Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income

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Abstract

This paper examines how risky labor income and retirement affect optimal portfolio choice. With idiosyncratic labor income risk, the optimal allocation to stocks is unambiguously larger for employed investors than for retired investors, consistent with the typical recommendations of investment advisors. Increasing idiosyncratic labor income risk raises investors' willingness to save and reduces their stock portfolio allocation towards the level of retired investors. Positive correlation between labor income and stock returns has a further negative effect and can actually reduce stockholdings below the level of retired investors.

Financial advisors typically recommend that their customers invest more in stocks than in safe assets when they are working, and to shift their investments towards safe assets when they retire (Jagannathan and Kocherlakota (1996), Malkiel (1996)). By contrast, the standard academic models of portfolio choice (Merton (1969, 1971), Samuelson (1969)) show that retirement is irrelevant for portfolio decisions if investment opportunities are constant and human capital is tradable. In this case the fraction of wealth optimally invested in risky assets should be constant over the lifetime of an individual.

Recent research, building on the pioneering work by Merton (1971), shows that time-varying investment opportunities result in portfolio rules with an intertemporal hedging component whose magnitude depends on the investment horizon of the investor (Balduzzi and Lynch (1997), Barberis (2000), Brandt (1999), Brennan, Schwartz, and Lagnado (1997), Campbell and Viceira (1999, 2000), Kim and Omberg (1996)). With time variation in investment opportunities, retirement and death play an instrumental role as events that exogenously fix the individual's investment horizon.

More importantly, retirement also marks the time at which individuals stop working and thereafter must live off their lifetime savings and (possibly) transfers. From this perspective, retirement matters for portfolio choice because moral hazard and adverse selection problems prevent most individuals from trading claims against their future labor income. This makes human capital a nontradable asset whose value depends on the retirement horizon of the individual, and labor income the yield on this asset.

In three important papers, Bodie, Merton, and Samuelson (1992), Heaton and Lucas (1997), and Koo (1995) incorporate nontradable labor income into the standard intertemporal model of portfolio choice and consumption. Bodie, Merton, and Samuelson (1992) show that, when future labor income is certain, it is optimal for employed investors to hold proportionally more stocks in their portfolios than it is for retired investors. This result holds whether labor supply is fixed or flexible. If labor supply is flexible, employed investors shift their portfolios toward stocks even more aggressively. However, arguably future labor income is uncertain for most individuals. Koo (1995) and Heaton and Lucas (1997) introduce risky labor income—in combination with portfolio constraints—in their analysis. Using numerical simulations, both papers find that investors hold most of their financial wealth in stocks, and they may even short the riskless bond to invest in stocks if they are not constrained. These papers ignore retirement by assuming that individuals work their entire lifetime, and they focus only on the case in which labor income is uncorrelated with stock returns.[1]

There is also a large literature that examines a closely related problem: The effect of a deterioration in background (or undiversifiable) risk on the investor's willingness to take on financial risk. Eeckhoudt, Gollier, and Schlesinger (1996), Gollier and Pratt (1996), Kimball (1993), and Pratt and Zeckhauser (1987) among others show that, under fairly general conditions on the utility function, investors reduce their holdings of risky assets when there is an increase in background risk. However, this work focuses on two-period models, which by construction ignore endogenous wealth accumulation. In a more realistic model with savings, the investor has two instruments available to weather increased income uncertainty: To save more and/or to reduce her exposure to portfolio risk. Elmendorf and Kimball (1999) and Koo (1995) address this issue in models that integrate the savings and portfolio decision. Koo (1995) shows numerically that an increase in the variance of permanent income shocks leads to a reduction in both the optimal portfolio allocation to stocks and the consumption-labor income ratio of power utility investors. When income shocks are only temporary, he finds that the effect on portfolio choice and consumption is negligible.[2] Using a two-period model that allows for endogenous consumption in the first period, Elmendorf and Kimball (1999) show that Koo's simulation results on portfolio choice hold for any utility function that exhibits decreasing absolute risk aversion and decreasing absolute prudence in the sense of Kimball (1990), while the results on savings require stricter conditions on the utility function—which are also satisfied by power utility.

This paper extends the previous literature in three directions. First, it incorporates retirement into a dynamic model of optimal consumption and portfolio choice with uninsurable labor income risk. Second, it explores the ability of stocks to hedge consumption from unexpected falls in labor income when risky labor income risk is correlated but not perfectly correlated with stock returns. Third, it derives an approximate analytical solution of the model. This analytical solution gives us economic insight into the nature of the problem and it greatly facilitates comparative statics analysis and calibration exercises. This is particularly useful to understand the effects of an uncompensated increase in labor income risk on savings and portfolio decisions.

To address these questions, the paper builds a stationary model in which it is possible to explore life-cycle effects on portfolio choice and savings, while preserving the analytical advantages of infinite-horizon models. Retirement is defined as a permanent zero-labor income episode.[3] Thus the individual has a strong motive to save for retirement. To generate a finite retirement horizon and a finite expected lifetime after retirement, the model follows Blanchard (1985) and assumes that retirement and death after retirement occur randomly with constant probabilities. The inverse of these probabilities are effectively the investor's expected retirement horizon and expected lifetime after retirement. This device is sufficient to capture the differences in the behavior of retired and employed investors. Moreover, by comparing the optimal allocations of investors with different retirement horizons, it is possible to understand portfolio allocations over the life-cycle because, if discount rates and the expected growth rate of labor income are constant over the life-cycle, the investor's retirement horizon is relevant for her portfolio decisions only in that it determines her remaining human capital.[4]

Because labor income is risky and not perfectly correlated with stock returns in this model, it is not possible to derive an exact analytical solution. However, following Campbell (1993) and Campbell and Viceira (1999), we can find an approximate analytical solution. This solution obtains after log-linearizing the budget constraint and the Euler equations of the problem, and using the method of undetermined coefficients to solve for policy functions that verify them. The Euler equations are replaced by a second order expansion, so that second-moment effects such as precautionary savings effects are accounted for in the solution. The approximate solution holds exactly in the case where labor income innovations are perfectly positively correlated with unexpected stock returns.

The paper is organized as follows. Section I specifies the model and explains the approximate analytical solution method. Sections II and III discuss the optimal consumption and portfolio allocations of retired and employed investors. Section IV presents a calibrated example that illustrates the properties of the solution using empirically plausible parameter values. Section V uses the model to understand the effects of increasing labor income risk on savings and portfolio choice. Section VI reports results on the numerical accuracy of the approximate analytical solution. Finally, Section VII concludes.

I. A Dynamic Model of Portfolio Choice and Savings

A. Specification of the Model

A.1. Assumptions on Labor Income and Human Capital

Labor income is uninsurable. That is, the investor cannot write claims against her future labor earnings. I also assume that labor income is exogenous or, equivalently, that labor is supplied inelastically and there is no endogenous human capital accumulation. However, the assumptions on labor income and investment opportunities (specified further below) imply that labor income is perfectly correlated with the value of human capital, and the return on human capital is given by the percent change in labor income (Fama and Schwert (1977)).

There are two states for labor income that occur with constant probabilities, employment and retirement. The employment state occurs with probability π^e. In this state, the individual receives a realization of the income process. The retirement state occurs with probability $\pi^r = 1 - \pi^e$, with $0 < \pi^r < 1$, and it is irreversible: If this state occurs, labor income is set to zero forever. After retirement, the individual faces each period a constant probability of death π^d.

In the employment state, labor income is subject to permanent, multiplicative shocks. One simple process that captures these shocks is the following:

$$Y_{t+1} = Y_t \exp\left\{g + \xi_{t+1}\right\}, \tag{1}$$

where $\xi_{t+1} \sim NIID(0, \sigma_\xi^2)$ and independent of the state for labor income. This model is based on Carroll's (1997) model for labor income, except that it ignores transitory shocks to labor income. The empirical evidence available suggests that individual labor income is subject to both permanent and transitory shocks (MaCurdy (1982), Abowd and Card (1989), Carroll (1992)). However, a version of this model (Viceira (1998)) that includes transitory shocks shows that they have very little impact on the relationship between retirement horizon and asset allocation.

A.2. Assumptions on the Investment Opportunity Set

To focus on the effects of labor income uncertainty on portfolio choice, and to keep the analysis simple, I assume that the investment opportunity set is constant. There are two tradable financial assets: Asset 1 is risky, with one-period log (continuously compounded) return given by $r_{1,t+1}$; asset f is riskless, with constant log return given by r_f. I refer to asset 1 as stocks, and to asset f as cash. The expected excess log return on the risky asset is constant, with

$$\mathrm{E}_t\left[r_{1,t+1} - r_f\right] = \mu.$$

The unexpected log return on the risky asset, denoted by u_{t+1}, is conditionally homoskedastic and serially uncorrelated, though it may be contemporaneously correlated with innovations in log labor income:

$$\mathrm{Var}_t\left(u_{t+1}\right) = \sigma_u^2, \qquad \mathrm{Cov}_t\left(u_{t+1}, \xi_{t+1}\right) = \sigma_{\xi u}\ . \tag{2}$$

Finally, I also assume that innovations in the risky asset return are independent of the state for labor income. This assumption is innocuous if we interpret the retirement state literally as retirement for age or permanent disability reasons, because we may safely regard these episodes as completely idiosyncratic, i.e. independent of the business cycle.[5]

A.3. Assumptions on Preferences and Time Horizon

The investor's preferences are described by a standard, time separable, power instantaneous utility function over consumption:

$$U\left(C_t\right) = \frac{C_t^{1-\gamma}}{1-\gamma}, \tag{3}$$

where $\gamma > 0$ is the coefficient of relative risk aversion and $1/\gamma$ is the elasticity of intertemporal substitution of consumption. The investor's discount factor is δ. She is infinitely-lived. However, a positive probability of retirement effectively shortens her horizon in the employment state to $1/\pi^r$ periods, and a positive probability of death makes her expected lifetime after retirement equal to $1/\pi^d$ periods.

A.4. The Problem

The investor faces the following intertemporal optimization problem. She wants to

$$\max_{\{C_t,\alpha_t\}_{t=0}^{\infty}} \mathrm{E}\left[\sum_{t=0}^{\infty} \delta^t U\left(C_t\right) \mid Y_0, u_0, \xi_0, \epsilon_0\right], \tag{4}$$

subject to the intertemporal budget constraint

$$W_{t+1} = \left(W_t + Y_t - C_t\right) R_{p,t+1}, \tag{5}$$

where W_{t+1} is financial wealth, defined as the value at the beginning of $(t+1)$ of financial assets carried over from period t, and $W_t + Y_t - C_t$ is savings, defined as the value of financial assets held at time t after receiving a realization of labor income and substracting the spending in consumption for that period. The one-period return on savings is given by

$$R_{p,t+1} = \alpha_t \left(R_{1,t+1} - R_f\right) + R_f \ , \tag{6}$$

where $R_{1,t+1} = \exp\{r_{1,t+1}\}$, $R_f = \exp\{r_f\}$, and α_t is the proportion of savings invested in the risky asset at time t.

