \rightarrow U(\gamma) \) is strictly increasing and one calculates
\[ U^{-1}\left(\frac{r_K}{\bar{v}}\right) = \frac{r_K(1 - \mu(1 - \pi_0)) - \mu \pi_0}{1 - \mu \pi_0 - \frac{r_K}{\bar{v}} \mu(1 - \pi_0)}. \]

The following string of inequalities results.
\[ \frac{\bar{v}}{\bar{v}} < r_K \leq \frac{\pi_0 \mu(2 - \mu)}{1 - (1 - \pi_0) \mu(1 - \mu)}. \] \hspace{1cm} (30)

Here the last inequality on the right follows from \( U^{-1}\left(\frac{r_K}{\bar{v}}\right) \leq U(0) \). The right-hand side of (30) converges to zero as \( \Delta \rightarrow 0 \) and this establishes a contradiction if \( \bar{v} > 0 \).

Claim 4. There is no sequential screening of valuations with \( \bar{v} > 0 \), whenever \( \Delta \) is small enough.

The case \( K = 1 \) is easy to exclude, because it would require that price be \( \bar{v} \) in the first period and this is suboptimal as \( \Delta \) vanishes, because with very high probability Buyer has not learned yet. Suppose therefore in the following that \( \bar{v} > 0 \) and focus on the case \( K = 2 \).

\(^{49}\)The function \( x \mapsto xe^{-rx}, \ x \geq 0 \) is strictly increasing, continuous and convex.
Optimality at $t = 1$ leads to
\[ U(0)((1 - \delta)\bar{v} + \delta r_2) + (1 - U(0))\delta r_2 \geq r_2. \]
This gives $r_2 \leq U(0)\bar{v}$. Since $U(0) \to 0$ as $\Delta \to 0$, a contradiction to the inequalities $r_2 > \bar{v} > 0$ results.

I now turn to the case $\bar{v} = 0$.

Claim 5. There is no PBE with sequential screening of valuations, whenever $\Delta$ is small if $\bar{v} = 0$, if in addition one requires stationarity and ‘divinity in bargaining’.

Assume there exists such a PBE, of maximal length on path of $K \geq 1$ for a sequence of $\Delta$ small and show that for any such PBE and $\Delta > 0$ small enough, the price quoted in the last period $K(\Delta)$ converges to a unique positive number, uniformly and independently of the ‘length’ $K(\Delta), \Delta \to 0$ of the equilibrium. Once this is established, it suffices to essentially repeat the argument in Claims 3 and 4 of the case $\bar{v} > 0$ to conclude the proof.

First note that due to ‘divinity in bargaining’ and stationarity, if the price $r_K$ is rejected, then in the period $K + 1$ Seller begins with belief of high type equal to zero and updates upon non-disclosure to the interim belief $U(0)$. Because $r_K$ is the reservation of type $\hat{v}$ after a history of rejection of $r_1, \ldots, r_{K-1}$ it holds $\hat{v} - r_K = \delta V^\text{cont}_K(\hat{v})$ where $V^\text{cont}_K(\hat{v})$ is the continuation payoff according to the PBE in question. This subgame is isomorphic to the whole game and so because of the stationarity assumption it holds $\hat{v} - r_K \geq \delta V_B$.

To ease notation, define for use in the following $p_k = r_{K-k}, k \leq K$. The reservation price relation for type $\bar{v}$ from the price dynamic leads to $\bar{v} - p_{K-1} = \delta^{K-l}(\bar{v} - p_0)$. The goal is to try to characterize $p_0$ explicitly.

Let $V^\text{cont}_k$ be the payoff of Buyer at the beginning of period $K - k$ on path, if she has not learned yet. It holds
\[ V_B = \mu \pi_0(\bar{v} - p_K) + 0 + (1 - \mu)\delta V^\text{cont}_{K-1}. \]
Moreover, the following recursion holds
\[ V^\text{cont}_k = \mu \pi_0(\bar{v} - p_k) + (1 - \mu)\delta V^\text{cont}_{k-1}. \]
Using the recursion repeatedly leads to
\[ V^\text{cont}_{K-1} = \mu \pi_0 \sum_{l=1}^{M} (\bar{v} - p_{K-l})((1 - \mu)\delta)^{l-1} + ((1 - \mu)\delta)^M V^\text{cont}_{K-M-1}. \]
Specialize to $M = K - 1$ to arrive at
\[ V^\text{cont}_{K-1} = \mu \pi_0 \delta^{K-1}(\bar{v} - p_0) \sum_{l=1}^{K-1} (1 - \mu)^{l-1} + ((1 - \mu)\delta)^{K-1} V^\text{cont}_0. \]
Here it holds $V^\text{cont}_0 = \mu \pi_0(\bar{v} - p_0) + (1 - \mu)(\hat{v} - p_0)$. Using this together with (31) and (32) leads after algebra to
\[ V_B = \delta^K \pi_0(\bar{v} - p_0)(1 - (1 - \mu)^{K+1}) + \delta^K (1 - \mu)^{K+1}(\hat{v} - p_0). \]
Combining this with the definition of the reservation price $p_0$ leads to the following equation for $p_0$:

$$\hat{v} - p_0 = \delta^{K+1} \pi_0 (\hat{v} - p_0)(1 - (1 - \mu)^{K+1}) + \delta^{K+1} (1 - \mu)^{K+1}(\hat{v} - p_0).$$

Here one has used the stationarity requirement. This can be solved for $p_0$ uniquely to give

$$p_0 = \frac{\pi_0 - \sigma \hat{v}}{1 - \sigma},$$

where $\sigma$ is given by

$$\sigma(\Delta, K, \pi_0) = \frac{\delta^{K+1}(1 - (1 - \mu)^{K+1})\pi_0}{1 - \delta^{K+1}(1 - \mu)^{K+1}}.$$ 

Note in particular that $\sigma \in (0, \pi_0)$ for all $\Delta$. $p_0$ remains bounded away from 0 as $\Delta \to 0$ as long as in the HFL $\sigma$ remains bounded away from $\pi_0$, as $\Delta \to 0$.

It is easy to see that $\frac{\sigma(\Delta, K, \pi_0)}{\pi_0}$ remains bounded away from one, as $\Delta \to 0$.

Namely, this follows from

$$\frac{\sigma(\Delta, K, \pi_0)}{\pi_0} = \frac{e^{-r\Delta (K(\Delta)+1)}(1 - e^{-\lambda \Delta (K(\Delta)+1)})}{1 - e^{-(r + \lambda)\Delta (K(\Delta)+1)}},$$

and the fact that $K(\Delta)\Delta$ remains bounded as $\Delta \to 0$. If $K(\Delta)\Delta$ remains bounded away from zero along a subsequence, then this is clear. Otherwise, one uses the elementary limit statement: $\frac{1 - e^{-rt}}{1 - e^{-\lambda t}} \to \frac{r}{r + \lambda}$ as $t \to 0$.

Proof of Proposition 5. Note that the proof of Claims 1,2,3 in the proof of Proposition 11 only uses the definition of sequential screening of valuations as well as the fact that $0 < \underline{v} < \overline{v}$. Therefore it can easily be adapted to give non-existence of sequential screening dynamics for the case of deterministic variable costs. This follows because with deterministic variable costs Buyer is never able to learn perfectly, because perfect learning is prohibitively costly. This results in Buyer types strictly above zero after every Seller-history, even in the case $\underline{v} = 0$.

By an analogous logic to the proof of the first part of Proposition 11, the result remains true for the case of stochastic fixed costs, provided that $\underline{v} > 0$.

A.3 High-price stationary equilibria and Buyer payoff near HFL

First, I give a formal definition of a high-price equilibrium.

Definition 4. Say that a PBE is a high-price equilibrium if, on path, whenever it is Seller’s turn to quote a price, he asks with probability one for the highest buyer type that has positive probability using public information at that moment in time.

Next I show that an equilibrium features zero Buyer payoff if and only if it is a high-price equilibrium.

Lemma 3. An equilibrium with zero Buyer payoff must be a high-price equilibrium.
Proof. To see this, suppose that Seller after an on-path history of some length \( t \), asks with positive probability for a price lower than the highest valuation she deems feasible with positive probability after that history. Buyer with the highest valuation can then achieve, with positive probability, a strictly positive payoff in the continuation game after that history. Namely, she can wait until period \( t \) and accept any price strictly below her reservation price. Since in an equilibrium of the game Buyer can always ensure a non-negative payoff after each non-terminal history which is on path, an overall positive surplus would result for Buyer, whenever the equilibrium is not a high-price equilibrium. □

Proof of Proposition 3. The proof is constructive. I find parameter restrictions which ensure that the Seller-optimality conditions (4) and (5) are satisfied. The last part of the proof specifies off-path beliefs.

Note that 
\[
U(\gamma) > \gamma
\]
and it is strictly increasing in \( \gamma \). Denote also 
\[
U(0) = \frac{\mu \pi_0}{1 - \mu (1 - \pi_0)}.
\]
Note also that the map \( \gamma \rightarrow B(\gamma, q) = \frac{\gamma (1 - q)}{(1 - q) + 1 - \gamma} \) has \( B(\gamma, q) < \gamma \) and that it is decreasing in \( q \) (equivalently increasing in \( (1 - q) \)) as well as increasing in \( \gamma \). In particular, 
\[
\lim_{\gamma \to 0} B(\gamma, q) = 0 \text{ uniformly in } q \in [0, 1].
\]
One can solve for the \( q \) in (3) to get
\[
q(\gamma) = \frac{\mu \pi_0 + \gamma \mu (1 - \pi_0)}{\gamma + (1 - \gamma) \mu \pi_0} \in (0, 1]. \tag{33}
\]
It holds \( q(0) = 1 \) and \( q(1) = \mu \) and \( q \) is strictly decreasing in \( \gamma \).

For future use, let us also note
\[
U(\gamma) q(\gamma) = \frac{\mu \pi_0 + \gamma \mu (1 - \pi_0)}{1 - (1 - \gamma) \mu (1 - \pi_0)}. \tag{34}
\]
\( U(\gamma) q(\gamma) \) is increasing in \( \mu \) and it is increasing in \( \gamma \).\(^{51}\) One has \( U(\gamma) q(\gamma) \big|_{\gamma=0} = U(0) \) as well as \( U(\gamma) q(\gamma) \to 0 \) as \( \mu \to 0 \), uniformly in \( \gamma \).

Moreover it holds
\[
\lim_{\mu \to 1^-} U(\gamma) q(\gamma) = \frac{\pi_0 + \gamma (1 - \pi_0)}{1 - (1 - \gamma) (1 - \pi_0)} = 1.
\]
The payoff function \( W(\gamma) \) of Seller from \( t = 2 \) on satisfies the recursion
\[
W(\gamma) = (\gamma + (1 - \gamma) \pi_0 \mu) \bar{v} \cdot q(\gamma)
+ (1 - \gamma) \mu (1 - \pi_0) \bar{v}
+ \delta((\gamma + (1 - \gamma) \mu \pi_0) (1 - q(\gamma)) + (1 - \gamma)(1 - \mu)) W(\gamma).
\]
After algebra this results in
\[
W(\gamma) = \frac{(\gamma + (1 - \gamma) \pi_0 \mu) \bar{v} \cdot q(\gamma) + (1 - \gamma) \mu (1 - \pi_0) \bar{v}}{1 - \delta((\gamma + (1 - \gamma) \mu \pi_0) (1 - q(\gamma)) + (1 - \gamma)(1 - \mu))}.
\]

\(^{50}\)Just note that \( B(\gamma, q) \leq \frac{\gamma}{1 + \gamma} \), for all \( q \in [0, 1] \).

\(^{51}\)Namely it holds
\[
\frac{d}{d\gamma} U(\gamma) q(\gamma) = \frac{\mu (1 - \pi_0)(1 - \mu)}{(1 - (1 - \gamma) \mu (1 - \pi_0))^2}.
\]
Because of the formula for $q(\gamma)$ this simplifies to
\[ W(\gamma) = \frac{\mu}{1 - \delta(1 - \mu)} \left( (\pi_0 + \gamma(1 - \pi_0))\bar{v} + (1 - \gamma)(1 - \pi_0)v \right) = \frac{\mu}{1 - \delta + \delta\mu} (\gamma\bar{v} + (1 - \gamma)v). \]

$W$ is strictly increasing in $\gamma$ with derivative
\[
\frac{dW(\gamma)}{d\gamma} = \frac{\mu(1 - \pi_0)(\bar{v} - v)}{1 - \delta(1 - \mu)}. \tag{52}
\]

Now I show that the breaking constraint for Seller-optimality is that from $t = 1$.

**Claim.** Whenever $q_1$ is such that Seller-optimality condition for $p = \bar{v}$ at $t = 1$ holds, seller optimality holds also for $t \geq 2$.

**Proof of the Claim.** Note that, because $U(\gamma)q(\gamma)$ is increasing, it holds $U(0)q_1 \leq U(0) \leq U(\gamma)q(\gamma)$. Moreover, it holds that $\delta W(\gamma) < \bar{v}$, uniformly for all $\gamma$. Namely,
\[
\delta W(\gamma) \leq \frac{\delta\mu\bar{v}}{1 - \delta + \delta\mu} < \bar{v}.
\]
Given that $\gamma \mapsto U(\gamma)q(\gamma)$ is strictly monotonic, the result in the claim follows.

**End of the proof of the Claim.**

Note that for $q_1 = 0$ Seller-optimality condition at $t = 1$ is never satisfied, whereas for $q_1 = 1$ it is satisfied whenever
\[
(C - high) \quad U(0)\bar{v} + (1 - U(0))\delta W(U(0)) > \hat{v}.
\]
By use of continuity this gives a sufficient condition for existence of the full-extraction PBE.

One can write $(C - high)$ as
\[
U(0)(1 - \delta + \delta\mu + \delta\mu(1 - U(0)))\bar{v} > (1 - \delta\mu(1 - U(0))^2)\hat{v}.
\]

As $\lambda \to \infty$ (equivalently $\mu \to 1$) it holds $U(0) \to 1$ and so $(C - high)$ in the limit $\mu \to 1$, and fixed other parameters, becomes the condition $\bar{v} > \hat{v}$. This is always true, by assumptions on the primitives.

It remains to specify off-path play. I focus on specifying play only after single deviations.\textsuperscript{53} For any more complicated deviations of the players, general existence theorems show existence of some continuation PBE after such histories.

\textsuperscript{52}In particular, it is also Lipschitz continuous in $(\mu, \gamma)$. We have $W(0) = \frac{\mu}{1 - \delta(1 - \mu)}$ and $W(1) = \frac{\mu\bar{v}}{1 - \delta(1 - \mu)}$.

But because $\gamma$ is bounded above, we know that the highest value of $W$ is actually
\[ W(U(0)) = \frac{\mu}{1 - \delta + \delta\mu} (U(0)\bar{v} + (1 - U(0))\hat{v}). \]

\textsuperscript{53}I specify off-path play only in this Proposition. For the sake of length, in other proofs in the following where stationary equilibria are constructed, I skip specifying off-path play whenever it follows an analogous logic to the one given here.
Seller off-path. Suppose that at the beginning of a period the current belief of Seller is \( \hat{\gamma} \neq 0, \gamma \) (in particular play is at a period \( t \geq 2 \)). Given belief about continuation play, and the fact that parameters are such that for given continuation play, Seller chooses optimally \( p = \bar{v} \) at \( \gamma = 0 \), it holds that: incentives to charge \( p = \bar{v} \) are even stricter when \( \hat{\gamma} > \gamma \) and \( t \geq 2 \) as well as for all \( \hat{\gamma} > 0 \) when \( t = 1 \).

Let us consider now the other case: \( \hat{\gamma} \in [0, \gamma) \) and \( t \geq 2 \). Because the beliefs need to be derived by equilibrium strategies of Buyer whenever possible, there are two cases to consider:

- Case 1: \( \hat{\gamma} \) came about after a past rejection of a price above \( \bar{v} \). In this case, in the very next period after the rejection the belief, from the specification of strategies, would be \( U(\gamma) \). Given continuation play, this would mean that \( \hat{\gamma} > \gamma \), a case already considered above.

- Case 2: \( \hat{\gamma} \) came about after a past rejection of a price \( p \in (\hat{v}, \bar{v}) \). In this case, in the next period Seller should have started with belief 0 and in any future period the starting belief of Seller should be the stationary \( \gamma \). Thus this case is covered by the specification of strategies on path.

Buyer off-path. - If Seller deviates to some price \( p \neq \bar{v} \) every Buyer type responds according to her reservation price strategy: if the price is strictly lower than the reservation price of her type Buyer accepts immediately, if it is strictly higher she rejects.

- If Buyer of type \( v \) has not disclosed in the past, she still remains indifferent between disclosing and not disclosing. Prescribe disclosure in the current period after such a history.

- If Buyer has disclosed good news and Seller has not asked yet for all the surplus, Buyer accepts any price weakly lower than \( \bar{v} \) and rejects any price strictly higher. This is optimal given the anticipation that the continuation payoff of Buyer is zero in the continuation game.

Proof of Proposition 4. First I prove a couple of auxiliary claims.

Claim 1. An equilibrium with zero Buyer payoff necessarily has Seller quote a price \( p = \bar{v} \) as long as bargaining goes on and there is no disclosure.

Proof of Claim 1. Suppose this is not the case and denote by \( t \) the first period in which, after no disclosure, the price quoted by Seller is lower than \( \bar{v} \) with positive probability. W.l.o.g. assume that the prices up to and including period \( t \) upon non-disclosure are in \((\hat{v}, \bar{v})\).\(^{54}\) Suppose that Buyer uses the following strategy: unless there is informational arrival and \( \theta = v \), wait until period \( t \) and in period \( t \) accept the current price if and only if it is weakly above Buyer’s period-\( t \) estimate of the value of the good. If information arrives before or at date \( t \) and \( \theta = v \), disclose immediately. For periods after \( t \), disclose only if \( \theta = v \) (and immediately) and otherwise accept only if price is strictly below current estimate of value. Under this strategy Buyer has a strictly positive payoff with positive probability in period \( t \), and otherwise non-negative payoff in all other histories. Overall, this leads to a contradiction to the assumption that the equilibrium payoff of Buyer is zero.

\(^{54}\)Otherwise it is easy to show Buyer receives positive payoff with positive probability already in period one.
End of proof of Claim 1.

Claim 2. Under any disclosure equilibrium in which Seller quotes \( p = \bar{v} \) on path, the overall payoff of Seller in the HFL is less than \( \frac{\lambda}{\lambda + r} \hat{v} \).

Proof of Claim 2. Fix such an equilibrium as in the statement of the claim for a \( \Delta > 0 \). Denote by \( A(\Delta) \) the random variable giving the agreement time and by \( L(\Delta) \) the random variable giving the time at which Buyer learns. Note that \( L(\Delta) \) is geometrically distributed. Because on path the price quoted upon non-disclosure is always \( \bar{v} \) it holds for both of \( \theta = v, \bar{v} \) that \( A(\Delta) \geq L(\Delta) \) almost surely. Moreover, Seller receives a payoff of either \( \bar{v} \) or \( v \) at time \( A(\Delta) \) under these strategies. One calculates

\[
\text{Seller-payoff} = \bar{v}E[\delta^{A(\Delta)}, \theta = \bar{v}] + vE[\delta^{A(\Delta)}, \theta = v] \\
\leq \bar{v}E[\delta^{L(\Delta)}, \theta = \bar{v}] + vE[\delta^{L(\Delta)}, \theta = v] \\
= \bar{v}\pi_0 E[\delta^{L(\Delta)}|\theta = \bar{v}] + v(1 - \pi_0)E[\delta^{L(\Delta)}|\theta = v] \\
= \bar{v}E[\delta^{L(\Delta)}].
\]

Here, the first inequality follows from \( A(\Delta) \geq L(\Delta) \) which holds almost surely under the assumptions made, whereas the last equality follows from the fact that the arrival of the opportunity to learn is independent of \( \theta \). Now I show that in the HFL \( E[\delta^{L(\Delta)}] \) converges to \( \frac{\lambda}{\lambda + r} \). Recall that \( \delta = \delta(\Delta) = e^{-r\Delta} \) and that \( L(\Delta) \) is geometrically distributed over \( 1, 2, \ldots \) with probability of success given by \( \mu = 1 - e^{-\lambda\Delta} \). It follows

\[
E[\delta^{L(\Delta)}] = \frac{\mu}{1 - \mu} \sum_{t=1}^{\infty} e^{-(r+\lambda)t} = \frac{\mu}{1 - \mu} \frac{e^{-(r+\lambda)\Delta}}{1 - e^{-(r+\lambda)\Delta}}.
\]

One uses that \( \frac{1 - e^{-x\Delta}}{\Delta} \rightarrow x \) as \( \Delta \rightarrow 0 \) for all \( x > 0 \) to finish the proof of the claim.

End of proof of Claim 2.

Now I finish the proof of the Proposition. The condition of optimality of \( p = \bar{v} \) in the first period under the assumption of a disclosure equilibrium is given by

\[
U(0)(\bar{v})q_1 \bar{v} + (1 - U(0)(\bar{v})q_1)\delta W \geq U(0)\{\hat{v}\}\hat{v}, \tag{35}
\]

where \( q_1 \) is the probability with which Buyer of type \( \bar{v} \) accepts the price \( \bar{v} \) in period 1 and \( W \) is the continuation payoff of Seller upon non-disclosure and rejection of price \( \bar{v} \) in period 1. Here \( U(0) \in \mathcal{P}(\{v, \bar{v}, \hat{v}\}) \) is the belief of Seller in \( t = 1 \) over Buyer types after non-disclosure. It is given explicitly by

\[
U(0)(\bar{v}) = \frac{\mu \pi_0}{1 - q_d \mu (1 - \pi_0)}, \\
U(0)(\hat{v}) = \frac{1 - \mu}{1 - q_d \mu (1 - \pi_0)}, \\
U(0)(v) = \frac{(1 - q_d) \mu (1 - \pi_0)}{1 - q_d \mu (1 - \pi_0)},
\]

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where $q_d \in [0,1]$ is the probability with which type $v$ discloses if Buyer learns in $t=1$ that $\theta = v$.

