Optimal Stopping with General Risk Preferences*

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Abstract

We give a full characterization of the continuation and stopping regions of optimal stopping of (time-homogeneous) diffusions for an agent with a general risk preference. We consider separately the case of a naive agent who is unaware of the possible time inconsistency in her behavior and the case of a sophisticated agent who is fully aware of such an inconsistency. We apply our general results to characterize optimal stopping behavior among several distinct classes of non-Expected Utility risk preference models. We also show that although some specific models related to probability weighting may exhibit extreme behavior in the sense of naive agents always continuing with positive probability or sophisticated agents never starting, many other risk preference models do not lead to such extreme behavior.

1 Introduction.

A vast literature in economics, finance and other fields studying decision making applies optimal stopping techniques to dynamic decision situations. Applications in economically relevant settings include models of gambling, investment, search behavior, but also models of experimentation and information acquisition.

In a general stopping decision problem an agent is facing a sequence of lotteries over prizes and deciding at each moment in time whether to stop the sequence of lotteries and leave with the cumulative realizations of the lotteries or to continue and face another prize lottery. In the classical literature one usually postulates the agent has an Expected Utility (EU) preference over prize lotteries. Violations of the main behavioral implication of EU, the Independence axiom, are well-documented in the empirical and experimental literature and many alternative non-EU models for decision making under risk have been developed in the literature. A few of these have recently been used to study optimal stopping behavior (see Barberis (2012), Ebert and Strack (2015), Ebert and Strack (2016) Xu and Zhou (2013) for the case of models with probability weighting) but a general treatment with minimal assumptions on the preference side of the model is lacking. A characterization for general preferences is valuable not simply as a robustness exercise on

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the results about probability weighting (there are many other distinct models of non-EU risk preferences), but would also shed light on the fundamental behavioral properties of a risk preference which drive optimal stopping behavior.

This paper characterizes behavior in optimal stopping problems of time-homogeneous diffusions under very weak technical assumptions on the risk preference of the agent. We only require that the risk preference of the agent is a continuous weak order and that it additionally satisfies first-order-stochastic-dominance-monotonicity (FOSD-monotonicity).

If the risk preference of the agent in the optimal stopping problem is not EU, the agent will not be dynamically consistent and knowing the risk preference of the agent is not sufficient for the full description of her behavior in dynamic problems. Here we focus on the two rules most used in the economic/finance literature to replace dynamic consistency for an agent without commitment: naiveté and sophistication.

1.1 Results.

General characterizations. The main contribution of this paper is the full characterization of the continuation and stopping regions of diffusions for general risk preferences for both naive and sophisticated agents as well as the application of these general results to a wide range of classes of non-EU risk preferences.

We focus first on the naive case: the agent is potentially dynamically inconsistent and is unaware of this. In this case the agent solves at each moment in time a static decision problem: she is picking the best distribution from the set of distributions implementable through stopping times. We give a full characterization of the continuation and stopping regions of a naive agent for a general risk preference. This gives as a byproduct the full characterization for an agent who has commitment. A preference condition called weak risk aversion plays a central role in the full characterization of naive behavior. Weak risk aversion says that an agent always weakly prefers to get the expected value of a lottery with probability one than face the lottery. We show that a necessary and sufficient condition for an agent to continue with positive probability when facing a martingale diffusion is that the agent’s risk preferences violate weak risk aversion at the current state of the diffusion. This result generalizes to arbitrary diffusions by a suitable rescaling of the prize space which depends on the diffusion and the starting point of the optimal stopping problem. This uses again the weak risk aversion concept but now applied to an agent whose preference has been appropriately ‘translated’.

Given the importance of weak risk aversion for the naive case, we also examine its connection for a general risk preference to the traditional concept of (strong) risk aversion, defined as aversion to mean-preserving spreads. Namely, we find the appropriate relaxation of Independence which when added to weak risk aversion makes it equivalent to the stronger concept of risk aversion.

We model the sophisticated case as a game between selves: the current self and the future selves. This is a difficult game to study as it has a continuum of players, besides the large and intractable strategy space of all possible stopping strategies. Consequently, we restrict the strategy space of the agent and the equilibrium concept of the game the sophisticated selves play: the sophisticated agent can only use pure Markov stopping policies and can randomize among them at time zero only. This is equivalent in our setting to restricting the agent’s policy choice to the set of simple threshold stopping times and randomizations thereof at time zero: the sophisticated agent stops whenever
the process leaves an interval of wealth levels and she can randomize at time zero among these intervals. Same setup, but without the mixing at time zero, is used in Ebert and Strack (2016) in the special case of Cumulative Prospect Theory (CPT) preferences. The mixing at time zero may be beneficial for an agent who has convex risk preferences. We also show that whenever the risk preference of the agent is quasi-convex the agent chooses not to randomize at time zero and that the restriction to simple threshold stopping times is without loss of generality, up to tie-breaking considerations.

In the case of a sophisticated agent the characterization of the continuation and stopping regions for a fixed diffusion depends on the local comparison of a technological constraint called the win probability of the diffusion, and of a preference constraint which we call the calibration function of the risk preference. The sophisticated agent is able to implement a simple threshold stopping time with bounds $a$ and $c$ if and only if the technological constraint is less binding than the preference constraint at each intermediate prize between $a$ and $c$: the win probability for the simple threshold stopping time has to be higher than the calibration function at each intermediate prize.

This result is useful in characterizing fully the preferences of those sophisticated agents who never start any diffusion. The preferences that lead to such ‘extreme’ behavior satisfy extreme sensitivity to risk: for any pair of high and low prizes, the slope of the calibration function at the low prize is infinite or the slope of the calibration function at the high prize is zero. It turns out that for many well-known models of risk preference it is easy to identify cases where extreme sensitivity to risk is violated and thus the sophisticated agent starts some diffusion.

Applications of the general results. We illustrate the general results for both the naive and the sophisticated case with different classes of risk preferences. These include well-known models of risk preferences which are either globally quasi-concave, globally quasi-convex or both or neither. To name a few, we apply the general results to Betweenness preferences (Dekel (1986) and Gul (1991)), Cautious EU preferences (Cerreia-Vioglio et al. (2015) as well as to Ordinal Independent preferences and their special case Rank-Dependent Utility (Green and Jullien (1987)). Additional results omitted due to space constraints are available upon request: e.g. quadratic preferences, CPE (Közsegí and Rabin (2007)). The general characterization results allow to find easy-to-verify sufficient conditions which ensure that any of naive or sophisticated agent stops or continues any particular diffusion.

Results suggest that extreme behavior like the naive agent always continuing with positive probability (Ebert and Strack (2015), Henderson et al. (2017)) and the sophisticated agent never starting (Ebert and Strack (2016)) irrespective of the diffusion faced can not be generically replicated outside of a certain subclass of models which satisfy Ordinal Independence. Moreover, for many specifications within the class of Ordinal Independence preferences neither of the extreme behaviors occur. Finally, we show that all other classes of non-EU preference models we consider here are similar to EU in the sense that a naive risk loving agent will stop very ‘unfavorable’ diffusions and a sophisticated risk averse agent will choose to start diffusions which are ‘favorable’ enough.

Related Literature. To the best of knowledge Karni and Safra (1990) is the first paper which considers an optimal stopping problem with general risk preferences. They look at a model in discrete time of an agent without recall, who faces a finite stream of sampling
opportunities from a known distribution. At each discrete moment in time the agent decides whether to draw from the sample or stop. They concentrate on the sophisticated case and solve for the optimal strategy without recall (Markov policy). They find that if the agent’s preferences are quasi-convex, the optimal stopping rule is deterministic and takes a simple threshold form. This is similar to our result about the sophisticated agent with quasi-convex preferences not randomizing at the start.

Most of the non-EU work in optimal stopping has focused on the case of CPT preferences. Barberis (2012) considers a finite horizon gambling model, where an agent who has either commitment or is otherwise naive or sophisticated about her time-inconsistency, faces a finite stream of binary lotteries and has preferences of the Cumulative Prospect Theory (CPT) sort. Ebert and Strack (2015) finds sufficient conditions for the extreme result of always continuing with positive probability any diffusion for the naive agent by finding sufficient conditions for the CPT preference to be weak risk averse in the sense of this paper. Ebert and Strack (2016) finds sufficient conditions for the extreme result of never starting any diffusion for the sophisticated agent. He et al. (2016) also consider an agent with CPT preferences who faces an infinite sequence of fair bets and either has commitment or is naive about any possible dynamic inconsistency. They find tight characterizations in terms of the functional forms of CPT for the behavior of the agent under commitment besides characterizing naive behavior under certain CPT specifications. Our results allow us to make more general statements across wider classes of preferences, e.g. a full characterization of when in the case of a CPT sophisticated agent the never-starting result holds for diffusions or show that other well-known risk preference models don’t exhibit the extreme behavior in the naive case.

In a new working paper Huang et al. (2017) consider the RDU agent and study both naive and sophisticated behavior in a continuous time and diffusion setting as ours. They analyze the sophisticated agent based on an iterative procedure whose result depends on the starting point (stopping time) of the procedure. This potentially adds to the number of equilibria instead of reducing them, whereas the simpler equilibrium concept for sophisticated agents used here yields a unique behavioral prediction up to tie-breaking issues. In the examples of Huang et al. (2017) which overlap with results and examples of this paper, the predictions in terms of behavior are the same.

The rest of the paper is organized as follows. Section 2 introduces formally the set up. Section 3 is devoted to general results. Section 4 contains the applications to different classes of risk preferences. The Appendices contain the proofs of the results in the main body of the paper and other technical details.

2 Set up and problem formulation.

Preferences. An agent is equipped with a weak order \( \succeq \) (i.e. complete and transitive order) over the space of Borel probability distributions over a prize space \([w, b]\) (her risk preference). We denote by \( \succ \) the irreflexive part of \( \succeq \). Almost all of the preference models we apply our general results to are defined for lottery prizes in a bounded interval as \([w, b]\). We hold this assumption as much more realistic than the alternative assumption of \([w, +\infty)\), which allows for arbitrarily large amounts of wealth to be feasible (as the saying goes ‘Stock prices, like trees, don’t grow to the sky’). In text we use interchangeably the name lottery for probability distributions over prizes and denote by \( \delta_x \) for \( x \in [w, b] \) the
degenerate probability distribution giving \( x \) with probability one. Throughout we assume the following for \( \succeq \).

**Assumption 1 - Continuity:** For all \( G \in \Delta([w, b]) \) the sets \( \{ F \in \Delta([w, b]) : F \succeq G \} \) and \( \{ F \in \Delta([w, b]) : F \preceq G \} \) are closed in the topology of convergence in distribution.

**Assumption 2 - FOSD-monotonicity** If \( F \) strictly FOSD-dominates \( G \), then \( F \succ G \).

Here \( F \) FOSD-dominates \( G \) if for all \( x \in [w, b] \) we have \( F(x) \leq G(x) \) and ‘strictly FOSD-dominates’ is the irreflexive part of the ‘FOSD-dominates’ relation.

It is well-known, that under Assumption 1 the agent has a utility function \( V : \Delta([w, b]) \rightarrow \mathbb{R} \) which is continuous in the topology of convergence in distribution (see e.g. Debreu and Hildenbrand (1983)). Under Assumption 2 this utility function is increasing with respect to the FOSD-order.

**Technology of the wealth process.** We assume the agent faces a sequence of lotteries in continuous time, which are generated by a diffusion

\[
dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = y_0.
\]

We call a pair \((X, y_0)\) consisting of a diffusion process and a starting point \(y_0 \in [w, b]\) a **stopping problem**.

Here \((W_t)_{t \in \mathbb{R}_+}\) is a Brownian motion and the drift \( \mu : [w, b] \rightarrow \mathbb{R} \) together with the volatility \( \sigma : [w, b] \rightarrow (0, +\infty) \) are assumed Lipschitz continuous. The process \( X_t, t \geq 0 \) lives in the Wiener space \( C([0, \infty), [w, b]) \) of continuous functions with image in \([w, b]\) and adapted to the (completed) filtration of the Brownian motion, which we denote by \( \mathcal{F} = (\mathcal{F}_t)_{t \geq 0} \). Lipschitz continuity can be relaxed without changing the results of this paper: see appendix A for more. We also assume that the diffusion is stopped, once it leaves \([w, b]\): the lower bound is a limited liability constraint of the agent, while the upper bound excludes gambles with arbitrarily large prizes. The assumption that the variance coefficient \( \sigma \) is bounded away from 0 means that the uncertainty the agent faces at each moment in time is non-negligible, independently of the current state of the diffusion. We call diffusions satisfying these properties **regular** diffusions. One can think of them as modeling the value of an asset or the wealth process of a gambler in a casino.

The diffusion model in (1) can accommodate costs of continuation or costs arising from impatience into the drift of the diffusion as long as they are time independent: for a given cost parameter \( c \), the net drift including costs of continuation/impatience would be modified to \((\mu(X_t) - c)dt\). There does not seem to exist a general theory of impatience for an arbitrary static risk preference as discussed in this paper so that any way of modeling impatience in our model is bound to have an ad-hoc flavor. Alternatively we can interpret the model as that of a trader or gambler active only within a period, like a day or week, too short for discounting to be relevant.

**Sophistication levels about future behavior.** The literature focuses on three possible behavioral assumptions about the beliefs of the agent regarding her future behavior (see Barberis (2012) for a related discussion).
First, the agent could have commitment. In this case, the agent’s choice maximizes her period-0 preferences and the optimal strategy of period-0 will be fully implemented.

Second, the agent could have no available commitment possibilities and also be naive about her dynamic inconsistency, i.e. she thinks at each period, she will follow through with her decided plan, but then she (mostly) doesn’t. In particular, a naive agent thinks at each moment in time she will behave as an agent with commitment.

Finally, the agent could again have no available commitment possibilities but is sophisticated about her dynamic inconsistency, i.e. she knows what future selves will actually do, when she is deciding about her current strategy. Sophistication requires the agent to exhibit a high degree of rationality, while not considering any normative revision to her preferences.

The behavior of an agent who has commitment can be fully characterized by the behavior of the naive agent at period zero, because this agent believes that she will act in the future as though she has commitment. On the other hand, due to the Markov assumption on the prize process $X$, if one knows the commitment solution for all starting wealth levels, one can construct the naive solution by pasting together the different commitment solutions as the prize process evolves with time. Therefore, since the two cases are essentially analyzed by the same procedure, except for Theorem 1 below, we defer commenting in detail on the case of commitment.