Depending on the realized state for Y_t, there are two possible sets of first order conditions for this intertemporal optimization problem. They imply the following pair of sets of Euler equations:

$$1 = \mathrm{E}_t \left\{ \left[\pi^e \delta^e \left(\frac{C^e_{t+1}}{C^e_t} \right)^{-\gamma} + (1 - \pi^e)\, \delta^r \left(\frac{C^r_{t+1}}{C^e_t} \right)^{-\gamma} \right] R_{i,t+1} \right\}, \tag{7}$$

and

$$1 = \mathrm{E}_t \left[\delta^r \left(\frac{C^r_{t+1}}{C^r_t} \right)^{-\gamma} R_{i,t+1} \right], \tag{8}$$

where $i = 1,\ f,\ p$, $\delta^e \equiv \delta$, and $\delta^r = (1 - \pi^r)\delta$. Note that the effect of introducing an expected finite lifetime after retirement is equivalent to modifying the investor's discount factor in the retirement state. Throughout the paper, a superscript $s = e, r$ on a variable denotes the labor income state that determines the value of that variable.[6]

B. Log-Linear Approximate Solution Method

There are no exact closed-form solutions for this intertemporal optimization problem, except when the retirement state occurs. In this case, the problem reduces to that of choosing optimal consumption and portfolio policies when the investment opportunity is constant, a classical problem in financial economics for which there is a well-known closed-form solution (Samuelson (1969), Merton (1969)).

To solve for the optimal policies in the employment state we can use numerical methods, in the spirit of the modern consumption literature (Deaton (1991), Carroll (1996, 1997), Gourinchas and Parker (1995)) and the infinite horizon models of portfolio choice with uninsurable labor income (Koo (1995), Heaton and Lucas (1997)). Alternatively, we can follow Campbell and Viceira (1999) and find an approximation to the problem that can be solved using the method of undetermined coefficients. This paper presents a solution to the problem based on this methodology. Section VI compares this solution with a numerical solution.

The solution method proceeds in three steps. First, it replaces the budget constraint and the Euler equations of the problem with log-linear approximations around the stationary state; in particular, it approximates the Euler equations using a second order expansion so that second-moment effects such as precautionary savings effects are accounted for in the solution. Second, it looks for optimal consumption and portfolio policies that verify these log-linear equations. Finally, it identifies the coefficients of the optimal policies using the method of undetermined coefficients.

We can log-linearize the problem because the assumptions on preferences, labor income, and the investment opportunity set ensure that along the optimal path financial wealth (W_t), savings ($W_t + Y_t - C_t$) and consumption are strictly positive, and the state variable, the log ratio of financial wealth to labor income (W_t^e / Y_t) is stationary.[7] Because the marginal utility of consumption approaches $+\infty$ when consumption approaches zero, and each period there is a strictly positive probability of a permanent zero-labor income event, the investor optimally chooses portfolio and consumption-savings rules to ensure strictly positive consumption next period. This implies that we must have $W_{t+1}^e > 0$ and $W_t + Y_t - C_t^e > 0$, so that the log of these objects is defined.[8] Similarly, in the retirement state we must also have $W_{t+1}^r > 0$ and $W_t - C_t^r > 0$. That is, there is no borrowing to finance current consumption along the optimal path.

The log-linear approximation to the intertemporal budget constraint in the employment state is given by[9]

$$w_{t+1}^e - y_{t+1} \approx k^e + \rho_w^e (w_t - y_t) - \rho_c^e (c_t^e - y_t) - \Delta y_{t+1} + r_{p,t+1}^e, \tag{9}$$

where lowercase letters denote variables in logs, Δ denotes the first-difference operator and k^e, ρ_w^e, and ρ_c^e are log-linearization constants given in Appendix A. These constants depend only on the long-term means of the log wealth-income ratio and the log consumption-income ratio, around which the log-linear budget constraint is approximated.

In the retirement state we have $Y_t = 0$ for all t, and the intertemporal budget constraint reduces to $W_{t+1}^r = (W_t - C_t^r)R_{p,t+1}^r$. Following Campbell (1993) we can write this budget constraint in log-linear form as:

$$w_{t+1}^r - w_t = k^r - \rho_c^r \left(c_t^r - w_t\right) + r_{p,t+1}^r, \tag{10}$$

where ρ_c^r and k^r are log-linearization constants given in Appendix A. It is important to note that equation (10) holds exactly, because the optimal consumption-wealth ratio is constant in the retirement state (see Proposition 1).

Campbell and Viceira (1999) derive an approximate expression for the log return on financial wealth which holds exactly in continuous time:

$$r_{p,t+1}^s = \alpha_t^s \left(r_{1,t+1} - r_f\right) + r_f + \frac{1}{2}\alpha_t^s \left(1 - \alpha_t^s\right)\sigma_u^2, \tag{11}$$

where $s = e, r$. Combining equations (9) or (10) and (11) we obtain a log budget constraint which is linear in both asset returns and labor income.

The Euler equations (7) and (8) are highly non-linear. However, we can find log-linear approximations for them. The Euler equation in the retirement state takes the following log-linear form:

$$0 = \log \delta^r - \gamma \mathrm{E}_t \left[c_{t+1}^r - c_t^r\right] + \mathrm{E}_t \left[r_{i,t+1}\right] + \frac{1}{2}\mathrm{Var}_t \left[r_{i,t+1} - \gamma\left(c_{t+1}^r - c_t^r\right)\right], \tag{12}$$

for $i = 1,\ f,\ p$. This log Euler equation holds exactly because consumption growth and returns are jointly conditionally lognormal in the retirement state.

The log-linear approximation to the Euler equation in the employment state is derived in

Appendix A. It is given by

$$0 = \sum_{s=e,r} \pi^s \left\{ \log \delta^s - \gamma \mathrm{E}_t \left[c^s_{t+1} - c^e_t \right] + \mathrm{E}_t \left[r_{i,t+1} \right] + \frac{1}{2} \mathrm{Var}_t \left[r_{i,t+1} - \gamma \left(c^s_{t+1} - c^e_t \right) \right] \right\}, \quad (13)$$

for $i = 1,\ f,\ p$. The log-linear Euler equation (13) is equal to the probability-weighted sum of the log-linear Euler equations we would obtain if the next-period state for labor income were known in advance.

We can now use this apparatus to characterize the investor's approximate optimal consumption and portfolio rules in each state. Since the retirement state is irreversible, the optimal policies in this state are independent of those in the employment state. By contrast, in the employment state the investor must take into account her retirement horizon when deciding how much to save and how much of her savings to put in the risky asset. Therefore, I first state the optimal policies in the retirement state and then proceed to state those in the employment state.

II. Optimal Consumption and Portfolio Choice of Retired Investors

When the retirement state occurs, labor income is set to zero forever and the investor must live off her financial wealth under constant investment opportunities. The investor then faces a decision problem classical in financial economics, for which an exact closed form solution exists (Samuelson (1969), Merton (1969)). In this case the solution method described above produces the exact solution up to the discrete-time approximation to the log return on wealth:

Proposition 1 *The optimal log consumption and portfolio rules in the retirement state are*

$$c^r_t = b^r_0 + b^r_1 w_t, \quad (14)$$

$$\alpha^r = \frac{\mu + \sigma_u^2/2}{\gamma b_1^r \sigma_u^2}, \tag{15}$$

where $b_1^r = 1$ *and*

$$b_0^r = \log\left(1 - \exp\left\{\left[\left(\frac{1}{\gamma} - b_1^r\right) \mathrm{E}\left[r_{p,t+1}^r\right] + \frac{1}{\gamma}\log\delta^r + \frac{1}{2\gamma}\left(1 - b_1^r\gamma\right)^2 \mathrm{Var}\left(r_{p,t+1}^r\right)\right]\right\}\right). \tag{16}$$

Proof. See Appendix B. ■

Proposition 1 shows that optimal log consumption of retired investors is a linear function of wealth. The slope of this function, b_1^r, is the elasticity of consumption with respect to financial wealth. This elasticity is exactly one in the retirement state, implying a constant consumption-wealth ratio. Equation (16) gives the log of this ratio.

The wealth elasticity of consumption also enters equation (15), which describes the optimal portfolio policy in the retirement state, α^r. I choose not to substitute $b_1^r = 1$ into the equation for α^r to emphasize the importance of this elasticity for portfolio choice. In particular, the product of the wealth elasticity of consumption times the relative risk aversion of the instantaneous utility function determines the relative risk aversion of the value function. This follows immediately from the familiar textbook equation $-J_{WW}/J_W = -(U_{CC}/U_C)C_W$, originally derived by Merton (1973):[10]

$$-\frac{J_{WW}}{J_W}W = \left(-\frac{U_{CC}}{U_C}C\right)\left(C_W\frac{W}{C}\right) \equiv \gamma b_1. \tag{17}$$

(Here I follow standard notation, so that J denotes the value function and subscripts denote partial derivatives).

Equation (15) shows the well-known result that the optimal fraction of savings invested in the risky asset is proportional to the risk premium and the reciprocal of the relative risk aversion of the value function. Since the investment opportunity set is constant, and the retired investor does not face labor income risk, there is no hedging component in α^r. For this reason, this portfolio rule is often called the "myopic portfolio rule" in the literature on portfolio choice.

The expression for b_0^r in equation (16) characterizes the constant consumption-wealth ratio of retired investors. It shows that consumption relative to wealth is increasing in the investor's discount rate $(-\log\delta)$. It is also increasing in the expected return on wealth for retired investors whose elasticity of intertemporal substitution $(1/\gamma)$ is smaller than the wealth elasticity of consumption. For these investors, the income effect of an increase in the expected return on financial wealth dominates the substitution effect and they choose to consume the extra wealth. By contrast, for those investors for whom $1/\gamma > b_1^r$, the substitution effect dominates, and they choose to save more and take advantage of the improvement in the investment opportunity set. Finally, there is also a precautionary savings effect on b_0^r given by the variance term in equation (16). The magnitude of this effect is proportional to the coefficient of relative risk aversion and the wealth elasticity of consumption.

III. Optimal Consumption and Portfolio Choice of Employed Investors

Given the optimal rules in the retirement state, we can now solve for the optimal rules in the employment state.