In particular, it holds $U(0)(\{v,\bar{v}\}), U(0)(\{\bar{v}\})q_1 \to 0$ as $\Delta \to 0$ uniformly in $q_1,q_d$. It holds
\[
\delta W = \left(\frac{\text{Seller-payoff} - q_d\mu(1 - \pi_0)v}{1 - q_d\mu(1 - \pi_0)} - U(0)(\bar{v})q_1\right) \frac{1}{1 - U(0)(\bar{v})q_1}.
\]

It follows that
\[
\limsup_{\Delta \to 0} \delta W \leq \frac{\lambda}{r + \hat{v}} < \hat{v}.
\]

Since the left-hand side of (35) in the limit converges to $\hat{v}$ this shows that the condition of optimality of $p = \bar{v}$ upon non-disclosure at $t=1$ cannot be satisfied for $\Delta \to 0$ (deviating to $\hat{v} - \epsilon$ for small and positive $\epsilon$ is strictly better for Seller near the HFL).

This finishes the proof of the Proposition. \qed

A.4 Proof of Theorem 1

I prove Theorem 1 through a series of Lemmas and Propositions. I focus first on the case of strongly stationary equilibria with mixed pricing before analysing pure pricing. This is because their analysis is more involved and several proof steps for the case of pure pricing are similar to the case of mixed pricing. The same organizational principle holds true for all results characterizing HFL of strongly stationary equilibria in the paper.

The following Lemma is helpful for all existence results of strongly stationary equilibria in the paper.

**Lemma 4.** The map $[0,1] \ni q \mapsto \frac{q}{1 - \delta(1 - \mu)(1 - (1-p)q)}$ is decreasing in $p$ and converges uniformly to 1, as $\Delta \to 0$.

**Proof.** One calculates that
\[
\frac{\partial}{\partial q} \left(\frac{q}{1 - \delta(1 - \mu)(1 - (1-p)q)}\right) = \frac{1 - \delta(1 - \mu)}{(1 - \delta(1 - \mu)(1 - (1-p)q))^2},
\]
which shows that this function is strictly increasing and strictly concave for $q \in [0,1]$. \qed

A.4.1 The case of mixed pricing

An auxiliary remark follows which is helpful in the proof of Theorem 1.

**Remark 3.** The function $g : (0,1) \to [0,1], p \mapsto \frac{\delta(1-p)}{1-\delta p}$ is decreasing in $p$ and converges uniformly to 1, as $\Delta \to 0$.

**Proof.** It suffices to note that $0 \leq 1 - \frac{\delta(1-p)}{1-\delta p} \leq 1 - \delta$. \qed

The prices $p_H,p_L$ are required to satisfy
\[
\bar{v} < p_L < p_H < \hat{v}, \text{ and } p_L < \hat{v}.
\]

Suppose Seller mixes among the prices $p_H,p_L$ with probability $(p,1-p)$. The triple $(p_H,p_L,p)$ necessarily satisfies

\[
\text{(36)}
\]
\[ \hat{v} - p_L = \delta(\mu \pi_0(p(\hat{v} - p_H) + (1 - p)(\hat{v} - p_L)) + (1 - \mu)(1 - p)(\hat{v} - p_L) + (1 - \mu)p\delta V_L), \]

where \( V_L \) is the continuation payoff from starting a period in the stationary phase with type \( \hat{v} \).

It holds due to the reservation pricing feature of PBEs (part 2 of Lemma 1) that \( \delta V_L = \hat{v} - p_L \). This leads to

\[ \hat{v} - p_L = \frac{\mu \pi_0(\hat{v} - p_H) + \delta(1 - \mu)(1 - p)(\hat{v} - p_L)}{1 - \delta p(1 - \mu)}. \]

Some algebra leads to

\[ \hat{v} - p_L = \frac{\mu \pi_0(\hat{v} - p_H)}{1 - \delta(1 - \mu)}, \]

so that in total the two reservation-pricing relations about the prices \( p_H, p_L \) are given by

\[ \frac{\hat{v} - p_L}{\hat{v} - p_H} = \frac{\mu \pi_0}{1 - \delta(1 - \mu)}, \quad \frac{\hat{v} - p_L}{\hat{v} - p_H} = \frac{1 - \delta p}{\delta(1 - p)}. \]

**Lemma 5.** 1) In any strongly stationary equilibrium with mixed pricing, \( p_H, p_L \) as a function of \( p \) are given by

\[ p_L(p) = \hat{v} - \frac{\mu \pi_0 \delta(1 - p)}{1 - \delta(1 - \mu) - \mu \pi_0 \frac{\delta(1 - p)}{1 - \delta p}}(\hat{v} - \hat{v}), \tag{37} \]

\[ p_H(p) = \hat{v} - \frac{(1 - \delta(1 - \mu)) \delta(1 - p)}{1 - \delta(1 - \mu) - \mu \pi_0 \frac{\delta(1 - p)}{1 - \delta p}}(\hat{v} - \hat{v}). \tag{38} \]

2) \( p_H, p_L \) are strictly increasing in \( p \in [0, 1] \).

3) The price spread \( ps(p) = p_H(p) - p_L(p) \) is given by

\[ ps(p) = \left[ 1 - \frac{(1 - \delta(1 - \mu)) \delta(1 - p)}{1 - \delta(1 - \mu) - \mu \pi_0 \frac{\delta(1 - p)}{1 - \delta p}} \right](\hat{v} - \hat{v}). \tag{39} \]

**Proof.** This is straightforward algebra. Solving the two reservation price relations of the two Buyer types as a function of the mixing probability \( p \) leads to (37) and (38). The rest is also straightforward calculations. \( \square \)

One checks easily by taking first derivatives, that the price spread \( p_H(p) - p_L(p) \) is strictly increasing in \( p \). Moreover, one checks easily that for every \( p \in (0, 1) \) it holds

\[ \hat{v} < p_L(p) < \hat{v} < p_H(p) < \hat{v}. \]

The boundary values of \( p_H, p_L \) are given by (\( p_H, p_L \) are continuous in \( p \in [0, 1] \)).

\[ p_L(p = 0) = \hat{v} - \frac{\mu \pi_0 \delta}{1 - \delta(1 - \mu) - \mu \pi_0 \delta} (\hat{v} - \hat{v}) \]
\[ p_H(p = 0) = \hat{v} \quad \text{and} \quad p_L(p = 1) = \hat{v}, \quad p_H(p = 1) = \bar{v}. \] (40)

As a final boundary value I note down the price spread at \( p = 0 \).

\[ p_H(0) - p_L(0) = (\bar{v} - v)(1 - \pi_0) \frac{(1 - \delta(1 - \mu))(1 - \delta)}{1 - \delta(1 - \mu) - \mu \pi_0 \delta}. \]

Rewriting Seller’s reservation price relation (7) leads to the definition of a function \( f(p, q) \) which satisfies

\[
\frac{f(p, q)}{\mu(1 - U(0))} = \pi_0 \left( 1 - \frac{\delta(1 - \mu)}{1 - \delta(1 - \mu)} \right) \frac{q}{1 - \delta(1 - \mu)(1 - (1 - p)q)} \delta \left( 1 - \delta \frac{p_L(p) - \pi_0 \hat{p} - (1 - \pi_0) \bar{v}}{\mu} \right).
\]

Existence of a two-price equilibrium with mixing probability \( p \) for Seller is tantamount to finding a root in \( q \) of \( f(p, q) \).

First let us show the following auxiliary lemma.

**Lemma 6.** It holds as \( \Delta \to 0 \) uniformly in \( p \in [0, 1] \) that

A. \( p_L(p) \to \psi := \frac{r \pi_0 \hat{v} + (r + \lambda)(1 - \pi_0) \bar{v}}{r + (1 - \pi_0) \lambda} \),

B. \( p_H(p) - p_L(p) \to 0 \).

**Proof.** A. Note that

\[
p_L(p) = \hat{v} - \frac{\mu \pi_0 \delta(1 - p)}{1 - \delta(1 - \mu) - \mu \pi_0 \delta(1 - p)} (\hat{v} - \hat{v}) = \hat{v} - \frac{\mu \pi_0 (1 + O(\Delta))}{1 - \delta(1 - \mu) - \mu \pi_0 (1 + O(\Delta))} (\hat{v} - \hat{v})
\]

\[
\to \hat{v} - \frac{\lambda \pi_0}{r + (1 - \pi_0) \lambda} (\bar{v} - \hat{v}) = \psi.
\]

Here the last step follows from algebra and the uniformity in \( p \) comes from Remark 3.

B. Use (39) from Lemma 5 to estimate that

\[
\frac{(1 - \delta(1 - \mu)) \delta(1 - p)}{1 - \delta(1 - \mu) - \mu \pi_0 \delta(1 - p)} = (1 + O(\Delta)) \frac{1 - \delta(1 - \mu)}{1 - \delta(1 - \mu) - \mu \pi_0 (1 + O(\Delta))} - \frac{\mu \pi_0}{\Delta}
\]

\[
\to 1.
\]

Here the uniformity of the convergence of \( \delta(1 - p) \) as well as the fact that \( \frac{1 - \delta}{\Delta} \to r, \frac{\mu}{\Delta} \to \lambda \) as \( \Delta \to 0 \) have been used. This and (39) establishes the result.

I note a sharper result for B. which is also used later. In particular, this Remark also proves that the price spread disappears in the HFL.
Remark 4. It holds 
$$\frac{ps(\Delta)}{\Delta} \to \frac{r+\lambda}{r+(1-\pi_0)\lambda} \frac{r}{1-p}\pi_0(1-\pi_0)(\bar{v} - \bar{v}).$$

Proof of Remark 4. Note that after some algebra

$$\frac{ps(p, \Delta)}{\Delta} = \pi_0(\bar{v} - \bar{v}) \left( 1 - \frac{1-\delta(1-\mu)}{\Delta} \frac{1-\delta(1-\mu)}{\Delta} - \frac{r}{\Delta} \pi_0 \right),$$

from which the result follows immediately, because $\bar{v} - \bar{v} = (1 - \pi_0)(\bar{v} - \bar{v})$.

Now I establish existence of mixed stationary equilibria for any $p \in (0, 1)$, whenever $\Delta$ small enough.

**Proposition 12.** For any $p \in (0, 1)$ there exist strongly stationary equilibria for all $\Delta$ small enough. Moreover, for fixed $\Delta$ and fixed $p$ such that existence is ensured, the prices $p_L, p_H$ and the mixing probability $q$ of the type $\hat{v}$ when she faces $p_L$ are unique.

**Proof.** Fix $p \in (0, 1)$.

**Existence.** Note that 
$$\frac{f(p, \Delta)}{\mu(1-U(0))} = \pi_0 \left( \frac{p_H(p) - p_L(p)}{1-\pi_0} \right) > 0$$

by construction. Thus, it suffices to show that $\frac{f(p, \Delta)}{\mu(1-U(0))} < 0$ for all $\Delta$ small enough. First note that

$$\frac{1-\delta(1-\mu)}{\delta \mu} p_L(p, \Delta) - \pi_0 \hat{p} - (1 - \pi_0) \psi \to \left( \frac{r+\lambda}{\lambda} - \pi_0 \right) \psi - (1 - \pi_0) \psi > 0. \quad (41)$$

The last inequality follows from the fact that $\psi > \psi$, as one can easily check by looking at the statement in part A. of Lemma 6. $(41)$ and Lemma 4 show that $[0, 1] \ni q \mapsto \frac{f(p, q)}{\mu(1-U(0))}$ is strictly decreasing and strictly convex in $q$. From the results in Lemma 6 (namely part B. there) one arrives at

$$\lim_{\Delta \to 0} \frac{f(p, q)}{\mu(1-U(0))} = -\frac{1}{1-p} \left( \frac{r}{\lambda} (\psi + (1-\pi_0)(\psi - \psi) \right) < 0. \quad (42)$$

Here, one uses that $\delta(1-\mu) = 1 + O(\Delta)$. Fixing some $\bar{\Delta}(p)$ where $\frac{f(p, q)}{\mu(1-U(0))} < 0$ with $\Delta < \bar{\Delta}(p)$ it follows that there exists a unique zero for $\frac{f(p, q)}{\mu(1-U(0))}$ whenever $\Delta < \bar{\Delta}(p)$, denoted $q(\Delta, p)$. Moreover, $q(\Delta, p)$ is in $(0, 1)$.

**Uniqueness.** The arguments above show uniqueness of $q(\Delta, p)$. Uniqueness of the prices $p_L(\Delta, p)$ and $p_H(\Delta, p)$ follows from $(38)$ and $(37)$ in Lemma 5.

Next I characterize the HFL of mixed pricing equilibria. This involves several steps. First, one calculates explicitly $q(\Delta, p)$ for fixed $p$ and all $\Delta$ small enough. From the condition $\frac{f(p, q)}{\mu(1-U(0))} = 0$ it follows that $q(\Delta, p)$ satisfies the relation

$$\frac{\pi_0}{1-\mu} ps(p) = \frac{q(\Delta, p)}{1 - \delta(1-\mu)(1 - (1-p)q(\Delta, p))} h(p, \Delta, \pi_0, \psi),$$

The function $h(p, \Delta, \pi_0, \psi)$ here is strictly positively valued and it becomes the constant function $h^* := \frac{r}{\lambda} (\psi + (1-\pi_0)(\psi - \psi))$ in the HFL. This follows from straightforward algebra. Solving explicitly for $q(\Delta, p)$ one arrives at
\[ q(\Delta, p) = \frac{(1 - \delta(1 - \mu))G(p, \Delta)}{1 - \delta(1 - \mu)(1 - p)G(p, \Delta)}, \]

where

\[ G(\Delta, p) = \frac{\pi_0}{1 - \mu h(p, \Delta, \pi_0, v)}. \]

Using Remark 4 one calculates that

\[ \frac{G(\Delta, p)}{\Delta} \rightarrow G^*(p) := \frac{r \pi_0}{1 - p h^* r + (1 - \pi_0)\lambda} \frac{r + \lambda}{\pi_0(1 - \pi_0)(\bar{v} - v)}. \]

It follows that

\[ \frac{q(\Delta, p)}{\Delta^2} = \frac{\frac{1 - \delta(1 - \mu) G(\Delta, p)}{\Delta}}{1 - \delta(1 - \mu)(1 - p)G(p, \Delta)}, \]

which leads to the HFL statement

\[ \frac{q(\Delta, p)}{\Delta^2} \rightarrow (r + \lambda)G^*(p), \quad \Delta \rightarrow 0. \tag{43} \]

The date of agreement is a geometric random variable with success probability \(1 - (1 - \mu)(1 - (1 - p)q(\Delta, p))\). One calculates using (43)

\[ \frac{1 - (1 - \mu)(1 - (1 - p)q(\Delta, p))}{\Delta} = \frac{\mu}{\Delta} + (1 - p)q(\Delta, p) \frac{1 - \delta(1 - \mu)(1 - (1 - p)q)}{\Delta^2} \cdot \Delta \rightarrow \lambda. \]

In all, the expected delay in real time converges to \(\frac{1}{\lambda}\), irrespective of \(p\).

Finally, one calculates the stationary payoffs of Buyer and Seller. Recall the relation

\[ V_\Delta(q, p) = \frac{\mu\pi_0 (p_H p + p_L (1 - p)) + (1 - \mu)(1 - p)qp_L + \mu(1 - \pi_0)v}{1 - \delta(1 - \mu)(1 - (1 - p)q)}, \]

for Seller’s stationary payoff. Using Lemma 6 and (43) one arrives easily at the limit

\[ V_S = \frac{\lambda}{r + \lambda} (\pi_0 \psi + (1 - \pi_0)v), \tag{44} \]

for the payoff of Seller in the HFL. The payoff of Buyer in any two-price stationary equilibria satisfies

\[ V_{\Delta,B}(q, p) = \frac{\mu\pi_0 (\bar{v} - \hat{p}) + (1 - \mu)(1 - p)q(\bar{v} - p_L)}{1 - \delta(1 - \mu)(1 - (1 - p)q)}. \]

By similar steps as the case of Seller’s payoff this converges in the HFL to

\[ V_B = \frac{\lambda}{r + \lambda} \pi_0 (\bar{v} - \psi). \tag{45} \]

Note that \(V_B, V_S\) are independent of \(p \in (0, 1)\). Straightforward algebra leads to the measure of inefficiency in the HFL given by

\[ \hat{v} - (V_B + V_S) = \frac{r}{r + \lambda} \hat{v}. \]
A.4.2 The case of pure pricing

For simplicity of exposition define in the following the function \( \zeta : [0, 1] \times (0, \infty) \to \mathbb{R}_+ \) by

\[
\zeta(q, \Delta) = \frac{q}{1 - \delta(1 - \mu)(1 - q)}.
\]

After algebra one can rewrite the Seller optimality condition in (6) as

\[
\frac{1 - U(0)}{U(0)} \zeta(q, \Delta) \frac{p_L - \delta V_{\Delta}(q, 0)}{1 - \delta}(1 - \delta(1 - \mu)(1 - q)) \geq \bar{v} - p_L.
\]

Lemma 7. 1) Fix any \( \Delta > 0 \). The reservation price relation of type \( \hat{v} \) is solvable for a unique \( p_L(\Delta) \). It holds \( p_L(\Delta) \to \psi \), as \( \Delta \to 0 \).

2) Let \( q(\Delta) \) satisfy \( \frac{q(\Delta)}{\Delta} \to \kappa \) for some \( \kappa \in [0, \infty] \). It holds

\[
\zeta(q(\Delta), \Delta) \to \frac{\kappa}{\kappa + r + \lambda}, \quad \Delta \to 0,
\]

where \( \frac{\kappa}{\kappa + r + \lambda} \) is to be understood as equal to 1, if \( \kappa = \infty \).

3) For any sequence \( q(\Delta), \Delta \to 0 \) it holds

\[
\frac{p_L(\Delta) - \delta V_{\Delta}(q(\Delta), 0)}{1 - \delta}(1 - \delta(1 - \mu)(1 - q(\Delta))) \to \psi + \frac{\lambda}{r}(1 - \pi_0)(\psi - \bar{v}) > 0, \quad \Delta \to 0.
\]

Proof. 1) One solves explicitly the reservation pricing relation

\[
\hat{v} - p_L = \frac{\delta \mu}{1 - \delta + \delta \mu} \pi_0 (\bar{v} - p_L),
\]

to get

\[
p_L(\Delta) = \frac{\hat{v} - \frac{\delta \mu}{1 - \delta + \delta \mu} \pi_0 \bar{v}}{1 - \frac{\delta \mu}{\delta \mu + (1 - \delta) \pi_0}}.
\]

Limit algebra leads to the conclusion that \( p_L(\Delta) \to \psi \).

2) It holds for every \( \Delta > 0 \) that

\[
\zeta(q(\Delta), \Delta) = \frac{q(\Delta)}{1 - \delta(1 - \mu)} + \frac{\bar{v}}{\delta(1 - \mu)}.
\]

The rest is simple limit algebra.

3) Simple algebra leads to

\[
\frac{p_L(\Delta) - \delta V_{\Delta}(q(\Delta), 0)}{1 - \delta}(1 - \delta(1 - \mu)(1 - q(\Delta)))
\]

\[
= p_L \frac{1 - \delta(1 - \mu)}{1 - \delta} - (1 - \mu)q(\Delta)p_L - \frac{\mu}{1 - \delta}(\pi_0 p_L + (1 - \pi_0)\bar{v}).
\]

The rest is simple limit algebra. \( \square \)

Lemma 7 leads to the following result about solvability of (46).

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Lemma 8. Fix any $v, \bar{v}, \pi_0, r, \lambda$ as in the setup of the model in section 2. Furthermore, fix any $q \in (0, 1)$.

Then (6) is solvable whenever $\Delta$ is small enough.

Proof. This is an easy consequence of the fact that $\frac{1-U(0)}{U(0)} = \frac{1-\mu}{\mu \pi_0} \to \infty$ as $\Delta \to 0$ and of results in Lemma 7.

Lemma 8 and the preceding calculations show existence of strongly stationary equilibria with pure pricing for any baseline game parameters in section 2. In particular, because $\frac{1-U(0)}{U(0)} \to \infty$, one can always pick a sequence of positive probabilities $q(\Delta) > 0, \Delta \to 0$ such that $\frac{q(\Delta)}{\Delta} \to \kappa$ as $\Delta \to 0$. This shows that for any $\kappa \in [0, \infty]$ there exists HFL of strongly stationary equilibria with pure pricing corresponding to $\kappa$.

Uniqueness up to the acceptance probability $q$ near the HFL for strongly stationary equilibria with mixed pricing is immediate from the arguments above.