**Strategies.** The set of pure strategies available to the agent in this setting is identified with the set of stopping times. These are automatically uniformly integrable because the prize process is bounded. For parts of the analysis we also assume an outside source of randomization in the form of a randomization device, independent of the Brownian motion driving the uncertainty of the prizes, which the agent can use to mix between possible pure strategies. Formally, we identify this with the measure space $([0,1], B([0,1]), \lambda))$, where $B([0,1])$ is the Borel sigma-Algebra of $[0,1]$ and $\lambda$ is the Lebesgue measure. The following definition gives the set of feasible strategies for the agent.

**Definition 1.** (1) (Pure Strategies) A stopping time $\tau$ is a $[0,\infty]$-valued random variable, such that for all $t \in [0,\infty)$, the event $\{\tau \leq t\}$ is contained in $F_t$.

(2) (Mixed Strategies) A randomized stopping time is a $(B([0,1]) \times F)$-progressively measurable function $\kappa : [0,1] \times C([0,\infty)) \rightarrow [0,\infty]$ such that for every $r \in [0,1]$, $\kappa(r,\omega)$ is a stopping time.

(3) A stopping time is called a simple threshold stopping time if it is given by

$$\tau_{a,c} = \inf\{t \geq 0 | X_t \leq a \text{ or } X_t \geq c\},$$

for some $a < c$, $a, c \in [w, b]$.

Randomized stopping times are well-known in mathematical game theory. Different papers use different names for different concepts of randomization with stopping times. See for example Shmaya and Solan (2014) who call our mixed strategies concept mixed stopping times and define another concept of randomization of stopping times and call it randomized stopping times. They show that under weak regularity conditions, the two concepts are equivalent for optimal stopping purposes in the sense that for a randomized stopping time of any of the two concepts there exists an appropriate stopping time of the
alternative randomization concept so that the joint distribution of paths of the process
and stopping time is the same. See also Laraki and Solan (2005) who in their Definition 1
use the same concept of randomized stopping times as in part (2) of the above definition.

Denote by $F_X$ the distribution induced by the random variable $X$ on the measure space
$([w, b], \mathcal{B}([w, b]))$, where $\mathcal{B}([w, b])$ is the Borel sigma-algebra of $[w, b]$. Then the distribution
induced by a stopping time $\tau$ is $F_{X_\tau}$. The distribution induced by a randomized stopping
time $\kappa$ is denoted again by $F_{X_\kappa}$ and is given by

$$F_{X_\kappa}(s) = \int_0^1 F_{X_{\kappa(r, \cdot)}}(s) dr. \quad (2)$$

The problem facing the naive agent at each moment in time and the agent with
commitment at time zero only can then be written as follows.

$$\sup_{\kappa \text{ randomized stopping time}} V(F_{X_\kappa}). \quad (3)$$

For the sophisticated agent without commitment we restrict her strategy space in (3)
at time zero to randomized stopping times $\kappa$, s.t. $\kappa(r, \omega)$ is a simple threshold stopping
time. After time 0 the agent can only pick simple threshold stopping times. This is for
tractability as we are working with a general risk preference and in subsection 3.2 we show
this assumption is without loss of generality if the agent has quasiconvex risk preferences.
Ultimately we are interested in behavior and the restricted strategy space already allows
to draw inferences on behavior even when her preference is not quasiconvex. Green (1987)
gives the following economic argument for quasiconvexity: there are wealth levels so that
an agent whose risk preferences are not quasi-convex can be manipulated into accepting
a compound lottery resulting in a final distribution over prizes which is dominated in the
FOSD-sense to the original wealth level she has. See also comment in the literature part
of section 1 on new approaches to the sophisticated agent for the special case of CPT
preferences.

**Stopping and continuation regions.** Whether naive or sophisticated without com-
mitment or naive without commitment, the behavior of the agent for a stopping problem
$(X, y_0)$ is encoded through a stopping region, where the agent stops with probability one
and a continuation region, where the agent stops with positive probability.

The following Definition formalizes the concepts and is valid for all agent types con-
sidered.

**Definition 2.** (1) For a diffusion $X$ and $y_0 \in [w, b]$ say that $y_0$ is in the stopping region
of $X$, if the agent stops with probability one in the stopping problem $(X, y_0)$.
(2) For a diffusion $X$ and $y_0 \in [w, b]$ say that $y_0$ is in the continuation region of $X$, if $y_0$
is not in the stopping region of $X$. 

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3 Characterization of behavior for a general risk preference.

3.1 The naive case and weak risk aversion

We define risk aversion in this paper as aversion to mean-preserving spreads (see e.g. Definition 6.D.2 and Example 6.D.2 in Mas-Collel et al. (1995); in other literature this concept is sometimes called strong risk aversion). One implication of risk aversion is that the agent always prefers to get the mean of the lottery for certain rather than face the lottery. Under the EU hypothesis the two concepts are equivalent (see e.g. Rothschild and Stiglitz (1970)).

Definition 3. Say that the agent is weakly risk averse (wRA) at $x \in (w, b)$ if for all lotteries $F$ with $\mathbb{E}[F] = x$ we have $F \preceq \delta_{\mathbb{E}[F]}$. If instead there exists $F$ with $\mathbb{E}[F] = x$ and $F \succ \delta_{\mathbb{E}[F]}$, we say the agent is not weakly risk averse (not wRA) at $x$.

For a stopping problem $(X, y_0)$ define the scaling function (see Revuz and Yor (2013) for more on the scaling function for homogeneous diffusions).

$$S(x, y_0) = \int_{y_0}^{x} \exp \left( -2 \int_{y_0}^{z} \frac{\mu(t)}{\sigma^2(t)} dt \right) dz. \quad (4)$$

It depends on the diffusion only through the normalized drift function $z \mapsto \frac{\mu(z)}{\sigma^2(z)}$. It shows how ‘favorable’ the diffusion facing the agent is.

Given a fixed diffusion $X$ as in (1) say that a stopping time implements a distribution $F \in \Delta([w, b])$ if the distribution of the random variable given by the stopped process $X_{\tau}$ is equal to $F$. The following Proposition essentially gathers together known results from the literature about the set of distributions that can be implemented through stopping.

Proposition 1. A. For a stopping problem $(X, y_0)$ with scaling function $S$, randomized stopping times can implement a distribution $F$ if and only if it is contained in

$$\mathcal{F}_X(y_0) = \{ F \in \Delta([w, b]) : \mathbb{E}_{x \sim F} [S(x, y_0)] = 0 \}.$$

$\mathcal{F}_X(y_0)$ is a convex, compact set (in the topology of convergence in distribution).

B. Every distribution in $\mathcal{F}_X(y_0)$ can be induced by a stopping time without randomization.

C. Binary distributions from $\mathcal{F}_X(y_0)$ can be induced by simple threshold stopping times. Moreover, the following are equivalent for both a naive and sophisticated agent in an optimal stopping problem.

(a) The agent uses a pure Markovian stopping policy if she starts.

(b) The agent picks a simple threshold stopping time if she starts.

Parts 1) and 2) follow from results in Ankirchner et al. (2015). From now on, we suppress the diffusion $X$ from the notation and write $\mathcal{F}(y_0)$ instead. It will be clear from
the context which diffusion $X$ underlies $F(y_0)$. Existence of an optimal strategy/optimal distribution out of $F(y_0)$ comes from Compactness of $F(y_0)$ and Assumption 1.

For $F \in \Delta([w, b])$ and $S : [w, b] \to \mathbb{R}$ strictly increasing, denote by $F \circ S^{-1}$ the distribution function from $\Delta([S(w), S(b)])$ given by $(F \circ S^{-1})(z) = F(S^{-1}(z))$ for all $z \in [S(w), S(b)]$. It is clear that for each $F \in \Delta([w, b])$ there is a corresponding $F \circ S^{-1} \in \Delta([S(w), S(b)])$ and vice versa.

As a last piece of convention, whenever $S(\cdot, y_0)$ is the scaling function of a stopping problem $(X, y_0)$, and $f$ is a real-valued function such that $\hat{f}(\cdot) = f \circ S^{-1}(\cdot, y_0)$ is well-defined, we will write for this function shortly $f \circ S^{-1}$, with the understanding that $(X, y_0)$ is clear from the context.

Now we can state the main Theorem for the naive case, which gives a complete characterization of naive behavior. We apply it extensively in Section 4.

**Theorem 1.**

1. The naive agent without commitment and the agent with commitment continue in the stopping problem $(X, y_0)$ if and only if the feasible set $F(y_0)$ contains a lottery which is strictly preferred to $\delta_{y_0}$.

2-a) If the agent is not weakly risk averse at $y_0 \in (w, b)$, then she continues with positive probability any Martingale diffusion started at $y_0$.

2-b) If the agent is weakly risk averse at $y_0$, then she stops all Martingale diffusions started at $y_0$ in finite time with probability one. Moreover, irrespective of any starting point $y \neq y_0$ she will stop with positive probability any stopping problem $(X, y)$, if $X$ is a Martingale diffusion.

3. The agent with utility $V$ stops a diffusion $X$ with scaling function $S$ at $y_0$ if and only if the agent, whose utility over lotteries $G$ in $\Delta([S(w, y_0), S(b, y_0)])$ is given by $V_S(G) = V(G \circ S)$, is weakly risk averse at 0.

The proof of Theorem 1 follows the logic of the proof of Theorem 2 in Ebert and Strack (2015). The difference is that their proof uses extensively the functional form of Cumulative Prospect Theory, whereas we have to work with the more general concept of weak risk aversion given the weak assumption on the agent’s preferences. Ebert and Strack (2015) are aware that their assumptions imply that their CPT agent satisfies weak risk aversion (see footnote 5 there). Here we show that weak risk aversion is also a necessary property to characterize optimal stopping for an agent with a general risk preference.

Intuitively, if an agent is weakly risk averse at $y_0$ she always prefers the certain amount $y_0$ equal to the expectation of a lottery induced by the Martingale diffusion, rather than facing the lottery. On the other hand, if she is not weakly risk averse at $y_0$ there exists a lottery with expectation $y_0$ she prefers to getting $y_0$ for sure. This lottery can be induced through a pure stopping strategy and so the agent continues with positive probability. This is the content of part 2) of Theorem 1. A general diffusion, not necessarily a Martingale, is equivalent in probabilistic terms to a Martingale diffusion where the prize space $[w, b]$ has been appropriately rescaled by the scaling function $S$. Thus this case be reduced to the Martingale one through rescaling.

3.1.1 Relation between weak risk Aversion and risk aversion

Here we clarify precisely for a general preference satisfying Assumptions 1 and 2 the behavioral relation between weak risk aversion which is a crucial concept for the charac-
terization of naive behavior (Theorem 1) and (strong) risk aversion defined as aversion to mean-preserving spreads.

The axiom needed to establish the relation is a relaxation of the Independence Axiom from Expected Utility theory. We restate the latter here for comparison and reader’s reference.

**Axiom: Independence**  For $G_1, G_2, H \in \Delta([w, b])$ with $G_1 \succeq G_2$ and any $\alpha \in [0, 1]$ we have that

$$\alpha H + (1 - \alpha)G_1 \succeq \alpha H + (1 - \alpha)G_2.$$  

The relaxation needed is as follows.

**Axiom: Mixture Monotonicity w.r.t. Certainty (MMC).**  Let for $i = 1, \ldots, n$ be $x_i \in [w, b]$, $F_i$ be lotteries with $\mathbb{E}[F_i] = x_i$ and $\alpha_i \geq 0$ with $\sum_i \alpha_i = 1$.

1) If $F_i \succeq \delta_{x_i}$ for all $i = 1, \ldots, n$, then

$$\sum_i \alpha_i F_i \succeq \sum_i \alpha_i \delta_{x_i}.$$  

2) If $F_i \succeq \delta_{x_i}$ for all $i = 1, \ldots, n$, then

$$\sum_i \alpha_i F_i \succeq \sum_i \alpha_i \delta_{x_i},$$  

Here, mixture operator is in the sense of distributions. This axiom says that one can aggregate preference comparisons as long as one (and the same) side of the comparisons concern the certain expected value of the other respective side of the comparisons. Obviously, MMC is implied by Independence.

The following Proposition states, that in the case of non-EU risk preferences, MMC and wRA are equivalent to risk aversion.

**Proposition 2.**  

i. The following are equivalent.

A. $\succeq$ satisfies risk aversion in the sense of aversion to mean-preserving spreads.

B. $\succeq$ satisfies Mixture Monotonicity w.r.t. Certainty and weak risk aversion everywhere.

ii. The following are equivalent.

A. $\succeq$ satisfies Mixture Monotonicity w.r.t. Certainty and strong not wRA everywhere:

for all $x \in (w, b)$ and $F \in \Delta([w, b])$ with $\mathbb{E}[F] = x$ we have $F \succeq \delta_{\mathbb{E}[F]}$.

B. $\succeq$ satisfies risk loving in the sense of preference for mean-preserving spreads.
3.2 The Sophisticated Case.

For the case of a sophisticated agent we restrict the set of policies for tractability reasons to the set of Markov policies and we model the behavior of the sophisticated agent as a *Markov Nash equilibrium*. The same assumption is made in Ebert and Strack (2016) who focus on CPT. We clarify in the end of the subsection how strong the Markovian assumption is by tying it to the risk preference of the agent.

**Definition 4.**

1) A pure Markov policy is a function \( s : [w, b] \rightarrow \{\text{continue}, \text{stop}\} \). It constitutes an equilibrium if at every point in time \( t \) it is optimal to take the decision \( s(X_t) \) given future selves use the strategy \( s \).

2) A mixed Markov stopping policy for the sophisticated agent is a probability distribution \( \sigma \) over pure Markov policies. It constitutes an equilibrium if
   - it is optimal for the self at time zero to randomize according to \( \sigma \) and
   - any realization \( s \) of \( \sigma \) constitutes an equilibrium according to 1).

A Markov Nash equilibrium in pure Markov policies always exists: choose with probability one at the start the pure Markov policy \( s_0 \equiv \text{stop} \) for all prizes. This instructs all selves to never start. The interesting question is to find conditions under which there is another equilibrium of the game played by the selves of the sophisticated agent, which is preferred by the self at time zero to the never starting one.

Randomization at time zero can be interpreted as the agent having partial commitment. The concept of randomization we use is not equivalent to other possible randomization concepts. For example, one could ask for time-homogeneous randomization which would be equivalent to studying Markov stopping policies of the type \( s : [w, b] \rightarrow [0, 1] \), where \( s(x) \) denotes the probability that the agent continues whenever she reaches \( x \). This precludes time-inhomogeneous behavior.