Proposition 2 *The approximate optimal log consumption and portfolio rules in the employment state are*

$$c_t^e - y_t = b_0^e + b_1^e \left(w_t - y_t \right), \tag{18}$$

$$\alpha^e = \frac{\mu + \sigma_u^2/2}{\gamma \bar{b}_1 \sigma_u^2} - \frac{\pi^e \left(1 - b_1^e \right)}{\bar{b}_1} \cdot \frac{\sigma_{\xi u}}{\sigma_u^2} \tag{19}$$

where $0 < b_1^e < 1$, b_0^e *is a constant given in (20) and* $\bar{b}_1 = \pi^e b_1^e + (1 - \pi^e)$.

Proof. See Appendix B. ■

Proposition 2 shows that the optimal portfolio rule in the employment state depends on the parameters of the consumption function, so it is not possible to understand optimal portfolio choice without first understanding optimal consumption. Therefore, I start by characterizing optimal consumption in the employment state.

A. Consumption of Employed Investors

Proposition 2 shows that the optimal log consumption-income ratio is a linear function of the log financial wealth-income ratio, the state variable of the problem. The slope of this equation (b_1^e) is the elasticity of consumption with respect to financial wealth, while $(1 - b_1^e)$ is the elasticity of consumption with respect to labor income. Because $0 < b_1^e < 1$, optimal consumption in the employment state is globally strictly quasiconcave—and strictly concave in each of its arguments.[11] This implies that the marginal propensity to consume out of wealth is decreasing in wealth, and the marginal propensity to consume out of labor income is decreasing in income.

The exponential of the intercept of the log consumption function (b_0^e) is a factor that scales the optimal consumption-income ratio up or down relative to the level determined by the current wealth-income ratio. This intercept is equal to

$$\begin{aligned} b_0^e \;=\; & -\frac{1}{k_1}\left[\left(\frac{1}{\gamma}-\bar{b}_1\right)\mathrm{E}\left[r_{p,t+1}^e\right]+\frac{1}{\gamma}\sum_{s=e,r}\pi^s\log\delta^s\right. \\ & \left.+\frac{1}{2\gamma}\,\mathrm{V}^e-\pi^e\left(1-b_1^e\right)g-\left(1-\pi^e\right)b_0^r-k^e\right], \end{aligned} \tag{20}$$

where V^e and k_1 are positive constants given in equations (B15) and (B20) in Appendix B. V^e is a precautionary savings term involving both the variance of the portfolio return and the variance of labor income growth.

Equation (20) shows that b_0^e is affected by the same factors as b_0^r. However, there are some important differences. First, b_0^e is an increasing function of g, the expected rate of growth in labor income. An investor who expects her future labor income to grow at a high rate can afford to consume a larger fraction of her current resources than an investor who does not. The effect of an increase in labor income growth on consumption is proportional to the income elasticity of consumption. Second, since $\bar{b}_1 < b_1^r$, the substitution effect of an increase in the expected return on wealth dominates the income effect at lower values of the elasticity of intertemporal substitution. Finally, precautionary savings also depend on the variance of labor income growth, through V^e.

The log financial wealth-labor income ratio determines the dynamics of the log consumption-labor income ratio in the employment state. Appendix B shows that $(w_t^e - y_t)$ follows a stationary process along the optimal path. This stationarity property is important, because the log budget constraint is log-linearized around the unconditional expectations of $(c_t^e - y_t)$ and $(w_t^e - y_t)$.

B. Portfolio Choice of Employed Investors

Proposition 2 also characterizes α^e, the optimal portfolio allocation to stocks for employed investors. Equation (19) shows that α^e has two components. The first component is similar to the optimal portfolio demand of retired investors: It is proportional to the risk premium on the risky asset, and inversely proportional to the product of the relative risk aversion coefficient times the wealth elasticity of consumption. Section II notes that this product determines the relative risk aversion of the value function. However, the relevant wealth elasticity here is $\bar{b}_1$, the average wealth elasticity of consumption across states.[12] Since $\bar{b}_1 < b_1^r$, this implies that employed investors are effectively less risk averse than retired investors. This result has strong implications for the optimal portfolio demand of employed investors versus retired investors:

Proposition 3 *When labor income risk is independent of asset return risk, employed investors hold a larger fraction of their savings in the risky asset than retired investors:*

$$\alpha^e > \alpha^r.$$

Moreover, $\lim_{\pi^e \to 0} \alpha^e = \alpha^r$.

Proof. When $\sigma_{\xi u} = 0$, $\alpha^e = \alpha^r / \overline{b}_1$. The proposition follows directly from the fact that $\overline{b}_1 = \pi^e b_1^e + (1 - \pi^e) b_1^r < b_1^r$ and $\overline{b}_1 \to b_1^r$ as $\pi^e \to 0$. ■

The intuition for this result is simple. Because employed investors have an alternative source of income, namely labor income, a one percent negative shock in their financial wealth does not fully translate into a one percent decrease in their consumption growth, while it does so for retired investors. For this reason, employed investors are willing to assume riskier portfolios than retired investors.

Proposition 3 extends the results of Bodie, Merton, and Samuelson (1992). They show that with non-stochastic, exogenous labor income, human capital is equivalent to an implicit investment in the riskless asset. Thus the share of financial wealth invested in risky assets must be larger than the share of total wealth (financial wealth plus human capital), and must decrease as the horizon shortens and human capital is depleted. Proposition 3 shows that a qualitatively similar horizon effect holds with risky, nontradable labor income, provided that this risk is idiosyncratic.[13] However, the calibration exercise in Section IV below shows that this horizon effect is less pronounced than in the riskless labor income case, because increasing idiosyncratic labor income risk shifts downwards the optimal portfolio allocation to risky assets. Therefore, the riskiness of labor income makes the investor behave *as if* she had an implicit investment in the riskless asset with a value *below* the unadjusted expected present value of labor income.

Proposition 3 also provides some intuition on why general equilibrium models of asset prices with idiosyncratic labor income have difficulties reproducing the historical equity premium when investors are assumed to have low risk aversion. This proposition shows that adding risky idiosyncratic labor income increases the demand for stocks relative to the standard model with no labor income. Therefore, any model in which stocks are held by households who also have labor income is likely to make the Equity Premium Puzzle of Mehra and Prescott (1985) worse. In a recent paper, Constantinides, Donaldson, and Mehra (1998) show that a general equilibrium, overlapping generations model with idiosyncratic labor income shocks can reproduce the historical equity premium only if young investors are prevented from investing in stocks. This paper provides intuition on why this constraint is necessary: Otherwise they would be heavily invested in stocks, hence reducing the equilibrium equity premium.

The second component of α^e is non-zero whenever the return on the risky asset is correlated with labor income. Therefore, it represents the hedging component of the optimal allocation to the risky asset. This hedging component is

$$-\frac{\pi^e\left(1-b_1^e\right)}{\bar{b}_1}\cdot\frac{\sigma_{\xi u}}{\sigma_u^2}, \tag{21}$$

which is proportional to the regression hedge ratio for labor income,[14] weighted by the income elasticity of consumption. The sign of the hedging demand is opposite to the sign of the correlation between shocks to labor income and unexpected returns on the risky asset. If innovations to labor income are negatively correlated with innovations to stock returns, stocks are desirable because they offer a good hedge against unfavorable innovations in labor earnings—since negative shocks to labor income are going to be accompanied, in general, by positive innovations in stock returns. This creates a positive hedging demand for stocks. If the correlation is positive, the opposite is true.

IV. A Calibrated Example

This section illustrates the analytical findings presented in Sections II and III using a calibration exercise. This exercise is based on an empirically plausible parameterization of the process for asset returns and the process for individual labor income. Section IV.A describes the choice of parameter values, and Section IV.B presents results for a wide array of values for the preference parameters and the retirement horizon.

The optimal policies in the employment state depend on ρ_w^e and ρ_c^e, the log-linearization constants in the budget constraint (9). But these constants are endogenous parameters, because they are monotonic functions of the mean financial wealth-labor income ratio. In fact, equations (18) and (19) for the optimal policies and the equations for ρ_w^e and ρ_c^e (given in Appendix A) define a nonlinear mapping of ρ_w^e and ρ_c^e onto themselves.

I solve for the fixed point in this mapping using a simple recursive algorithm: For a given set of parameter values, I first compute $\{b_0^r, \alpha^r\}$ (which do not depend on any loglinearization constants) and I choose initial values for the log-linearization constants.[15] Next I compute $\{b_0^e,\ b_1^e,\ b_2^e,\ \alpha^e\}$ and I use them to obtain a new set of values for ρ_w^e and ρ_c^e. From this new set of log-linearization constants we can find new values for the optimal policies in the employment state. This recursion continues until convergence is achieved. The convergence criterion I use is that the sum of the absolute deviations of each log-linearization constant in two consecutive iterations is less than 10^{-4}. Convergence is usually achieved in less than 10 iterations.

A. Baseline Parameter Values

The values for the parameters describing the investment opportunity set are based on the historical estimates of the average equity premium, the short-term real interest rate and the variance of excess stock returns in the US stock market. Table 8.1 in Campbell, Lo and MacKinlay (1997) report these estimates. The return on the riskless asset R_f is set to two percent per year. The standard deviation of unexpected log excess returns (σ_u) is set to 18 percent per year. The log excess return on the risky asset (μ) is set to 4.21 percent per year to match the historical six percent excess return on equities.

The values for the parameters describing the labor income process are based on the most recent microeconometric estimates of equation (1) available. The baseline value for the standard deviation of innovations in log labor income (σ_ξ) is set to 10 percent per year. Expected log income growth (g) is set so that expected income growth $\mathrm{E}_t[Y_{t+1}/Y_t]$ equals three percent per year. These values are consistent with estimates reported in Chamberlain and Hirano (1999) and Carroll and Samwick (1995) using data on individuals.[16]

The only remaining parameter describing the stochastic structure of the model is the correlation between innovations in log labor income and innovations in stock returns. I consider two values for $\text{Corr}_t(r_{1,t+1}, \Delta y_{t+1})$, zero and 25 percent. The zero correlation value represents the important benchmark case of idiosyncratic labor income risk, while the 25 percent correlation value is useful to illustrate the interaction of hedging, retirement horizon, and risk aversion on optimal portfolio demand.