Next I characterize HFL payoffs depending on $\kappa$. It holds for Seller payoff in strongly stationary equilibrium with pure pricing

$$V_S(\kappa) = \begin{cases} \hat{v}, & \text{if } \kappa = \infty, \\ \frac{\lambda \pi_0 \bar{v} + (\lambda \pi_0 + \kappa) \psi}{r + \lambda + \kappa}, & \text{if } \kappa \in [0, \infty). \end{cases}$$

For Buyer payoff one calculates

$$V_B(\kappa) = \begin{cases} \hat{v} - \psi, & \text{if } \kappa = \infty, \\ \frac{\lambda \pi_0 \bar{v} + (\lambda \pi_0 + \kappa) \psi}{r + \lambda + \kappa}, & \text{if } \kappa \in [0, \infty). \end{cases}$$

The sum of Buyer and Seller payoffs in the HFL is

$$\begin{cases} \hat{v}, & \text{if } \kappa = \infty, \\ \frac{\kappa + \lambda}{\kappa + \lambda + r} \hat{v}, & \text{if } \kappa \in [0, \infty). \end{cases}$$

Note that the sum of Buyer and Seller payoffs in the HFL is an increasing function of $\psi$ and converges to its corresponding value $\hat{v}$ as $\kappa$ moves along a finite sequence which converges to $\infty$.

Finally, it remains to calculate the expected delay in the HFL for a sequence of strongly stationary equilibria with pure pricing corresponding to some $\kappa \in (0, \infty]$. Equilibrium

\[\text{Equilibrium for } \kappa = 0 \text{ one can also use the strongly stationary equilibria with mixed pricing of the previous subsection, whereas for } \kappa = \infty \text{ one can pick } q(\Delta) = q_0 \in (0, 1] \text{ with the understanding that if } q_0 = 1 \text{ the equilibrium satisfies ‘divinity in bargaining’.}\]
construction shows that the agreement date is a geometric random variable with success probability \(1 - (1 - \mu)(1 - q(\Delta))\). One calculates thus for the expected delay

\[
\frac{\Delta}{1 - (1 - \mu)(1 - q(\Delta))} = \frac{1}{\frac{\mu}{\Delta} + \frac{q(\Delta)}{\Delta}(1 - \mu)} \rightarrow \frac{1}{\lambda + \gamma}, \quad \Delta \to 0,
\]

where the limit value is to be understood as zero if \(\gamma = \infty\).

This finishes the proof of Theorem 1.

### B  Proofs for section 3

#### B.1 ‘Positive Buyer payoff in every equilibrium with costs’ and its corollaries

I show a slightly more general statement than that of Theorem 2. The generalization consists in

- I also look at the case of endogenous choice of intensity (see section 4.1)
- I assume that Buyer can pick any two-dimensional experiments.

The two-parametric experiments are modeled as follows.

**Remark 5.** A general experiment is given by \(\mathcal{E} : \{v, \bar{v}\} \to \Delta(\{H, L\})\). \(s \in \{H, L\}\) is the signal Buyer sees after performing the experiment. An experiment \(\mathcal{E}\) is fully identified with the two accuracy parameters \(a_H = \mathbb{P}(s = H|\theta = \bar{v})\) and \(a_L = \mathbb{P}(s = L|\theta = v)\). The experiment is uninformative if and only if \(\frac{a_H}{1-a_L} = 1 \iff \frac{a_L}{1-a_H} = 1\).

Because otherwise one can always relabel signals, one can assume w.l.o.g. that the region of possible accuracy parameters is given by

\[

\nabla = \{(a_H, a_L) : a_H, a_L \in [0, 1], a_H + a_L \geq 1\}.

I restrict in the rest of this subsection of the appendix to experiments parametrized by pairs \((a_H, a_L)\) in \(\nabla\).

Define the function \(L : \text{int}(\nabla) \to \{(l_1, l_2) \in (1, \infty)^2, l_1 + l_2 < l_1l_2 + 1\}\) given by

\[
L(a_H, a_L) = \left(\frac{a_H}{1-a_L}, \frac{a_L}{1-a_H}\right).
\]

This map is a diffeomorphism.\(^{56}\) Note that the two variables \((l_1, l_2)\) are in \((1, \infty)\) and independent of each other, as long as they are different from 1 and satisfy the condition \(l_1 + l_2 > l_1l_2 + 1\).

They correspond to the informativeness of the two signals \(H, L\) w.r.t. the two states. To see this, note e.g. that

\[
l_1 = \frac{a_H}{1-a_L} = \frac{P(s = H|\theta = \bar{v})}{P(s = H|\theta = v)}.
\]

Let \(v(H)\) be the valuation of the good, starting from prior \(\pi_0\), if \(s = H\) and analogously \(v(L)\) the valuation if \(s = L\). It holds

\[
v(H) = \frac{a_H\pi_0\bar{v} + (1 - a_L)(1 - \pi_0)v}{a_H\pi_0 + (1 - a_L)(1 - \pi_0)}, \quad v(L) = \frac{(1 - a_H)\pi_0\bar{v} + a_L(1 - \pi_0)v}{(1 - a_H)\pi_0 + a_L(1 - \pi_0)},
\]

\(^{56}\)See section on (non-)concavity of value of information in the online appendix for a proof.
for the Bayesian estimates of the value of good, upon observing a high (H) or low (L) signal.

One calculates

\[ v(H) - \hat{v} = \pi_0(1 - \pi_0)(\bar{v} - v) \left( \frac{1}{\pi_0 + \frac{1}{1-a_L}} \right) \]

and

\[ \hat{v} - v(L) = \pi_0(1 - \pi_0)(\bar{v} - v) \left( \frac{1}{\frac{1}{1-a_H} + \pi_0} \right). \]

As straightforward algebra shows, it holds that \( v(H) - \hat{v} \) is increasing and concave in \( \frac{a_H}{1-a_L} \)
and \( \hat{v} - v(L) \) is increasing and concave in \( \frac{a_L}{1-a_H} \).

It is therefore economically meaningful to put the costs on the pair \( \left( \frac{a_H}{1-a_L}, \frac{a_L}{1-a_H} \right) \). Namely, one can interpret the differences \( v(H) - \hat{v} \) and \( \hat{v} - v(L) \) as ‘outputs’ from a production process of information in which the ‘inputs’ are precisely \( l_1, l_2 \).

Say \( c : [1, \infty)^2 \to \mathbb{R} \) is a cost function on informativeness if it satisfies

A. \( c(1, 1) = 0 \),
B. \( c \) is strictly convex and increasing in each argument,
C. \( \lim_{t \to \infty} c'(t, b) = \lim_{t \to \infty} c'(a, t) = +\infty \) for every \( a, b \in [1, \infty) \).

Finally, say that \( C : [\frac{1}{2}, 1]^2 \to \mathbb{R}_+ \) is a cost function if it holds \( C(a_H, a_L) = c(L(a_H, a_L)) \) for all pairs \( (a_H, a_L) \in [\frac{1}{2}, 1]^2 \) and a \( c \) which is a cost function on informativeness.

In the model with deterministic variable costs assume for this section of the appendix only, that Buyer possesses a cost function as defined in Remark 5.

**Proposition 13.** There is no high-price equilibrium in either of the following cases:
- deterministic variable accuracy costs with \( \partial_a C(\frac{1}{2}, \frac{1}{2}) = 0 \) for \( a = a_H \) or \( a = a_L \)
- stochastic fixed accuracy costs.

**Proof.** The proof is by contradiction. Pick a Seller-history \( h \) on path. Because it is an on-path history and because of part 4) of Lemma 1 (learning is possible only once), one can focus on \( h \) such that there has not been any disclosure until that point in time and in which Buyer has rejected all prices up to that point in time.

Assume first, \( h \) is the shortest on-path history that has Seller put positive probability on Buyer having received some news.

Let \( \bar{w}(h) \) be the highest type feasible from the perspective of Seller after history \( h \). Thus, it corresponds to an agent who has learned. According to the conjecture, the reservation price \( \bar{p}(h) \) of Buyer who has received good news, is quoted with probability one. This is because reservation prices move co-monotonically with the type of Buyer (see Corollary 1), and because learning happens once. Because of learning once, the type of Buyer won’t change over time so going forward \( \bar{p}(h) \) is the only price being quoted as long as no agreement is reached. Thus, equilibrium payoff for this Buyer type going forward is also \( \bar{w}(h) - \bar{p}(h) \). From the reservation pricing relation of Buyer with good news, \( \bar{w}(h) - \bar{p}(h) = \delta(\bar{w}(h) - \bar{p}(h)) \), one sees that this type has zero continuation payoff, i.e. \( \bar{w}(h) = \bar{p}(h) \).
Now look at the history $h'$ which precedes $h$ by one period and has Buyer get a chance to learn with positive probability. Assume that $h'$ exists.

Seller after no-disclosure at $h'$ thinks that Buyer has not learned yet (this is because of the definition of $h$). Thus, Seller charges at the end of that period the reservation price of $\hat v$, which is based on the specified continuation play in the equilibrium.

Consider first the model with stochastic fixed costs of accuracy. I show that Buyer has incentives to learn some of the time. This is because with positive probability the opportunity to learn will arrive and the costs will be small enough to justify learning $\theta$ conclusively. In the case of good news, Buyer makes a profit of at least $\bar v - \hat v$, whereas in the case of bad news the payoff is zero.\footnote{Here, one uses that reservation prices of a type are weakly below their valuation.} Thus, when $c$ is so low that

$$\pi_0(\bar v - \hat v) - c > 0, \quad (47)$$

Buyer has strict incentives to learn. Note that (35) happens with positive probability under the assumptions on $F$ in section 3.

Consider next the model with deterministic variable costs on accuracy, with two-parametric experiments and so that the assumption in the statement is satisfied.

I show that for some pair $(a_H, a_L)$ it holds $\pi_0 a_H(\bar v - \hat v) + (1 - \pi_0)(1 - a_L)(\bar v - \hat v) - C(a_H, a_L) > 0$.

To see this, suppose the costs satisfy $\partial_a C(\frac{1}{2}, \frac{1}{2}) = 0$ for either $a = a_H, a_L$. Assume it for $a = a_H$, the other case being analogous. Then for $a_H, a_L$ near $\frac{1}{2}$ given by $a_H = \frac{1}{2} + \epsilon, a_L = \frac{1}{2}$ with some small $\epsilon > 0$, one has

$$\pi_0 a_H(\bar v - \hat v) + (1 - \pi_0)(1 - a_L)(\bar v - \hat v) - C(a_H, a_L) = \epsilon \pi_0(1 - \pi_0)(\bar v - \hat v) - C(\frac{1}{2} + \epsilon, \frac{1}{2})$$

$$= \epsilon \pi_0(1 - \pi_0)(\bar v - \hat v) - O(\epsilon^2). \quad (48)$$

Here the last equality uses Taylor formula. Thus, there are incentives to learn at least just very little, just before the last period in $h$.

Thus, it has to be that $h$ corresponds to a history started in period one (i.e. $h'$ does not exist). In particular, the equilibrium must prescribe that Buyer chooses to learn with positive probability in period one, if she gets the chance to learn already in that period.

Take first the model where intensity is exogenous. Then Buyer has zero benefit from learning because of the conjectured high-price structure of Seller's continuation strategy. Since learning is costly, Buyer does not learn so the conjecture of Seller about Buyer learning with positive probability already in the first period is wrong. This is a contradiction.

Consider next the model in which intensity is endogenously chosen at a cost (see assumptions in section 4.1). The same argument leads to a contradiction, because, given that the benefit of learning is zero in the continuation play, and picking a positive intensity is costly, Buyer does not pick a positive intensity at all.

Overall, the assumption of a high-price equilibrium leads to a contradiction under the assumptions on costs made in the statement of the Proposition.

\[\square\]

**Proof of Theorem 2.** Recall Lemma 3. It suffices to show the following claim.

\footnote{I.e. signal structure is informative only in the case of good news but by very little.}
**Claim.** For any $\Delta > 0$ there are no high-price equilibria, if in the case of deterministic variable costs the set of available experiments to Buyer is constrained to the one-parametric one in section 3.

To see this, one adapts the proof of Proposition 13 to show that there are no high-equilibria with costly learning in the set up of restricted experiments from section 3. The only change necessary is in the case of deterministic variable costs. Assume that instead of general experiments, Buyer only has access to the one-parametric ones from section 3. One looks at a deviation to $a = \frac{1}{2} + \epsilon$ and one replaces (48) with

$$
\pi_0\left(\frac{1}{2} + \epsilon\right)(\bar{v} - \hat{v}) + (1 - \pi_0)\left(\frac{1}{2} - \epsilon\right)(\bar{v} - \hat{v}) - c\left(\frac{1}{2} + \epsilon\right) = \epsilon\pi_0(1 - \pi_0)(\bar{v} - \hat{v}) - O(\epsilon^2).
$$

Here one uses that $\frac{1}{2} + \epsilon = 1 + \frac{4\epsilon}{1 - 2\epsilon} = 1 + O(\epsilon)$, as $\epsilon \rightarrow 0$. The remaining formal arguments to conclude the proof of the Claim are verbatim the same as in the proof of Proposition 13. The combination of Lemma 3 and of the Claim finishes the proof of Theorem 2.

Theorem 2 and Proposition 13 have several important implications. Call an equilibrium a no-learning equilibrium if on path, Buyer learns with probability zero.

**Corollary 3.** There is no no-learning equilibrium under the conditions of Proposition 13.

**Proof.** This follows from the arguments in the proof of Proposition 13. In an equilibrium in which Buyer never learns, the price quoted by Seller in every period, as long as the bargaining goes on, is given by $\hat{v}$. But under this requirement, it occurs with positive probability that Buyer has strict incentives to learn whenever she gets the chance.

**Corollary 4.** In the presence of costs there is no stationary high-price equilibrium.

**Proof.** This follows immediately from Proposition 13 when specializing to stationary equilibria.

In particular, none of the stationary high-price equilibria from Proposition 3 survives the introduction of learning costs.

**B.2 Proof of Proposition 6**

The proof is split according to the assumptions on learning costs (deterministic variable or stochastic fixed) and makes use of a series of auxiliary results stated in the following in the form of Lemmas.

As in the case of costless learning I exhibit first the proof for the case of mixed pricing and then add details for the case of pure pricing.

**B.2.1 The case of mixed pricing**

**The case of deterministic variable costs.** To save on notation introduce the shortcut $C(a) = c(I(a))$ for $a \in [\frac{1}{2}, 1)$.

**Lemma 9.** There exists $\epsilon = \epsilon(c, \pi_0, \hat{v}) > 0$ such that for all $\hat{p}$ and $\Delta$ with $|\hat{v} - \hat{p}| < \epsilon$, $\Delta < \epsilon$ the reservation pricing relations of Buyer with good news and Buyer of type $\hat{v}$ are uniquely solvable.
Proof. Corollary 3 implies that any PBE with the parametric assumptions made on costs in section 3 involves some amount of learning by Buyer.

Focus in the following only on \( \hat{p} \) such that \( BL(\hat{p}) := V_A(a(\hat{p}), \hat{p}) - C(a(\hat{p})) > 0 \). Because \( BL(\hat{v}) > 0 \) and continuity, the requirement \( BL(\hat{p}) > 0 \) is fulfilled in a small enough open neighborhood of \( \hat{v} \).

Fix an average price \( \hat{p} \) such that \( v < \hat{p} < \hat{v} \). With the assumption on costs it follows that \( a(\hat{p}) > \frac{1}{2} \). Therefore, the possible Buyer valuations \( \hat{w}, \hat{w} \) are also functions of \( \hat{p} \), i.e. one writes \( \hat{w}(\hat{p}), \hat{w}(\hat{p}) \). From the reservation price relation for type \( \hat{w} \) one can write \( p_L(\hat{p}) = (1 - \delta)\hat{w}(\hat{p}) + \delta \hat{p} \), whereas from the reservation price relation of type \( \hat{v} \) one can write

\[
p_L(\hat{p}) = \hat{v} - \frac{\delta \mu}{1 - \delta + \delta \mu} (V(a(\hat{p}), \hat{p}) - C(a(\hat{p}))).
\]

Note that the reservation price relation for \( \hat{v} \) does not deliver \( p_L < \hat{p} \). It only delivers \( p_L < \hat{v} \). To ensure that for given \( \hat{p} \) it holds for \( p_L(\hat{p}) \) that \( p_L(\hat{p}) < \hat{p} \), ask for \( \hat{p} \) near enough to \( \hat{v} \) it holds that \( V_A(a(\hat{v}), \hat{v}) - C(a(\hat{v})) > V_A(\frac{1}{2}, \hat{v}) - C(\frac{1}{2}) = \frac{1}{2}(\hat{v} - \hat{v}) = 0 \). Therefore in the following restrict to a neighborhood \( \mathcal{N} \) of \( \hat{v} \) such that

\[
\hat{v} - \hat{p} < \frac{1}{2} \frac{\lambda}{\lambda + r} (V_A(a(\hat{p}), \hat{p}) - C(a(\hat{p}))). \tag{49}
\]

In addition \( \mathcal{N} \) is required to satisfy \( BL(\hat{p}) > 0 \) for all \( \hat{p} \in \mathcal{N} \).

Fulfillment of (49) for all \( \hat{p} \) near \( \hat{v} \) is ensured because the inequality is true for \( \hat{p} = \hat{v} \) and the involved functions are continuous.\(^{60}\)

Now let \( \Delta > 0 \) be small enough and pick \( \mathcal{N}_1 \), a compact non-empty subinterval of \( \mathcal{N} \) such that for all \( \hat{p} \in \mathcal{N}_1 \) and \( \Delta \leq \Delta \) it holds

\[
\hat{v} - \hat{p} < \frac{\delta \mu}{1 - \delta + \delta \mu} (V_A(a(\hat{p}), \hat{p}) - C(a(\hat{p}))).
\]

This is again possible due to continuity and the fact that \( \frac{\delta \mu}{1 - \delta + \delta \mu} \to \frac{\lambda}{\lambda + r} \), as \( \Delta \to 0 \).

This, together with the trivial bound \( \frac{\delta \mu}{1 - \delta + \delta \mu} \leq 1 \), leads to \( p_L(\hat{v}) < \min\{\hat{p}, \hat{v}\} \), as solved from the reservation price relation \( \hat{v} \). This finishes the proof of the Lemma.

Focus in the following on \( \Delta \) small and \( \hat{p} \) with \( |\hat{v} - \hat{p}| < \epsilon \), as needed in statement of Lemma 9.

Once \( p_L(\hat{p}) \) is determined with the property \( p_L(\hat{p}) < \hat{p} \), the stationary mixing probability of Seller upon non-disclosure is uniquely determined the formula \( p(\hat{p}) = \frac{\rho - p_L(\hat{p})}{\rho H(\hat{p}) - p_L(\hat{p})} \). This proves that the sufficient statistic for the construction of the strongly stationary equilibria is \( \hat{p} \).

**Lemma 10.** It holds $w(\hat{p}) < \hat{p}$ in a suitable open neighborhood of $\hat{v}$.

**Proof.** To see this, note that $w(\hat{p})$ is a continuous function of $\hat{p}$ and that $w(\hat{v}) < \hat{v}$, because valuation $w$ is induced by bad news.

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\(^{59}\)The inequality is strict because of the fact that $C$ is strictly convex and Corollary 3.

\(^{60}\)Continuity of $(v, \hat{v}) \ni \hat{p} \mapsto V(a(\hat{p}), \hat{p}) - C(a(\hat{v}))$ follows from Berge’s maximum theorem.
In the following take $\mathcal{N}$, an open neighborhood around $\hat{v}$ and $\Delta \leq \bar{\Delta}$ with $\bar{\Delta} > 0$ such that

- $BL(\hat{p}) > 0, \hat{p} \in \mathcal{N}$,
- Lemmas 9 and 10 are true for the neighborhood $\mathcal{N}$ of $\hat{v}$ and $\Delta \leq \bar{\Delta}$.

It holds automatically that $V(a, \hat{p}) < \hat{v}$ whenever $\hat{p} > 0$. To see this, use the definition to get $V_A(a, \hat{p}) < \pi_0 \hat{v} + (1 - \pi_0)q - (\pi_0 a + (1 - \pi_0)(1 - a))\hat{p} < \hat{v}$. This is a uniform bound which does not use the parametric form of the costs nor the value of $\lambda, \Delta$. One sharpens this estimate by noticing that $\pi_0 a + (1 - \pi_0)(1 - a) \geq c(\pi_0) := \min\{\pi_0, 1 - \pi_0\}$. This leads to the inequality

$$V_A(a, \hat{p}) \leq \hat{v} - c(\pi_0)\hat{p}. \quad (50)$$

This inequality is uniform in the specification of costs, $\lambda, \Delta$. It also does not depend on the reservation price relation for type $\hat{v}$. Therefore, by using the reservation pricing relation for the type $\hat{v}$ one arrives at the uniform estimate

$$p_L(\hat{p}) \geq \frac{1 - \delta}{1 - \delta + \delta \mu} \hat{v} + \frac{\delta \mu}{1 - \delta + \delta \mu} c(\pi_0)\hat{p}, \quad (51)$$

whenever $\hat{p} \in \mathcal{N}$.