To analyze when there exists a non-trivial equilibrium take a fixed \( y_0 \in (w, b) \). The proof of Proposition 1 implies that when a Markov policy at \( y_0 \) for the sophisticated agent exists that is better than always stopping, there is a simple threshold stopping time, which has the form \( \tau_{a,c} \) (see 3) in Definition 1) with some \( w \leq a < c \leq b \) and is so that the agent continues at each \( y \in (a, c) \) and stops otherwise. Any mixed Markov stopping policy corresponds then to a randomized stopping time \( \kappa \) according to Definition 1, where for each \( r \in [0, 1] \), \( \kappa(r, \cdot) \) is a simple threshold stopping time. Note that randomized stopping times may lead to time-inhomogeneous behavior because the stopping strategy started at zero prescribes a different action in the case of a return to the starting point, whenever the randomized stopping time is non-constant in the first argument. Nevertheless, from the perspective of period zero, the set of such randomized stopping times exhausts the set \( \mathcal{F}(y_0) \) of prize distributions achievable by stopping strategies (see Lemma 3 in the appendix). To find out which distributions out of \( \mathcal{F}(y_0) \) can be sustained through future selves we first focus on characterizing the simple threshold stopping times which are equilibria according to 1) of Definition 4.

Given a diffusion \( X \) as in (1), started at some point \( y \in (a, c) \) and \( \tau_{a,c} \) a simple threshold stopping time, the self at \( y \) faces lottery of the form \( L(1-p(y), a, c) = (1-p(y))\delta_a + p(y)\delta_c \), where \( p(y) \) is given by

\[
1 - p_{a,c}^X(y) = \frac{S(c, y_0) - S(y, y_0)}{S(c, y_0) - S(a, y_0)},
\] (5)
where $S(y, y_0)$ is the scale function for the stopping problem $(X, y_0)$ (see for example Revuz and Yor (2013), pp. 303). $p^X_{a, c}$ is strictly increasing with $p^X_{a, c}(a) = 0, p^X_{a, c}(c) = 1$.

**Definition 5.** For a fixed diffusion $X$ whose dynamic is given by (1) and interval $(a, c) \subset [w, b]$ the function $p^X_{a, c} : (a, c) \rightarrow [0, 1]$ given by $p^X_{a, c}(y) = \frac{S(y, y_0) - S(a, y_0)}{S(c, y_0) - S(a, y_0)}$ is called the win probability of $X$ for the interval $(a, c)$.

We now turn to the preference constraint.

**Definition 6.** The curve $q_{a, c} : [a, c] \rightarrow [0, 1]$ given by

\[ L(1 - q_{a, c}(y), a, c) \sim \delta_y, \quad y \in [a, c]. \]

is called the calibration function of the agent for $(a, c)$.

The calibration function of an agent for $(a, c)$ gives for each element $y \in (a, c)$ in the interval the probability of $c$ which ‘calibrates’ $y$, i.e. the probability $q(y)$ in the lottery $L(1 - q(y), a, c)$ which makes the agent indifferent between $L(1 - q(y), a, c)$ and $\delta_y$. By our assumptions $q_{a, c}$ is continuous and strictly increasing. The two definitions above allow us to formulate the necessary and sufficient condition for never starting any diffusion in terms of a general risk preference.

**Definition 7.** Say that the agent exhibits extreme sensitivity to risk at $y_0$ if and only if the following holds:

\[ (C) \text{ for every } a < y_0 < c \text{ the slope of } q_{a, c} : [a, c] \rightarrow [0, 1] \text{ is } +\infty \text{ at } a \text{ or } 0 \text{ at } c. \]

To characterize formally the optimal solution for the sophisticated agent, we introduce the set of feasible binary lotteries for the sophisticated agent for a diffusion with normalized drift $z \rightarrow \frac{\mu(z)}{\sigma^2(z)}$.

**Definition 8.** Let

\[ \mathcal{F}_{X, \text{soph}}(y_0) = \{ L(1 - p(y_0), a, c) : p(z) = p^X_{a, c}(z) \geq q_{a, c}(z), \text{ for all } z \in [a, c] \}. \]

Denote by $\text{conv}(\mathcal{F}_{X, \text{soph}}(y_0))$ its closed convex hull. This is the set of distributions from which time zero self picks his favorite. Both $\mathcal{F}_{X, \text{soph}}(y_0)$ and $\text{conv}(\mathcal{F}_{X, \text{soph}}(y_0))$ are compact (usually strict) subsets of $\mathcal{F}_{X}(y_0)$.

The following Theorem gives the full characterization of behavior in the case of the sophisticated agent. We apply it extensively in Section 4 to a variety of risk preferences.

**Theorem 2.** For a sophisticated agent at current wealth $y_0$ the following hold true.

**A.** $y_0$ is contained in the stopping region for all diffusions if and only if the agent exhibits extreme sensitivity to risk at $y_0$.

**B.** $y_0$ is contained in the continuation region of $X$ if and only if there exists $a, c$ with $w \leq a < y_0 < c \leq b$ such that $p^X_{a, c}$ dominates pointwise $q_{a, c}$. Otherwise $y_0$ is contained in the stopping region of $X$. 

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C. Under the tie-breaking rule that a sophisticated agent will start whenever she is indifferent between starting and stopping the solution to the stopping problem \((X, y_0)\) of the sophisticated agent is given by

\[
\max_{F \in \text{conv}(\mathcal{F}_X, \text{soph}(y_0))} V(F), \text{ so that } V(F) \geq V(\delta_{y_0}).
\]

Intuitively, the calibration function for an interval is a preference constraint the win probability of a diffusion has to overcome so that the sophisticated agent can set up a consistent plan, which will be followed by all possible future selves. The higher the calibration function for an interval, the higher the constraint the diffusion has to overcome for the sophisticated agent to find it optimal to start.

3.3 On the optimality of simple threshold stopping times.

In general, whether a strategy prescribes stopping at a particular value of \(X\) at a particular time \(t\) might depend on the whole path of \(X\) up to time \(t\). This is the case if the optimal continuation lottery a naive agent chooses is not binary. The latter are not implemented through simple threshold stopping times.

We note here a condition on the preference \(\succeq\) which implies the existence of optimal policies which satisfy the Markovian property for all stopping problems. It follows that in those cases we can restrict the set of feasible strategies to simple threshold stopping times. In section 4 and in the online appendix, we apply our general results to several preferences satisfying this condition.

**Definition 9.** \(\succeq\) is called **quasi-convex** if for \(F, F', F'' \in \Delta([w, b])\)

\[
F, F' \preccurlyeq F'' \quad \text{implies} \quad \alpha F + (1 - \alpha) F' \preccurlyeq F'', \quad \text{for all } \alpha \in (0, 1)
\]

and

\[
F \preccurlyeq F'', F' \prec F'' \quad \text{implies} \quad \alpha F + (1 - \alpha) F' \prec F'', \quad \text{for all } \alpha \in (0, 1).
\]

This implies quasi-convexity of the utility function \(V\). In the literature, quasi-convexity is defined as \(L \sim L' \Rightarrow \alpha L + (1 - \alpha) L' \preccurlyeq L\). Under our Continuity assumption this definition is equivalent with the first part of Definition 9.

When preferences are quasi-convex it is sufficient to check violations of wRA only for binary bets, as the following Proposition shows.

**Proposition 3.** Assume that \(\succeq\) is quasi-convex. Then the agent exhibits wRA at \(y_0 \in [w, b]\) if and only if for all binary lotteries \(F\) with mean \(y_0\) it holds

\[
F \preccurlyeq \delta_{E[F]}.
\]

The agent violates wRA at \(y_0\) if and only if there exists a binary lottery \(F\) with mean \(x\) such that

\[
F \succ \delta_{E[F]}.
\]

Moreover, up to tie-breaking considerations, a quasi-convex agent of all three types: naive, sophisticated with commitment and sophisticated without commitment will use simple threshold stopping times whenever she starts.
An agent with quasi-convex preferences is averse to mixtures of lotteries in particular also to mixing binary lotteries. But binary lotteries are ‘extremal’ in the sense that any lottery with finite support is a mixture of binary lotteries. This also shows that a sophisticated agent with quasiconvex preferences has no incentive to randomize at time zero.

Note, that for a preference which is represented by a quasiconcave utility function it is in general not true, that optimal stopping policies for both naive agent and the agent with commitment are simple threshold stopping times (see example 4.3 in Section 4).

4 Applications

The well-known Allais paradox and other related paradoxes challenge the main behavioral implication of EU: the independence axiom (see Machina (1982) and Machina (1989) for a discussion). As a response to violations of Independence, a wide variety of risk preference models have been offered as alternatives in the decision theory and behavioral economics literature. Except for a narrow focus on CPT preferences there is little work in the economics literature on the optimal stopping problem with non-standard risk preferences.

Here we use our general characterization results to discuss features of optimal stopping behavior across a large class of well-known non-EU risk preference models. We first consider preferences which satisfy Betweenness and among them, we focus on the special case of Disappointment Aversion (DA) preferences. We then consider a class of convex risk preferences called Cautious EU (CEU), introduced and axiomatized in Cerreia-Vioglio et al. (2015). We then focus on preferences which satisfy Ordinal Independence. These include many well-known models such as those of probability weighting like RDU and CPT.

Due to space constraints we skip here some other applications which are available upon request: these are quadratic utility as introduced in Machina (1982) and axiomatized in Chew et al. (1991). Within the quadratic class, we also look at CPE, a behavioral model first introduced in Köszegi and Rabin (2007), and which enjoys popularity in the applied behavioral theory literature.

There are two main take-aways of interest to optimal stopping applications from the results in this section. First, unless one is ready to violate our two basic assumptions of continuity and monotonicity on the static risk preference from section 2, they suggest that some specific classes of ordinal independence preferences are the only preferences among the classes we consider which exhibit extreme behavior in the following sense: a naive agent always continues any diffusion with positive probability and a sophisticated agent always stops any diffusion. Second, all other static models of risk preference we consider are closer to EU in the sense that, when the diffusion process is ‘favorable’ enough, even a sophisticated risk averse agent will find it profitable to continue with positive probability and when the diffusion process is ‘unfavorable’ enough, even a naive risk loving agent will stop.

4.1 Betweenness preferences

Dekel (1986) considers relaxing the Independence axiom of EU to Betweenness: \( F \succ G \) \( (F \sim G) \) implies \( F \succ \lambda F + (1 - \lambda)G \succ G \) \( (F \sim \lambda F + (1 - \lambda)G \sim G) \).
Betweenness preferences are quasi-convex, because their indifference curves are linear, though in general not parallel. In particular, due to Proposition 3 we can focus analysis on simple threshold stopping times.

A special case of Betweenness preferences are DA preferences as defined in Gul (1991). They find application in finance to explain the equity premium puzzle and to generate counter-cyclical risk aversion (see e.g. Routledge and Zin (2016)). They are encoded by a pair \((u, \beta)\) consisting of a Bernoulli utility function \(u : [w, b] \to \mathbb{R}\) and a ‘disappointment’ parameter \(\beta \in (-1, \infty)\), which measures the degree with which the probability of those prizes in the support of a lottery is overvalued, which are less preferred by the agent than the lottery itself (‘disappointing’ prizes). An agent with \(\beta > 0\) is said to exhibit Disappointment Aversion, whereas an agent with \(\beta < 0\) is said to be elation loving. The case \(\beta = 0\) corresponds to EU. Here we exhibit the representation of DA preferences for binary lotteries only (see e.g. Theorem 1 in Gul (1991) and its surrounding discussion for the general representation of DA preferences). For binary lotteries of the form \(L(1-p, x, y)\) with \(x \leq y\) the representation of DA is given by

\[
V((1-p)\delta_x + p\delta_y) = \frac{p}{1 + (1-p)\beta} u(y) + \frac{(1-p)(1+\beta)}{1 + (1-p)\beta} u(x).
\]  

(6)

As a first application, for the case of the naive DA agent and \(X\) a geometric Brownian motion \((\mu(x) = \mu x, \sigma(x) = \sigma x)\) or an arithmetic Brownian motion \((\mu(x) = \mu, \sigma(x) = \sigma)\) we find the following.

**Proposition 4.** For either of the two following cases

A. \(X\) is a geometric Brownian motion with parameters \(\mu \in \mathbb{R}, \sigma \in \mathbb{R}_+\) and the worst prize \(w\) is positive,

B. \(X\) is an arithmetic Brownian motion with parameters \(\mu \in \mathbb{R}, \sigma \in \mathbb{R}_+\),

the continuation region of \(X\) for a naive DA agent consists of

\[
C_X = \left\{ y_0 \in (w, b) : \text{there exists } x < y_0 < y \text{ with } \frac{e^{2\mu(y-y_0)} - 1}{1 - e^{-2\mu(y-y_0)}} \frac{u(y) - u(y_0)}{u(y_0) - u(x)} > 1 + \beta \right\}.
\]

The continuation region \(C_X\) is non-empty and strictly smaller than the whole prize space \([w, b]\) for a wide variety of parameter values. All else equal, a higher \(\beta\), which corresponds to higher Disappointment Aversion, shrinks the continuation region as does a more concave \(u\) or a lower normalized drift \(\frac{\mu}{\sigma^2}\).

We now turn to the sophisticated case.

**Proposition 5.** If the DA agent is sophisticated and in the representation \((u, \beta)\) with \(\beta > -1\) \(u\) is strictly increasing and differentiable in \((w, b)\), there is some diffusion with non-empty continuation region.

The following example illustrates further the behavior of a sophisticated DA agent. Just as Example 4.2 below, it illustrates the fact that, similarly to an EU agent, a risk averse sophisticated DA agent never stops diffusions which have a ‘favorable’ enough dynamic.
Example 4.1: Sophisticated, risk averse DA agent who never stops. Assume that \( u(x) = x, \beta \in (0, +\infty) \) and that \( X \) follows a Brownian motion (or a geometric Brownian motion); with variance \( \sigma > 0 (\sigma X_t, \sigma > 0) \) and drift \( \mu \in \mathbb{R} (\mu X_t) \). Assume also, that the worst prize \( w \) is greater than zero. According to Gul (1991) this agent is risk averse.

In the appendix we first find that the feasibility of a simple threshold stopping times \( \tau_{a,c} \) depends on \( d_0 = \frac{\mu}{\sigma^2}(c - a) \) and \( \beta \) and not on the starting point \( y_0 \): \( \tau_{a,c} \) is feasible for the sophisticated agent if and only if \( d_0 \) is high enough in comparison to \( \beta \). In particular, for \( \mu \leq 0 \) the sophisticated DA agent will not start. We show in appendix C, that for a region of parameter values the agent’s optimal stopping time is \( \tau_{w,b} \), i.e. the agent gambles till either ruin or the highest prize.