B. Horizon and Hedging Effects on Portfolio Choice and Savings

Tables I and II report the optimal policies for relative risk aversion coefficients γ=$\{2, 3, 5, 8, 10, 12\}$ and expected number of years until retirement $(1 - \pi^e)^{-1}$=$\{35, 30, 25, 20, 15, 10, 5\}$. Panel A in each table presents results when labor income is idiosyncratic, and Panel B presents results when there is a 25 percent correlation between unexpected stock returns and shocks to labor income. The expected lifetime after retirement is set to 10 years, and the time preference rate is set to 10 percent per year, so that $\delta = 1/1.10$. I choose this rate of time preference to ensure that the investor is "impatient," in the sense that she would optimally choose not to save in the absence of uncertainty in labor income and asset returns—given the values I assume for g, μ and r_f. I also show the effect of varying δ on portfolio choice and savings at the end of this section.

Panel A in Table I reports α^e and α^r when labor income risk is uncorrelated with stock market risk. As predicted by Proposition 3, the share of stocks in savings is systematically larger in the employment state than in the retirement state. Moreover, this share also decreases as the expected number of years until retirement falls. To understand this pattern, it is useful to reiterate the intuition for Proposition 3: When labor income risk is idiosyncratic, non-retired investors choose their portfolios as if their human capital resembles a forced investment on the riskless asset. This investment is larger for investors with longer horizons, because expected future labor income (relative to its current level) is increasing in the expected retirement horizon.[17] Thus it is optimal for investors with longer horizons to hold a larger fraction of their financial wealth in stocks. Panel A also shows that investors go long in both the riskless asset and stocks, except those with low risk aversion, who short the riskless asset. This is a reflection of the high historical equity premium.

The interaction between hedging, retirement horizons, and risk aversion is illustrated in Panel B of Table I. This panel shows that a small, positive correlation between labor income risk and stock market risk has significant negative effects on the optimal portfolio demand for stocks. As in Panel A, the fraction of financial wealth invested in stocks goes down with risk aversion. However, the ratio of the hedging component to total demand actually increases (in absolute value) with relative risk aversion. These effects are more pronounced the longer is the expected retirement horizon. For highly risk averse investors ($\gamma = 12$), it may even offset the horizon effect, so that the optimal portfolio share of the risky asset stays roughly the same as the expected retirement horizon falls.[18] Even for modestly risk averse investors, hedging represents a significant fraction of total portfolio demand at long retirement horizons. For highly risk averse investors with long retirement horizons, the hedging component of demand represent more than 50 percent of the total demand for stocks.

Table II reports the exponentiated long-run mean of the log financial wealth-permanent income ratio along the optimal path in the employment state. This mean describes the target assets, relative to permanent income, that an employed investor would optimally accumulate in the employment state. The optimal target is in general large. It increases as the expected retirement horizon falls and it is also increasing in γ. Retirement and precautionary saving motives explain this pattern of variation across the table. Investors facing a high probability of retirement want to accumulate more assets than investors facing a low probability. But investors with long retirement horizons have a stronger precautionary saving motive than investors with short horizons, because for them a negative shock to labor income has a larger downward effect on expected future labor income. Equation (B15) shows that this effect is compounded by γ, which represents prudence as well as risk aversion. Hence highly prudent investors with long retirement horizons have a larger long-term target for financial wealth than investors who are less prudent.

Panel B in Table II shows that positive correlation between shocks to labor income and unexpected returns slightly lowers the financial wealth-permanent income target ratio. A positive correlation results in a more conservative portfolio policy, which in turn reduces $\text{Var}_t(r^e_{p,t+1})$, and the precautionary saving motive against negative shocks to financial wealth.

Table III explores the interaction between labor income growth and expected retirement horizon. It reports optimal portfolio allocations and long-run asset holdings for employed investors with $\gamma = 3$, expected labor income growth rates ($\text{E}_t[Y_{t+1}/Y_t]$) between zero percent and eight percent per year, and expected retirement horizons from 35 to 5 years.[19] This analysis is interesting because expected labor income growth is not constant over the life-cycle. Cocco, Gomes, and Maenhout (1997) and Gakidis (1998) find that labor income growth typically presents an inverted-U shape as a function of the retirement horizon. We can easily explore the implications of this pattern in labor income growth within this model.

Table III shows that α^e is an increasing function of $\text{E}_t[Y_{t+1}/Y_t]$ at any expected retirement horizon. However, if we compare the optimal allocations for investors with long and short retirement horizons and low income growth with the optimal allocations of investors with medium retirement horizons and high income growth, we find an inverted-U shape for α^e. This is the shape that Cocco, Gomes, and Maenhout (1997) and Gakidis (1998) find in their numerical calibrations of the standard life-cycle model. The intuition behind this pattern is twofold: First, an inverted-U pattern in labor income growth produces a similar pattern in the value of human capital relative to the current level of labor income under constant discounting;[20] second,we have shown that investors behave as if human capital is closer to an implicit investment on the riskless asset than to an investment on stocks when labor income risk is idiosyncratic. This implies that α^e is going to present the same pattern as the value of human capital across retirement horizons.

Finally, the effect on portfolio choice of varying δ is shown in Figure 1 for an investor with $\gamma = 3$. Increasing δ (reducing the time preference rate $-\log\delta$) reduces α^e: More patient investors invest a smaller fraction of their savings in stocks than less patient investors. They invest proportionally less in the risky asset as a result of their savings behavior. More patient investors accumulate more financial wealth[21] and for them labor income becomes relatively less important. Thus they are more similar to retired investors.

V. The Effect of Income Uncertainty on Portfolio Choice and Savings

The dynamic nature of this model gives us an interesting framework to explore the effects of an uncompensated increase in labor income risk on portfolio choice and savings, and the approximate solution method allows us to explore these implications for a wide range of parameter values. But we first need a working definition of an uncompensated increase in labor income risk. Since labor-income cannot be negative, it seems natural to consider the effect on portfolio choice of a mean-preserving increase in the variance of labor income.[22] To emphasize the characterization of labor income as unavoidable risk, I assume throughout this section that labor income growth is uncorrelated with asset returns, so that $\alpha^e = (\mu + \sigma_u^2/2)/\gamma\bar{b}_1\sigma_u^2$.

Figures 2 and 3 summarize the effects on portfolio choice and savings of a mean-preserving increase in the variance of labor income. They plot α^e and $\exp\{\mathrm{E}[w_t^e - y_t^P]\}$ as a function of the standard deviation of labor income growth (σ_ξ) for $\gamma = \{3, 8\}$ and an expected retirement horizon of 25 years. I consider mean-preserving increases of σ_ξ in the interval $[0, 0.20]$.[23]

Figure 2 shows that α^e declines and slowly approaches α^r, the optimal allocation to stocks of a retired investor, as the variance of labor income growth increases.[24] This effect comes through a positive effect of σ_ξ on the wealth elasticity of consumption or, equivalently, a positive effect of σ_ξ on the relative risk aversion of the value function (see equation (17)). Kimball (1992) shows that labor income risk increases the absolute risk aversion of the value function because it increases the marginal propensity to consume out of wealth. Figure 1 shows that it also increases the relative risk aversion of the value function.

Figure 3 illustrates the effect of the increase in the volatility of labor income on the optimal level of savings. A mean-preserving increase in σ_ξ has a positive effect on asset accumulation. As the level of savings increases, labor income becomes less important relative to financial wealth as a source of consumption for the investor. Hence, the employed investor becomes more like a retired investor and behaves like her when making portfolio decisions.

Figures 2 and 3 also have interesting implications about the relative effect of labor income risk on savings and portfolio choice. When γ is high or moderately high, α^e appears to be more responsive to mean-preserving changes in the variance of labor income than savings. When γ is low, however, this pattern may reverse.

VI. Numerical Accuracy of the Solution

The model presented in this paper has no exact solution, except when labor income innovations are perfectly correlated with return innovations, and the ratio of proportionality between their standard deviations equals α^r (Bodie, Merton, and Samuelson (1992)). In this special case, the approximate analytical solution method also delivers the exact solution.[25] In all other cases we can assess the accuracy of the approximate solution by comparing it with a solution to the model using numerical methods.

The Bellman equation for the problem in the employment state is

$$J^e\left(W,Y\right) = \max_{\alpha^e, C^e} \left\{ U\left(C^e\right) + \mathrm{E}_t\left[\pi^e \delta^e J^e\left(W',Y'\right) + \left(1-\pi^e\right)\delta^r J^r\left(W'\right)\right]\right\}, \tag{22}$$

where $J^e(\cdot,\cdot)$ and $J^r(\cdot,\cdot)$ denote the value function in the employment and retirement states, and I simplify notation by eliminating time subscripts and using an apostrophe to denote variables evaluated at time $(t+1)$. The policy functions in the retirement state imply that $J^r\left(W\right) = \left(1-\gamma\right)^{-1}\exp\left\{\left(1-\gamma\right)b_0^r\right\}W^{1-\gamma}$, where b_0^r is given in equation (16). This function is homogeneous of degree $(1-\gamma)$. The value function in the employment state also has this property (Koo (1995)). This allows us to reduce the number of state variables from two (W and Y) to one (the ratio W/Y).

The numerical method used here is an adaptation of the method described in Campbell, Cocco, Gomes, Maenhout, and Viceira (1998). This method parameterizes the policy functions α^e and $c^e - y = \log(C^e/Y)$ as polynomials of the state variable $x = \log(W/Y)$,

$$\begin{aligned} \alpha^e &= a_0 + a_1 x + ... + a_p x^p, \\ c^e - y &= b_0 + b_1 x + ... + b_q x^q, \end{aligned} \tag{23}$$

and uses a Newton-Raphson algorithm to find the coefficients of these polynomials that maximize equation (22). Appendix C describes in full detail the solution algorithm. The approximate analytical solution is also polynomial, with $p = 0$ and $q = 1$.

To solve the model numerically we need a grid of values for the state variable. There is no obvious way to choose this grid, because the state variable is itself endogenous. However, the approximate analytical solution is helpful to choose this grid. I choose an interval for $W/Y = \exp\{x\}$ centered at $\exp\{\overline{x}\}$, where $\overline{x}$ is the unconditional mean of x implied by the approximate analytical solution (see Appendix B.3). The bounds of the interval are ± 3 conditional standard deviations away from this point.

Figures 4 and 5 report the numerical and the approximate analytical solution for α^e and C/Y for $\gamma = 3$, $\sigma_\xi = 10$ percent, $\delta = 1/1.10$ and expected retirement horizons of 5, 20 and 30 years. The dashed horizontal lines correspond to the approximate closed form solution, and the solid lines to the numerical solution. This solution assumes $p = 1$ and $q = 3$.