From (51) and the reservation pricing relation for Buyer with good news $\bar{w}$ one arrives at the estimate

$$p_H(\hat{p}) - p_L(\hat{p}) \leq (1 - \delta)\bar{w}(\hat{p}) + \delta \left(1 - \frac{\mu}{1 - \delta + \delta \mu} c(\pi_0)\right) \hat{p} - \frac{1 - \delta}{1 - \delta + \delta \mu} \hat{v}. \quad (52)$$

Seller’s indifference condition reduces to the study of zeros of the function

$$f(\hat{p}, q) = U(0)(p_H(\hat{p}) - p_L(\hat{p})) + (1 - U(0))q(\delta V_{\Delta}(q, \hat{p}) - p_L(\hat{p})).$$

Here $V_{\Delta}(q, \hat{p})$ is Seller’s payoff which is given in equilibrium by

$$V_{\Delta}(q, \hat{p}) = \frac{\mu GN(a(\hat{p}))\hat{p} + (1 - \mu)(1 - p)qp_L(\hat{p}) + \mu(1 - GN(a(\hat{p}))\bar{w}(\hat{p}))}{1 - \delta(1 - \mu)(1 - (1 - p)q)}, \quad (53)$$

where the shortcut $GN(a)$ denotes the stationary probability of good news given by $GN(a) = a\pi_0 + (1 - a)(1 - \pi_0)$. To see (53), note that $V_{\Delta}(q, \hat{p})$ satisfies the recursion

$$V_{\Delta}(q, \hat{p}) = \mu GN(a(\hat{p}))\hat{p} + (1 - \mu)(1 - p)qp_L + \mu(1 - GN(a(\hat{p})))\bar{w}(\hat{p}) + \delta(1 - \mu)(1 - (1 - p)q)V_{\Delta}(q, \hat{p}).$$

One calculates

$$\delta V_{\Delta}(q, \hat{p}) - p_L(\hat{p}) = \frac{\delta \mu GN(a(\hat{p}))\hat{p} + (\delta(1 - \mu) - 1)p_L(\hat{p}) + \delta \mu(1 - GN(a(\hat{p})))\bar{w}(\hat{p})}{1 - \delta(1 - \mu)(1 - (1 - p(\hat{p}))q)} \quad (54)$$

$$\leq \frac{\delta \mu \hat{p} + (\delta(1 - \mu) - 1)p_L(\hat{p})}{1 - \delta(1 - \mu)(1 - (1 - p(\hat{p}))q)} \leq \frac{\delta \mu(1 - c(\pi_0))\hat{p} - (1 - \delta)\hat{v}}{1 - \delta(1 - \mu)(1 - (1 - p(\hat{p}))q)}.$$  

---

$^{61}$Note that $c(\pi_0) > 0$. 

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Here the first inequality uses that \( w(\hat{v}) < \hat{v} \) and that \( \hat{p} \) is close to \( \hat{v} \) (recall the restrictions on \( \mathcal{N} \)). The second inequality in (54) uses (51). Impose now the following assumption on \( \delta, \mu \):

\[
\text{agents are not too patient: } \frac{\delta \mu}{1-\delta}(1-c(\pi_0)) < 1. \tag{55}
\]

Note that in the high-frequency limit this assumption corresponds to \( \lambda(1-c(\pi_0)) < r \). A sufficient condition for satisfying (55) irrespective of the prior is \( r > \lambda \), i.e. the discount rate is higher than the arrival rate of opportunities to learn.

Combine (55) with (54) to get that uniformly for all \( \hat{p} \in \mathcal{N}, \Delta \leq \Delta \) and all \( \delta, \mu \) satisfying (55) that

\[
q(\delta V(q, \hat{p}) - p_L(\hat{p})) \leq \frac{q}{1-\delta(1-\mu)(1-(1-p(\hat{p})))}(\delta \mu(1-c(\pi_0))\hat{p} - (1-\delta)\hat{v}) \\
\leq q(\delta \mu(1-c(\pi_0))\hat{p} - (1-\delta)\hat{v}).
\]

Look now at \( \frac{f(\hat{p}, q)}{\mu(1-U(0))} = \frac{\pi_0}{1-\mu}(p^H(\hat{p})-p_L(\hat{p})) + \frac{q}{\mu}(\delta V(q, \hat{p}) - p_L(\hat{p})) \). Overall the following estimate results for all \( \hat{p} \in \mathcal{N}, \Delta \leq \Delta \) and all \( \delta, \mu \) satisfying (55)

\[
\frac{f(\hat{p}, q)}{\mu(1-U(0))} \leq \frac{\pi_0}{1-\mu} \left( (1-\delta)\hat{v}(\hat{p}) + \delta \left( 1 - \frac{\mu}{1-\delta + \delta \mu}c(\pi_0) \right) \hat{p} - \frac{1-\delta}{1-\delta + \delta \mu} \hat{v} \right) \\
+ q \left( \delta (1-c(\pi_0))\hat{p} - \frac{(1-\delta)}{\mu} \hat{v} \right). \tag{56}
\]

I make further assumptions on \( r, \lambda \) and redefine \( \mathcal{N}, \Delta \) appropriately so that the right-hand side of (56) becomes negative in near the HFL. For this, one sets first \( q = 1 \) and looks at \( \Delta \to 0 \). In the HFL the right-hand side of (56) becomes

\[
\pi_0(1 - \frac{\lambda}{r + \lambda}c(\pi_0))\hat{p} - \frac{r}{r + \lambda} \pi_0 \hat{v} + (1-c(\pi_0))\hat{p} - \frac{r}{\lambda} \hat{v}. \tag{57}
\]

The coefficient in front of \( \hat{p} \) is positive so that if one replaces \( \hat{p} = \hat{v} \) it is sufficient in the limit to require the restriction

\[
\pi_0(1 - \frac{\lambda}{r + \lambda}c(\pi_0)) + (1-c(\pi_0)) < \frac{r}{r + \lambda} \pi_0 + \frac{r}{\lambda}, \tag{58}
\]

and in addition, that \( \hat{p} \) is near enough to \( \hat{v} \) so that (57) remains valid for these \( \hat{p} \) close to \( \hat{v} \). This additional restriction is on top of the other previous restrictions set above on \( \mathcal{N} \).

Suppose first that \( c(\pi_0) = \pi_0 \) which is equivalent to \( \pi_0 \leq \frac{1}{2} \). Then the left-hand side of (58) is strictly lower than the right-hand side due to (55). To see this, replace \( c(\pi_0) \) in (58) to arrive at the sufficient condition \( 1 - \frac{r}{\lambda} < \frac{r}{r + \lambda} \pi_0 + \frac{\lambda}{r + \lambda} \pi_0^2 \). To get the result uniformly on \( \pi_0 \leq \frac{1}{2} \) require \( r \geq \lambda \) for the case \( \pi_0 \leq \frac{1}{2} \).

Consider now \( c(\pi_0) = 1 - \pi_0 \) which is equivalent to \( \pi_0 \geq \frac{1}{2} \). Plugging in \( c(\pi_0) = 1 - \pi_0 \) in (58), this inequality becomes \( \frac{\lambda}{r + \lambda} \pi_0^2 + \pi_0 < \frac{r}{\lambda} \). Here the left-hand side \( \frac{\lambda}{r + \lambda} \pi_0^2 + \pi_0 \) is decreasing in \( \frac{\lambda}{r} \), whereas the right-hand side is increasing in \( \frac{\lambda}{r} \) (it being the identity map on \( \frac{\lambda}{r} \)). To get a condition which is uniform for all \( \pi_0 > \frac{1}{2} \) one needs the condition \( \frac{\lambda}{r + \lambda} + 1 < \frac{r}{\lambda} \). Algebraic manipulation shows that this is equivalent to \( \frac{r}{\lambda} > \sqrt{2} \).

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Require thus overall, that \( \Delta \leq \hat{\Delta} \leq \bar{\Delta} \) for some \( \hat{\Delta} > 0 \) appropriate such that

\[
\frac{\delta \mu}{1 - \delta} < 1 \text{ if } \pi_0 \leq \frac{1}{2}, \quad \frac{\delta \mu}{1 - \delta} < \frac{1}{\sqrt{2}} \text{ if } \pi_0 > \frac{1}{2}.
\]

(59)

The corresponding HFL assumption for (59) is the one required in the statement of Proposition 6.\(^{62}\)

With regards to the continuous function \( f : (0, 1) \times (\bar{v}, \bar{\bar{v}}) \rightarrow \mathbb{R} \) the above analysis has shown the following two facts.

- **A.** There exists an open neighborhood of \( \hat{v} \) such that for all \( \hat{\bar{p}} \) in that neighborhood it holds
  \[
  \limsup_{\Delta \to 0} f(1, \hat{\bar{p}}) < 0.
  \]

- **B.** Fix any \( \hat{\bar{p}} \in (\bar{v}, \bar{\bar{v}}) \) such that solvability of \( p_L, p_H \) is guaranteed (Lemma 9). Then for any \( \mu \in (0, 1) \) and \( \delta < 1 \) it holds
  \[
  \lim_{q \to 0} f(q, \hat{\bar{p}}) = U(0)(p_H(\hat{\bar{p}}) - p_L(\hat{\bar{p}})) > 0.
  \]

Pick first some neighborhood \( \mathcal{N} \) of \( \hat{v}, \hat{\Delta} \) so that Lemmas 9 and 10 are ensured for \( \hat{\bar{p}} \in \mathcal{N}, \hat{\Delta} > \Delta > 0 \). Redefine \( \hat{\Delta} > 0 \) small such that for all \( \Delta < \hat{\Delta} \) (59) holds true. This implies then that \( f(1, \hat{\bar{p}}) < 0 \) for all \( \hat{\bar{p}} \in \mathcal{N} \).\(^{63}\) Using B. and the intermediate-value theorem for continuous functions, one finds the required zero \( q(\Delta, \hat{\bar{p}}) \in (0, 1) \). This establishes that equilibria exist for the sufficient statistic \( \hat{\bar{p}} \in \mathcal{N} \) and \( \Delta \leq \hat{\Delta} \).

Lemma 4 can be used to show easily that for all \( \Delta > 0 \) small, the function \( \frac{f(\hat{\bar{p}}, q)}{\mu(1-U(0))} \) is strictly decreasing and convex in \( q \). This shows uniqueness of \( q(\Delta, \hat{\bar{p}}) \) for fixed \( \hat{\bar{p}}, \Delta \) as above. This finishes the proof of 1) and 2) from Proposition 6 for the case of deterministic variable costs.

**The case of stochastic fixed costs.** Note that the two reservation pricing relations for Buyer imply

\[
\hat{v} - p_L \leq \frac{\delta \mu}{1 - \delta + \delta \mu}(\pi_0(\bar{v} - \hat{\bar{p}})) < \pi_0(\bar{v} - \hat{\bar{p}}).
\]

In particular, \( \mu > 0 \) for any pair of prices \((p_H, p_L)\) satisfying the reservation pricing relations of types \( \bar{v}, \hat{v} \).

**Remark 6.** The function \( x \rightarrow \mathbb{E}[c|c \leq x] \) is weakly increasing in \( x \) for all \( F \) that are ‘well-behaved’, i.e. are limits in total variation norm of distributions having a positive and smooth density w.r.t. Lebesgue measure over \( \mathbb{R}_+ \).\(^{64}\)

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\(^{62}\)To see this for the case \( \pi_0 > \frac{1}{2} \), note that the function \( g : (0, \infty) \rightarrow \mathbb{R}_+ \) given by \( g(t) = \frac{t^{1/\pi}}{1 + t^{1/\pi}} \) has derivative \( g'(t) = \frac{1 - t^{1/\pi}}{1 + t^{1/\pi}} \) and so is strictly decreasing and that \( g(\sqrt{2}) = 2 \).

\(^{63}\)This is because the function \( f(1, \hat{\bar{p}}) \) is continuous also in \( \Delta \) (via its continuity on \( \mu \) and \( r \)) which are in turn continuous functions of \( \Delta \).

\(^{64}\) ‘Smooth’ in the following means infinitely often continuously differentiable.
Proof of Remark 6. Step 1. First, verify the claim for an $F$ that has a smooth density $f$ over $(0, \infty)$. It holds

$$\frac{d}{dx} \mathbb{E}[c|c \leq x] = \frac{d}{dx} \left( \int_0^x cf(c)dc \right) = \frac{xf(x)F(x) - f(x)\int_0^x cf(c)dc}{F(x)^2} \frac{f(x)}{F(x)} (x - \mathbb{E}[c|c \leq x]).$$

This derivative is strictly positive for all $x > 0$ because of the assumption that $F$ puts positive probability on arbitrarily low costs (from section 3).

Step 2. Let $F$ be continuous with finite first moment and support on $[0, \infty)$ and assume that $F(0) = 0$.

Claim. There exists a sequence of distributions $F_n$ which have smooth densities over $(0, \infty)$ as in Step 1 and so that $F_n$ converges to $F$ uniformly.

To see the Claim, note the following facts.

A. the set of distribution functions with finite support is dense in the topology of weak convergence of probability measures,\(^{65}\)

B. if $F_n$ over $\mathbb{R}$ converges weakly to a continuous distribution $F$, then the convergence is uniform,\(^{66}\)

C. if $F_n$ converges weakly to $F$ and $F$ is continuous then $F_n(\cdot|\cdot \leq x)$ converges weakly to $F(\cdot|\cdot \leq x)$, for any $x > 0$ (the convergence is even uniform),

[Proof. To see this, recall that $F(x) > 0$ for any $x > 0$ by the assumptions in section 3. In particular, $(0, \infty) \ni c \mapsto F(c|c \leq x)$ is a continuous distribution function. Pick any $y < x$. Then obviously $F_n(x) \rightarrow F(x)$ and $F_n(y) \rightarrow F(y)$ as $n \rightarrow \infty$. This leads to the result stated in C.]

D. any distribution $G$ which is a step function can be approximated from above and below pointwise by a sequence of distribution functions which are infinitely differentiable functions;

[Proof. To see this, first approximate the step function through a continuous function by interpolating ‘near’ the (finitely many) discontinuities of $G$, to get two continuous distribution functions $G', G''$ which differ from $G$ only in intervals of size less than $\epsilon$ around the discontinuities, are continuous and satisfy $G''(x) \leq G(x) \leq G'(x)$ for all $x \geq 0$. Then one performs a ‘mollification’ procedure on $G', G''$ around the finitely many points of their non-differentiabilities to get smooth distribution functions $H'', H'$ which satisfy $H''(x) \leq G''(x) \leq G(x) \leq G'(x) \leq H'(x)$ for all $x \geq 0$. See chapter 5 and in particular section 5.3 in Evans (2010) for more on mollification arguments.]

Now I finish the proof of Step 2. above and thus also of the Remark.

From Step 1 one knows that $\mathbb{E}_H[c|c \leq x]$ are increasing in $x$ (even strictly) for $H$ of the type $H', H''$ as given in the proof of D above. One can use the argument in D.

---

\(^{65}\)This is easy to see and uses the generally known fact of undergraduate analysis: any bounded, measurable function can be approximated from below by step functions with finitely many discontinuities.\(^{66}\)See e.g. exercise 3.2.9. in Durrett (2010).
above to show that monotonicity of \( x \mapsto \mathbb{E}_F[c \leq x] \) holds true for \( F \) a step function. To see this, suppose there is \( y > x > 0 \) and \( F \) a step distribution function so that \( \mathbb{E}_F[c \leq y] < \mathbb{E}_F[c \leq x] \). It holds in general with the Fubini Theorem that

\[
\mathbb{E}_F[c, c \leq y] = \mathbb{E}_F \left[ \int_0^c dt, c \leq y \right] = \int_0^y F(y) - F(t) dt.
\] (60)

Because of monotonicity of the approximation in point D. above and (60) one sees easily that:

for \( z = x, y \) and step function \( F, \quad \mathbb{E}_F[c, c \leq z] = \lim_{n \to \infty} \mathbb{E}_{H_n}[c \leq z], \) (61)

for a sequence \( \{H_n : n \geq 1\} \) of distribution functions which are smooth.

Ultimately, (61) and the assumption \( \mathbb{E}_F[c, c \leq y] < \mathbb{E}_F[c, c \leq x] \) lead to contradiction of the monotonicity in Step 1.

It follows that \( x \mapsto \mathbb{E}_F[c, c \leq x] \) is increasing for \( F \) a step distribution function.

Finally, by points A., B. and C. above, one establishes monotonicity of \( \mathbb{E}_F[c, c \leq x] \) for general \( F \) by an approximation argument.

\[ \square \]

Denote in the following \( BL(\hat{p}, p_L) = \pi_0(\hat{v} - \hat{p}) - \mathbb{E}[c | c \leq \pi_0(\hat{v} - \hat{p}) - (\hat{v} - p_L)] \). Note that this definition is independent of \( \Delta \).

Because \( F \) puts positive probability on arbitrarily small costs, \( BL(\hat{p}, p_L) \) is strictly positive for any \( p_L < \hat{v}, \hat{p} < \hat{v} \). Note that \( BL(\hat{p}, p_L) \) is decreasing in \( p_L \) and the direction of the monotonicity in \( \hat{p} \) is ambiguous. Finally, note that

\[ BL(\hat{p}, p_L) \leq \pi_0(\hat{v} - \hat{p}) \leq \hat{v} - c(\pi_0)\hat{p}. \] (62)

This helps in proving again an estimate precisely as in (51), whenever one has solvability of the reservation price relation for type \( \hat{v} \).

Moreover, because of the assumption that \( F \) puts positive probability on a neighborhood of zero it follows that

\[ BL(\hat{p}, p_L) - (\hat{v} - p_L) > 0, \quad \text{for every } \hat{p} < \hat{v}, p_L < \hat{v}. \] (63)

The two reservation pricing relations give a system of equations to be solved for the pair \((p_H, p_L)\).

For \( \hat{p} \in (\underline{v}, \bar{v}) \) and \( p_L \in (\underline{\bar{v}}, p_L) \) define for ease in notation \( \bar{\mu}(\hat{p}, p_L) = F(\pi_0(\hat{v} - \hat{p}) - (\hat{v} - p_L)) \). Denote \( p_L^{\text{min}}(\hat{p}) = \hat{v} - \pi_0(\hat{v} - \hat{p}) = (1 - \pi_0)\underline{v} + \pi_0 \hat{p} - \bar{p} \).

It holds \( \bar{\mu}(\hat{p}, p_L^{\text{min}}(\hat{p})) = 0 \) and \( \bar{\mu}(\hat{p}, p_L) = F((1 - \pi_0)(\hat{p} - \underline{v})) > 0 \). Note that \( \bar{\mu}(\hat{p}, p_L) \) is falling in \( \hat{p} \) and increasing in \( p_L \).

Lemma 11. 1) Suppose \( \mu_0(\Delta) = 1 - e^{-\Delta \lambda} \). The reservation pricing equation for \( \hat{v} \) is solvable for \( p_L \) whenever \( \hat{p} \) is in a suitable open neighborhood \( \mathcal{N} \) of \( \hat{v} \) and \( \Delta \) is smaller than some \( \Delta > 0 \).

Moreover, whenever solvable, the solution \( p_L(\hat{p}) \) is unique for each pair \((\Delta, \hat{p})\) and weakly increasing in \( \hat{p} \) for fixed \( \Delta \).

\[ ^{67} \text{Formally, this follows because } \mathbb{E}[c | c \leq x] < x \text{ for } x > 0 \text{ under the assumption that } F \text{ puts a positive weight on a neighborhood of zero (recall assumptions in section 3).} \]
2) Let \( \hat{p} \in \mathcal{N} \), where \( \mathcal{N} \) comes from 1). As \( \Delta \to 0 \), \( p_L(\hat{p}) \) converges to a \( \tilde{p}_L(\hat{p}) \) that satisfies

\[
\hat{v} - \tilde{p}_L(\hat{p}) = \frac{\lambda \hat{\mu}(\hat{p}, \tilde{p}_L(\hat{p}))}{r + \lambda \hat{\mu}(\hat{p}, \tilde{p}_L(\hat{p}))} BL(\hat{p}, \tilde{p}_L(\hat{p})).
\]  

(64)

**Proof.** 1) Existence.

One needs to solve for \( p_L \) in

\[
\hat{v} - p_L = \frac{\delta \mu}{1 - \delta + \delta \mu} BL(\hat{p}, p_L),
\]

(65)

for given \( \hat{p} \) and \( \Delta > 0 \). Moreover \( p_L \leq \hat{v} \). I relax this second requirement in the first part of the proof of existence and make sure it is satisfied in the end.

Recall \( p^{\text{min}}_L(\hat{p}) = \hat{v} - \pi_0(\hat{v} - \hat{p}) = (1 - \pi_0)\hat{v} + \pi_0 \hat{p} < \hat{p} \). Note that automatically from the definition of \( p^{\text{min}}_L(\hat{p}) \) it follows \( p^{\text{min}}_L(\hat{p}) < \hat{v} \), because \( \hat{p} < \hat{v} \). It follows that the right-hand side of the reservation price relation for type \( \hat{v} \) is zero if one plugs \( p_L = p^{\text{min}}_L(\hat{p}) \), whereas the left-hand side is positive. On the other hand, when setting \( p_L = \hat{p} \) the left hand side becomes \( \hat{v} - \hat{p} \) and the right hand side becomes \( \frac{\delta \mu \hat{\mu}(\hat{p}, \hat{p})}{1 - \delta + \delta \mu \hat{\mu}(\hat{p}, \hat{p})} BL(\hat{p}, \hat{p}) \). Note that both \( \hat{v} - \hat{p} \) and \( \frac{\delta \mu \hat{\mu}(\hat{p}, \hat{p})}{1 - \delta + \delta \mu \hat{\mu}(\hat{p}, \hat{p})} BL(\hat{p}, \hat{p}) \) are continuous in the parameters \( \hat{p} \in (\underline{v}, \bar{v}), \Delta \in [0, \infty) \).