4.2 Cautious Expected Utility

We say that the preference has a Cautious Expected Utility (CEU) representation if there exists a compact, convex set \( U \) of strictly increasing and continuous functions \( u : [w, b] \rightarrow \mathbb{R} \) such that the preferences are represented by

\[
V(F) = \inf_{u \in U} u^{-1}(E_{x \sim F}[u(x)]). \quad (7)
\]

One usually assumes that all \( u \in U \) are normalized: \( u(w) = 0, u(b) = 1 \). These preferences are convex, they can explain the Allais Paradox and exhibit the Certainty Effect, which is related to the common ratio version of the Allais paradox. CEU were introduced and axiomatized in Cerreia-Vioglio et al. (2015).

One knows that Cautious EU representations where the set \( U \) has only finitely many elements don’t satisfy Betweenness (Cerreia-Vioglio et al. (2018)). The main behavioral axiom satisfied by a cautious EU preference is Negative Certainty Independence (NCI): for all \( x \in [w, b], F, G \in \Delta([w, b]) \) and \( \lambda \in [0, 1] \)

\[
F \succeq \delta_x \implies \lambda F + (1 - \lambda)G \succeq \lambda \delta_x + (1 - \lambda)G.
\]

NCI says that if the certain outcome \( x \) is not able to compensate for \( F \) despite the riskiness in \( F \), then mixing both \( F \) and \( \delta_x \) with another lottery \( G \) will further (weakly) lower the appeal of \( x \) vis-a-vis \( F \).

The following Proposition is a partial characterization of the optimal stopping behavior of a naive CEU agent.

Proposition 6. Assume a naive agent has a Cautious EU representation parametrized by the set of Bernoulli functions \( U \), assume \( 0 \in (w, b) \) and that she faces a stopping problem \( (X, y_0) \) with respective scaling function \( S \).

A. If for all the functions \( u \in U \) the function \( u \circ S^{-1} \) is concave, then \( y_0 \) is in the stopping region of \( X \).

B. If for all the functions \( u \in U \) the function \( u \circ S^{-1} \) is convex, then \( y_0 \) is in the continuation region of \( X \).

C. (S-shaped case) If all functions \( u \) in \( U \) are convex for \( x < 0 \), concave for \( x > 0 \), \( u(0) = 0 \) and have a one-sided derivative at \( b \) with \( u'(b) \geq \frac{u(b) - u(w)}{b - w} \), then the agent violates \( wRA \) everywhere and so, if \( X \) is a Martingale diffusion, the continuation region of \( X \) is \( (w, b) \).
This result shows that cautious EU preferences can accommodate a wide range of behavior, just as EU agents and DA agents.

To gain more intuition for Proposition 6, consider the case of $X$ a geometric Brownian motion. Then $S^{-1}$ is concave and becomes more concave for lower, negative $\frac{\mu}{\sigma^2}$, i.e. for more unfavorable diffusions. $u \circ S^{-1}$ balances then the risk aversion of an EU-agent with Bernoulli utility $u$ with the (un)favorability of the diffusion represented by the degree of concavity of $S^{-1}$. Given that the agent is cautious, she stops if $u \circ S^{-1}$ is concave for all $u$. Thus, part 2. of the Proposition says that even for a risk loving naive CEU agent, if a geometric Brownian motion $X$ is unfavorable enough, the agent will not start.

We close the analysis of CEU preferences by giving sufficient conditions for the sophisticated CEU agent to actually start some diffusion. These conditions are rather weak so that extreme sensitivity is generically not satisfied for CEU models.

**Proposition 7.** Assume that the set $U$ is a compact set of strictly increasing functions and that the one-sided derivatives of all $u \in U$ are bounded uniformly away from zero and infinity:

$$\sup_{u \in U} \sup_{x \in [w,b]} u'(x) < +\infty, \quad \inf_{u \in U} \inf_{x \in [w,b]} u'(x) > 0.$$  

Then for all $y_0 \in [w,b]$ there are diffusions $X$ such that $y_0$ is in the continuation region of $X$.

### 4.3 Ordinal Independence

Ordinal Independence (OI) preferences were introduced in Green and Jullien (1987) and encompass many models in the literature, including the well-known Rank-Dependent Utility and its special case Cumulative Prospect Theory (CPT). The main behavioral assumption behind OI is that whenever two distributions $F$ and $G$ have a coinciding left or right tail, then their preference ranking should not depend on how the common tail looks like.

A typical utility function over lotteries conforming to OI is as follows.

$$V(F) = \int_{w}^{b} h(x, 1 - F(x)) dx.$$  

Here, $h : [w, b] \times [0, 1] \to \mathbb{R}_+$ is any continuous function which is strictly increasing in the second argument with $h(x, 0) = 0$ for every $x \in [w, b]$.

Some sufficient conditions for never starting for the sophisticated OI agent are as follows.

**Proposition 8.** A sophisticated OI agent never starts at $y_0$ whenever either of the following conditions is satisfied.

A. $\frac{\partial h}{\partial y}(x, 0+) = 0$ for all $x \in [w, b]$.

B. $\frac{\partial h}{\partial y}(y_0, 1-) = +\infty$.

Intuitively, the conditions together say that the agent is relatively insensitive to small changes in probabilities when these are near one and/or she is extremely sensitive to small changes in probabilities when these are near zero.
Rank-Dependent Utility (RDU). RDU is the most well-known special case of OI. Here \( h \) has product form, i.e. it can be written as \( h(x, p) = u'(x)\nu(p) \) (we show in the appendix that an OI preference is RDU if and only if the ‘local’ utility \( h \) of the OI preference is of product form). They can be written as

\[
V(F) = -\int_w^b u(x)d\nu(1-F(x)) = u(w) + \int_w^b u'(x)\nu(1-F(x))dx,
\]

where \( \nu : [0,1] \rightarrow [0,1] \) is a strictly increasing, onto function and \( u \) is any continuous, increasing function.

Assume now that \( u : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is differentiable, strictly increasing and that \( \nu \) is differentiable as well. For the case of the sophisticated RDU agent, Proposition 8 immediately gives the full characterization for never-starting any diffusion. Corollary 2 in the appendix shows that the cases covered in Proposition 8 are not already direct consequences of Corollary 1.

**Corollary 1 (RDU-Never Starting).** For a sophisticated RDU agent the stopping region of any diffusion is the whole prize space \([w,b]\) if and only if one of the following conditions is satisfied

A. \( \nu'(0+) = 0 \),

B. \( \nu'(1-) = +\infty \).

This Proposition implies that for the never-starting result it is necessary that \( \nu \) not be concave everywhere, so that the agent cannot be globally risk loving and her static preferences cannot be globally convex. Many specifications of probability weighting in the applied finance/economics literature satisfy the requirements of the Corollary. We mention

\[
\nu(p) = \frac{p^\alpha}{(p^\alpha + (1-p)^\alpha)^{\frac{1}{\alpha}}}, \quad \nu(p) = \frac{ap^\delta}{ap^\delta + (1-p)\delta}, \quad \nu(p) = \exp\left((-\ln(p))^\alpha\right), \quad p \in [0,1]
\]

where \( \alpha \in (0,1), a, \delta > 0, \delta \neq 1 \) (see Prelec (1998) for a lengthy discussion).

The next example is a partial converse to Corollary 1: there are many cases where a sophisticated RDU agent exhibits similar qualitative behavior as EU, in the sense that, a risk averse sophisticated agent never stops diffusions which have a ‘favorable enough’ dynamic. In the following we suppress the sub- and-superscripts \( a,c,X \) in the notation of \( p_{a,c}^X \). The omission will be clear from the context.

**Example 4.2: Sophisticated RDU agent who never stops ‘favorable’ enough diffusions.** Assume RDU with \( u(x) = x, w \geq 0 \) and any strictly convex and twice continuously differentiable \( \nu \) with \( \nu'(0+) > 0, \nu'(1-) < +\infty \). Note that this agent is risk averse and her utility function \( V \) over lotteries in \( \Delta([w,b]) \) is quasi-convex and thus we can make use of Proposition 3. We look at simple geometric Brownian motion with \( \mu \in \mathbb{R}, \sigma > 0 \) and show that there are specifications such that the sophisticated agent chooses \( \tau_{w,b} \) as an optimal stopping rule (bang-bang).

For fixed \( y_0 \) the diffusion chosen gives for the win probability for \( y_0 \in (a,c) \)

\[
p(y) = \frac{e^{-a\frac{2y}{\sigma^2}} - e^{-y\frac{2a}{\sigma^2}}}{e^{-a\frac{2y}{\sigma^2}} - e^{-c\frac{2a}{\sigma^2}}}.\]
The sophisticated agent solves then the following problem, if she starts:

$$\max_{a,c,v \leq a \leq y_0 < c \leq b} \nu(1-p(y_0))a + (1-\nu(1-p(y_0)))c, \text{ such that } \frac{c-y}{c-a} \geq \nu(1-p(y)),$$

Focus on the case $\mu > 0$. The agent will always start such a geometric Brownian motion and we show, that she will choose $\tau_{w,b}$ as an optimal policy whenever the normalized drift $\frac{\mu}{\sigma^2}$ is large enough. How large $\frac{\mu}{\sigma^2}$ has to be to induce never stopping depends on $y_0$ and the shape of the probability distortion $\nu$. The higher $y_0$ is, the lower the values of $\frac{\mu}{\sigma^2}$ needed, all else equal. In particular, there are stopping problems, for which this class of RDU agents continues to gamble even if in the neighborhood of the worst wealth level $w$, despite being sophisticated and risk averse.

To get a feeling for the stopping times a naive RDU agent chooses, consider the following example. It is based on a modification of the technical results from Xu and Zhou (2013) which the interested reader can find in the appendix (see Proposition 10 in the appendix).

**Example 4.3: Naive RDU agent who always picks non-binary distributions.** Consider an RDU agent with

$$V(F) = \int_0^b u(x)d(\nu(F(x))), \quad (10)$$

for some finite $b > 0$ large enough and with $u(x) = x^r, r \in (0,1]$ and $\nu(x) = x^\alpha, \alpha \in (0,1)$. In particular, this agent has quasi-concave risk preferences (see Wakker (1994)). Thus we shouldn’t expect the naive agent to choose simple threshold stopping times in the optimal stopping of a diffusion. Assume the agent faces a geometric Brownian motion

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = y_0 > 0.$$

Define $\gamma = \frac{r \sigma^2}{\sigma^2 - 2 \mu}$ and $\beta = \frac{\sigma^2 - 2 \mu}{\sigma^2}$ and assume that the stopping problem and the preference parameters are such that $\gamma \in (0, \alpha)$. One finds that the agent’s optimal distribution out of $F(y_0)$ is

$$F_{y_0}(x) = \begin{cases} 
1, & \text{if } x = b \\
1 - (1 - x_b) \left(\frac{x}{b}\right)^{-\frac{\beta}{1-\alpha}}, & \text{if } b > x \geq b(1 - x_b)^{\frac{1-\alpha}{\beta}} \\
0, & \text{if } 0 \leq x < b(1 - x_b)^{\frac{1-\alpha}{\beta}},
\end{cases} \quad (11)$$

where $x_b$ is a function of $y_0$. This distribution is a truncated Pareto distribution with an atom at $b$. One possible optimal stopping time inducing $F_{y_0}$ is the so-called Azema-Yor stopping time.

$$\tau_{AY,y_0} = \inf\{t \geq 0 | X_t \leq \min\{b, (\frac{x + r}{1 - r})^{\frac{1}{\beta}} \max_{0 \leq s \leq t} X_s\}\}. \quad (12)$$

When using this strategy, the commitment agent sells as soon as the price has fallen some percentage below the historical maximum. The fact that under $\mu > 0$ there is positive probability that $X_t$ reaches $b$ before it falls below $(\frac{x + r}{1 - r})^{\frac{1}{\beta}} \max_{0 \leq s \leq t} X_s$ explains the atom
at \( b \), i.e. that the optimal distribution has a jump at \( b \): the probability of \( b \) is \( 1 - x_b \). The formulas in the appendix show that \( 1 - x_b \) is increasing in \( y_0 \).

If we consider the sophisticated agent with preferences as above but \( r = 1 \) and facing a stopping problem with \( \mu < 0 \) and so that \( \gamma < \alpha \), we know that the sophisticated agent will never start at any \( y_0 \). This, and the fact that the naive agent starts with positive probability, allows in principle an outside analyst in our model to infer the level of sophistication of the agent under the parameter restrictions assumed.

We finish by recording the following result about a naive agent with OI preferences whose proof uses some new results (to the best of knowledge) about OI and RDU preferences recorded in the appendix.

**Proposition 9.** For every possible interval of prizes \([w, b]\) there are OI preferences which are not RDU so that the naive agent continues with positive probability at each \( y_0 \in (w, b) \).

Ebert and Strack (2015) and Henderson et al. (2017) show that naive agents with particular specifications of CPT preferences always continue with positive probability no matter the diffusion they face. They use the following specification of CPT.

\[
CPT(X) = \int_{\mathbb{R}^+} \nu^+ (\mathbb{P}(U(X) > y)) dy - \int_{\mathbb{R}^-} \nu^- (\mathbb{P}(U(X) < y)) dy,
\]

where \( U : \mathbb{R} \rightarrow \mathbb{R} \) is continuous, strictly monotonic, has left and right derivatives, \( \partial_- U(x), \partial_+ U(x) \) at every wealth level \( x \) with \( \lambda = \sup_{x \in \mathbb{R}} \frac{\partial_- U(x)}{\partial_+ U(x)} < \infty \) and \( \nu^-(q) = 1 - \nu^+(1 - q) \) with \( \nu^+(0) = 0 \) and \( \nu^+(1) = 1 \). As readily checked these preferences also satisfy a RDU representation.

Proposition 9 complements the results in Ebert and Strack (2015) and Henderson et al. (2017) by showing that the class of preferences where this extreme behavior occurs is strictly larger than CPT (or RDU for that matter).

## 5 On the Stochastic Process.

**Proof. Proof of Proposition 1.** 1)-2) The first statement of the Lemma is a corollary of Ankirchner et al. (2015). We elaborate this in the following.