Figure 4 shows that the numerical solution for α^e is a slowly decreasing function of x, the log financial wealth-income ratio. As financial wealth becomes large relative to income, the employed investor becomes more like a retired investor, and so does her optimal portfolio policy. That is, α^e approaches α^r. The numerical solution tells us that this convergence is slow. This is why the (constant) approximate analytical solution is relatively close to the numerical solution. The numerical solution also reproduces the decreasing pattern in the optimal allocation to stocks as the expected retirement horizon shortens. The plots in Figure 5 suggest that the approximate analytical solution for the consumption policy also seems to work reasonably well.

VII. Conclusion

This paper has shown under which conditions a stylized dynamic model of rational portfolio choice and consumption with risky labor income and retirement supports the popular notion that people should invest more in stocks when they are working than when they are retired. When labor income risk is idiosyncratic, this advice is always welfare-maximizing, and the optimal fraction of savings invested in the risky asset is positively related to the retirement horizon of the investor. This effect on portfolio choice is determined by the investor's elasticity of consumption with respect to financial wealth which, together with the relative risk aversion of the instantaneous utility over consumption, determines the relative risk aversion of the value function. Investors subject to labor income risk have wealth elasticities strictly smaller than one, while retired investors have unit wealth elasticities. Thus the impact of a negative one percent shock to the risky asset return on consumption growth is always smaller than one percent for an employed investor, while it is exactly one percent for a retired investor. Because employed investors have an additional source of income, they can afford more aggressive portfolio policies than retired investors.

If labor income innovations are correlated with unexpected returns on a risky asset, a rational investor modifies her optimal portfolio to take advantage of the hedging properties of the risky asset derived from this non-zero correlation. In particular, if this correlation is positive, it is optimal for an employed investor to have a negative hedging demand for the risky asset, since shorting the risky asset hedges consumption from unexpected falls in labor income. A large enough positive correlation may even cause the hedging component of portfolio demand to offset the non-hedging part, and it may be optimal for employed investors to invest a smaller fraction of their savings in risky assets than retired investors. This paper shows that for highly risk averse investors, a small positive correlation is enough to create this result.

This paper provides intuition about the finding in the standard life-cycle model that the fraction of savings invested in the risky asset is increasing with age in the early part of the life-cycle (Cocco et al. (1997) and Gakidis (1998)). It shows that the optimal portfolio allocation to stocks is positively related to both expected labor income growth and expected retirement horizon when labor income risk is idiosyncratic. That is, investors shift their financial wealth towards stocks when their human capital is large. In the standard life-cycle model, labor income growth exhibits a hump-shaped form, which induces a similar pattern in the ratio of human capital to labor income and in the fraction of savings optimally invested in stocks.

Finally, this paper has also investigated the effects of an increase in labor income risk on savings and optimal portfolio holdings of the risky asset. It finds that a mean-preserving increase in the variance of labor income growth reduces the investor's willingness to hold the risky asset and increases her willingness to save. Moreover, the level of savings is more responsive to changes in labor income risk than portfolio demand for risky assets when investors are moderately or highly risk averse. That is, when wealth accumulation is endogenous, the investor reacts to an increase in labor income risk primarily by increasing her savings and only secondarily by reducing her exposure to portfolio risk. This paper also shows that the optimal portfolio share of the risky asset of an employed investor approaches the optimal portfolio share of a retired investor as the volatility of labor income growth increases. The increase in labor income volatility has two effects: First, it makes labor income look more like the risky asset than the riskless asset, so that the investor is less willing to assume portfolio risk; second, it causes an increase in the level of savings that reduces the importance of labor income relative to financial wealth as a source of consumption for the employed investor, who becomes more like a retired investor when making her portfolio decisions.

The approach of this paper can be extended in several directions. First, we could consider multiple risky assets in the model, to investigate whether labor income risk and retirement can solve the "asset allocation puzzle" of Canner, Mankiw, and Weil (1997). They find that the ratio of bonds to equities in the optimal portfolio increases with the coefficient of relative risk aversion, which is consistent with conventional portfolio advice but inconsistent with static mean-variance analysis. Second, it would be interesting to allow for an unemployment state in the model. In a further extension of the model not shown here for space considerations, I have considered an additional zero-labor-income state in which labor income is set to zero, but may or may not be zero in the future. The model then becomes the standard buffer-stock model of consumption and precautionary savings, allowing for portfolio choice. Adding an additional state of temporary unemployment does not alter in a fundamental way any of the results in the paper: Because the unemployment state is temporary, and the probability of occurrence is constant over time, the model predicts that it is optimal for the investor to have the same portfolio policy in the employment and unemployment states. A more realistic assumption would be to allow for non-i.i.d. transition probabilities between the employment and unemployment states, in such a way that the probability of unemployment in one period is higher if the previous period the investor was unemployed than if he was not. This non-diagonal transition matrix would introduce persistence in unemployment and might lead to more plausible optimal portfolio policies. Finally, it is important to endogenize the labor supply decision in the model. Presumably a flexible labor supply will accentuate the horizon effect, as in the model of Bodie, Merton, and Samuelson (1992) with riskless labor income. But it may also have interesting effects on the hedging demand for the risky asset.

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Appendix A. Log-Linearization of the Budget Constraint and the Euler Equations

A.1. Derivation of the Log-Linear Intertemporal Budget Constraint

In the employment state we can rewrite equation (5) as

$$\frac{W_{t+1}^e}{Y_{t+1}} = \left(1 + \frac{W_t}{Y_t} - \frac{C_t^e}{Y_t}\right)\left(\frac{Y_t}{Y_{t+1}}\right) R_{p,t+1}^e,$$

or, in logs,

$$w_{t+1}^e - y_{t+1} = \log\left(\exp\left\{w_t - y_t\right\} - \exp\left\{c_t^e - y_t\right\}\right) - \Delta y_{t+1} + r_{p,t+1}^e, \tag{A1}$$

where I write W_{t+1}^e and $R_{p,t+1}^e$(or w_{t+1}^e and $R_{p,t+1}^e$) because financial wealth at $(t+1)$ depends on the state-dependent consumption and portfolio decisions made at t.

We can now linearize equation (A1) by taking a first-order Taylor expansion around $(c_t^e - y_t) = \mathrm{E}[c_t^e - y_t]$ and $(w_t^e - y_t) = \mathrm{E}[w_t^e - y_t]$. This gives equation (9) in text, where

$$\rho_w^e = \frac{\exp\left\{\mathrm{E}[w_t^e - y_t]\right\}}{1 + \exp\left\{\mathrm{E}[w_t^e - y_t]\right\} - \exp\left\{\mathrm{E}[c_t^e - y_t]\right\}}, \tag{A2}$$

$$\rho_c^e = \frac{\exp\left\{\mathrm{E}[c_t^e - y_t]\right\}}{1 + \exp\left\{\mathrm{E}[w_t^e - y_t]\right\} - \exp\left\{\mathrm{E}[c_t^e - y_t]\right\}}, \tag{A3}$$

and

$$k^e = -(1 - \rho_w^e + \rho_c^e)\log(1 - \rho_w^e + \rho_c^e) - \rho_w^e \log(\rho_w^e) + \rho_c^e \log(\rho_c^e). \tag{A4}$$

Note that $\rho_w^e, \rho_c^e > 0$ because $W_t + Y_t - C_t^e > 0$ along the optimal path.

The log-linearization of the intertemporal budget constraint in the retirement state follows Campbell (1993), who log-linearizes the budget constraint around $\mathrm{E}[c^r - w_t]$. The log-linearization constants are $\rho_c^r = \exp\{\mathrm{E}[c^r - w_t]\}/(1 - \exp\{\mathrm{E}[c^r - w_t]\})$ and $k^r = -(1 + \rho_c^r)\log(1 + \rho_c^r) + \rho_c^r \log(\rho_c^r)$. ■

A.2. Derivation of the Log Euler Equations

We can write equation (7) as

$$\begin{aligned} 1 &= \pi^e \, \mathrm{E}_t \left[\exp\left\{\log\delta - \gamma\left(c^e_{t+1} - c^e_t\right) + r_{i,t+1}\right\}\right] \\ &\quad + (1-\pi^e)\, \mathrm{E}_t \left[\exp\left\{\log\delta - \gamma\left(c^r_{t+1} - c^e_t\right) + r_{i,t+1}\right\}\right] \\ &= \pi \, \mathrm{E}_t \left[\exp\left\{x_{t+1}\right\}\right] + (1-\pi)\ \mathrm{E}_t \left[\exp\left\{y_{t+1}\right\}\right]\ , \end{aligned} \tag{A5}$$

where the notational correspondence between the first and second line is obvious. Taking a second order Taylor expansion of $\exp\{x_{t+1}\}$ and $\exp\{y_{t+1}\}$ around $\overline{x}_t = \mathrm{E}_t[x_{t+1}]$ and $\overline{y}_t = \mathrm{E}_t[y_{t+1}]$ we can write:

$$\begin{aligned} 1 &\approx \pi \, \mathrm{E}_t \left[\exp\{\overline{x}_t\}\left(1 + (x_{t+1} - \overline{x}_t) + \frac{1}{2}(x_{t+1} - \overline{x}_t)^2\right)\right] \\ &\quad + (1-\pi)\ \mathrm{E}_t \left[\exp\{\overline{y}_t\}\left(1 + (y_{t+1} - \overline{y}_t) + \frac{1}{2}(y_{t+1} - \overline{y}_t)^2\right)\right] \\ &= \pi \exp\{\overline{x}_t\}\left(1 + \frac{1}{2}\ \mathrm{Var}_t\left(x_{t+1}\right)\right) + (1-\pi)\exp\{\overline{y}_t\}\left(1 + \frac{1}{2}\ \mathrm{Var}_t\left(y_{t+1}\right)\right)\ . \end{aligned} \tag{A6}$$

Finally, a first-order Taylor expansion around zero gives the result:

$$1 \approx \pi\left(1 + \overline{x}_t + \frac{1}{2}\ \mathrm{Var}_t\left(x_{t+1}\right)\right) + (1-\pi)\left(1 + \overline{y}_t + \frac{1}{2}\ \mathrm{Var}_t\left(y_{t+1}\right)\right)\ . \tag{A7}$$

The derivation of the second log Euler equation mimics the one above. ■

Appendix B: Optimal Rules

B.1. Proof of Proposition 1

To prove Proposition 1, I first guess that the optimal policies take the form $\alpha^r_t = \alpha^r$ and $c^r_t = b^r_0 + b^r_1 w_t$. Next I show that this guess verifies the log-linear Euler equation (12) and the log budget constraint (10), provided that α^r, b^r_0 and b^r_1 satisfy three equations whose coefficients are functions of the primitive parameters that define the preference and stochastic structure of the problem.