Evaluated at \( \hat{p} = \hat{v} \) and \( \Delta = 0 \) the expression \( \hat{v} - \hat{p} \) is zero, whereas \( \frac{\delta \mu \hat{\mu}(\hat{p}, \hat{p})}{1 - \delta + \delta \mu \hat{\mu}(\hat{p}, \hat{p})} BL(\hat{p}, \hat{p}) \) is \( \frac{\lambda \hat{\mu}(\hat{v}, \hat{v})}{r + \lambda \hat{\mu}(\hat{v}, \hat{v})} > 0 \). Thus, there exists an open neighborhood of \( \hat{v} \), denoted by \( \mathcal{N} \) and a \( \Delta > 0 \) such that for all \( \hat{p} \in \mathcal{N} \) and \( 0 < \Delta < \bar{\Delta} \) it holds true that \( \hat{v} - \hat{p} < \frac{\lambda \hat{\mu}(\hat{v}, \hat{v})}{r + \lambda \hat{\mu}(\hat{v}, \hat{v})} BL(\hat{p}, \hat{p}) \). Overall the intermediate-value theorem for continuous functions gives existence of \( p_L(\Delta, \hat{p}) \in (p^{\text{min}}_L(\hat{p}), \hat{p}) \) that satisfies (65) for \( \hat{p} \in \mathcal{N}, \Delta \leq \bar{\Delta} \).

Finally, for existence one needs to ensure that \( p_L(\hat{p}) < \hat{v} \) for the \( \hat{p} \) from a small, suitable neighborhood of \( \hat{v} \). Note that \( BL(\hat{p}, p_L) \) is decreasing in \( p_L \) and that \( BL(\hat{v}, \hat{v}) = \pi_0(\hat{v} - \hat{v}) - \mathbb{E}[c | c \leq \pi_0(\hat{v} - \hat{v})] > 0 \). Therefore, there exists a small neighborhood of \( \hat{v} \) such that \( BL(\hat{p}, p_L) > BL(\hat{p}, \hat{p}) > 0 \) for the \( \hat{p} \) in this neighborhood. Note that this argument is independent of \( \Delta > 0 \) because the function \( (\hat{p}, p_L) \mapsto BL(\hat{p}, p_L) \) does not depend on \( \Delta \).

Restrict in the following \( \mathcal{N} \) to a strict, non-empty subset of itself so that in addition it is satisfied: \( BL(\hat{p}, \hat{p}) > 0 \) for all \( \hat{p} \in \mathcal{N} \). It follows from the reservation price relation for type \( \hat{v} \) that \( p_L(\Delta, \hat{p}) < \hat{v} \) for such \( \hat{p} \), independently of \( \Delta \).

To see uniqueness, note that the reservation price relation for type \( \hat{v} \) can be written as

\[
1 - \delta + \delta \mu_0 \hat{\mu}_0(\hat{p}, p_L) (\hat{v} - p_L) + \mathbb{E}[c | c \leq \pi_0(\hat{v} - \hat{p}) - (\hat{v} - p_L)] = \pi_0(\hat{v} - \hat{p}).
\]

This can be transformed into

\[
(1 - \delta)(\hat{v} - p_L) = \delta \mu_0 \mathbb{E}[(\pi_0(\hat{v} - \hat{p}) - (\hat{v} - p_L) - c), c \leq \pi_0(\hat{v} - \hat{p}) - (\hat{v} - p_L)].
\]

(66)

Note that the right-hand side is strictly increasing in \( p_L \), whereas the left-hand side is strictly decreasing in \( p_L \). This gives uniqueness of \( p_L(\hat{p}) \).\(^{68}\)

Monotonicity of \( p_L(\hat{p}) \) is shown in two steps. First, rewrite the reservation price relation of type \( \hat{v} \) as

\[
1 - \delta \mu_0 (\hat{v} - p_L) = \mathbb{E}_F[p_L - \pi_0 \hat{p} - (1 - \pi_0)\underline{v} - c, c \leq p_L - \pi_0 \hat{p} - (1 - \pi_0)\underline{v}].
\]

(67)

\(^{68}\)Here, for ease of notation, I suppress the dependence on \( \Delta \) without loss of meaning.
Step 1. Suppose first that \( F \) has smooth density. The left-hand side of (67) is not directly dependent of \( \hat{p} \) and is strictly increasing in \( p_L \), for any fixed \( \hat{p} \). One can use the implicit function theorem locally, because of the differentiability of the functions involved. The right-hand side is strictly increasing in \( p_L - \pi_0 \hat{p} \) and the derivative of \( p_L(\hat{p}) - \pi_0 \hat{p} \) with respect to \( \hat{p} \) is given locally by \( \frac{dp}{d\hat{p}}(\hat{p}) - \pi_0 \). Thus, the implicit function theorem together with the chain rule for differentiation delivers a relation of the type \( \frac{d\text{LHS}}{dp_L} = \frac{d\text{RHS}}{dp_L} \left( \frac{dp}{d\hat{p}}(\hat{p}) - \pi_0 \right) \). Because \( \frac{d\text{RHS}}{dp_L} > 0, \frac{d\text{LHS}}{dp_L} < 0 \), this delivers a positive derivative \( \frac{dp}{d\hat{p}}(\hat{p}) \).

Step 2. Suppose now that \( F \) does not have a smooth density.

Claim. Let \( F_n \) converge weakly to \( F \), with \( F_n, n \geq 0 \), \( F \) satisfying the conditions about stochastic fixed costs in the main body of the paper. Then \( G_n := F_n(c \in \cdot, c \leq x_n) \) converges to \( F(c \in \cdot, c \leq x) \), whenever \( x_n \to x, n \to \infty, x > 0 \).

Proof of Claim. Recall from the proof of Remark 6 that for any distribution \( F \) it holds

\[
\mathbb{E}_F[c, c \leq y] = \mathbb{E}_F \left[ \int_0^c dt, c \leq y \right] = \int_{\mathbb{R}_+} (F(y) - F(t)) \mathbf{1}_{\{t \leq y\}} dt. \tag{68}
\]

Now, since \( x_n \to x \), the sequence of functions \( t \mapsto (F(x_n) - F(t)) \mathbf{1}_{\{t \leq x_n\}} \) is bounded and has support contained on a compact set \( K \) of \( \mathbb{R}_+ \). Moreover, due to weak convergence, it follows that \( (F(x_n) - F(t)) \mathbf{1}_{\{t \leq x_n\}} \to (F(x) - F(t)) \mathbf{1}_{\{t \leq x\}} \) for all \( t \in K \). One applies finally Lebesgue dominated convergence to get the result.

\[ \square \]

The Claim shows that the reservation price relation of type \( \hat{v} \) is ‘stable’ with respect to weak limits of \( F \). Recall the proof of Remark 6. Arguments there imply, that for any \( F \) satisfying the conditions for stochastic fixed costs in section 3, one can pick a sequence \( F_n \) with the same properties, converging weakly to \( F \), and satisfying the additional requirement that \( F_n \) have smooth densities w.r.t. Lebesgue measure.

Take then such a sequence \( F_n \) and apply Step 1 to each \( F_n \). The uniqueness of the solution \( p_L(\Delta, \hat{p}) \) for fixed \( \Delta \) and any \( F \), delivers that \( p_L^n(\Delta, \hat{p}) \to p_L(\Delta, \hat{p}), n \to \infty \). It follows that \( \hat{p} \mapsto p_L(\Delta, \hat{p}) \) is weakly increasing.

2) To see that all the limit points of \( p_L(\Delta, \hat{p}) \) are the same, divide (66) by \( \Delta \) and take \( \Delta \to 0 \) to arrive at

\[
r(\hat{v} - p_L) = \lambda \mathbb{E}[(\pi_0(\hat{v} - \hat{p}) - (\hat{v} - p_L) - c), c \leq \pi_0(\hat{v} - \hat{p}) - (\hat{v} - p_L)].
\]

This is a relation that has to be satisfied for all limit points of \( p_L(\Delta, \hat{p}) \) and thus the limit \( \bar{p}_L(\hat{p}) \) exists, because the right-hand side is strictly increasing in \( p_L \), whereas the left-hand side is strictly decreasing in \( p_L \).

\[ \square \]

\[ ^{69} \text{Here, LHS and RHS denote respectively the left-hand side and right-hand side of (67).} \]

\[ ^{70} \text{To see this, recall that from A. and B. in that proof, } F \text{ can be approximated uniformly by distribution functions that are step functions, and then use C. there to get a pointwise approximation of } F \text{ through a sequence of } F_n \text{ which are smooth. This uses the fact that the approximation of step functions through step functions is done monotonically and the differences between the step function and the approximands are in ‘small’ sets around a finite number of discontinuities.} \]
Let $V_\Delta(q, \hat{p})$ be stationary payoff of Seller.\footnote{The recursion leading to Seller’s payoff, is exactly the same as for the case of no costs, or deterministic variable costs.} By the same logic as in the case of deterministic costs it satisfies the recursion

$$\delta V_\Delta(q, \hat{p}) = \mu \pi_0 \hat{p} + (1 - \mu)(1 - p)qp_L + \mu(1 - \pi_0)\bar{v} + \delta(1 - \mu)(1 - (1 - p)q)V_\Delta(q, \hat{p}),$$

which leads to

$$\delta V_\Delta(q, \hat{p}) - p_L(\hat{p}) = \frac{\delta \mu \pi_0 \hat{p} + (\delta(1 - \mu) - 1)p_L(\hat{p}) + \delta \mu (1 - \pi_0)\bar{v}}{1 - \delta(1 - \mu)(1 - (1 - p)q)}.$$

Analogous to the cases of costless information and the case of deterministic variable costs, one considers the function $g(q, \hat{p}) = \frac{f(q, \hat{p})}{\delta \mu (1 - U(0))} = \frac{\pi_0}{\delta (1 - \mu)}(p_H(\hat{p}) - p_L(\hat{p})) + \hat{q} \frac{\delta V(q, \hat{p}) - p_L(\hat{p})}{\delta \mu}$. It holds that $g(0, \hat{p}) > 0$ for all $\Delta > 0$ and $\hat{p}$ from a neighborhood $\mathcal{N}$ as required in Lemma 11.

In order to find sufficient conditions for $g(1, \hat{p}) < 0$ one uses (65) from the proof of Lemma 11 to get through the use of (62) an analogous estimate as (52) from the case of deterministic variable costs. This leads to the estimate for the price spread

$$p_H(\hat{p}) - p_L(\hat{p}) \leq (1 - \delta)\bar{v} + \delta \left(1 - \frac{\mu}{1 - \delta + \delta \mu}c(\pi_0)\right)\hat{p} - \frac{1 - \delta}{1 - \delta + \delta \mu}\hat{v}. \quad (69)$$

All ingredients are present to apply the same procedure for existence of strongly stationary equilibria as for the case of deterministic variable costs. One uses a similar chain of estimates as for the case of deterministic variable costs to show that $g(1, \hat{p}) < 0$, whenever $\hat{p}$ is near enough to $\hat{v}$ and $\Delta$ is small enough. The replacements needed are $\hat{w}(\hat{p}) \rightsquigarrow \hat{w}, \hat{v}(\hat{p}) \rightsquigarrow \hat{v}$. Moreover, one has to replace $\frac{\mu(\Delta)}{\Delta}$ from the deterministic variable model with $\frac{\mu_0 \hat{p}_0 \mu_0 (\hat{p}_0, p_L(\hat{p}_0))}{\Delta}$ throughout. Near the HFL one gets sufficient conditions ensuring existence just as for the case of deterministic variable costs of the type $\frac{r}{\chi}$ is high enough’. These are implied by the same conditions of the type $\frac{r}{\chi}$ is high enough’ that appear in the case of deterministic variable costs.

This finishes the proof of Proposition 6 for the case of stochastic fixed costs of accuracy.

### B.3 Proof of Theorem 3

**The case of deterministic variable costs.** In the following whenever it is said uniformly in $\hat{p}$, it is meant that the statement holds uniformly for all $\hat{p} \in \mathcal{N}$ where the neighborhood $\mathcal{N}$ comes from Proposition 6.

Note first the following easy-to- prove facts:

- $p_H(\hat{p})$ converges in HFL uniformly to the identity function $id(\hat{p}) = \hat{p}$. The difference $p_H(\hat{p}) - \hat{p}$ is $O(\Delta)$, uniformly in $\hat{p}$.

- Because information choice does not depend on $\mu, \delta$ and $V(a(\hat{p}), \hat{p}) - C(a(\hat{p}))$ is bounded uniformly in $\hat{p}$, it holds

  $$p_L(\hat{p}) \rightarrow p_L(\hat{p}) := \hat{v} - \frac{\lambda}{\lambda + r}(V(a(\hat{p}), \hat{p}) - C(a(\hat{p}))),$$

  in HFL uniformly in $\hat{p}$.\footnote{The recursion leading to Seller’s payoff, is exactly the same as for the case of no costs, or deterministic variable costs.}
Note that \( p_L(\hat{p}) \) is decreasing in \( \hat{p} \). For future use I express the HFL of \( p_L(\hat{p}) \) in a more helpful form. Recall that \( V(a(\hat{p}), \hat{p}) = GN(a(\hat{p}))(\bar{w}(\hat{p}) - \hat{p}) \). One can use this to express the HFL of \( p_L(\hat{p}) \) as

\[
p_L(\hat{p}) = \hat{v} - \frac{\lambda}{\lambda + r}(GN(a(\hat{p}))(\bar{w}(\hat{p}) - \hat{p}) - C(a(\hat{p}))).
\]

- It follows from the first two claims that

\[
p_H(\hat{p}) - p_L(\hat{p}) \rightarrow \frac{\lambda}{\lambda + r}(V_A(a(\hat{p}), \hat{p}) - C(a(\hat{p}))) - (\hat{v} - \hat{p}), \text{ in HFL uniformly in } \hat{p}. \quad (70)
\]

Denote this limit by \( ps(\hat{p}) \) where ‘ps’ stands for price spread.

Note that in HFL, uniformly in \( \hat{p} \) it holds \( \hat{v} - \hat{p} < \hat{v} - \bar{p}_L(\hat{p}) = \frac{\lambda}{\lambda + r}(V_A(a(\hat{p}), \hat{p}) - C(a(\hat{p}))) \), where the equality follows from the reservation pricing relation for type \( \hat{v} \). This, and the proof of Proposition 6 (especially the proof of Lemma 9) for the case of deterministic variable costs ensures that the price spread remains bounded away from zero for all \( \hat{p} \) from the neighborhood \( \mathcal{N} \).

Another simple, but important implication of (70) is that it shows that the HFL of a sequence of mixed pricing equilibria cannot correspond to a pure pricing equilibrium, whenever the limit average price \( \hat{p} \) satisfies \( \hat{p} \geq \hat{v} \).

- From the formula \( p(\hat{p}) = \frac{\hat{p} - p_L(\hat{p})}{p_H(\hat{p}) - p_L(\hat{p})} \) for the mixing probability of Seller one sees convergence to 1 of \( p(\hat{p}) \). This convergence is uniform in \( \hat{p} \), as one can see from the calculation

\[
1 - p(\hat{p}) = \frac{p_H(\hat{p}) - \hat{p}}{p_H(\hat{p}) - p_L(\hat{p})} = (1 - \delta) \frac{\bar{w}(\hat{p}) - \hat{p}}{p_H(\hat{p}) - p_L(\hat{p})}.
\]

Now I calculate explicitly \( q(\Delta, \hat{p}) \), the probability with which the \( \hat{v} \)-type ends the game when facing price \( p_L(\hat{p}) \). Recall that \( q \) is determined through the equation \( \frac{f(\hat{p}, q)}{\mu(1 - U(0))} = 0 \). Define the quantities \( A(\Delta, \hat{p}) = \frac{\pi_0}{1 - p}(p_H(\Delta, \hat{p}) - p_L(\Delta, \hat{p})) \) and \( B(\Delta, \hat{p}) = \delta GN(a(\hat{p}))(\hat{p} - L(\hat{p}, \Delta)) + \delta(1 - GN(a(\hat{p})))w(\hat{p}) \). By using the HFL of \( p_L \) one calculates the HFL of \( B(\Delta, \hat{p}) \) to be

\[
B(\hat{p}) := -\frac{r}{\lambda} \hat{v} - C(a(\hat{p})),
\]

where the fact that \( GN(a(\hat{p}))\bar{w}(\hat{p}) + (1 - GN(a(\hat{p})))\bar{w}(\hat{p}) = \hat{v} \) (Martingale property of beliefs) has been used.

Note that \( B(\hat{p}) \) is uniformly bounded away from zero and negative. One easily calculates the HFL of \( A \) to be

\[
A(\hat{p}) := \pi_0 ps(\hat{p}).
\]

It follows that

\[
\frac{f(\hat{p}, q)}{\mu(1 - U(0))} = A(\Delta, \hat{p}) + \frac{q}{1 - \delta(1 - \mu) + \delta(1 - \mu)(1 - p(\Delta, \hat{p}))} B(\Delta, \hat{p}).
\]

One can define \( C(\Delta, \hat{p}) = -\frac{A(\Delta, \hat{p})}{B(\Delta, \hat{p})} \) and see that \( q(\Delta, \hat{p}) \) can be solved in close form as

\[
q(\Delta, \hat{p}) = \frac{(1 - \delta(1 - \mu))C(\Delta, \hat{p})}{1 - \delta(1 - \mu)(1 - p(\Delta, \hat{p})) C(\Delta, \hat{p})}.
\]
The HFL of $C$ is easily calculated to be

$$C(\hat{p}) = \frac{\pi_0 \lambda \cdot ps(\hat{v})}{r\hat{v} + \lambda C(a(\hat{p}))}.$$  

In particular, $C(\hat{p})$ is bounded away from zero for all $\hat{p} \in \mathcal{N}$. Now recall that $1 - p(\Delta, \hat{p}) = O(\Delta)$ and that $\frac{1-\delta(1-\mu)}{\Delta} \rightarrow r + \lambda$ to get that in the HFL

$$\frac{q(\Delta)}{r + \lambda} \rightarrow (r + \lambda)C(\hat{p}).$$

Next, I calculate Seller’s payoff in the HFL. Recalling (53) one looks first at the denominator and concludes that

$$1 - \delta(1-\mu)\frac{(1 - (1-p(\Delta, \hat{p}))q(\Delta, \hat{p}))}{\Delta}$$

Looking at the numerator of Seller’s payoff one notes that

$$\mu GN(a(\hat{p}))\hat{p} + (1-\mu)(1-p(\Delta, \hat{p}))q(\Delta, \hat{p})p_L(\hat{p}, \Delta) + \mu(1-GN(a(\hat{p})))w(\hat{p})$$

Overall this delivers for Seller’s payoff in the HFL

$$V_S(\hat{p}) = \frac{\lambda}{r + \lambda} (GN(a(\hat{p}))\hat{p} + (1-GN(a(\hat{p})))w(\hat{p})).$$

Turning to Buyer’s payoff in the HFL: $V_B(\Delta, \hat{p})$ satisfies the recursion

$$V_B(\Delta, \hat{p}) = \mu(V_A(a(\hat{p}), \hat{p}) - C(a(\hat{p}))) + (1-\mu)(1-p(\Delta, \hat{p}))q(\Delta, \hat{p})\hat{v} - p_L(\Delta, \hat{p}))$$

which can be solved for

$$V_B(\Delta, \hat{p}) = \frac{\mu(V_A(a(\hat{p}), \hat{p}) - C(a(\hat{p}))) + (1-\mu)(1-p(\Delta, \hat{p}))q(\Delta, \hat{p})\hat{v} - p_L(\Delta, \hat{p}))}{1 - \delta(1-\mu)\frac{(1 - (1-p(\Delta, \hat{p}))q(\Delta, \hat{p}))}{\Delta}}.$$  

The same (limit-)algebra as in the case of Seller delivers the HFL

$$V_B(\hat{p}) = \frac{\lambda}{r + \lambda} (GN(a(\hat{p}))(\hat{w}(\hat{p}) - \hat{p}) - C(a(\hat{p}))).$$

The sum of the payoffs in the HFL is

$$V_B(\hat{p}) + V_S(\hat{p}) = \frac{\lambda}{\lambda + r} (\hat{v} - C(a(\hat{p}))),$$

where again the Martingale property of beliefs has been used. Finally, I turn to the expected delay in the HFL.
Note that the date of agreement is a geometric random variable with success probability $1 - (1 - \mu)(1 - p(\Delta, \hat{p})) q(\Delta, \hat{p})$. One calculates

$$
1 - (1 - \mu)(1 - (1 - p(\Delta, \hat{p})) q(\Delta, \hat{p})) \quad \Delta = \mu \Delta + (1 - \mu)(1 - p(\Delta, \hat{p})) q(\Delta, \hat{p})
$$

given the HFL behavior of $q(\Delta, \hat{p})$ and $p(\Delta, \hat{p})$.

This finishes the proof of Theorem 3 for the case of deterministic variable costs.

**The case of stochastic fixed costs.** Pick a $\hat{p}$ as needed in Proposition 6 for existence near the HFL. By exactly the same steps as for the case of deterministic variable costs one arrives at similar results, but for the only changes that $w(\hat{p}) \sim v$, $\lambda \sim \lambda \bar{\mu}$ and $GN(a(\hat{p})) \sim \pi_0$.