For the case of Martingale diffusions \( (\mu \equiv 0) \), we combine the statement of their Theorem 2 in Section 1 and the first statement in their Proposition 2 in the second Section. These state that for Martingale diffusions, the set of implementable distributions through stopping times (pure strategies) is precisely

\[
\mathcal{F}_X(y_0) = \{ F \in \Delta([w, b]) : \mathbb{E}_{x \sim F}[x] = y_0 \}.
\]

Section 6 of their paper from the beginning till the statement of Theorem 6, but excluding it, discusses how the result for the martingale case can be used to arrive at the result for general diffusions. The scaling function \( S \) rescales the diffusion to a Martingale diffusion and the interval of prizes \([w, b]\) to \([S(w, y_0), S(b, y_0)]\). Thus the set of implementable distributions through stopping times (strategies) is

\[
\mathcal{F}_X(y_0) = \{ F \in \Delta([w, b]) : \mathbb{E}_{x \sim F}[S(x, y_0)] = y_0 \}.
\]

This set is convex. This and equation (2) imply that no randomized stopping time (mixed strategy) can achieve a distribution which can’t be achieved through a stopping time (pure
strategy). Finally, since \( S \) is a bounded, continuous function, \( \mathcal{F}_X(y_0) \) is closed in \( \Delta([w,b]) \), which is compact in the topology of convergence in distribution, since \([w,b]\) is a compact interval.

3) It is clear that binary distributions can be induced by simple threshold stopping times, since if the support of \( F \) is \( \{a,c\} \), then \( X_{\tau_{a,c}} \in \{a,c\} \) with probability one.

We show that a pure Markovian stopping policy corresponds to a unique threshold stopping time. For the second part, let \( s : [w,b] \rightarrow \{\text{stop, continue}\} \) be a pure Markovian policy and assume that current wealth is \( y_0 \). If \( s(y_0) = \text{continue} \), pick \( a = \sup\{y \in [w,y_0] : s(y) = \text{stop}\} \) and \( c = \inf\{y \in [y_0,b] : s(y) = \text{stop}\} \). Then the strategy \( s \) can be implemented by the simple threshold stopping time \( \tau_{a,c} \). Moreover, any stopping time which implements \( s \) has to be a simple threshold stopping time and it has to be equal to the \( \tau_{a,c} \) defined above. This follows from the continuity of the paths of the diffusion.

For the other direction, let \( \tau_{a,c} \) be a simple threshold stopping time. Define \( s(x) \) to be \textit{continue} if \( x \in (a,c) \) and \textit{stop} otherwise. This is obviously a pure Markovian stopping policy and it is implemented by \( \tau_{a,c} \).

Ankirchner et al. (2015) look at weak solutions in their paper, but this is w.l.o.g. in our setting because strong solutions, (which are ensured by our assumptions on the diffusion), are also weak solutions and the embeddings found in their paper (i.e. the respective stopping times) are always measurable w.r.t. the filtration of the weak solution assumed in their set up.

We also note down the Stroock-Varadhan support theorem which is needed in the proof of Theorem 1 (see for example the version in Pinsky (1995), p. 65). There it is stated for the general version of diffusions with values in \( \mathbb{R} \), but it can be extended easily to the case of diffusions stopped when leaving a bounded interval. To formalize this, consider the space of continuous functions with domain \([0,\infty)\) and values in \([w,b] : C([0,\infty],[w,b])\). Every path of the diffusion \( X \) is an element from this set. Equip this space with the Maximum norm.

**Lemma 1 (Stroock-Varadhan).** Assume the diffusion starts at \( y_0 \in (w,b) \) and consider

\[
C_{y_0} := \{ f \in C([0,\infty),[w,b]) : f(0) = y_0 \}.
\]

The set of possible paths of the diffusion \( X \) started at \( y_0 \) is dense in \( C_{y_0} \) w.r.t. Maximum norm.

**Proof. Proof of Lemma 1.** Let \( f \in C([0,\infty),[w,b]) \) with \( f(0) = y_0 \) and consider it as a function in the larger space \( C([0,\infty),\mathbb{R}) \). Then according to the standard version of the Stroock-Varadhan Support Theorem (Theorem 6.1. in Pinsky (1995)) there exists a sequence of diffusion paths \( f_n \) with \( f_n \) converging to \( f \) in the maximum norm for all compact subintervals of \([0,\infty)\). For an arbitrary function in \( C([0,\infty),\mathbb{R}) \) define the hitting times \( \tau_x = \inf\{t \geq 0 : f(t) = x\} \). Consider the Lipschitz continuous mapping \( \pi : C([0,\infty),\mathbb{R}) \rightarrow C([0,\infty),[w,b]) \) given by

\[
\pi(f)(t) = \begin{cases} 
  f(t) & \text{if } t \leq \min\{\tau_w, \tau_b\} \\
  w & \text{if } t > \tau_w \\
  b & \text{if } t > \tau_b. 
\end{cases}
\]

This is an onto map of \( C([0,\infty),\mathbb{R}) \) into \( C([0,\infty),[w,b]) \). The paths \( f_n \) are mapped to paths \( \pi(f_n) \) of the diffusion stopped when leaving \([w,b]\). Moreover, Lipschitz continuity of \( \pi \) implies that \( \pi(f_n) \) converges to \( \pi(f) \) in the topology of uniform convergence in compact subsets of \([0,\infty)\). \(\square\)

**Remark 1.** 1) The assumptions for the SDE (1) ensure that it has a strong unique solution (path-wise uniqueness). Strulovici and Szydlowski (2015) make the point, that for modeling economic situations only strong solutions of SDEs should be considered, as weak solutions are
defined on larger spaces and thus account for more (uncontrolled) uncertainty than is intended to be modeled by the analyst (the analyst usually assumes all uncertainty is modeled through $W_t$).

2) Our assumption of Lipschitz continuity of the coefficients of the SDE in (1) can be relaxed along the lines of Le Gall (1983) (see Lemme 1.0, Corollaire 1.1, Corollaire 1.2 there and the subsequent discussion) and the general result for the naive agent can be proven for this more general class of diffusions. Details are available upon request.

3) Our assumptions on (1) imply that there exists a constant $\epsilon > 0$ s.t. the variance process $\sigma$ is always above $\epsilon$. This fact is the only technical assumption needed in the proof of Theorem 2 to arrive at the characterization of the behavior of the sophisticated agent, besides conditions ensuring unique strong solutions for (1). This assumption corresponds to cases 2 and 3 in Theorem 1.3 of Le Gall (1983) and thus is consistent with the existence of unique strong solutions for (1) as well.

On Stopping Policies vs. Stopping times In the literature one sometimes finds an alternative description of stopping times through the history dependent stopping policies they induce. Call a function $s : \cup_{t \geq 0} F_t \rightarrow [0,1]$ a stopping policy. For each $t \geq 0$ and event $A \in F_t$, $s(A)$ is the probability the agent continues after history $A$. If the stopping policy is pure, then $s$ says whether the agent continues or stops with probability one after history $A$. Markov stopping policies can be rewritten as functions $s : [w,b] \rightarrow \Delta(\{0,1\})$ and they are pure when they can be rewritten as functions $s : \cup_{t \geq 0} F_t \rightarrow \{0,1\}$.

Formally, we say that $s$ is a Markov stopping policy if for every $t \geq s \geq 0, A \in F_t, B \in F_s$ such that the trace sigma-algebras $A \cap \sigma(X_t), B \cap \sigma(X_s)$ are equal, it holds $s(A) = s(B)$.

6 Proofs for section 3

6.1 Proof of Theorem 1
We will use in the proofs an equivalent way to rewrite the condition $F \preceq \delta_{E[F]}$ for every lottery $F$ with $E[F] = x$: $F_{x+\epsilon} \preceq \delta_x$ for every zero-mean random variable $\epsilon$ such that the random variable $x + \epsilon$ has support within $[w,b]$. We suppress $X$ whenever it’s clear from context.

Proof. 1) Since $F(y_0)$ is the feasible set of lotteries induced by stopping times chosen by the agent this is obvious.

2-a) Assume first that $X$ is a martingale and let for a fixed $y_0 \in (w,b)$ be $\epsilon$ the zero-mean lottery, such that

$$V(F_{y_0+\epsilon}) > V(\delta_{y_0}).$$

Assume that this $y_0$ is current wealth. If the agent has a stopping time, such that $X_\tau \sim y_0 + \epsilon$, then the agent won’t stop. But $\epsilon$ has bounded support and $\sigma$ is bounded away uniformly from zero, so the conditions of Lemma 1 are fulfilled and the existence of the stopping time is assured. The construction in Ankirchner et al. (2015) is done for every filtration for which a weak solution exists but this is w.l.o.g. as we comment in appendix A.

The support of $x+\epsilon$ being bounded and it having mean $x$, it follows that $\{X_{t\wedge \tau}\}_t$ is a bounded Martingale converging in $L^1$ (and a.s. too). Therefore, the embedding from Ankirchner et al. (2015) is also uniformly integrable.

2-b) Note that we have

$$V(\delta_{y_0}) \geq V(F_{y_0+\epsilon}),$$

for all integrable zero-mean $\epsilon$, provided $x + \epsilon$ has support in $[w,b]$. Assume agent doesn’t stop when some martingale diffusion $X$ starts at $y_0$. This means there exists a stopping time $\tau$ such
that \( X_\tau \sim F \) for some \( F \in \mathcal{F}(y_0) \) with \( V(F) > V(\delta_{y_0}) \). But then \( X_\tau - x \) fulfills the definitions of \( \epsilon \) in (14) and we have a contradiction.

The second claim follows Lemma 1. Lemma 1 from appendix A implies, that for every \( f \in C([0, \infty), [w, b]) \) with \( f(0) = y, \ell > 0, \epsilon > 0, y \in [w, b] \), under the assumptions made on the diffusion it holds

\[
\mathbb{P}_{y}(\sup_{0 \leq s \leq \ell} |X(s) - f(s)| < \epsilon) > 0.
\]  

(15)

This can be seen as follows: The set \( B_\ell^f(\epsilon) = \{ g \in C_y : \sup_{0 \leq s \leq \ell} |g(s) - f(s)| < \epsilon \} \) is an open neighborhood of \( f \) in the metric space \( C_y \) (equipped with the maximum-norm). Lemma 1 shows that the paths of the diffusion \( X \) are dense in \( C_y \) (w.r.t. the topology induced by the maximum-norm). In particular, the measure \( \mathbb{P}_{y} \) has full support on \( C_y \). This, the definition of the support of a measure and the openness of the set \( B_\ell^f(\epsilon) \) imply (15). To complete the proof, take for \( y_0 \in (w, b) \) a continuous function \( f \) with \( f(\frac{1}{3}) = y_0 - \frac{\epsilon}{4}|y_0| \) and \( f(\frac{2}{3}) = y_0 + \frac{\epsilon}{4}|y_0| \) with \( \epsilon > 0 \) small enough, we have that the probability that \( X \) reaches \( y_0 \) when starting from \( y \) is positive.

3) For the case that \( X \) is a Martingale the result is trivial as then \( S(x, y_0) = x \). For the case that \( X \) is not a martingale, we eliminate the drift using the scale function (an increasing homeomorphism)

\[
S(y, y_0) = \int_{y_0}^{y} \exp \left( -2 \int_{x}^{t} \frac{\mu(z)}{\sigma^2(z)} dz \right) dt.
\]

Then \( M = S(X) \) is a martingale and fulfills

\[
dM_t = \hat{\sigma}(M_t) dW_t, \quad M_0 = 0.
\]

with \( \hat{\sigma} = (S' \cdot \sigma) \circ S^{-1} \). This can be seen by applying Ito’s formula. Note that \( \hat{\sigma} \) is bounded away from zero locally, because \( S' \), \( \sigma \) are. Define \( \rho = (y_0 + \epsilon) \circ S^{-1} \). Then \( \rho \) has again a bounded support. It holds that \( X_\tau \overset{d}{=} (y_0 + \epsilon) \) is equivalent to \( M_\tau \overset{d}{=} \rho \). It follows from this, that \( \mathbb{E} [\rho] = 0 \), because \( X \) being a bounded process, so is \( M \) and we can use again the bounded convergence Theorem for Martingales. Lemma 1 is again applicable.

Note for the following argument that

\[
\Delta([S(w, y_0), S(b, y_0)]) = \{ L = \alpha \circ S^{-1} : \alpha \in \Delta([w, b]) \}.
\]

It follows, that the agent will continue at \( y_0 \), if the agent whose utility is given for each \( L \) of the form \( L = \alpha \circ S^{-1} \) for some \( \alpha \in \Delta([w, b]) \), by

\[
V_S(L) = V \circ S^{-1}(L) = V(\alpha),
\]

continues at 0. This holds, since the latter agent prefers \( L \) to 0 if and only if the agent with utility function \( V \) prefers \( \alpha \) to \( y_0 \). \( \square \)

6.1.1 Proof of Proposition 2.

Proof. 1) We show that (b) implies (a) first. Assume \( \geq \) satisfies \( \triangleright \) and MMC.

Let \( F \) be a mean preserving spread of \( G \). Then there exists a probability kernel \( K : [w, b] \times [w, b] \to [0, 1] \) (\( K(z, \cdot) \) is measurable for each \( z \in [w, b] \) and \( z \mapsto K(z, y) \) is a probability distribution function for all \( y \in [w, b] \)) such that

\[
F(A) = \int \left( \int_A K(dz, y) \right) dG(y),
\]
and
\[ \int zK(dz, y) = y, \text{ for all } y \in [w, b]. \]

This characterization follows from the arguments in Example 6.D.2 in Chapter 6 of Mas-Collel et al. (1995).

In particular, we have that the distribution \( K(\cdot, y) \) is different from \( \delta_y \) for \( y \in [w, b] \) only by a zero-mean bet. It follows from wRA, that
\[ K(\cdot, y) \preceq \delta_y, \text{ for all } y \in [w, b]. \tag{16} \]

Now MMC implies that \( F \preceq G \) if \( G \) is a step function. For a general distribution \( G \) there is a sequence of step functions \( G_n \), which are also probability distributions, such that \( G_n \) converges weakly to \( G \). It follows for
\[ F_n(z) = \int K(z, y)dG_n(y), z \in [w, b]. \]

that \( F_n \) is a mean preserving spread of \( G_n \) and that therefore \( F_n \succeq G_n \) due to the previous argument. We now use the following

**Fact 1.** If a sequence of distributions \( G_n, n \geq 1 \) converges weakly to \( G \), then for all upper-semicontinuous functions \( f : [w, b] \to \mathbb{R} \) we have
\[
\limsup_{n \to \infty} \int f(z)dG_n(z) \leq \int f(z)dG(z). \tag{17}
\]

**Proof.** Proof of Fact 1. For \( f \) an indicator of a closed set, the result follows from the so-called Portmanteau Theorem (Theorem 4.25 in Kallenberg (2006)). Otherwise, the result is standard, once it is recalled that an upper-semicontinuous function over a compact set has a maximum and thus is bounded from above. A reference is for example Theorem 1.3.4 in Van Der Vaart and Wellner (1996).