Optimal Portfolio Rule in the Retirement State

Substracting the log Euler equation (12) for $i = f$ from the log Euler equation (12) for $i = 1$, we find that the log excess return verifies the following equation

$$\mathrm{E}_t\, r_{1,t+1} - r_f + \frac{1}{2}\,\mathrm{Var}_t\left(r_{1,t+1}\right) = \gamma\,\mathrm{Cov}_t\left(c^r_{t+1} - c^r_t, r_{1,t+1}\right), \tag{B1}$$

or

$$\mu + \frac{1}{2}\sigma_u^2 = \gamma b_1^r \alpha_t^r \sigma_u^2. \tag{B2}$$

The left-hand side (LHS) of equation (B2) follows directly from the assumptions on the investment opportunity set. The right-hand side (RHS) obtains from the log budget constraint (10) by noting that the guess on the log consumption-wealth ratio implies

$$c^r_{t+1} - c^r_t = b_1^r(w^r_{t+1} - w_t). \tag{B3}$$

Optimal Consumption Rule in the Retirement State

To derive the optimal consumption rule, note that the log Euler equation (12) in the retirement state when $i = p$ yields the following equation for expected log consumption growth:

$$\mathrm{E}_t\left[c^r_{t+1} - c^r_t\right] = \frac{1}{\gamma}\left[\mathrm{E}\left[r^r_{p,t+1}\right] + \log\delta + \frac{1}{2}\,\mathrm{Var}\left[r^r_{p,t+1} - \gamma\left(c^r_{t+1} - c^r_t\right)\right]\right], \tag{B4}$$

where I suppress the subscript t from the conditional moments on the RHS because $\alpha_t^r \equiv \alpha^r$ implies that they are constant. Moreover, equation (B3) implies $\mathrm{Var}_t[r^r_{p,t+1} - \gamma(c^r_{t+1} - c^r_t)] = (1 - b_1^r\gamma)^2\,\mathrm{Var}(r^r_{p,t+1})$.

On the other hand, from equation (B3) and the log budget constraint (10) we have

$$\begin{aligned}\mathrm{E}_t\left[c^r_{t+1} - c^r_t\right] &= b_1^r\,\mathrm{E}_t\left[w^r_{t+1} - w_t\right] \\ &= b_1^r\,\mathrm{E}\left[r^r_{p,t+1}\right] - b_1^r\rho_c^r b_0^r + b_1^r k^r + b_1^r\rho_c^r\left(1 - b_1^r\right)w_t.\end{aligned} \tag{B5}$$

Equalizing the right-hand side of equations (B4) and (B5) and identifying coefficients, we obtain two equations. The first one implies $b_1^r = 1$, while the second one implies

$$b_0^r = -\left(\frac{1}{b_1^r\rho_c^r}\right)\left[\left(\frac{1}{\gamma} - b_1^r\right)\mathrm{E}\left[r^r_{p,t+1}\right] + \frac{1}{\gamma}\log\delta + \frac{1}{2\gamma}\left(1 - b_1^r\gamma\right)^2\mathrm{Var}\left(r^r_{p,t+1}\right) - b_1^r k^r\right]. \tag{B6}$$

Since $b_1^r = 1$, we have $b_0^r \equiv \mathrm{E}[c_t^r - w_t] = \log(\rho_c^r) - \log(1 + \rho_c^r)$. We can easily substitute the log-linearization constants ρ_c^r and k^r out from equation (B6) and obtain equation (16) in text.

To see the equivalence of this solution and the solution in Samuleson (1969), note that Samuelson solution for $T \to \infty$ is $C_t^r / W_t = 1 - (\delta^r \, \mathrm{E}[R_{p,t+1}^{1-\gamma}])^{1/\gamma}$(see Ingersoll (1987), p. 243). This expression reduces to equation (16) after taking expectations under the approximation that $R_{p,t+1}$ is lognormal. ■

B.2. Proof of Proposition 2

The proof of Proposition 2 follows the same lines as the proof of Proposition 1. First, we first guess the functional form of the optimal policies in the employment state:

$$\begin{aligned} c_t^e - y_t &= b_0^e + b_1^e \left(w_t - y_t \right), \\ \alpha_t^e &= \alpha^e. \end{aligned} \tag{B7}$$

To prove Proposition 2, it is useful to write the optimal consumption policy in the retirement state, equation (B6), in the same form as equation (B7):

$$c_{t+1}^r - y_{t+1} = b_0^r + b_1^r \left(w_{t+1} - y_{t+1} \right), \tag{B8}$$

where $b_1^r = 1$.

Optimal Portfolio Rule in the Employment State

Substracting the log Euler equation (13) for $i = f$ from the log Euler equation (13) for $i = 1$ yields:

$$\mathrm{E}_t \, r_{1,t+1} - r_f + \frac{1}{2} \mathrm{Var}_t \left(r_{1,t+1} \right) = \gamma \left[\pi^e \, \mathrm{Cov}_t \left(c_{t+1}^e - c_t^e, r_{1,t+1} \right) + \left(1 - \pi^e \right) \mathrm{Cov}_t \left(c_{t+1}^r - c_t^e, r_{1,t+1} \right) \right]. \tag{B9}$$

But equations (B7) and (B8), the log-linear intertemporal budget constraint (9) and the trivial equality

$$c_{t+1}^s - c_t^e = (c_{t+1}^s - y_{t+1}) - (c_t^e - y_t) + (y_{t+1} - y_t), \tag{B10}$$

imply that

$$\begin{aligned} \mathrm{Cov}_t\left(c^s_{t+1}-c^e_t, r_{1,t+1}\right) &= \mathrm{Cov}_t\left(b^s_1 r^e_{p,t+1}+\left(1-b^s_1\right)\left(y_{t+1}-y_t\right), r_{1,t+1}\right) \\ &= b^s_1\alpha^e_t\sigma^2_u+\left(1-b^s_1\right)\sigma_{\xi u}, \end{aligned} \tag{B11}$$

for $s=e$, r. The second line follows from the assumptions on asset returns and labor income. Substituting back into equation (B9) and using equation (B2) we find

$$\mu+\frac{1}{2}\sigma^2_u=\gamma\left[\left(\pi^e b^e_1+\left(1-\pi^e\right)\right)\alpha^e_t\sigma^2_u+\pi^e\left(1-b^e_1\right)\sigma_{\xi u}\right], \tag{B12}$$

from which equation (19) in text obtains immediately.

Optimal Consumption Rule in the Employment State

The log-Euler equation (13) for $i=p$ and the trivial equality (B10) imply

$$\sum_{s=e,r}\pi^s\,\mathrm{E}_t\left[c^s_{t+1}-y_{t+1}\right]=\left(c^e_t-y_t\right)+\Upsilon^e_t, \tag{B13}$$

where

$$\Upsilon^e_t\equiv\Upsilon^e=\frac{1}{\gamma}\left(\mathrm{E}\left[r^e_{p,t+1}\right]+\frac{1}{2}\,\mathrm{V}^e+\sum_{s=e,r}\pi^s\log\delta^s\right)-g, \tag{B14}$$

and

$$\begin{aligned} \mathrm{V}^e \;=\; & \left[\pi^e(1-b^e_1\gamma)^2+(1-\pi^e)(1-b^r_1\gamma)^2\right]\mathrm{Var}[r^e_{p,t+1}]+\pi^e\gamma(1-b^e_1)\,\mathrm{Var}[\Delta y_{t+1}] \\ & -2\pi^e\gamma(1-\gamma b^e_1)(1-b^e_1)\,\mathrm{Cov}[r^e_{p,t+1},\Delta y_{t+1}]. \end{aligned} \tag{B15}$$

If we substitute equations (B7) and (B8) in equation (B13) we obtain

$$\overline{b}_0+\overline{b}_1\,\mathrm{E}_t\left[w_{t+1}-y_{t+1}\right]=\Upsilon^e+b^e_0+b^e_1\left(w_t-y_t\right), \tag{B16}$$

where $\overline{b}_0=\pi^e b^e_0+(1-\pi^e)b^r_0$ and $\overline{b}_1=\pi^e b^e_1+(1-\pi^e)$. Further substitution of the log budget constraint in the employment state (9) and guess (B7) in the LHS of equation (B16) yields

$$\overline{b}_0+\overline{b}_1\left(\rho^e_w-\rho^e_c b^e_1\right)\left(w_t-y_t\right)+\overline{b}_1\left(k^e-\rho^e_c b^e_0-g+\mathrm{E}\,r^e_{p,t+1}\right)=\Upsilon^e+b^e_0+b^e_1\left(w^e_t-y_t\right). \tag{B17}$$

Identifying coefficients on both sides of this equation we get the following two-equation system:

$$\begin{aligned}\bar{b}_1\left(\rho_w^e - \rho_c^e b_1^e\right) &= b_1^e, \\ \bar{b}_0 + \bar{b}_1\left(k^e - \rho_c^e b_0^e - g + \mathrm{E}\, r_{p,t+1}^e\right) &= \Upsilon^e + b_0^e. \end{aligned} \quad \text{(B18)}$$

We can solve this system recursively, since the first equation depends only on b_1^e and the second on b_1^e and b_0^e. Simple algebraic manipulation of the first equation gives the following quadratic equation for b_1^e:

$$0 = \pi^e \rho_c^e \left(b_1^e\right)^2 + \left[1 - \pi^e \rho_w^e + \left(1 - \pi^e\right)\rho_c^e\right] b_1^e - \left(1 - \pi^e\right)\rho_w^e. \quad \text{(B19)}$$

The expression for b_0^e is given in equation (20) in text, with

$$k_1 = \left(1 - \pi^e\right) + \rho_c^e \bar{b}_1 > 0. \quad \text{(B20)}$$

Concavity of the Consumption Function

Proving this result is equivalent to characterize the roots of equation (B19). Rewrite equation (B19) as

$$0 = \left(b_1^e\right)^2 + \frac{1}{\pi^e \rho_c^e}\left[1 - \pi^e \rho_w^e + \left(1 - \pi^e\right)\rho_c^e\right] b_1^e - \frac{1}{\pi^e \rho_c^e}\left(1 - \pi^e\right)\rho_w^e. \quad \text{(B21)}$$

Since $\rho_w^e, \rho_c^e > 0$, and by assumption $0 < \pi^e < 1$, the discriminant of this quadratic equation is

$$\left(\frac{1 - \pi^e \rho_w^e + \left(1 - \pi^e\right)\rho_c^e}{\pi^e \rho_c^e}\right)^2 + 4\frac{\left(1 - \pi^e\right)\rho_w^e}{\pi^e \rho_c^e} \geq 0,$$

so the roots of the equation are always real. Moreover, from basic results in quadratic equations, the product of the roots of equation (B21) is $-\left(1 - \pi^e\right)\rho_w^e / \pi^e \rho_c^e < 0$, so one of the roots must be positive and the other must be negative. I discard the negative root because it implies that the level of consumption is a decreasing function of financial wealth at all levels of income—so that the individual is better off with less wealth. Therefore, $b_1^e > \dot{0}$.