One arrives at

$$
V_B(\hat{p}) = \frac{\lambda \bar{\mu}(\hat{p})}{r + \lambda \bar{\mu}(\hat{p})} (\pi_0(\hat{v} - \hat{p}) - \mathbb{E}_F[|c| c \leq \pi_0(\hat{v} - \hat{p}) - (\hat{v} - \bar{p}_L(\hat{p}))])
$$

and

$$
V_S(\hat{p}) = \frac{\lambda \bar{\mu}(\hat{p})}{r + \lambda \bar{\mu}(\hat{p})} (\pi_0 \hat{p} + (1 - \pi_0) \bar{v}^*).
$$

**B.3.1 On multiplicity of equilibria with mixed pricing in the HFL of costly learning**

In contrast to the case of costless learning the equilibrium multiplicity of strongly stationary equilibria with mixed pricing survives the HFL. This multiplicity remains when accuracy costs become vanishingly small. To see this, fix $\pi_0, r, \lambda$ satisfying the conditions in the statement of Proposition 6 and of Theorem 3. Suppose that the respective costs $c$ become arbitrarily small or $F$ approaches the zero-cost distribution. The proof of Proposition 6 shows that in this situation, one can pick the respective existence neighborhoods $\mathcal{N}$ of $\hat{v}$ independently of the information costs.

The intuition for the multiplicity in the HFL is as follows. If Seller decides to quote a lower $\hat{p}$, this leads *ceteris paribus* to a higher option value from learning for Buyer of type $\hat{v}$. Because learning is costly, this lowers the reservation price of the type $\hat{v}$. In addition, the lower $\hat{p}$ makes Buyer who has received good news more willing to accept. Moreover, due to higher incentives for learning, a lower price spread is needed to incentivize learning in the limit. Given lower reservation prices for Buyer who has not learned or Buyer who has learned good news, Seller is then indeed only able to extract a lower average price $\hat{p}$.

The following Lemma contains a formal result that underlies the intuition of this equilibrium multiplicity.

**Lemma 12.** Look at the HLF of the strongly stationary equilibria for $\hat{p} \in \mathcal{N}$, the existence neighborhood of $\hat{v}$ from Proposition 6.

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72In the case of costless information there is no reason for Seller to keep a price spread in the limit. Buyer always waits for the arrival of costless information and thus there is no gain in the HFL from screening between Buyer who has not learned and the one who has learned good news.
Lemma 13. 1) Fix any \( p_L(\hat{p}) \) given by \( \bar{p}_L(\hat{p}) \) is strictly increasing in \( \hat{p} \). The same holds for the limit of the spread \( ps(\hat{p}) \). Clearly, the limit of \( p_H(\hat{p}) \) is increasing, being equal to \( \hat{p} \).

Proof. I give only the proof for the case of deterministic costs. The proof of monotonicity of \( \bar{p}_L(\hat{p}) \) in the case of stochastic fixed costs follows from Lemma 11, whereas the proof of the monotonicity of the price spread for the case of stochastic fixed costs is analogous to the proof of the case of deterministic variable costs.

Recall that the price spread in the HLF is given by

\[
\frac{\lambda}{\lambda + r} (V_A(a(\hat{p}), \hat{p}) - C(a(\hat{p}))) - (\hat{v} - \hat{p}).
\]

Taking a derivative of this expression w.r.t. \( \hat{p} \) and using the envelope theorem results in the expression

\[
\frac{\partial}{\partial \hat{p}} ps(\hat{p}) = 1 - \frac{\lambda}{\lambda + r} GN(a(\hat{p})) > 0.
\]

Moreover, taking a derivative of the formula for \( \bar{p}_L(\hat{p}) \), given by \( p_L(\hat{p}) = \hat{v} - \frac{\lambda}{\lambda + r} (V(a(\hat{p}), \hat{p}) - C(a(\hat{p}))) \) delivers that \( \bar{p}_L(\hat{p}) \) increasing in \( \hat{p} \). \( \square \)

Thus, a higher \( \hat{p} \) co-moves with a higher \( \bar{p}_L(\hat{p}) \), higher limit of \( p_H(\hat{p}) \) as well as a higher price spread.

Finally, I give the arguments showing that multiplicity persists as accuracy costs become vanishingly small.

Remark 7. Deterministic variable costs.

Note that Lemmas 9 and 10 require the neighborhood \( N \) to be dependent on the costs \( c \). It is easy to see from the proofs, that whenever passing from some costs \( c_1 \) to some \( c_2 \) with \( c_1 > c_2 \) (for \( a \neq \frac{1}{2} \)), the neighborhood \( N \) can be chosen to be strictly smaller (in the sense of set-inclusion) for the costs \( c_2 \) than \( c_1 \).

Stochastic fixed costs. This is similar to the case of deterministic variable costs, except that one replaces the inequality \( c_1 \geq c_2 \) with \( F_1 > F_{OSD} F_2 \).

B.3.2 Proof of Proposition 6 and Theorem 3 in the case of pure pricing

Given the analysis of the case of mixed pricing and because of the analogy in the proof between the two cases accuracy costs, I shorten exposition of the case of pure pricing by only spelling out the proof for the case of deterministic variable costs. In the case of stochastic fixed costs, the only changes are the replacements \( \bar{w}(\bar{p}) \sim \bar{v} \), \( \bar{w}(\bar{p}) \sim \bar{v} \), \( GN(a(\bar{p})) \sim \pi_0 \), \( \mu \sim \mu_0 \bar{u}(\bar{p}, \Delta) \), \( \lambda \sim \lambda \bar{u}(\bar{p}) \) and \( c(I(a(\bar{p}))) \sim \mathbb{E}[c|c \leq (1 - \pi_0)(\bar{p} - \bar{v})] \).

As a first step, one writes after algebra the Seller optimality condition as follows.

\[
\frac{1 - U(0)}{U(0)} \zeta(q, \Delta) p_L - \delta V_A(q, 0) \frac{1}{1 - \delta} (1 - \delta (1 - \mu) (1 - q)) \geq \bar{w}(\bar{p}, \Delta) - p_L. \tag{74}
\]

Lemma 13. 1) Fix any \( \Delta > 0 \). The reservation price relation of type \( \hat{v} \) is solvable for some \( p_L(\Delta) = \hat{p}(\Delta) \).

2) It holds for any limit point \( \bar{p} \) of \( p_L(\Delta) \) as \( \Delta \to 0 \) that \( \bar{p} > \bar{w}(\bar{p}) \).

3) For any sequence \( q(\Delta), \Delta \to 0 \) with limit point \( \bar{q} \in [0, 1] \) and any limit point \( \bar{p} \) of \( p_L(\Delta) \) from part 1) and 2) it holds
\[
\frac{p_L(\Delta) - \delta V_\Delta(q(\Delta), 0)}{1 - \delta} (1 - \delta(1 - \mu)(1 - q(\Delta))) \\
\rightarrow \left(1 + \frac{\lambda}{r} - \bar{q}\right) \bar{p} - \frac{\lambda}{r} (GN(a(\bar{p}))\bar{p} + (1 - GN(a(\bar{p}))w(\bar{p}))) > 0, \quad \Delta \rightarrow 0.
\]

Proof. 1) One needs to show existence of \(p_L \in (\bar{v}, \hat{v})\) that satisfies

\[
\hat{v} - p_L = \frac{\delta \mu}{1 - \delta + \delta \mu} (GN(a(p_L))(\bar{w}(p_L) - p_L) - c(I(a(p_L)))).
\]

If \(p_L = \hat{v}\) then the left-hand side of (75) is zero, whereas the right-hand side is strictly positive, as shown in the proof of Proposition 6 for the case of mixed pricing. If \(p_L > 0\) but very close to 0, then the inequality is reversed because of the inequality (50) shown in the case of mixed pricing.\(^{73}\)

This, and the intermediate-value theorem for continuous functions shows existence. 2) The HFL of (75) is given by

\[
\hat{v} - \bar{p} = \frac{\lambda}{\lambda + r} (GN(a(\bar{p}))(\bar{w}(\bar{p}) - \bar{p}) - c(I(a(\bar{p})))).
\]

This can be transformed through simple algebra into

\[
\bar{p} = \frac{r \hat{v} + \lambda(1 - GN(a(\bar{p})))\bar{w}(\bar{p}) + \lambda c(I(a(\bar{p})))}{r + \lambda(1 - GN(a(\bar{p})))}.
\]

The right-hand side of (76) can be estimated from below as follows.

\[
\frac{r \hat{v} + \lambda(1 - GN(a(\bar{p})))\bar{w}(\bar{p}) + \lambda c(I(a(\bar{p})))}{r + \lambda(1 - GN(a(\bar{p})))} > \frac{r \hat{v} + \lambda(1 - GN(a(\bar{p})))\bar{w}(\bar{p})}{r + \lambda(1 - GN(a(\bar{p})))} > w(\bar{p}).
\]

3) Simple algebra delivers

\[
\frac{p_L(\Delta) - \delta V_\Delta(q(\Delta), 0)}{1 - \delta} (1 - \delta(1 - \mu)(1 - q(\Delta)))
\]

\[
= p_L(\Delta)\frac{1 - \delta(1 - \mu)}{1 - \delta} - (1 - \mu)q(\Delta)p_L(\Delta)
\]

\[
- \frac{\mu}{1 - \delta} (GN(a(p_L(\Delta)))p_L(\Delta) + (1 - GN(a(p_L(\Delta))))\bar{w}(p_L(\Delta))).
\]

The limit statement follows from simple limit algebra. The limit is strictly positive because it can be rewritten as

\[
\left(1 + \frac{\lambda}{r} - \bar{q}\right) \bar{p} - \frac{\lambda}{r} (GN(a(\bar{p}))\bar{p} + (1 - GN(a(\bar{p}))\bar{w}(\bar{p})))
\]

\[
= (1 - \bar{q})\bar{p} + \frac{\lambda}{r} (1 - GN(a(\bar{p}))(\bar{p} - \bar{w}(\bar{p})).
\]

This finishes the proof of the Lemma. \(\square\)

\(^{73}\)Note that (50) only depends on the average price quoted upon non-disclosure and not the price distribution.
From here, the proof of existence and HFL characterization of strongly stationary equilibria with pure pricing follows the same steps as the corresponding case for costless learning (see proof arguments in subsection A.4.2). I note here down Buyer and Seller payoffs in the HFL with price upon non-disclosure $\bar{p}$ as well as their sum.

$$V_B(\bar{p}, \kappa) = \lambda (GN(a(\bar{p}))(\bar{w}(\bar{p}) - \bar{p}) - c(I(a(\bar{p})))) + \kappa(\bar{v} - \bar{p})$$

for Buyer payoff in any HFL with $\kappa$ and price upon non-disclosure equal to $\bar{p}$.

$$V_S(\bar{p}, \kappa) = (\lambda GN(a(\bar{p})) + \kappa) \bar{p} + \lambda(1 - GN(a(\bar{p}))w(\bar{p}))$$

for Seller payoff in any HFL with $\kappa$ and price upon non-disclosure equal to $\bar{p}$. The sum of payoffs in a HFL with $\kappa$ and price upon non-disclosure equal to $\bar{p}$ is calculated to be

$$\hat{v} - \frac{\lambda c(I(a(\bar{p}))) + r\hat{v}}{r + \lambda + \kappa}. ($$77$$)

C  Proofs for sections 4.1 and 4.2

C.1 Results with costly choice of intensity

C.1.1 The case of mixed pricing

Proposition 7 contains only some of the results included in the Propositions of this section of the appendix. The results presented here have the same structure as the results for the case of exogenous intensity: first existence results near the HFL and then the analysis in the HFL.

Throughout the proofs of this section I focus first on the special case $C(\Delta, \mu) = \Delta \cdot f(\mu)$ for ease of exposition and then comment on the changes needed for the general case of costs $C(\Delta, \mu)$ which satisfy the conditions stated in section 4.1.

C.1.2 Deterministic variable costs on accuracy with endogenous intensity

To save on notation, in the following I suppress the dependence of $C$ on $\Delta$ and I recall it only when looking at arguments near the HFL or in cases where the dependence on $\Delta$ is important for the argument. Thus, with some abuse of notation, when I write $C'(\mu)$ I actually mean $\frac{\partial}{\partial \mu} C(\Delta, \mu)$.

At the beginning of the period of a strongly stationary equilibrium Buyer who has not learned yet solves the following maximization problem

$$\max_{\mu} \mu(V_A(a(\hat{p}), \hat{p}) - C(a(\hat{p}))) + (1 - \mu)(\hat{v} - p_L) - C(\mu).$$

On the equilibrium path, after the information acquisition, it is true that $V(a(\hat{p})) - C(a(\hat{p})) > \hat{v} - p_L$. Therefore, the FOC condition delivers a unique $\mu$ characterized implicitly by the FOC condition

$$(V_A(a(\hat{p}), \hat{p}) - C(a(\hat{p}))) - (\hat{v} - p_L) = C'(\mu).$$  \hspace{2cm} (77)$$
This gives a unique \( \mu(\hat{p}, p_L) \) \( \in (0, 1) \) and therefore defines a map \( (\hat{p}, p_L) \mapsto \mu(\hat{p}, p_L) \) as an intensity-reaction function of Buyer. One solves explicitly \( \mu(\hat{p}, p_L) = C^{r-1}((V_A(a(\hat{p}))) - (\hat{v} - p_L)) \) and it is trivial to see that the reaction function of Buyer is smooth in \( (\hat{p}, p_L) \).

Envelope theorem for the stage of accuracy choice in a strongly stationary equilibrium with deterministic variable costs delivers \( \frac{\partial}{\partial \bar{p}} \{V_A(a(\hat{p})) - C(a(\hat{p}))\} < 0. \) (77) implies for the partial derivatives of \( \mu \):

\[
\frac{\partial}{\partial \hat{p}} \mu(\hat{p}, p_L) < 0, \quad \frac{\partial}{\partial p_L} \mu(\hat{p}, p_L) > 0.
\]

The reservation pricing relations for \( \bar{w}, \bar{v} \) remain the same as in the model with exogenous intensity, except for the fact that \( \mu \) is determined at the preceding stage of intensity choice and is therefore a function of \( \hat{p} \), besides of \( \Delta \).

Just as in the case of deterministic variable costs with exogenous \( \lambda \), I restrict in the following the analysis to \( \hat{p} \in (\bar{w}, \bar{v}) \) such that \( BL(\hat{p}) = V_A(a(\hat{p})), \hat{p} - C(a(\hat{p})) > 0 \).

Denote \( p_L^{\min}(\hat{p}) = \hat{v} - BL(\hat{p}) \), where \( a(\hat{p}) \), as always, is the accuracy-reaction function of Buyer once she gets an opportunity to learn. Recall the estimate \( BL(\hat{p}) < \hat{v} - c(\pi_0)\hat{p} \) shown in the case of exogenous \( \lambda \). This estimate remains true in this more general set up as well. This is a consequence of stationary play on path.

Note that \( \mu(\hat{p}, p_L^{\min}(\hat{p})) = 0 \) for all \( \hat{p} \). Therefore, \( \mu(\hat{p}, p_L) > 0 \) if and only if \( p_L > p_L^{\min} \).

**Lemma 14.** Let \( \Gamma = \{ \hat{p} \in (\bar{w}, \bar{v}) : \hat{v} - \hat{p} < BL(\hat{p}) \} \).\(^74\) For every compact and non-empty interval \( I \) contained in \( \Gamma \), there exists a constant \( \kappa(I) > 0 \) such that for all \( \hat{p} \in (\bar{w}, \bar{v}] \) it holds \( \mu_{\Delta}(\hat{p}, p_L) \geq \kappa(I) \Delta \) for all \( 0 < \Delta \leq \Delta \), where \( \Delta > 0 \) is uniformly on \( \hat{p} \in I \).

**Proof.** Recall that \( \mu_{\Delta}(\hat{p}, p_L) \) satisfies

\[
\frac{1}{1 - \mu} f' \left( -\frac{\log(1 - \mu)}{\Delta} \right) = BL(\hat{p}) - (\hat{v} - p_L).
\]

Because the right-hand side of the FOC condition is uniformly bounded across all \( \hat{p} \leq \bar{v} \) and \( p_L \leq \hat{v} \) it holds for the unique solution \( \mu_{\Delta}(\hat{p}, p_L) \) (for any \( \hat{p} \in (\bar{w}, \bar{v}), p_L \leq \hat{v} \)) that all limit points of \( \mu_{\Delta}(\hat{p}, p_L) \) with respect to \( \Delta \) are zero, and so overall \( \mu_{\Delta}(\hat{p}, p_L) \to 0 \), as \( \Delta \to 0 \). Now pick a \( \Delta \) such that \( \mu_{\Delta}(\bar{w}, \bar{v}) \leq \frac{1}{2} \) for all \( \Delta < \Delta \). One uses the uniform upper bound \( \mu_{\Delta}(\hat{p}, p_L) \leq \mu_{\Delta}(\bar{w}, \bar{v}) \), so that the convergence of \( \mu_{\Delta}(\hat{p}, p_L) \) to zero is uniform. This delivers the uniform estimates

\[
f'(\frac{2\mu_{\Delta}(\hat{p}, p_L)}{\Delta}) \geq f'\left( -\frac{\log(1 - \mu_{\Delta}(\hat{p}, p_L))}{\Delta} \right) \geq (1 - \mu_{\Delta}(\bar{w}, \bar{v})) (BL(\hat{p}) - (\hat{v} - \hat{p})).
\]

Here the first estimate uses the elementary inequality \( -\log(1 - \mu_{\Delta}(\hat{p}, p_L)) \leq 2\mu_{\Delta}(\hat{p}, p_L) \) for all \( \hat{p} \), which holds whenever \( \Delta < \Delta \) because of the uniform estimate \( \mu_{\Delta}(\hat{p}, p_L) \leq \mu_{\Delta}(\bar{w}, \bar{v}) \) and the Taylor series of the logarithm.

The function \( \hat{p} \mapsto BL(\hat{p}) - (\hat{v} - \hat{p}) \) is continuous and strictly positive on \( \Gamma \) and so has a positive minimum \( m \) on \( I \). It thus follows

\[
f'(\frac{2\mu_{\Delta}(\hat{p}, p_L)}{\Delta}) \geq (1 - \mu_{\Delta}(\bar{w}, \bar{v})) m \geq \frac{1}{2} m, \text{ for all } \hat{p} \in I, \Delta < \Delta \).
\]

This delivers the result, recalling that \( f' \) is strictly positive for positive arguments and also strictly increasing. \( \Box \)

\(^74\)Note that the interior of \( \Gamma \) is non-empty as it contains \( \bar{v} \).
Remark 8. In the more general case of intensity costs satisfying (18) one uses instead that relation to prove the same statement as in Lemma 14. The only fact used in the new proof is the uniformity of the limit in (18). This is used twice. Once to prove that \( \mu(\delta,p_L) \) remains bounded as \( \Delta \to 0 \), for any \( \delta,p_L \) and the second time to show that the same expression is also bounded from below. The details are very similar to the ones in the proof of Lemma 14 and so are skipped.

Lemma 15. 1-a) Fix any \( \Delta > 0 \). The reservation pricing relation for type \( \hat{v} \) is solvable for \( p_L \) for any \( \hat{p} \geq \hat{v} \).

1-b) Let \( I \) as in Lemma 14 and let \( \kappa(I) \) be the corresponding positive constant delivered by Lemma 14. Then there is a \( \Delta(I) > 0 \) such that when \( 0 < \Delta < \Delta(I) \), the reservation pricing relation for type \( \hat{v} \) is solvable for all \( \hat{p} \) in \( I \) that satisfy

\[
\hat{v} - \hat{p} < \frac{1}{2} \frac{\kappa(I)}{r + \kappa(I)} BL(\hat{p}).
\] (78)

Moreover, whenever the reservation pricing relation for type \( \hat{v} \) is solvable for some \( \hat{p} \), the solution is unique for fixed \( \Delta \) and \( \hat{p} \). \( p_L(\Delta, \hat{p}) \) is increasing in \( \hat{p} \) for fixed \( \Delta \).

2) There exists an open neighborhood of the form \((\hat{p}, \hat{v})\) of \( \hat{v} \) and a \( \Delta > 0 \) such that the reservation pricing relation for type \( \hat{v} \) is solvable for \( p_L \) as long as \( \hat{v} \geq \hat{p} > \bar{p} \), whenever \( \Delta < \Delta \) and for all \( \Delta > 0 \) as long as \( \hat{p} \geq \hat{v} \). Moreover, the solution \( p_L(\Delta, \hat{p}) \) is unique and continuous in the parameters \( \Delta, \hat{p} \) as long as \( \Delta < \Delta \) and \( \hat{p} \in (\bar{p}, \hat{v}) \).

Proof. Fix a \( \hat{p} \in (\bar{p}, \hat{v}) \). One needs to solve for \( p_L \) in

\[
\hat{v} - p_L = \frac{\delta \mu(\hat{p}, p_L)}{1 - \delta + \delta \mu(\hat{p}, p_L)} BL(\hat{p}).
\] (79)

Here necessarily \( p_L \leq \min\{\hat{p}, \hat{v}\} \). Suppose \( \hat{v} \leq \hat{p} \). Then, if one sets \( p_L = \hat{v} \) on the left hand side it follows that the right-hand side of (79) is strictly larger because it is positive. If one sets \( p_L = p_L^{\min}(\hat{p}) \) then right-hand side becomes zero, whereas left-hand side is strictly positive.\(^{76}\) It follows that there exists a \( p_L \in (p_L^{\min}(\hat{p}), \hat{v}) \) such that (79) is satisfied.