Weak convergence of \( G_n \) to \( G \) and the fact that \( K(\cdot, y) \) is upper-semicontinuous for all \( y \in [w, b] \), it being a probability distribution, implies together with the Fact just proved, that
\[
\limsup_{n \to \infty} F_n(y) = \limsup_{n \to \infty} \mathbb{E}_{x \sim G_n}[K(x, y)] \leq \mathbb{E}_{x \sim G}[K(x, y)] = F(y), \quad y \in [w, b] \tag{18}
\]
Assume by contradiction that \( F \succ G \). Due to continuity, there is a natural number \( N \) s.t. \( F \succ G_n \) for all \( n \geq N \). Furthermore, the space \( \Delta([w, b]) \) being compact w.r.t. convergence in distribution, we have a subsequence \( F_{n_k}, k \geq 1 \) converging weakly to some \( \bar{F} \in \Delta([w, b]) \). Now we note the following fact.

**Fact 2.** For a distribution \( F \in \Delta([w, b]) \) the set of discontinuities is countable and each open interval of \([w, b]\) has a point where \( F \) is continuous.

**Proof.** Proof of Fact 2. It suffices to show that the number of discontinuities is countable as the second part follows from the fact that any open, non-empty interval has uncountably many points. Any discontinuity of \( f \) is in
\[ \cup_{n \geq 1} A_n := \cup_{n \geq 1} \{ x \in [w, b] : f(x+) - f(x-) > \frac{1}{n} \}. \]

Since \( f(b) - f(w) = 1 \), any of the \( A_n \) sets cannot contain infinitely many elements. \( \square \)
We use this claim on the implication of (18) which implies that
\[ \hat{F}(y) \leq F(y), \] for all continuity points \( y \) of \( \hat{F} \).

Given this, we note that for arbitrary \( y \in [w, b] \) we have
\[ \hat{F}(y) = \lim_{\epsilon_n \to 0^+} \hat{F}(y + \epsilon_n) \leq \lim_{\epsilon_n \to 0^+} F(y + \epsilon_n) = F(y). \]

Here, in the first equality we have used Fact 2 to construct a sequence of positive numbers \( \epsilon_n \) going to zero, s.t. \( \hat{F} \) is continuous in the points \( y + \epsilon_n \) as well as right-continuity of \( \hat{F} \), whereas the inequality follows from (19) and the last equality again from right-continuity of \( F \).

This gives that \( \hat{F} \) FOSD-dominates \( F \). In particular, for all \( n_k \geq N \) we have \( \hat{F} \succ N \). But \( F_n \) converges weakly to \( \hat{F} \), so that there exists \( n_k, k \) large enough with \( F_n \) as well. This is a contradiction to the assumption. It follows that \( F \lesssim G \) and thus the conclusion for arbitrary \( F \) as well.

Now we show that (a) implies (b). Risk aversion implies directly wRA. Given this, the case \( F_{x_i} \) never occurs in the strict form, so that for MMC we can focus on the case \( F_{x_i} \). But it is easy to see that
\[ \sum_i \alpha_i F_{x_i + \epsilon_i} \]
is a mean preserving spread of the distribution of the lottery \( \sum_i \alpha_i \delta_{x_i} \).

2) The proof is analogous to the proof of 1).

6.2 Proof of Theorem 2

Proof. Let \( C_{inc}^{2,L}([w, b]) \) be the subset of twice continuously differentiable, strictly increasing functions \( f : [w, b] \to \mathbb{R} \) such that the second derivative \( f'' \) is also Lipschitz continuous. For a fixed interval \( (a, c) \), which contains current wealth \( y_0 \), there is a correspondence between diffusions started in \( (a, c) \) and the set
\[ \mathcal{P}_{a,c} = \{ p : [a, c] \to [0, 1] | p(a) = 0, p(c) = 1, p \in C_{inc}^{2,L}([a, c]), p'(a+) < +\infty, p'(c-) > 0 \}. \]

This is the set of all possible win probabilities \( p_{a,c}^X \) for \( (a, c) \) as one varies through regular diffusions \( X \). In this setting, the set of technologically feasible lotteries, given a fixed diffusion \( X \) with normalized drift \( \frac{\mu(z)}{\sigma(z)} \), which corresponds uniquely to some scaling function \( S \) is given by
\[ \mathcal{F}_X(y_0) = \{ L(1 - p_{a,c}^X(y_0), a, c) : a < y_0 < c, 1 - p_{a,c}^X(y_0) = \frac{S(c)-S(y_0)}{S(c)-S(a)} \}. \]

Note, that for all diffusions: \( \mathcal{F}_X(y_0) \cup \{ \delta_{y_0} \} = \text{ext}(\mathcal{F}_X(y_0)) \), the set of extreme points of \( \mathcal{F}_X(y_0) \).

For fixed \( y_0, S \), due to continuity of \( S \) it is easy to see that \( \mathcal{F}_X(y_0) \) is closed and thus also compact. We establish the following useful correspondence between \( C_{inc}^{2,L} \) and diffusions.

Lemma 2. Each scaling function \( S \) of a diffusion with normalized drift \( \frac{\mu}{\sigma^2} \) is a member of \( C_{inc}^{2,L}([w, b]) \). Conversely, for each function \( S \) in \( C_{inc}^{2,L}([w, b]) \) such that \( S(y_0) = 0 \) for some \( y_0 \in [w, b] \) there exists a diffusion such that \( S \) is its scaling function.

Proof. Proof. Checking the first statement is routine, except for perhaps the Lipschitz continuity of \( S''(\cdot, y_0) \). We give arguments for this in the following. Note that it holds
\[ S''(x, y_0) = (-2) \exp \left( -2 \int_{y_0}^x \frac{\mu(t)}{\sigma^2(t)} dt \right) \frac{\mu(x)}{\sigma^2(x)}. \]
We note first that $x \mapsto \frac{1}{\sigma^2(x)}$ is Lipschitz continuous. This uses the fact that $\sigma$ as a function is bounded away from zero and from above. To close the argument, we use the fact that the product of two bounded, Lipschitz continuous functions is again Lipschitz continuous two times: first this delivers that $x \mapsto \frac{\mu(x)}{\sigma^2(x)}$ is Lipschitz continuous and second, that since $x \mapsto \exp \left( -2 \int_{y_0}^{x} \frac{\mu(t)}{\sigma^2(t)} \, dt \right)$ is Lipschitz continuous and bounded, the second derivative $x \mapsto S''(x, y_0)$ is Lipschitz continuous as well.

For the second statement of the Lemma, take $S \in C_{inc}^{2,L}([w, b])$ and $y_0 \in [w, b]$ such that $S(y_0) = 0$. Then, obviously

$$S(y) = \int_{y_0}^{y} S'(z) \, dz, \quad y \in [w, b]$$

and $S'(z) = \exp(-\log(S(z)))$. Consider then $\mu : [w, b] \mapsto \mathbb{R}$ given by

$$\mu(z) = -\frac{S''(z)}{2S'(z)}.$$  

This function is Lipschitz continuous, since $S$ is in $C_{inc}^{2,L}([w, b])$ and it is trivial to check that for any diffusion with normalized drift equal to $\mu$ the stopping problem $(X, y_0)$ has scaling function equal to $S$. \hfill \square

The requirement of never starting any diffusion $X$, identified with its respective $p$, boils down to:

for every $a < x < c$ and every $p \in \mathcal{P}_{a,c}$ the following holds:

$$\{y \in (a, c)|L(1 - p(y), a, c) \leq \delta_y\} \neq \emptyset.$$  

Recall now that for fixed $a < x < c$ we have $p(y) = \frac{S'(c) - S'(y)}{S'(c) - S'(a)}$ and thus $p'(y)$ is proportional to $-S'(y)$. But note also that

$$S'(y) = \exp \left( -\int_{x}^{y} \frac{2\mu(z)}{\sigma^2(z)} \, dz \right).$$

Due to our regularity assumption on the diffusion processes, we know that $S'(y)$ is always bounded away from zero and infinity on each interval $(a, c)$ with $x \in (a, c)$. Now the statements of the theorem follow straightforwardly with the help of the following simple: for any strictly increasing, continuous $f : [a, c] \mapsto [0, 1]$ with $f(a) = 0, f(c) = 1$ there exists some $p \in \mathcal{P}_{a,c}$ with $p(x) > f(x), x \in (a, c)$ if and only if $f'(a+) = +\infty$, or $f'(c-) = 0$. \hfill \square

### 6.3 Proof of Proposition 3.

Before proving Proposition 3 we prove a technical Lemma which is needed in the proof and is of technical interest on its own. In the following the mixture operation is the mixture operation on distributions over $[w, b]$ and we suppress notation of the diffusion $X$.

**Lemma 3.** 1) The set $\mathcal{F}(y_0)$ is a compact, convex set.
2) Every finite support element of $\mathcal{F}(y_0)$ can be written as a convex combination of $\delta_{y_0}$ and binary lotteries from $\mathcal{F}(y_0)$.
3) The subset of finite support measures in $\mathcal{F}(y_0)$ is dense.

**Proof. Proof. Step 1.** Let us show the result first for Brownian motion. In this case

$$\mathcal{F}^{bm}(y_0) = \{F \in \Delta([w, b]) : \mathbb{E}_{x \sim F}[x] = y_0\}.$$  

1) This is clearly a convex set. Compactness follows from the fact that the set $\mathcal{F}(y_0)$ is closed. The latter fact follows from the continuity of the function $F \mapsto \mathbb{E}[F]$, which is clear by the
properties of weak convergence of probability measures and the fact that the function \( id : [w, b] \to [w, b] \) with \( id(x) = x \) is continuous and bounded.

2) Let \( F \) be a finite support distribution in \( \mathcal{F}(y_0) \) and denote by \( p \) the finite support lottery it induces over \([w, b]\). Assume first that \( p(y_0) = 0 \). The result then follows from Lemma C.1. and Corollary C.2. in the appendix of Xu and Zhou (2013). Their set of feasible distributions \( \mathcal{D} \) is defined through an inequality, because the space of possible prizes there is \([0, +\infty)\), but a look at their proof shows that the proof is valid word-for-word in the case of our model as well, where the prize space is \([w, b]\).

3) Take a sequence of finite support distributions \( F_n \) on \([w, b]\) s.t. \( F_n \) converges weakly to \( F \). Because the function \( id : [w, b] \to [w, b] \) with \( id(x) = x \) is continuous and bounded it follows for \( x_n := \mathbb{E}[F_n] \) that \( x_n \to x \).

Take \( 0 < \epsilon < \frac{1}{2} \min\{|x-w|, |x-b|\} \). From now on, consider only sequences \( F_n \) s.t. \( x_n \in (x-\epsilon, x+\epsilon) \) for all \( n \). If \( x_n \geq x \) define \( z_n = -\epsilon + x \) and \( x_n < x \) define \( z_n = +\epsilon + x \). Define then \( \lambda_n \in [0,1] \) s.t. \( x = \lambda_n x_n + (1-\lambda_n) z_n \) for all \( n \).

Looking at \( G_n = \lambda_n F_n + (1-\lambda_n) \delta_z \) it follows \( \lambda_n \to 1, n \to \infty \) from \( x_n \to x \) and so that \( G_n \to F \) weakly. But \( \mathbb{E}[G_n] = x \) by construction and \( G_n \) are again probability distributions with finite support.

Step 2. We now take an arbitrary diffusion which satisfies our regularity assumption. Due to Ankirchner et al. (2015) we know

\[
\mathcal{F}(y_0) = \{ F \in \Delta([w, b]) : \mathbb{E}_{x \sim F}[S(x, y_0)] = 0 \}.
\]

Note that the argument in Step 1 didn’t use the particular form of the interval \([w, b]\). So in the following for the set \( \mathcal{F}_{bn}^m(y_0) \) defined in Step 1 use \([S^{-1}(w), S^{-1}(b)]\). This interval contains obviously zero \((S^{-1}(y_0))\). Define the map \( \psi : \mathcal{F}(y_0) \to \mathcal{F}_{bn}^m(0) \), given by \( \psi(F)(x) = F(S^{-1}(x)) \). It is easy to see that this map is continuous, linear and a homeomorphism between the two compact sets \( \mathcal{F}(y_0) \) and \( \mathcal{F}_{bn}^m(0) \). Using this fact and Step 1 we are done.

**Proof.** **Proof of Proposition 3.** We prove: if the agent prefers the certain expected value of a binary lottery to the binary lottery itself and is quasi-convex then she exhibits wRA everywhere. The case of not wRA is similar. Fix \( x \in (w, b) \) and a finite sequence of distributions \( F_i, i = 1, \ldots, n \) with \( \mathbb{E}[F_i] = x \) such that each has a support of two elements only and so that

\[
F_i \preceq \delta_x, \tag{20}
\]

Proper quasi-convexity then implies, that for all \( \alpha_i \geq 0, i = 1, \ldots, n \) with \( \sum_{i=1}^{n} \alpha_i = 1 \) we have

\[
\sum_{i} \alpha_i F_i \preceq \delta_x.
\]

By varying the \( F_i \)-s and the \( \alpha_i \)-s, it is easy to see that we can extend (20) to all finite support distributions with mean \( x \). This uses Lemma 3. Due to Assumption 1 and Continuity (20) can then be extended to all distributions \( F \) with mean \( x \).

The above argument can be modified to show, that the agent whose current wealth is \( y_0 \) and whose preferences satisfy quasi-convexity will always find it optimal to choose either \( \delta_{y_0} \) or a binary lottery out of \( \mathcal{F}(y_0) \). To see this, let w.l.o.g. \( F^* \neq \delta_{y_0} \) be an element which maximizes preference over \( \mathcal{F}(y_0) \). This implies in particular, that the agent starts. Assume that \( L \prec F^* \) for all binary \( L \in \mathcal{F}(y_0) \). From quasi-convexity of preference, completeness and Lemma 3 it follows for all \( L \) in \( \mathcal{F}(y_0) \) with finite support that \( L \prec F^* \). This can be seen as follows: if it weren’t true then for some finite support \( L \) with \( F^* \preceq L \) and decomposition \( L = \sum_{i=1}^{n} \alpha_i L_i \) where \( \alpha_i \geq 0, L_i \in \mathcal{F}(y_0) \) have support of at most two elements, we can pick \( L_i \) with \( F^* \preceq L_i \), contradicting either \( L \prec F^* \) for all binary \( L \in \mathcal{F}(y_0) \) or that \( F^* \neq \delta_{y_0} \).
Take a sequence of finite support \( L_n \in \mathcal{F}(y_0) \) with \( L_n \) converging to \( F^* \). The existence of this sequence is assured due to Lemma 3. Again due to quasi-convexity and completeness of preference, there exists some \( B_n \in \mathcal{F}(y_0) \), binary lottery, or \( B_n = \delta_{y_0} \) such that \( L_n \preceq B_n \) for all \( n \). It follows in all \( L_n \preceq B_n \prec F^* \). \( \mathcal{F}(y_0) \) being compact, there exists a converging subsequence of \( B_n \). In particular, its limit \( B \) has to be either a binary lottery or \( \delta_{y_0} \). Continuity of preference then implies, that \( B \sim F^* \). In all, the maximand in \( \mathcal{F}(y_0) \) can always be chosen to be a binary lottery in the case of quasi-convex preferences. \( \square \)

7 Proofs for section 4

Proof. Proof of Proposition 8.