I now prove that $b_1^e < 1$ by showing that $b_1^e > 1$ is inconsistent with having strictly positive savings along the optimal path ($W_t + Y_t - C_t^e > 0$). If $b_1^e \geq 1$ then

$$b_1^e = \frac{-\left[1 - \pi^e \rho_w^e + (1 - \pi^e)\,\rho_c^e\right] + \sqrt{\left[1 - \pi^e \rho_w^e + (1 - \pi^e)\,\rho_c^e\right]^2 + 4\pi^e (1 - \pi^e)\,\rho_w^e \rho_c^e}}{2\pi^e \rho_c^e} \geq 1, \tag{B22}$$

so that

$$\sqrt{\left[1 - \pi^e \rho_w^e + (1 - \pi^e)\,\rho_c^e\right]^2 + 4\pi^e (1 - \pi^e)\,\rho_w^e \rho_c^e} \geq \left[1 - \pi^e \rho_w^e + (1 - \pi^e)\,\rho_c^e\right] + 2\pi^e \rho_c^e.$$

Squaring the terms on both sides and simplifying we obtain the following inequality:

$$0 \geq 1 - \rho_w^e + \rho_c^e. \tag{B23}$$

But equations (A2) and (A3) imply that

$$0 < 1 - \rho_w^e + \rho_c^e, \tag{B24}$$

which is inconsistent with equation (B23). Therefore, we must have $b_1^e < 1$.

To find the limiting value of b_1^e when $\pi^e \to 1$, take limits on the left-hand side of equation (B22). It is immediate to see that $\lim_{\pi^e \to 1} b_1^e = 0$, as claimed. ■

B.3. Dynamics of the Wealth-Income Ratio

The conditional expectation of $(w_{t+1}^e - y_{t+1})$ one-period ahead obtains from reordering the terms in equation (B16) and noting that $b_0^e - \bar{b}_0 = (1 - \pi^e)(b_0^e - b_0^r)$:

$$\mathrm{E}_t\left[w_{t+1}^e - y_{t+1}\right] = \frac{1}{\bar{b}_1}\left[\Upsilon^e + (1 - \pi^e)\left(b_0^e - b_0^r\right)\right] + \frac{b_1^e}{\bar{b}_1}\left(w_t^e - y_t\right), \tag{B25}$$

where $0 < b_1^e/\bar{b}_1 < 1$. Therefore, $(w_t - y_t)$ follows a stationary ARMA(1,1) process with long-run mean $\left[\Upsilon^e + (1 - \pi^e)\left(b_0^e - b_0^r\right)\right] / (1 - \pi^e)\left(1 - b_1^e\right)$.

Equation (B25) suggests that, as $\pi^e \to 1$, $(w_t^e - y_t)$ approaches a random walk with drift Υ^e. If $\Upsilon^e < 0$, the investor eventually runs down her assets to zero—i.e., $(w_{t+l} - y_{t+l})$ approaches

$-\infty$. It is interesting to note that $\Upsilon^e < 0$ is a generalization, allowing for asset return uncertainty and portfolio choice, of the sufficient condition for the existence of a contraction mapping in the models of Deaton (1991, equations (26) and (27)), Carroll (1996, equation (30)) and Carroll (1997, equation (5)). Nevertheless, it is important to note that the model cannot say anything about what happens right at the limit, when $\pi^e = 1$, because it is built on the assumption that $0 < \pi^e < 1$. It predicts only what happens near the limit. ■

Appendix C: Numerical Solution Algorithm

The solution algorithm follows the following iterative procedure:

1. For a given set of parameters defining preferences and the processes for labor income and asset returns, the approximate analytical solution is computed. This solution is used to choose a grid of states for x and the initial values for the coefficients of the polynomials (23) The value function that corresponds to these values is then computed using a numerical algorithm that looks for the fixed point in the Bellman equation.

2. The gradient (Λ) and hessian (H) of the value function are evaluated numerically at each state given a set of values for the coefficients of the polynomials (23). The numerical derivatives of the value function are computed by perturbing the coefficients of the polynomials, finding the value function corresponding to the perturbed parameters at each state and aggregating across states using an equally weighted average of the value function at each state.

3. The coefficients of the polynomials (23) are updated using a Newton-Rahpson algorithm. The new coefficients are given by

$$v^{(i+1)} = v^{(i)} - \lambda H^{-1}\Lambda, \tag{C1}$$

where $v^{(i)} = (a_0^{(i)}, ..., a_p^{(i)}, b_0^{(i)}, ..., b_q^{(i)})'$ is the vector of coefficients in iteration (i), and τ controls the step size to avoid divergence problems. It is set initially to 2/3. Given

$v^{(i+1)}$, the new value function is computed and compared to the previous one. If there is no improvement, λ is reduced by multiplying it by itself, and a new $v^{(i+1)}$ and its corresponding value function is computed. The process is repeated as long as there is no improvement in the value function.

4. The value function at $v^{(i+1)}$ is aggregated across states using an equally-weighted average and compared to the value function at $v^{(i)}$ aggregated in the same way. If the absolute percentage change in the value function is larger than 10^{-5}, steps 3 and 4 are repeated until convergence.

Footnotes

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[1]Merton (1977) and Weil (1994) do consider the effect on portfolio choice of a non-zero correlation between labor income and stock returns, but they use a two-period model that ignores the savings decision. Svensson and Werner (1993) are able to characterize analytically the optimal policies. However, they solve the model only when investors have constant absolute risk aversion which implies, counterfactually, a zero wealth elasticity of risky investment.

[2]Letendre and Smith (1998) corroborate this result.

[3]Alternatively, we can assume that if the investor receives transfers at retirement, they are not large enough to keep the investor's marginal utility of consumption from being infinitely large if her consumption is limited to these transfers.

[4]In a recent paper, Gertler (1997) argues that this device can be used to capture life-cycle behavior while delivering an analytically tractable model.

[5]However, if we think of the retirement state as an episode of permanent unemployment for economic reasons, we may want to consider a model in which the probability of such an event is correlated with innovations in the risky asset.

[6]Note that for some variables the timing of the variable and the timing of s are different.

For example, the return on financial wealth at time $t+1$ depends on the portfolio decisions made at time t, so $R_{p,t+1}$ depends on the realized state for labor income at t.

[7]Appendix B shows that $\log(W_t^e/Y_t)$ is stationary along the optimal path.

[8]Note that the intertemporal budget constraint implies that $W_t^e + Y_t - C_t^e > 0$ is a necessary condition for next-period's financial wealth to be positive.

[9]See Appendix A for the derivation of all the approximations.

[10]See Merton (1973), equation (17). Merton notes that this follows immediately from the envelope condition $J_W = U_C$.

[11]In a model with no portfolio choice, Carroll and Kimball (1996) show that optimal consumption is a strictly concave function under power utility when there is both labor income risk and asset return risk. This property holds regardless of the statistical characterization of these risks.

[12]To see why α^e depends on $\bar{b}_1$ and not just on b_1^e note that the effects on wealth of a portfolio decision made at time t are not known until $t+1$, when the return on the risky asset is realized. Since there is uncertainty about the state of labor income next period, the investor must take this into account when making her portfolio decision.

[13]In fact, labor income whose risk is purely idiosyncratic would be discounted at the riskless rate if the investor were allowed to trade her human capital.

[14]Note that $\sigma_{\xi u}/\sigma_u^2$ is the slope of the regression of labor income shocks onto unexpected stock returns.

[15]I choose the initial values for ρ_w^e and ρ_c^e so that ρ_w^e, $\rho_c^e > 0$ and $1 - \rho_w^e + \rho_c^e > 0$. This ensures that k^e is defined.

[16]Chamberlain and Hirano (1999) estimate the implied predictive distribution of σ_ξ using PSID data and find that the median of this distribution is between $10 - 11$ percent for the full sample and the subsamples of high-school graduates and college graduates. Carroll and

Samwick (1995) report a full sample estimate of σ_ξ which is somewhat larger, around 15 percent.

[17]If r_Y is the log discount rate on labor income, equation (1) implies that the present discounted value of expected future labor income is $Y_t/(1-\pi^e \exp\{g+\sigma_\xi^2 - r_Y\})$, which is increasing in π^e —or, equivalently, in $(1-\pi^e)^{-1}$, the expected retirement horizon.

[18]For values of γ larger than 12, it is possible to show that horizon effect even reverses, and α^e increases as the expected retirement horizon falls. Guiso et al. (1996) and Heaton and Lucas (2000) have found a positive effect of age on stock holdings for a sample of Italian and American households.

[19]All other parameters take their baseline values (including a zero correlation between labor income innovations and unexpected asset returns).

[20]See footnote 17.

[21]Results not shown here to economize space, but readily available upon request, show that the financial wealth-permanent income target ratio is increasing in δ.

[22]When labor income is lognormally distributed, a mean-preserving increase in the variance of labor income is a second-degree stochastic dominance shift in the distribution of labor income. Eeckhoudt, Gollier, and Schlesinger (1996) study this form of deterioration in background risk in a two-period model with no savings decision.

[23]Note that it is rather difficult to evaluate $\partial\alpha^e/\partial\sigma_\xi$ analytically, because this dependence is highly nonlinear. However, in a two-period version of this model, Viceira (1998) shows that this derivative is negative provided that the investor is sufficiently risk averse, and it is strictly positive when $\gamma \to 0$.

[24]Some empirical results in the literature are consistent with the slowly decreasing pattern of α^e as a function of σ_ξ. In their study of portfolio holdings of Italian households, Guiso et al. (1996) divide their sample in two groups, low- and high-income risk, and find that there are small differences in their holdings of risky assets as a percentage of total financial assets.

[25]Bodie, Merton, and Samuelson (1992) show that $\alpha^e = \alpha^r$ when $\sigma_\xi = \alpha^r \sigma_u$ and $\sigma_{\xi u} = \sigma_\xi \sigma_u$. Simple substitution of these conditions in equation (19) shows that this result also holds for the approximate solutions given in PropositItions 1 and 2.