Suppose now that \( \hat{p} < \hat{v} \). The condition (78) implies that \( \hat{v} - \hat{p} < BL(\hat{p}) \), which in turn implies that \( p_L^{\min}(\hat{p}) < \hat{p} \). It holds again that if one sets \( p_L = p_L^{\min} \) the right-hand side in (79) is zero, and the left-hand side is strictly positive. Now note that the function \( \delta \mapsto \frac{\delta \mu(\hat{p}, \hat{p})}{1 - \delta + \delta \mu(\hat{p}, \hat{p})} \) is increasing in \( \mu(\hat{p}, \hat{p}) \). Pick a compact interval \( I \) in the interior of \( \Gamma \) where \( \Gamma \) is defined in the statement of Lemma 14. Using Lemma 14 one picks a \( \bar{\Delta}(I) \) such that for all \( \mu(\hat{p}, \hat{p}) \geq \kappa(I) \), whenever \( \hat{p} \in I, 0 < \bar{\Delta} < \Delta(I) \). The condition (78) together with the fact that \( B(\hat{p}) \) is bounded away from zero and from above for \( \hat{p} \in I \), imply immediately

\[
\hat{v} - \hat{p} < \frac{\delta \kappa(I)}{1 - \delta + \delta \kappa(I)} BL(\hat{p}),
\] (80)

\(^{75}\)There is nothing special about the constant \( \frac{1}{2} \). The proof would go through when replacing (78) with an estimate of the type

\[
\hat{v} - \hat{p} < \frac{\kappa(I)}{r + \kappa(I)} BL(\hat{p}),
\]

as long as \( \bar{a} \in (0, 1) \).

\(^{76}\)Note also that \( p_L^{\min}(\hat{p}) < \hat{p} \) automatically in this case.
if one picks \( 0 < \bar{\Delta}_1(I) \leq \bar{\Delta}(I) \) small enough. \( \Delta_1(I) \) has to ensure that one can go over from (78) to (80) for all \( \hat{p} \in I \). (80) together with the estimate \( \mu_\Delta(\hat{p}, \hat{p}) \geq \kappa(I)\Delta \) delivers that for \( \hat{p} \in I \) and \( \Delta < \Delta_1(I) \) it holds

\[
\hat{v} - \hat{p} < \frac{\delta \mu_\Delta(\hat{p}, \hat{p})}{1 - \delta + \delta \mu_\Delta(\hat{p}, \hat{p})} BL(\hat{p}).
\]

Now the intermediate-value theorem for continuous functions delivers existence.

To see uniqueness for fixed \( \Delta, \hat{p} \) note that one can easily rewrite (79) as

\[
\frac{1 - \delta + \delta \mu_\Delta(\hat{p}, p_L)}{\delta \mu_\Delta(\hat{p}, p_L)} (\hat{v} - p_L) = BL(\hat{p}).
\] (81)

The left-hand side of (81) is strictly decreasing in \( p_L \) and so uniqueness follows immediately. Since both sides of (81) are strictly decreasing in respectively \( \hat{p} \) and \( p_L \) it follows that \( p_L(\hat{p}) \) is strictly increasing in \( \hat{p} \).

2) This is an easy consequence of 1) and its proof arguments. For the case \( \hat{p} < \hat{v} \), pick an \( I \) as in Lemma 14 and for the corresponding \( \kappa(I) \) delivered from that Lemma, pick \( \hat{p} \) such that (78) in the proof of 1-b) above is ensured for any \( \hat{p} \in (\hat{p}, \hat{v}) \).

To continue with the proof of existence near HFL I establish first the following auxiliary result.

**Lemma 16.** For any fixed \( \hat{p} \) such that \( BL(\hat{p}) > 0 \) and \( p_L < \min\{\hat{p}, \hat{v}\} \) such that \( \mu_\Delta(\hat{p}, p_L) \) satisfies (77) and \( p_L \) solves the reservation price relation for the type \( \hat{v} \) for \( \hat{p} \) and \( \Delta \), the sequence \( \frac{\mu_\Delta(\hat{p}, p_L)}{\Delta} \) converges to a strictly positive limit which is a function of \( \hat{p} \).

**Proof.** For ease of notation I drop the arguments in this proof \( \mu \) depends on.

Replacing the reservation price relation of type \( \hat{v} \) into (77) leads to the equation

\[
BL(\hat{p}) \frac{1 - \delta}{1 - \delta + \delta \mu} = \frac{1}{1 - \mu} f'(\frac{-\log(1 - \mu)}{\Delta}).
\] (82)

From the estimate \( f'(\frac{-\log(1 - \mu)}{\Delta}) \leq BL(\hat{p}) \) and \( \mu \leq -\log(1 - \mu) \) one sees that \( \frac{\mu}{\Delta} \) remains bounded from above. On the other hand, \( \frac{\mu}{\Delta} \) is clearly bounded from below so that overall one has that the sequence is bounded. Let now \( \lambda(\hat{p}) \) be a limit point of \( \frac{\mu}{\Delta} \) as \( \Delta \to 0 \). It follows from (82) that \( \lambda(\hat{p}) \) satisfies

\[
f'(\lambda) \frac{r + \lambda}{r} = BL(\hat{p}).
\] (83)

Here one sees that \( \lambda \) only depends on \( \hat{p} \) and it is unique because the left-hand side of (83) is strictly increasing in \( \lambda \). It easily follows, that \( \lambda(\hat{p}) \) is strictly decreasing in \( \hat{p} \). Positivity of the limit is straightforward from the assumptions on \( f \).

**Remark 9.** In the case of general costs of intensity satisfying (18), the proof of Lemma 16 follows the same steps, except that one replaces in (82) the right-hand side with \( \frac{\partial}{\partial \mu} C(\Delta, \Delta, \frac{\mu}{\Delta}) \).
In the following, for a neighborhood $N$ around $\hat{v}$ so that the reservation pricing of type $\hat{v}$ is solvable for $p_L < \min\{\hat{p}, \hat{v}\}$ for all small $\Delta$, I write $\lambda(\hat{p})$ for the unique solution $\lambda$ to the equation (83).

A careful analysis of the proof of existence near HFL of the strongly stationary equilibria shows that the steps of the proof carry through by just replacing $\lambda \leadsto \lambda(\hat{p})$ when the following conditions for $\hat{p}$ from a suitable open neighborhood of $\hat{v}$ are required.

$$\lambda(\hat{p}) \hat{p} (1 - c(\pi_0)) < r \hat{v}, \quad (84)$$

and in addition, depending on whether $\pi_0 \leq \frac{1}{2}$ or $\pi_0 > \frac{1}{2}$ require\(^{77}\)

if $\pi_0 \leq \frac{1}{2}$, then $r > \lambda(\hat{p})$, if $\pi_0 > \frac{1}{2}$, then $r > \sqrt{2} \lambda(\hat{p}). \quad (85)$

If $\hat{p}$ is chosen close enough to $\hat{v}$, the conditions in (85) are sufficient to ensure (84). Another sufficient condition can be given by requiring (85) for $\hat{p} = \hat{v}$ and then requiring $\hat{p}$ in a neighborhood of $\hat{v}$ such that (84) and (85) are then satisfied in that neighborhood. In this way, using (83) and the properties of $f$ one arrives at the following sufficient conditions for (85) evaluated at $\hat{p} = \hat{v}$.

$$\text{if } \pi_0 \leq \frac{1}{2}, \text{ then } f'(r) > \frac{1}{2} BL(\hat{v}), \quad \text{if } \pi_0 > \frac{1}{2}, \text{ then } f'\left(\frac{1}{\sqrt{2}} r\right) > \frac{\sqrt{2}}{\sqrt{2} + 1} BL(\hat{v}). \quad (86)$$

Recalling the uniform estimate $BL(\hat{p}) < \pi_0 \bar{v}$ one can strengthen these even more by requiring a sufficient condition on $f'$ alone.

$$\text{if } \pi_0 \leq \frac{1}{2}, \text{ then } f'(r) > \frac{1}{2} \pi_0 \bar{v}, \quad \text{if } \pi_0 > \frac{1}{2}, \text{ then } f'\left(\frac{1}{\sqrt{2}} r\right) > \frac{\sqrt{2}}{\sqrt{2} + 1} \pi_0 \bar{v}. \quad (87)$$

**Remark 10.** One can always ensure that (87) and other weaker variants of it are satisfied for some $f$, because the class of functions $f$ yielding $\mathcal{C}$ through the requirement (18) is a cone within the space of convex, strictly increasing, differentiable functions with a zero at zero and derivative zero at zero. Thus, if a $f$ as needed in (18) does not work for (87), one can always go over to some rescaling of $f$ by a factor $\alpha \in (0, 1)$.

The following Propositions result.

**Proposition 14.** [Existence of mixed SSE with deterministic variable accuracy costs and endogenous intensity of learning] Pick any $r, \pi_0, \underline{v}, \bar{v}$. Assume that (87) holds true.

Then there is a neighborhood $N$ of $\hat{v}$ and an $\epsilon > 0$ such that for all $\Delta < \epsilon$ and $\hat{p} \in N$ there exist stationary two-price disclosure equilibria with average price $\hat{p}$. Moreover, for any fixed average price $\hat{p}$ the quantities $\mu(\Delta, \hat{p}), a(\hat{p}), q(\hat{p}, \Delta), p_L(\hat{p}, \Delta), p_H(\hat{p}, \Delta), p(\hat{p}, \Delta)$ are uniquely determined.

Next follows the analysis in the HFL.

**Proposition 15.** For any $\hat{p}$ in the neighborhood $N$ of $\hat{v}$ coming from Proposition 14 the following hold true in HFL.

\(^{77}\)Note that (85) can be weakened even more by steps similar to the ones in the proof of Proposition 6. I skip these for the sake of length.
A. Expected delay in real time is equal to \( \frac{1}{\lambda(p)} \) and is increasing in \( \hat{p} \).

B. The price spread \( ps(\hat{p}) \) is bounded away from zero but the low price is charged with vanishingly small probability.

C. Denoting by \( a(\hat{p}) \) the reaction function of Buyer and \( GN(a(\hat{p})) \) the resulting probability of good news, Buyer’s and Seller’s payoffs are given by

\[
V_B(\hat{p}) = \frac{\lambda(\hat{p})}{r + \lambda(\hat{p})} (GN(a(\hat{p}))(\bar{w}(\hat{p}) - \hat{p}) - C(a(\hat{p}))) - \frac{f(\lambda(\hat{p}))}{r + \lambda(\hat{p})}.
\]

and

\[
V_S(\hat{p}) = \frac{\lambda(\hat{p})}{r + \lambda(\hat{p})} (GN(a(\hat{p})))\hat{p} + (1 - GN(a(\hat{p})))\bar{w}(\hat{p})).
\]

D. The shortfall in efficiency (i.e. the difference between \( \hat{v} \) and sum of payoffs) is given by

\[
\frac{r}{r + \lambda(\hat{p})} \hat{v} + \frac{\lambda(\hat{p})}{r + \lambda(\hat{p})} C(a(\hat{p})) + \frac{f(\lambda(\hat{p}))}{r + \lambda(\hat{p})}
\]

and is always strictly positive.

**Details on proofs of Proposition 14 and 15.** One finds first a neighborhood \( \mathcal{N} \) of \( \hat{v} \) and \( \bar{\Delta} > 0 \) such that \( w(\hat{p}) < \hat{p} \) for \( \hat{p} \in \mathcal{N} \) and so that the reservation pricing for type \( \hat{v} \) is solvable for \( p_L \) as a function of \( \hat{p} \in \mathcal{N} \) whenever \( \Delta < \bar{\Delta} \). The estimates (51) and (52) remain true in this setting, when one also allows the additional dependence of \( \mu \) on \( \hat{p} \).

The estimate (54) remains the same with the added dependency on \( \hat{p} \) for \( \mu \). From here the proof follows the same steps as for the case of exogenous intensity.

One has to ensure that in the HFL the equivalent of (55) in the proof of the case of exogenous \( \lambda \) remains true. One also has to ensure that the relation (57) for the \( \hat{p} \) near enough to \( \hat{v} \) is valid. The arguments are the same as in the case of exogenous \( \lambda \), except that now \( \lambda \rightarrow \lambda(\hat{p}) \) and \( \lambda(\hat{p}) \) satisfies (83).

Once one picks a neighborhood of \( \hat{v} \) \( \mathcal{N} \) and \( \bar{\Delta} > 0 \) (where \( \bar{\Delta} > 0 \) is small enough to work for all \( \hat{p} \in \mathcal{N} \)) so that the corresponding \( \hat{p} \)-dependent version of (55) and the near-HFL version of the corresponding \( \hat{p} \)-dependent version of (57) is ensured for \( \hat{p} \in \mathcal{N} \) and \( \Delta < \bar{\Delta} \), the proof of existence follows precisely the same steps as in Proposition 6 (case of deterministic variable costs).

For the HFL limit, one just follows the same steps as in the proof of Theorem 3 (case of deterministic variable costs) and uses the fact that \( \frac{\mu(\hat{p},p_L(\hat{p}))}{\Delta} \rightarrow \lambda(\hat{p}), \) as \( \Delta \rightarrow 0 \) instead of its exogenous version \( \mu(\Delta) \rightarrow \lambda, \) as \( \Delta \rightarrow 0 \). \( \square \)

C.2 Stochastic fixed costs of information with endogenous intensity

First note, that \( \bar{\mu}(\hat{p},p_L)(BL(\hat{p},p_L) - (\hat{v} - p_L)) \) for \( \hat{p} \in (\hat{v}, \bar{v}) \) and \( p_L < \min\{\hat{p}, \bar{v}\} \) is again increasing in \( p_L \) and decreasing in \( \hat{p} \). To see this, just note that

\[
\bar{\mu}(\hat{p},p_L)(BL(\hat{p},p_L) - (\hat{v} - p_L)) = \mathbb{E} [\pi_0(\bar{v} - \hat{p}) - (\hat{v} - p_L) - c, c \leq \pi_0(\bar{v} - \hat{p}) - (\hat{v} - p_L)].
\]
The FOC for the optimal choice of intensity $\mu_0(\Delta, \hat{p})$ at the beginning of a period satisfies

$$\bar{\mu}(\hat{p}, p_L)(BL(\hat{p}, p_L) - (\hat{v} - p_L)) = \frac{\partial}{\partial \mu_0} C(\Delta, \mu_0).$$

(90)

To ease on notation in the following, I suppress again the dependence of $C$ on $\Delta$ when this is not explicitly needed for the argument.

Throughout I use the fact that the function

$$\mathbb{E}_{c \sim F}[x - c, c \leq x]$$

is strictly increasing in $x \in \text{supp}(F)$, where $\text{supp}(F)$ denotes the support of $F$. This implies that in terms of the best response of Buyer, it holds $\mu_0(\Delta, \hat{p}, p_L) = \rho(\Delta, p_L - \pi_0\hat{p})$ for some function $\rho$ which is strictly increasing in both of its arguments. In particular, for fixed $\hat{p}, p_L$ the sequence $\frac{\mu_0(\Delta, \hat{p}, p_L)}{\Delta}$ has a limit since it is bounded. Thus, (90) delivers a unique $\mu_0(\Delta, \hat{p}, p_L) \in (0, 1)$ for every $\Delta > 0, \hat{p} \in (\hat{v}, \hat{v})$. $p_L^\text{min}(\hat{p}) < p_L \leq \min\{\hat{v}, \hat{p}\}$. $\mu_0(\Delta, \hat{p}, p_L) \in (0, 1)$ is strictly decreasing in $\hat{p}$ and strictly increasing in $p_L$.

Moreover, note that for fixed $\hat{p}, p_L$ the sequence $\frac{\rho(\Delta, p_L - \pi_0\hat{p})}{\Delta}$ converges to some $\bar{\rho}(p_L - \pi_0\hat{p})$ with $\bar{\rho}(\cdot)$ increasing.

In all, this delivers a function $\mu_\Delta(\hat{p}, p_L) = \mu_0(\Delta, \hat{p}, p_L) \bar{\mu}(\hat{p}, p_L)$ which is continuous in its arguments and strictly increasing in $p_L$ as well as strictly decreasing in $\hat{p}$. Furthermore it holds again $\mu_\Delta(\hat{p}, p_L^\text{min}(\hat{p})) = 0$ with $p_L^\text{min}(\hat{p}) = \hat{v} - \pi_0(\hat{v} - \hat{p})$.

Finally, note that the left-hand side of (90) is independent of $\Delta$ and therefore, with the same arguments as in the case of deterministic variable costs, the sequence $\frac{\mu_0(\Delta, \hat{p}, p_L)}{\Delta}$ is bounded away from zero, and converges to a function which is denoted henceforth by $\lambda(\hat{p}, p_L)$.

**Lemma 17.**

1) a) Fix any $\Delta > 0$. The reservation pricing relation for type $\hat{v}$ is always solvable for $p_L$ for any $\hat{p} \geq \hat{v}$.

1) b) There exists a $\hat{p}$ such that the reservation pricing relation for type $\hat{v}$ is always solvable for $\hat{p} \in (\hat{p}, \hat{v})$.

Moreover, whenever the reservation pricing relation for type $\hat{v}$ is solvable for some $\hat{p}$, the solution is unique for fixed $\Delta$ and $\hat{p}$. The solution $p_L(\Delta, \hat{p})$ is increasing in $\hat{p}$ for fixed $\Delta$.

2) There exists an open neighborhood of the form $(\hat{p}, \hat{v})$ of $\hat{v}$ and a $\tilde{\Delta} > 0$ such that the reservation pricing relation for type $\hat{v}$ is solvable for $p_L$ as long as $\hat{v} \geq \hat{p} > \tilde{p}$, whenever $\Delta < \tilde{\Delta}$ and for all $\Delta > 0$ as long as $\hat{p} \geq \tilde{v}$. Moreover, the solution $p_L(\Delta, \hat{p})$ is unique and continuous in the parameters $\Delta, \hat{p}$ as long as $\Delta < \tilde{\Delta}$ and $\hat{p} \in (\hat{p}, \hat{v})$. Finally, $p_L(\hat{p}, \Delta)$ is strictly increasing in $\hat{p}$.

3) As $\Delta \to 0$ the solution $p_L(\Delta, \hat{p})$ converges to $\tilde{p}_L(\hat{p})$ which satisfies the relation

$$\hat{v} - \tilde{p}_L(\hat{p}) = \frac{\lambda(\hat{p})\bar{\mu}(\hat{p}, \tilde{p}_L(\hat{p}))}{r + \lambda(\hat{p})\bar{\mu}(\hat{p}, \tilde{p}_L(\hat{p}))}BL(\hat{p}, \tilde{p}_L(\hat{p})).$$

**Proof.**

1) a) is proven just as in Lemma 15.

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78Note here that $\pi_0(\hat{v} - \hat{p}) - (\hat{v} - p_L) = p_L - \pi_0\hat{p} - (1 - \pi_0)\hat{v}$ which is strictly positive whenever $p_L > p_L^\text{min} = (1 - \pi_0)\hat{v} + \pi_0\hat{p}$. 

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Lemma 17. Look at the case \( \hat{p} < \hat{v} \). I just need to establish one direction of the inequality to justify the use of the intermediate-value theorem. Note that \( BL(\hat{p}, \hat{p}) = \pi_0(\bar{v} - \hat{p}) - \mathbb{E}[c | c \leq (1 - \pi_0)(\hat{p} - \bar{v})] \), \( \mu(\Delta, \hat{p}, \hat{p}) = \mu_0(\Delta, \hat{p}, \hat{p}) F((1 - \pi_0)(\hat{p} - \bar{v})) \), where \( \mu_0 \) satisfies

\[
\mathbb{E}[(1 - \pi_0)(\hat{p} - \bar{v}) - c, c \leq (1 - \pi_0)(\hat{p} - \bar{v})] = C'(\mu_0(\hat{p}, \hat{p})).
\]

In particular, \( \mu(\Delta, \hat{p}, \hat{p}) > 0 \) for all \( \hat{p} \in (\bar{v}, \hat{v}) \). Clearly \( \mu(\Delta, \hat{p}, \hat{p}) \) and \( \mu(\Delta, \hat{p}, \hat{p}) \) are increasing and continuous in \( \hat{p} \). Take some \( p < \hat{v} \). Then \( \lambda(\hat{p}, \hat{p}) \geq \lambda(p, p) \), \( \mu(\hat{p}, \hat{p}) \geq \mu(p, p) \) for all \( \hat{p} \in (p, \hat{v}) \). First pick \( p_0 \in (p, \hat{v}) \) such that

\[
\hat{v} - \hat{p} < \frac{1}{2} \frac{\lambda(p, p) \hat{v} + (p, p)}{2 + \lambda(p, p) \hat{v} + (p, p)} BL(p, p),
\]

for \( \hat{v} > \hat{p} > p_0 \). Now pick \( \Delta > 0 \) such that for \( \Delta < \Delta \) it holds true

\[
\frac{1}{2} \frac{\lambda(p, p) \mu(p, p)}{r + \lambda(p, p) \mu(p, p)} < \frac{\delta \mu_0(\Delta, p, p) \mu(p, p)}{1 - \delta + \delta \mu_0(\Delta, p, p) \mu(p, p)}
\]

Combining and using monotonicity delivers for \( \hat{p} \in (p_0, \hat{v}), \Delta < \Delta \)

\[
\hat{v} - \hat{p} < \frac{\delta \mu_0(\Delta, \hat{p}, \hat{p}) \mu(\hat{p}, \hat{p})}{1 - \delta + \delta \mu_0(\Delta, p, p) \mu(p, p)} BL(\hat{p}, \hat{p}).
\]

Thus, for all \( \hat{p} \in (p_0, \hat{v}), \Delta < \Delta \) solvability follows for some \( p_L(\Delta, \hat{p}) \in (p^{min}_L(\hat{p}), \hat{p}) \).

Uniqueness and part 2), except for monotonicity, follow exactly as in Lemma 15. Monotonicity follows the same steps as in the proof of Lemma 11 and is thus skipped.