We find the calibration function for a typical OI preference. Given \( a < y_0 < c \) with all three prizes in \([w, b]\) we calculate that it satisfies the identity

\[
\int_a^c h(x, q_{a,c}(y))dx = \int_a^y h(x, 1)dx.
\]

Taking one-sided derivatives w.r.t. \( y \) we arrive at the identity

\[
q'_{a,c}(y) = \frac{h(y, 1)}{\int_a^y h_p(x, q_{a,c}(y))dx}.
\]

We can now use Theorem 2 to establish that extreme sensitivity in the case of OI preferences is precisely equivalent to the two conditions in the statement of the Proposition. \( \square \)

Proof. Details for Example 4.1. The continuation condition for the stopping time \( \tau_{w,b} \) is

\[
p(y)b + (1 + \beta)(1 - p(y))w \geq y(1 + (1 - p(y))\beta), \quad y \in (w, b).
\]

Rewrite (21) as

\[
p(y) \geq \frac{(1 + \beta)(y - a)}{c - a + \beta(y - a)}.
\]

Define the function \( g(y, a, c) \) as the LHS of (22). Given that \( p(y) \) and \( g(y, a, c) \) are both strictly concave and strictly increasing for \( y \in (a, c) \), \( \tau_{a,c} \) is feasible if and only if its respective win probability function \( p(\cdot) \) satisfies

\[
\frac{d}{dy} p(y)|_{y=a} > \frac{d}{dy} g(y, a, c)|_{y=a}, \quad \frac{d}{dy} p(y)|_{y=c} < \frac{d}{dy} g(y, a, c)|_{y=c}.
\]

These are ensured if and only if

\[
\frac{2\mu}{\sigma^2} (c - a) \frac{1}{1 - e^{-(c-a)\frac{2\mu}{\sigma^2}}} > 1 + \beta, \quad \frac{2\mu}{\sigma^2} (c - a) \frac{1}{e^{(c-a)\frac{2\mu}{\sigma^2}} - 1} < 1.
\]

Since the function \( f_1 : [0, \infty) \rightarrow \mathbb{R} t \mapsto \frac{t}{e^t - 1} \) is increasing in \( t \) and the function \( f_2 : [0, \infty) \rightarrow \mathbb{R}, t \mapsto \frac{1}{e^t - 1} \) is decreasing in \( t \) we see that \( d_0 \) is determined as the maximum of \( d_1 \) and \( d_2 \) which satisfy respectively \( f_1(d_1) = 1 + \beta \) and \( f_2(d_2) = 1 \).

To get a sufficient condition which implies that \( \tau_{w,b} \) is optimal for the agent we require first the bang-bang lottery to be preferred to current wealth.

A. For \( q = \frac{1 - e^{-(y_0 - w)\frac{2\mu}{\sigma^2}}}{1 - e^{-(b-w)\frac{2\mu}{\sigma^2}}} \) it holds \( \frac{q b + (1 + \beta)(1 - q)w}{1 + (1 - q)\beta} > y_0 \).  

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Now we find conditions that ensure that given any fixed policy $\tau_{a,c}$ with $w < a < y_0 < c < b$, the agent who has started would always want to increase $c$ and lower $a$.

We rewrite the objective function

$$Obj(a, c) = \frac{p(y_0)c}{1 + (1 - p(y_0))\beta} + \frac{(1 + \beta)(1 - p(y_0))a}{1 + (1 - p(y_0))\beta}$$

by making the change of variables $d = c - a$ into

$$\hat{Obj}(a, d) = a + \frac{p(y_0)d}{1 + (1 - p(y_0))\beta}.$$ 

Note that now the win probability of the diffusion can be written as

$$p(y_0) = \frac{1 - e^{-(y_0-a)\frac{2\mu}{\sigma^2}}}{1 - e^{-d\frac{2\mu}{\sigma^2}}}.$$

We calculate

$$\frac{d\hat{Obj}}{da} = 1 - \frac{2\mu}{\sigma^2} \frac{(1 + \beta)d}{(1 + (1 - p(y_0))\beta)^2} \frac{e^{-(y_0-a)\frac{2\mu}{\sigma^2}}}{1 - e^{-d\frac{2\mu}{\sigma^2}}}.$$

and

$$\frac{d\hat{Obj}}{dd} = \frac{p(y_0)}{1 + (1 - p(y_0))\beta} \left[ 1 - \frac{2\mu}{\sigma^2} \frac{(1 + \beta)d}{(1 + (1 - p(y_0))\beta)^2} \frac{e^{-(y_0-a)\frac{2\mu}{\sigma^2}}}{1 - e^{-d\frac{2\mu}{\sigma^2}}} \right].$$

One can then see the following facts:

**B.** Whenever $\frac{2\mu}{\sigma^2}$ is large enough: if the agent starts a simple threshold stopping time $\tau_{a,c}$, i.e. in particular $a < y_0$ then she wants optimally to set $d = c - a$ as large as possible. This uses the fact that $p(y_0)$ for the interval $(a, c)$ remains bounded away from zero as $d$ is varied, whenever $a < y_0$.

**C.** Whenever $\frac{2\mu}{\sigma^2}$ is large enough: if the agent starts a simple threshold stopping time $\tau_{a,c}$, i.e. in particular $a < y_0$ and $d$ is large enough she wants to optimally set $a$ as small as possible.

Combining now 1., 2. and 3. it is easy to see that for large enough $\frac{d\mu}{\sigma^2}(b - w)$ the Gul sophisticated agent chooses optimally $\tau_{w,b}$, i.e. she never stops. \hfill \square

**Proof. Details for Example 4.2.** For $\mu > 0$, $1 - p(y)$ is a convex, decreasing function, and one can check by taking derivatives (see (30)) that $\nu(1 - p(y))$ is convex and decreasing in $y$. Here we have used that $S$ is a concave function for $\mu > 0$. It is given by

$$S(x, y_0) = \frac{\sigma^2}{2\mu} \left( 1 - \exp(-2\frac{\mu}{\sigma^2}(x - y_0)) \right).$$

Moreover, clearly $\nu(1 - p(a)) = 1, \nu(1 - p(c)) = 0$. It follows, that the agent will start any geometric Brownian motion with $\mu > 0$. More so, the set of possible stopping times for the agent are all $\tau_{a,c}$ with $w \leq a < y_0 < c \leq b$. Given that the continuation requirement is void, the problem of the agent reduces to

$$\max_{a: w \leq a \leq y_0 \leq c \leq b} \nu(1 - p(y_0))a + (1 - \nu(1 - p(y_0)))c.$$
Denote the objective function by

\[ \text{Obj}(a, c, y_0) = \nu(1 - p(y_0))a + (1 - \nu(1 - p(y_0)))c. \quad (23) \]

Its derivative w.r.t. \( a \) is

\[
\frac{d\text{Obj}}{da} = \nu'(1 - p(y_0)) \frac{d}{da} (1 - p(y_0)) (a - c) + \nu(1 - p(y_0)),
\]

(24)

while its derivative w.r.t. \( c \) is

\[
\frac{d\text{Obj}}{dc} = \nu'(1 - p(y_0)) \frac{d}{dc} (1 - p(y_0)) (a - c) + (1 - \nu(1 - p(y_0))).
\]

As long as respectively \( \frac{d\text{Obj}}{da} < 0 \) or \( \frac{d\text{Obj}}{dc} > 0 \), it is profitable to lower \( a \) and raise \( c \). Note, that

\[
\frac{d}{da} (1 - p(y)) = \frac{2\mu}{\sigma^2} e^{-\frac{2\mu}{\sigma^2}(1 - p(y))} > 0,
\]

(26)

and that

\[
\frac{d}{dc} (1 - p(y)) = \frac{2\mu}{\sigma^2} \frac{1}{e^{(c-a)\frac{2\mu}{\sigma^2}} - 1} p(y) = \frac{2\mu}{\sigma^2} \frac{e^{(c-a)\frac{2\mu}{\sigma^2}} - e^{(c-y)\frac{2\mu}{\sigma^2}}}{(e^{(c-a)\frac{2\mu}{\sigma^2}} - 1)^2} > 0.
\]

(27)

Plugging (26) and (27) in the derivative expressions (24) and (25), it follows after routine cancellations that a sufficient condition for \( \frac{d\text{Obj}}{da} > 0 \) and \( \frac{d\text{Obj}}{dc} < 0 \) for all \( a < y_0 < c \) is for \( \frac{\mu}{\sigma^2} \) to be large enough (with \( a \) fixed). The argument establishing this is independent of the shape of \( w \) as long as \( \frac{\mu}{\sigma^2} \) is large enough. It follows, that whenever \( r_{w, b} \) is feasible, it will be chosen as long as \( \frac{\mu}{\sigma^2} \) is large enough. \( \square \)

**Proof.** Details for Example 4.3.

The commitment case for geometric Brownian motion with \( w = 0, b = +\infty \), i.e. arbitrarily high prizes possible, has been studied in Xu and Zhou (2013). One can see from the paper that their results can be adapted to the case of \( b < +\infty \). In particular, Theorem 5.1 there, holds true again with \( b < +\infty \). To expand on details: the main change in the case \( b < +\infty \) in the Xu and Zhou (2013) set-up is that the set of possible distributions which can be implemented by uniformly integrable stopping times is given now by

\[
\mathcal{D} = \{ F : [w, b] \to [0, 1] : F \text{ is a CDF and } E[F[S(\cdot, y_0)]] = 0 \}.
\]

This is the same set as in Lemma 3.2 in Xu and Zhou (2013), except that the inequality \( E[F[S(\cdot, y_0)]] \leq 0 \) is strengthened to an equality. The authors work with the geometric Brownian motion transformed into a Martingale, but that restriction is w.l.o.g. by the same steps as we have used here when proving the general results for naive agents. Moreover, the weak inequalities in Lemma 3.2 of Xu and Zhou (2013) are due to their choice of \( b = +\infty \) (see Ankirchner et al. (2015) for more on this too). The proof of Theorem 5.1 in Xu and Zhou (2013) can be repeated word for word for the case \( b < +\infty \) as well and the defining equation (5.3) in Xu and Zhou (2013) for the optimal quantile function becomes

\[
G^*(x) = \min\{b, G^*_{XZ}(x)\}.
\]

We state the version of the Proposition we need in the following. \( \beta = -\frac{2\mu}{\sigma^2} + 1 \) as defined in the main body of Xu and Zhou (2013).
Proposition 10. Assume that \( v(x) = u(x^{\frac{1}{\alpha}}) \) is concave and with a strictly monotone derivative, as well as \( \nu \). If there exists a \( \lambda \geq 0 \) such that \((v')^{-1}\left(\frac{\lambda}{\nu'(1-x)}\right) > 0, \forall x \in (0,1) \) and
\[
\int_0^1 \min\left\{ (v')^{-1}\left(\frac{\lambda}{\nu'(1-x)}\right), b \right\} \, dx = y_0^\beta, \tag{28}
\]
then \( G(x) = \min\left\{ (v')^{-1}\left(\frac{\lambda}{\nu'(1-x)}\right), b \right\} \) is the quantile function of an optimal distribution for the agent with commitment in the optimal stopping problem \((X, y_0)\).

For our specification, denoting \( x_b \in (0,1) \) the smallest number with \( G(x_b) = b \), the relation between \( x_b \) and \( \lambda \), if \( \lambda \) as in Proposition 10 exists is
\[
x_b = 1 - \left(\frac{\alpha}{\lambda}\right)^{\frac{1}{1-\alpha}} b^{-\frac{\beta-x}{1-\alpha}}.
\]
There is one-to-one relation between \( x_b \) and \( \lambda > 0 \).

The feasibility condition can then be written in terms of \( x_b \) as
\[
\frac{\beta - r}{\beta \alpha - r} (1 - x_b)^{1-\alpha} (1 - (1 - x_b)^{\frac{\alpha - \beta - r}{\beta - r}}) + (1 - x_b) = \frac{y_0^\beta}{b^\beta}.
\] \tag{29}

It is easily seen that for all \( y_0 \in (0,b) \) an \( x_b \in (0,1) \) to fulfill the above equation exists and thus also a respective \( \lambda > 0 \) exists. The quantile function of the optimal distribution as given by Proposition 10 is
\[
G(x) = b \min\left\{ 1, \left(\frac{1 - x_b}{1 - x}\right)^{\frac{1}{\beta - r}} \right\}.
\]

Using the relation between \( \lambda \) and \( x_b \) one gets by inverting the quantile function for the optimal distribution
\[
F_{y_0}(x) = \begin{cases} 
1, & \text{if } x = b \\
1 - (1 - x_b) \left(\frac{x}{b}\right)^{-\frac{\beta-x}{1-\alpha}}, & \text{if } b > x \geq (1 - x_b)^{\frac{1}{\beta - r}} \\
0, & \text{if } 0 \leq x < (1 - x_b)^{\frac{1}{\beta - r}},
\end{cases}
\]

In particular, due to the one-to-one correspondence between quantile functions and distributions, the optimal distribution chosen by the naive agent is a Pareto distribution truncated at \( b < +\infty \) and with an atom at \( b \).

Proof of the last statement for the sophisticated agent. We prove the following slightly more general claim.

Claim. A sophisticated RDU agent never starts if \( u : [w, b] \to \mathbb{R}_+ \) is (weakly) concave, \( \mu \) is negative and \( \nu \) is strictly concave.

Proof of the Claim. The respective preference constraint that has to be satisfied for a sophisticated agent to implement a simple threshold stopping time is
\[
\frac{u(c) - u(y)}{u(c) - u(a)} \geq \nu(1 - p(y)).
\]

Here \( p(y) \) is the win probability for an interval \((a,c)\) (we omit for notational simplicity the superscripts and subscripts denoting resp. the diffusion and the interval). One can easily show that \( 1 - p(y) \) is a strictly concave and decreasing function whenever \( \mu < 0 \). The right-hand side of the above inequality is thus decreasing and strictly concave, since
\[
\frac{d^2}{dy^2} (\nu(1 - p(y))) = \nu''(1 - p(y)) \frac{S'(y, y_0)^2}{(S(c, y_0) - S(a, y_0))^2} - \nu'(1 - p(y)) \frac{S''(y, y_0)}{S(c, y_0) - S(a, y_0)} < 0 \tag{30}
\]

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Here we have used that \( S \) is a strictly convex function for \( \mu < 0 \). Since the left-hand side is (weakly) convex and decreasing and both sides coincide at \( y = a, c \) and are continuous, it follows that the required inequality can never be satisfied under the conditions stated in the Claim. End of Proof of Claim.