3) is an easy consequence of the continuity of the functions involved and of the fact that \( \mu(\Delta, \hat{p}, p_L(\Delta)) \) is continuous in all of the arguments and converges as \( \Delta \rightarrow 0 \).

Combining (90) and the solvability of the reservation price relation of type \( \hat{v} \) from Lemma 17 one proves again that there exists a neighborhood of \( \hat{v} \) such that \( \mu_0(\Delta, \hat{p}, p_L(\Delta)) \rightarrow \lambda(\hat{p}) \) (proof is analogous to the one of Lemma 16).

Define

\[
BL(\hat{p}) = \mathbb{E} [{\pi_0}(\hat{v} - \hat{p}) - (\hat{v} - p_L(\hat{p})) - c, c \leq {\pi_0}(\hat{v} - \hat{p}) - (\hat{v} - p_L(\hat{p}))].
\]

One combines (90) with solvability of the reservation price relation of the type \( \hat{v} \) to write the relation

\[
BL(\hat{p}) = f'(\lambda(\hat{p})),
\]

as

\[
r(\hat{v} - p_L(\hat{p})) = f'(\lambda(\hat{p}))(\lambda(\hat{p})).
\]

Lemma 17 delivers that \( \hat{p} \mapsto \lambda(\hat{p}) \) is strictly decreasing in \( \hat{p} \) when the reservation price relation of the type \( \hat{v} \) is satisfied. Thus, a higher average price has a detrimental effect on the intensity chosen to learn. I note also for future use that it follows \( p_L(\hat{p}) - \pi_0 \hat{p} \) is decreasing in \( \hat{p} \). Since this is a sufficient statistic for all of \( \mu, \mu, BL \), it also follows that all of these are decreasing in \( \hat{p} \).
From here on the proof remains very similar to the case of deterministic variable costs. One does the replacements \( \lambda(\hat{p}) \sim \lambda(\hat{p})\bar{\mu}(\hat{p}) \), where the shortcut \( \bar{\mu}(\hat{p}) := \bar{\mu}(\hat{p}, p_{L}(\hat{p})) \) has been used. To find conditions which are not minimal, but are not directly dependent on the distribution of costs \( F \) one can just estimate \( \bar{\mu}(\hat{p}) \leq 1 \) to get, depending on whether \( \pi_{0} \leq \frac{1}{2} \) or \( \pi_{0} > \frac{1}{2} \), a condition just as (59) in the proof of Proposition 6 for the case of deterministic variable costs. Finally, one uses (92) to estimate \( r(\hat{v} - p_{L}(\hat{p})) < \hat{v} - p_{L}^{\min}(\hat{p}) = \pi_{0}(\hat{v} - \hat{p}) < \pi_{0}\hat{v} \). This delivers the following sufficient condition for existence in the neighborhood of \( \hat{v} \).

\[
\text{if } \pi_{0} \leq \frac{1}{2}, \text{ then } f'(r) > \pi_{0}\hat{v}, \quad \text{if } \pi_{0} > \frac{1}{2}, \text{ then } f'(r) > \frac{r}{\sqrt{2}} > 2\pi_{0}\hat{v}. \tag{93}
\]

Note that (93) is independent of the specification of \( F \).

The analysis delivers the following Propositions.

**Proposition 16.** [Existence with stochastic fixed accuracy costs and endogenous intensity of learning] Pick any \( r, \pi_{0}, \bar{\nu}, \bar{v}. \) Assume that (93) holds.

Then there is a neighborhood \( \mathcal{N} \) of \( \hat{v} \) and an \( \epsilon > 0 \) such that for all \( \Delta < \epsilon \) and \( \hat{p} \in \mathcal{N} \) there exist strongly stationary equilibria with average price \( \hat{p} \). Moreover, for any fixed average price \( \hat{p} \) the quantities \( \mu(\Delta, \hat{p}), \mu_{0}(\Delta, \hat{p}), q(\hat{p}, \Delta), p_{L}(\hat{p}, \Delta), p_{H}(\hat{p}, \Delta), p(\hat{p}, \Delta) \) are uniquely determined.

And now to the HFL of this case. The proof of the following Proposition is a simple adaptation of the proof from the case of exogenous intensity, with the added dependence of \( \lambda \) on \( \hat{p} \).

**Proposition 17.** Pick any \( r, \pi_{0}, \bar{\nu}, \bar{v}. \) Assume that (93) holds. Denote \( \bar{\mu}(\hat{p}) = F(\pi_{0}(\bar{v} - \hat{p}) - (\bar{v} - p_{L}(\hat{p}))) \).

For any \( \hat{p} \) as in Proposition it holds in the HFL

A. Expected delay in real time is equal to \( \frac{1}{\lambda(\hat{p})\bar{\mu}(\hat{p})} \) and it is increasing in \( \hat{p} \).

B. The price spread \( ps(\hat{p}) \) is bounded away from zero but the low price is charged with vanishingly small probability

C. Buyer’s and Seller’s payoffs are given by

\[
V_{B}(\hat{p}) = \frac{\lambda(\hat{p})\bar{\mu}(\hat{p})}{r + \lambda(\hat{p})\bar{\mu}(\hat{p})} \left[ \pi_{0}(\bar{v} - \hat{p} - \mathbb{E}[c|c \leq (\bar{v} - \hat{p}) - (\bar{v} - p_{L}(\hat{p}))]) - \frac{f(\lambda(\hat{p}))}{r + \lambda(\hat{p})} \right]. \tag{94}
\]

and

\[
V_{S}(\hat{p}) = \frac{\lambda(\hat{p})\bar{\mu}(\hat{p})}{r + \lambda(\hat{p})\bar{\mu}(\hat{p})} \left( \pi_{0}\hat{p} + (1 - \pi_{0})\bar{\nu} \right). \tag{95}
\]

D. The shortfall in efficiency (i.e. the difference between \( \bar{v} \) and sum of payoffs) is given by

\[
\frac{r}{r + \lambda(\hat{p})\bar{\mu}(\hat{p})} \bar{v} + \frac{\lambda(\hat{p})\bar{\mu}(\hat{p})}{r + \lambda(\hat{p})\bar{\mu}(\hat{p})} \pi_{0}\mathbb{E}[c|c \leq \pi_{0}(\bar{v} - \hat{p}) - (\bar{v} - p_{L}(\hat{p}))] + \frac{f(\lambda(\hat{p}))}{r + \lambda(\hat{p})},
\]

and is positive.

\(^{79}\)Again, the stated conditions are not minimal conditions on the parameters and their relaxation follows in a similar way to the relaxation of the existence conditions in the case of deterministic variable costs on accuracy.
C.2.1 The case of pure pricing

The case of strongly stationary equilibria with pure pricing is a straightforward combination of arguments from the case of mixed pricing and of the case of costless intensity. For brevity’s sake I only give a sketch of the arguments for the solvability of the $\hat{v}$-indifference condition.

In the case of deterministic variable costs intensity choice satisfies (recall that $\mu = 1 - e^{-\lambda \Delta}$)

$$BL(\hat{p}) - (\hat{v} - \hat{p}) = C'(\mu).$$

In the case of stochastic fixed costs it satisfies

$$\bar{\mu}(\hat{p}, \hat{p}) (BL(\hat{p}, \hat{p}) - (\hat{v} - \hat{p})) = C'(\mu).$$

In both cases this results in an equilibrium value for $\mu$ which I denote by $\mu(\hat{p}, \hat{p})$ using the notation of subsection C.1.1.

The $\hat{v}$-indifference relation in the case of pure pricing is given by

$$\hat{v} - \hat{p} = \frac{\delta \mu(\hat{p}, \hat{p})}{1 - \delta + \delta \mu(\hat{p}, \hat{p})}BL(\hat{p}).$$

Solvability of the indifference relation for the case of deterministic variable costs for some $\hat{p} \in (v, \hat{v})$ follows very closely that of subsection B.3.2. Therefore I skip it. In the case of stochastic fixed costs, note that the right-hand side of (96) is strictly positive if $\hat{p} = \hat{v}$, while the left-hand side is zero. In case $\hat{p} = v$ the left-hand side is strictly positive, whereas the right-hand side is zero, because $\bar{\mu}(v, v) = F((1 - \pi_0)(v - v)) = 0$. From here, the proof follows very closely a combination of the cases of pure pricing for exogenous intensity and the case of mixed pricing with endogenous intensity.

C.2.2 Comparative statics for information acquisition

In this subsection I don’t comment separately on the cases of pure and mixed pricing, because the proofs are verbatim the same for both cases.

The results stated in Proposition 8 are part of the statements in the following two Propositions.

**Proposition 18.** 1) Suppose there are two strongly stationary equilibria in the HFL with the same average price $\hat{p}$ and all parameters the same except for $r_1 > r_2$. Then the equilibrium intensity is higher for $r_1$ than $r_2$. Equilibrium accuracy is the same in both cases.

2) Suppose there are two strongly stationary equilibria with the same average price $\hat{p}$ and all parameters the same except for $\pi_1 > \pi_2$ and $\frac{\hat{v} + \bar{\pi}}{2} > \hat{p}$, whereas it is higher for $\pi_2$ if $\frac{\hat{v} + \bar{\pi}}{2} < \hat{p}$. Equilibrium intensity is always higher for $\pi_1$.

**Proof.** Recall that in this case, the benefit of learning in the HFL, $BL(\hat{p})$ is dependent only on $v, \bar{v}, \pi_0$. It follows from the relation determining $\lambda(\hat{p})$, given in (83), that for fixed $\hat{p}$, $\lambda$ is increasing in $r$ and also in $\pi_0$. Next, look at accuracy $a(\hat{p})$. Recall that it does not depend on any other parameters, except $\hat{p}, \pi_0, v, \bar{v}$. Using the first order condition related to the incentive constraint (OL-intensive) one arrives easily at the required result. $\square$
Next the comparative statics for the case of stochastic fixed costs of accuracy.

**Proposition 19.** 1) Suppose there are two strongly stationary equilibria in the HFL with the same average price \( \hat{p} \) and all parameters the same, except for \( r_1 > r_2 \). Then the equilibrium intensity is higher for \( r_1 \) than \( r_2 \).

2) Suppose there are two strongly stationary equilibria with the same average price \( \hat{p} \) and all parameters the same, except for \( \pi_0^1 > \pi_0^2 \). Then the equilibrium intensity is higher for \( \pi_0^1 \).

3) Suppose in the case of stochastic fixed costs there are two strongly stationary equilibria with the same average price \( \hat{p} \) and all parameters the same, except for \( F_1 >_{FOSD} F_2 \). Then \( \lambda_1 \) is lower than \( \lambda_2 \).

**Proof.** By using the definition (91) one writes the relation in part 3) of the statement of Lemma 17 as follows.

\[
\frac{r}{\lambda} (\hat{v} - p_L) = BL(\hat{p}, p_L). \tag{97}
\]

Recall that the choice of \( \lambda \) at the beginning of a period satisfies the FOC condition in the HFL.

\[
BL(\hat{p}, p_L) = f'(\lambda) \tag{98}
\]

1) Suppose that the environment changes so that there is a higher \( r \), but the other parameters stay the same. There are two choice variables that can adjust to keep the reservation pricing relation for type \( \hat{v} \) satisfied, i.e. relation (97) intact. Either \( \lambda \) can increase to reinstate the balance in (97), or if \( \lambda \) weakly falls, then \( p_L \) necessarily goes up. But, other else equal, this leads to a higher level of \( BL(\hat{p}, p_L) \), which, via (98) also leads to an increase of \( \lambda \). Overall it follows the equilibrium intensity \( \lambda \) must increase with \( r \) in the HFL.

2) Suppose that \( \pi_0 \) increases but other parameters are kept the same. Because \( BL(\hat{p}, p_L) \) is strictly increasing in \( \pi_0 \), all else kept equal, \( \lambda \) and/or \( p_L \) necessarily change to reinstate (97). Suppose \( p_L \) falls enough so that \( BL \) remains the same in both situations. Then \( \lambda \) necessarily increases. Suppose next, that \( p_L \) falls but by little enough so that \( BL \) increases overall in the situation with the higher prior. Via (98) the equilibrium value of \( \lambda \) necessarily increases in this case. Suppose for the remaining case, that \( p_L \) remains the same after the change in \( \pi_0 \), so that the (97) is reinstated through a fall in \( \lambda \). But in this case, \( BL \) necessarily increases as well, so that via (98), \( \lambda \) has to increase as well. Thus, this last case cannot arise because it leads to a contradiction.

Overall, it follows that the equilibrium intensity \( \lambda \) in the HFL necessarily increases with the prior.

3) Suppose that \( F \) increases in the FOSD-sense. Then for fixed \( \hat{p}, p_L \), \( BL \) decreases. If \( p_L \) adjusts upwards so that \( BL \) overall does not change, (97) implies that the equilibrium value \( \lambda \) necessarily decreases. If \( p_L \) adjusts upwards so that overall \( BL \) still decreases, then equilibrium \( \lambda \) falls because of (98). Suppose instead that \( p_L \) adjusts downwards to a shift of \( F \) in the FOSD-sense. Then overall, \( BL \) decreases. This implies via (97) that \( \lambda \) necessarily increases, because \( \hat{v} - p_L \) increases in this case. But (98) implies that \( \lambda \) has to decrease. This is a contradiction.

Overall, the equilibrium intensity \( \lambda \) in the HFL necessarily decreases whenever \( F \) increases in the FOSD-sense. \( \square \)
C.3 Proofs for subsections 4.2 and 4.3

C.3.1 Proofs for the extension to pre-learning negotiations

Proof of Proposition 9. I show first that there is no PBE in which game continues past \( t = 0 \). Suppose for the sake of contradiction there is such a PBE. In particular, it has to prescribe an offer \( p \) on path by Seller with positive probability, which is subsequently rejected with positive probability by Buyer. Let \( V_B(p) \) be the payoff of Buyer in the continuation bargaining game after she has rejected \( p \). It holds \( \hat{v} - p \leq \delta V_B(p) \). Moreover, the payoff from offering \( p \) of Seller is then at most \( \delta V_S(p) \) where \( V_S(p) \) is Seller-payoff in the continuation bargaining game played after Buyer has rejected \( p \). It holds \( \delta V_S(p) \leq \delta(\hat{v} - V_B(p)) \leq \delta \hat{v} + p - \hat{v} = p - (1 - \delta)\hat{v} \). Suppose Seller deviates instead to \( p' = \hat{v} - \delta V_B(p) - \epsilon \) for some \( \epsilon > 0 \) very small. Then if Buyer accepts she receives more than \( \delta V_B(p) \) and Seller receives \( p' > \delta(\hat{v} - V_B(p)) \), i.e. strictly more than \( \delta V_S(p) \). To arrive at the desired contradiction, assume now in addition that \( p \) is so that

- it is the price on path offered with positive probability and rejected with positive probability, with the highest \( V_B(p) \) upon rejection.

Whenever the set of prices offered on path and rejected with positive probability is compact, this assumption is w.l.o.g. in that, a price \( p \) can always be chosen to satisfy it. Otherwise, one can use an approximation of the supremum and pick \( \epsilon(p_n) > 0 \) very small, for \( p_n, n \geq 1 \) a sequence of prices such that they are offered and rejected on path with positive probability and so that \( V_B(p_n), n \geq 1 \) converge to the supremum in question. In that case one arrives at a contradiction similarly.

Efficiency of equilibria results trivially from the fact that agreement is always at \( t = 0 \).

C.3.2 Discussion of the case of negative values

First, I discuss any necessary adaptations of the proof arguments for existence and characterization of strongly stationary equilibria from the case \( v \geq 0 \) to the case \( v < 0 \).\(^{80}\)

When both disclosure and walking away are possible, then Buyer has zero continuation payoff whenever bad news arrives. To see this, suppose that in equilibrium the accuracy chosen leads to a bad-news valuation which is non-negative. Then Buyer can disclose bad news immediately and receive a zero continuation payoff. If Buyer instead picks an accuracy such that it leads to a negative valuation after bad news then Buyer can walk away. This again results in zero continuation payoff.

Overall, Buyer’s information acquisition decision is analogous to the case of \( v \geq 0 \), because payoff upon exercising the strategic outside option has the same structure as in the baseline model. Seller’s payoff in equilibria is different, depending on whether Buyer’s type can become negative or not. To include both cases, one needs to just do the following replacement \( w \rightarrow \max\{w, 0\} \) in Seller’s payoffs in this case.\(^{81}\) Otherwise the analysis remains the same.

Note that one does not need to put additional restrictions on parameters to ensure existence, besides the one that \( BL(\hat{p}) > 0 \), i.e. option value from learning in the HFL is

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\(^{80}\) The general results for all PBE for any \( \Delta > 0 \) hold true verbatim in this variation of the model. Details are available upon request.

\(^{81}\) Here \( w \) stands for \( w(\hat{p}) \) or \( v \) depending on type of accuracy costs.
positive. This is always possible for \( \hat{\rho} > 0 \) near \( \hat{v} \), because any amount of learning leads to a valuation after good news \( \bar{w} \) which is strictly positive.\(^{82}\)

**Proof of Proposition 10.** 1) is trivial so I focus on parts 2) and 3).

2) The sum of Buyer and Seller payoffs in the HFL is given by

\[
V_S(\hat{p}) + V_B(\hat{p}) = \frac{\lambda}{\lambda + r} \left( GN(a(\hat{p}))\bar{w}(\hat{p}) + (1 - GN(a(\hat{p})))\bar{w}^+(\hat{p}) - C(a(\hat{p})) \right).
\]

The change in welfare due to the possibility to learn is given by

\[
\frac{\lambda}{r + \lambda} C(a(\hat{p})) + \hat{v} - \frac{\lambda}{r + \lambda} \max\{\hat{v}, GN(a(\hat{p}))\bar{w}(\hat{p})\}.
\]

To show that learning can be beneficial from a welfare point of view, assume \( \pi_0 > \frac{1}{2} \), choose \( r \approx \sqrt{2}\lambda \) and let the exploitation costs \( c \) be very small where needed, i.e. \( c \approx 0 \) in the topology of pointwise convergence. The welfare change with respect to the case that learning is impossible is bounded from above by

\[
\frac{\lambda}{r + \lambda} o(1) + \hat{v} - \frac{1}{\sqrt{2} + 1} \pi_0 \bar{v}.
\]

Note that the condition \( \hat{v} < \frac{1}{\sqrt{2} + 1} \pi_0 \bar{v} \) is equivalent to

\[
\frac{\sqrt{2}}{\sqrt{2} + 1} \frac{\pi_0}{1 - \pi_0} < -\frac{\bar{v}}{\bar{v}},
\]

whereas the condition that \( \hat{v} > 0 \) is equivalent to

\[
\frac{\pi_0}{1 - \pi_0} > -\frac{\bar{v}}{\bar{v}}.
\]

Overall, it follows that under the condition that

\[
\frac{\sqrt{2}}{\sqrt{2} + 1} \frac{\pi_0}{1 - \pi_0} < -\frac{\bar{v}}{\bar{v}} < \frac{\pi_0}{1 - \pi_0},
\]

whenever exploitation costs are small enough and \( r \approx \sqrt{2}\lambda \), learning is beneficial from a welfare perspective.

3) The sum of Buyer and Seller payoffs in the HFL is given by

\[
V_S(\hat{p}) + V_B(\hat{p}) = \frac{\lambda \hat{\mu}(\hat{p})}{r + \lambda \hat{\mu}(\hat{p})} \left( \pi_0 \bar{v} - \mathbb{E}[c|c \leq \pi_0(\bar{v} - \hat{p}) - (\hat{v} - p_L(\hat{p}))] \right).
\]

The efficiency loss with respect to the case where learning is impossible (i.e. \( \hat{v} - (V_S + V_B) \)) is given by

\[
\frac{r}{r + \lambda \hat{\mu}(\hat{p})} \pi_0 \bar{v} + \frac{\lambda \hat{\mu}(\hat{p})}{r + \lambda \hat{\mu}(\hat{p})} \mathbb{E}[c|c \leq \pi_0(\bar{v} - \hat{p}) - (\hat{v} - p_L(\hat{p}))] + (1 - \pi_0)\bar{v}
\]

\[
= \frac{r}{r + \lambda \hat{\mu}(\hat{p})} \hat{v} + \frac{\lambda \hat{\mu}(\hat{p})}{r + \lambda \hat{\mu}(\hat{p})} \left( \mathbb{E}[c|c \leq \pi_0(\bar{v} - \hat{p}) - (\hat{v} - p_L(\hat{p}))] + (1 - \pi_0)\bar{v} \right).
\]

\(^{82}\)In the case of pure pricing equilibria \( \hat{p} \) is positive and of course strictly below \( \hat{v} \).
Fix \( \hat{p} = \hat{v} \) and consider a sequence of \( F_n \) that are smooth and converge uniformly to the distribution of the Dirac measure on zero, i.e. I let the accuracy costs converge to zero approach zero uniformly. Then \( \hat{\mu}(\hat{v}) \) has a limit point which is strictly above zero. In fact, it is 1, because there should not be any divergence between the learning and exploration rate in the limit \( n \to \infty \) as costs of accuracy become vanishingly small. So that near the limit one gets approximately an inefficiency given by

\[
\frac{r}{r + \lambda} \pi_0 \bar{v} + (1 - \pi_0) \psi.
\]

Note that this is strictly negative whenever

\[
\frac{r}{r + \lambda} \pi_0 \bar{v} < - (1 - \pi_0) \psi < \pi_0 \bar{v},
\]

which is exactly the condition given in the statement of Proposition 10. This finishes the proof of the Proposition.

\( \Box \)