Returning to the parametric functions of the example: The claim covers the case \( r = 1 \). The result remains valid for all \( r < 1 \) but near enough 1, because the slope of the left-hand side of (30) doesn’t depend on \( r \) (the necessary parameter restriction consisting of \( \gamma < \alpha \) remains valid whenever \( \mu \) is negative enough).

Proof. Proof of Proposition 4. Fix a stopping problem \((X, y_0)\) of the type prescribed in the statement. Note that in both of diffusion classes considered, the scaling function for the stopping problem \((X, y_0)\) is as follows.

\[
S(x, y_0) = \frac{\sigma^2}{2\mu} \left( 1 - \exp\left( -2 \frac{\mu}{\sigma^2} (x - y_0) \right) \right).
\]

Because the preference is quasi-convex (it satisfies Betweenness), we can restrict our analysis of feasibility of stopping times to simple threshold ones and only look at elements from \( F(y_0) \) which are binary lotteries. The feasibility condition for a lottery \( L(p, y, x) \) with \( x < y_0 < y \) can then be written as

\[
1 = (1 - p)e^{-2 \frac{\mu}{\sigma^2} (x - y_0)} + pe^{-2 \frac{\mu}{\sigma^2} (y - y_0)}.
\]

We solve here for \( p \) and plug the resulting equation in the formula of the utility of Gul preferences for binary lotteries, given by (6). It holds \( V(L(p, y, x)) > u(y_0) \) if and only if \( y_0 \) is not in the stopping region of \( X \). Rearranging gives the condition

\[
\frac{e^{2 \frac{\mu}{\sigma^2} (y_0 - x)} - 1}{1 - e^{-2 \frac{\mu}{\sigma^2} (y - y_0)}} \frac{u(y) - u(y_0)}{u(y_0) - u(x)} > 1 + \beta.
\]

Proof. Proof of Proposition 5. For fixed \( a < y_0 < c \), that the slope of the \( q := q_{a,c} \) function fulfills

\[
u'(y)(1 + (1 - q(y)))\beta - u(y)q'(y)\beta = u(c)q'(y) - (1 + \beta)u(a)q'(y).
\]

In particular, it follows

\[
q'(c-) = \frac{u'(c-)}{(1 + \beta)(u(c) - u(a))}, \quad q'(a+) = \frac{u'(a+)(1 + \beta)}{u(c) - u(a)}.
\]

Result now follows directly from (31) and from Theorem 2.

Proof. Proof of Proposition 6. We first prove the following claim. For a stopping problem \((X, y_0)\) define
Theorem 3 of Cerreia-Vioglio et al. (2015) 1.-2. follow directly.

Proof. Proof of Proposition 7. It is easy to see that
\[
\mathbb{E}_F[u] = \mathbb{E}_{F \circ S^{-1}}[u \circ S^{-1}], \quad F \in \Delta([w, b]), u \in U.
\]
From this it follows
\[
V(F) = \inf_{u \in U} u^{-1}(\mathbb{E}_F[u]) = \inf_{u \circ S^{-1} \in U} S^{-1} \circ (u \circ S^{-1})^{-1} \circ (\mathbb{E}_{F \circ S^{-1}}[u \circ S^{-1}])
\]
\[
= S^{-1} \left( \inf_{u \circ S^{-1} \in U} (u \circ S^{-1})^{-1} \circ (\mathbb{E}_{F \circ S^{-1}}[u \circ S^{-1}]) \right) = S^{-1} \left( V(S \circ F^{-1}) \right).
\]
Note that the set \( \hat{U} \) as defined in the statement of the claim, is again convex and compact. This is because the map
\[
\psi : C([w, b]) \rightarrow C([S(w, y_0), S(b, y_0)]), \quad \psi(u) = u \circ S^{-1}
\]
is a homeomorphism between the two spaces of continuous functions, considered as normed spaces with the maximum norm.

It follows that \( V_S \) has a CEU representation with space of Bernoulli utilities given by \( \hat{U} \).

From part 3. of Theorem 1 it follows that \( y_0 \) is in the stopping region of \( X \) if and only if \( V_S \) exhibits wRA at zero. With the characterization of risk averse and risk loving behavior from Theorem 3 of Cerreia-Vioglio et al. (2015) 1.-2. follow directly.

3. Let \( y_0 \in (w, b) \) arbitrary. Note that for every \( u \in U \) the condition \( u'(b) \geq \frac{u(b) - u(w)}{b - w} \) implies that there exists \( a(y_0) < b, c(y_0) > y_0 \) such that the line connecting \((a(y_0), u(a(y_0))) \) and \((c(y_0), u(a(y_0))) \) lies above the graph of \( u \) restricted to the interval \((a(y_0), c(y_0)) \). Then choosing probability \( p \) such that \( L = L(p, a(y_0), c(y_0)) \) has mean \( y_0 \) we have that \( \mathbb{E}_L[u] > u(y_0) \). This shows that \( V(L(p, a(y_0), c(y_0)) > V(\delta_{y_0}) = y_0 \). It follows that the agent violates wRA at \( y_0 \).

The result now follows from part 2-a) of Theorem 1.

4. Due to the claim, we can check whether part 3. of Theorem 1 is satisfied for the preference represented by \( V_S \) at the point \( 0 \in [S(w, y_0), S(b, y_0)] \). Writing out the condition required there finishes the proof.

Proof. Proof of Proposition 7. Note first, that to ensure FOSD-monotonicity of \( V \), the minimal set \( U \), in the sense of inclusion, in the representation (see section 2.5 of and in particular Theorem 2 in Cerreia-Vioglio et al. (2015) for the meaning of ‘minimal set \( U \)’) has to contain only strictly increasing functions. Given that \( U \) is compact, for each \( y_0 \in (w, b) \) there exists \( u_y \) some element in the \( U \) such that \( V((1 - q(y))\delta_w + q(y)\delta_b) = w_y^{-1}(q(y)) = y \). It follows \( q(y) = u_y(y) \). We know that \( q(y) \) is non-decreasing so that it has one-sided derivatives at \( w, b \). It follows
\[
\frac{q(b) - q(y)}{b - y} = \frac{1 - u_y(y)}{b - y} = \frac{1}{b - y} \int_y^b u_y'(s)ds,
\]
where we have used that the functions \( u \in U \) are absolutely continuous due to being non-decreasing. The assumption in the statement implies then that \( q'(b-) > 0 \). One can show similarly, that \( q'(w-) < +\infty \). The result follows then from Theorem 2.
7.1 Proof of Proposition 9.

Here we prove Proposition 9 by proving first a couple of results we didn’t find in the literature on the relation between OI and RDU models.

**Proposition 11.** An OI preference defined by the continuous function \( h : [w, b] \times [0, 1] \rightarrow \mathbb{R}_+ \) which is strictly increasing in the second argument with \( h(x, 0) = 0 \) for every \( x \in [w, b] \) are RDU if and only if \( h \) is multiplicatively separable in both arguments.

**Proof.** Green and Jullien (1987) have shown that any RDU functional can be rewritten in an OI functional form so we focus on the other direction. W.l.o.g. we can rewrite the RDU functional of the preference as follows:

\[
W(F) = \int_0^1 u(x)w(1 - F(x))dx, \quad F \in \Delta([w, b])
\]

for functions \( u, w \) which are strictly increasing and continuous as well as \( w \) satisfies \( w(0) = 0, w(1) = 1 \). In the following we also assume for simplicity that \( h, u, w \) are piece-wise continuously differentiable. Let’s focus on distributions \( F \) with support of at most two elements: \( L(p, a, b) = p\delta_a + (1 - p)\delta_b \) for some \( p \in [0, 1], a, b \in [0, 1] \). We get

\[
\psi \left( \int_0^a u(x)dx + \int_a^b u(x)w(1 - p)dx \right) = \int_0^a h(x, 1)dx + \int_a^b h(x, 1 - p)dx.
\]

Differentiate this identity w.r.t. \( p \) on both sides to get

\[
\int_a^b \partial_p h(x, 1 - p)dx = \psi' \left( \int_0^a u(x)dx + \int_a^b u(x)w(1 - p)dx \right) \left[ \int_a^b u(x)dx \right] \cdot w'(1 - p).
\]

Divide now by \((b - a)\) both sides and let \( b, a \rightarrow x \) for any \( x \in [0, 1] \). We arrive at

\[
\partial_p h(x, p) = [w'(1 - p)] \left[ u(x)\psi' \left( \int_0^x u(z)dz \right) \right].
\]

Thus there exist a continuous function \( U : [0, 1] \rightarrow \mathbb{R} \) so that \( \partial_p h(x, p) = w'(1 - p)U(x) \) for all \( x \in [0, 1] \). We find then using that \( h(x, 0) = 0 \)

\[
h(x, p) = \int_0^p \partial_p h(x, r)dr = \left[ \int_0^p w'(1 - r)dr \right] \cdot U(x) = w(1 - p)U(x).
\]

Thus \( h \) has product form as well. This finishes the proof. \( \square \)

Given the identification of RDU preferences with product-form functions \( h \) within the OI class, we prove in the following the following Corollary about RDU preferences.

**Corollary 2.** For any two RDU functionals \( V_1, V_2 \) encoded by \((u_1, w_1)\) and \((u_2, w_2)\) and \( a \in (0, 1) \) it holds that \( aV_1 + (1 - a)V_2 \) is an RDU preference if and only if either of the following conditions is satisfied.

A. \( u_1 \) is a positive rescaling of \( u_2 \);

B. \( w_1 = w_2 \).

**Proof.** The direction from the conditions to the conclusion is trivial integral algebra. For the other direction note that Proposition 11 implies that \( h \) in the OI representation corresponds to an RDU preference only if the following condition holds true.

\[
(C) \quad \partial_p \left[ \frac{h(x, p)}{h(y, p)} \right] = 0, \text{ for all } p \in [0, 1), x, y \in [w, b].
\]
Thus we would be done if we show that \( h(x, p) := au_1(x)w_1(1-p) + (1-a)w_2(x)w_2(1-p) \) fulfills (C) only if any of the two conditions 1. and 2. of the statement of the Corollary are true. By straightforward calculation and algebra we arrive at

\[
\frac{\partial_p}{\partial y} \left( \frac{h(x, p)}{h(y, p)} \right) = -a(1-a) \frac{[w_1'(1-p)w_2(1-p) - w_2'(1-p)w_1(1-p)] \cdot [u_1(x)u_2(y) - u_1(y)u_2(x)]}{h(y, p)^2}
\]

This shows that if condition 1. doesn't hold true, then one needs \( w_1'(1-p)w_2(1-p) - w_2'(1-p)w_1(1-p) = 0 \) for all \( p \) a.e. which upon integration, together with the boundary conditions of \( w_1, w_2 \), results in condition 2. from the statement of the Corollary.

**Proof. Proof of Proposition 9.** We split the proof in different steps.

**Step 1.** Assume that we have two continuous CPT preferences encoded respectively by \((U_1, w_1^+, w_1^-)\) and \((U_2, w_2^+, w_2^-)\) which satisfy Assumption 1 in Ebert and Strack (2015) with the same reference point \( r \) and potentially different \( \lambda_i, i = 1, 2 \) but always so that Assumption 2 in Ebert and Strack (2015) holds for a common \( p \in (0, 1) \). This is always possible by continuity. Moreover, we assume that the preferences are so that \( U_1 \) is not a positive scaling of \( U_2 \), we require that \( U_1(r) = U_2(r) = 0 \) and finally we also require that \( w_1^+ \) differs from \( w_2^+ \) at some point \( q \in (0, 1) \). Again this is always possible because of the continuity of the functions involved. We use the notation of Ebert and Strack (2015) in denoting the respective functional representing the CPT preference by \( CPT_i, i = 1, 2 \). (see (13) in main body of the paper).

Pick any \( a \in (0, 1) \). It follows from Corollary 2 that the risk preference defined by the functional

\[
V^a(F) = a \cdot CPT_1(F) + (1-a) \cdot CPT_2(F), \quad F \in \Delta([w, b])
\]

is not an RDU preference.

**Step 2.** One can check now that the proof of Theorem 1 in Ebert and Strack (2015) goes through with very small modifications. Namely, the fact that Assumption 1 holds for the same \( r \) for both \( U_1 \) and \( U_2 \) and that Assumption 2 holds for the same \( p \) allows one to pick for every \( x \in (w, b) \) the same (small-stakes) lottery \( L(p, a_n, b_n) \in \Delta([w, b]) \) for both \( CPT_1 \) and \( CPT_2 \), namely the one from the proof of Theorem 1 in Ebert and Strack (2015) separately for the case \( CPT_1 \) and \( CPT_2 \). Thus, the following Claim similar to the statement of Theorem 1 in Ebert and Strack (2015) holds true for the preference given by \( V^a \).

**Claim 1.** For every wealth level \( x \in (w, b) \) there exists an attractive zero-mean binary lottery that is arbitrarily small.

Using part 2-a) of Theorem 1 we know then that the naive agent with risk preference \( V^a \) will continue with positive probability any Martingale diffusion started at \( x \in (w, b) \). This will hold true for every \( x \in (w, b) \).

**Step 3.** For the case of a general diffusion with scaling function \( S \) we follow the recipe of part 3) in Theorem 1 and look at the risk preference over \( \Delta([S(w, y_0), S(b, y_0)]) \) given by

\[
V^a_S(G) := V^a(G \circ S) = a \cdot CPT_1^i(G \circ S) + (1-a) \cdot CPT_2^i(G \circ G).
\]

Here \( CPT_1^i \) is the continuous CPT/RDU preference on \( \Delta([S(w, y_0), S(b, y_0)]) \) given respectively by the Bernoulli utility \( U_i \circ S^{-1}(-, y_0) \) and probability weighting functions \( w_1^+, w_1^-, i = 1, 2 \). But note that now we can use Step 2 in precisely the same way as the proof of Theorem 2 in Ebert and Strack (2015) to reduce this case to the case of a Martingale diffusion (look at the last paragraphs in the proof of Theorem 2 in pg. 1631 of Ebert and Strack (2015)). This completes the proof.

\[\square\]
References