Table I

Optimal Percentage Portfolio Share of Stocks in the Employment and Retirement States ($\alpha \times 100$)

Portfolio shares of stocks are given in percentage points and are based on the following baseline parameter values: $\delta = 1/(1+0.10)$, $E_t[Y_{t+1}/Y_t] = \exp\{g+\sigma_\xi^2/2\} = 1.03$, $\sigma_\xi = 0.10$, $R_f = 1.02$, $E_t[R_{1,t+1}/R_f] = \exp\{\mu + \sigma_u^2/2\} = 1.06$, $\sigma_u = 0.18$. The expected lifetime after retirement is 10 years. All covariances are set to zero, except in Panel B, that allows for a 25 percent correlation between stock returns ($r_{1,t+1}$) and labor income growth (Δy_{t+1}). The column labelled "Ret." reports the optimal allocation in the retirement state. The lower part of Panel B reports $(-(1-\bar{b}_1)\sigma_{\xi,u}/\sigma_u^2)/\alpha$.

RRA	Expected Time Until Retirement (Years)							Ret.($\boldsymbol{\alpha}^r$)
	35	30	25	20	15	10	5	
	Panel A: Corr($r_{1,t+1}, \Delta y_{t+1}$) = 0%							
2	292.45	250.93	218.24	190.04	164.46	140.06	114.77	89.92
3	148.16	135.48	123.18	111.08	99.02	86.63	73.16	59.95
5	76.29	71.23	66.02	60.63	54.97	48.92	42.25	35.97
8	43.50	41.11	38.54	35.77	32.75	29.46	25.79	22.48
10	33.57	31.89	30.03	28.00	25.74	23.25	20.46	17.98
12	27.22	25.97	24.55	22.96	21.18	19.18	16.96	14.99
	Panel B: Corr($r_{1,t+1}, \Delta y_{t+1}$) = 25%							
2	255.52	225.06	198.87	175.42	153.72	132.82	111.11	89.92
3	130.46	120.26	110.32	100.58	90.85	80.93	70.23	59.95
5	62.62	59.14	55.60	51.95	48.17	44.20	39.89	35.97
8	31.35	30.28	29.14	27.93	26.64	25.26	23.77	22.48
10	21.94	21.48	20.99	20.45	19.87	19.24	18.56	17.98
12	15.99	15.88	15.76	15.62	15.47	15.31	15.13	14.99
	Hedging Demand (Percentage of Total Demand)							
2	-11.84	-10.97	-10.01	-8.90	-7.58	-5.90	-3.48	0.00
3	-16.30	-15.12	-13.77	-12.18	-10.26	-7.82	-4.42	0.00
5	-26.77	-24.65	-22.21	-19.35	-15.93	-11.71	-6.19	0.00
8	-45.75	-41.64	-36.95	-31.57	-25.25	-17.79	-8.80	0.00
10	-61.20	-55.26	-48.52	-40.89	-32.12	-22.09	-10.56	0.00
12	-79.69	-71.26	-61.86	-51.40	-39.70	-26.65	-12.35	0.00

Table II

Long-Run Expected Holdings of Financial Assets Relative to Permanent Income in the Employment State ($\exp\{E[w_t^e - y_t]\}$)

The numbers in the table report the exponentiated mean optimal log financial wealth-labor income ratio in the employment state. They are based on the following baseline parameter values: $\delta = 1/(1+0.10)$, $E_t[Y_{t+1}/Y_t] = \exp\{g + \sigma_\xi^2/2\} = 1.03$, $\sigma_\xi = 0.10$, $R_f = 1.02$, $E_t[R_{1,t+1}/R_f] = \exp\{\mu + \sigma_u^2/2\} = 1.06$, $\sigma_u = 0.18$. The expected lifetime after retirement is 10 years. All covariances are set to zero, except in Panel B, that allows for a 25 percent correlation between stock returns and labor income growth.

RRA	Expected Time Until Retirement (Years)						
	35	30	25	20	15	10	5
	Panel A: Corr($r_{1,t+1}, \Delta y_{t+1}$) = 0%						
2	3.44	3.95	4.62	5.46	6.52	7.85	9.46
3	5.62	6.35	7.23	8.30	9.58	11.12	12.91
5	9.45	10.32	11.34	12.54	13.97	15.68	17.68
8	14.08	14.84	15.75	16.86	18.22	19.94	22.10
10	16.66	17.26	18.01	18.97	20.23	21.89	24.09
12	18.95	19.36	19.92	20.71	21.82	23.40	25.61
	Panel B: Corr($r_{1,t+1}, \Delta y_{t+1}$) = 25%						
2	3.34	3.87	4.54	5.38	6.43	7.76	9.38
3	5.41	6.14	7.01	8.06	9.34	10.89	12.74
5	8.97	9.81	10.80	11.98	13.42	15.18	17.36
8	13.18	13.90	14.78	15.88	17.29	19.15	21.62
10	15.49	16.05	16.78	17.76	19.09	20.94	23.52
12	17.55	17.92	18.48	19.31	20.53	22.34	24.99

Table III

Optimal Percentage Portfolio Share of Stocks as a Function of Expected Retirement Horizon and Expected Labor Income Growth

The numbers in the table report percentage portfolio shares of stocks in the employment state across different expected retirement horizons ($1/\pi^e$) and expected labor income growth rates ($E_t[Y_{t+1}/Y_t] = \exp\{g + \sigma_\xi^2/2\}$). They are based on the following baseline parameter values: $\delta = 1/(1+0.10)$, $\gamma = 3$, $\sigma_\xi = 0.10$, $R_f = 1.02$, $E_t[R_{1,t+1}/R_f] = \exp\{\mu + \sigma_u^2/2\} = 1.06$, $\sigma_u = 0.18$. The expected lifetime after retirement is 10 years. All covariances are set to zero.

$E_t[Y_{t+1}/Y_t]$	Expected Time Until Retirement (Years)						
	35	30	25	20	15	10	5
1.08	345.48	276.71	224.13	181.02	144.56	112.83	83.80
1.06	248.77	211.34	179.21	150.97	125.64	102.25	79.61
1.04	177.10	158.14	140.44	123.70	107.66	91.81	75.33
1.02	123.00	115.14	107.23	99.10	90.63	81.49	70.96
1.00	81.59	80.30	78.82	76.94	74.55	71.34	66.48

Figures

Figure 1. Optimal portfolio allocation to stocks in the employment state plotted against the time preference rate. The three lines within the graph correspond to different expected retirement horizons: 30 years (solid line), 20 years (dashed line) and 10 years (dash/dot). These allocations are given in percentage points and they are based on the following baseline parameter values: $\mathrm{E}_t[Y_{t+1}/Y_t] = \exp\{g + \sigma_\xi^2/2\} = 1.03$, $\sigma_\xi = 0.10$, $R_f = 1.02$, $\mathrm{E}_t[R_{1,t+1}/R_f] = \exp\{\mu + \sigma_u^2/2\} = 1.06$, $\sigma_u = 0.18$. The expected lifetime after retirement is 10 years. All covariances are set to zero.

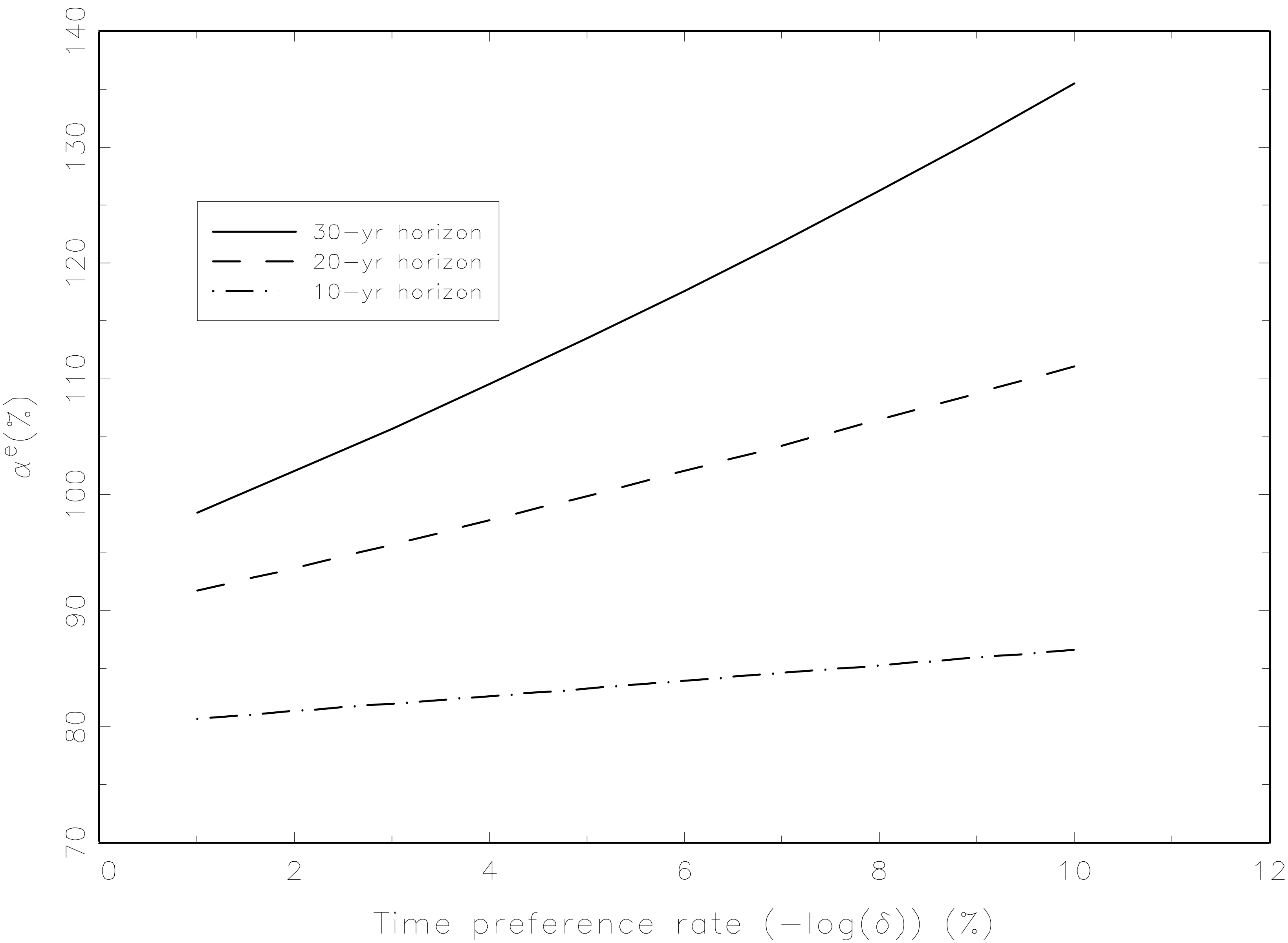
Figure 2. Optimal portfolio allocation to stocks in the employment state plotted against the standard deviation of log labor income growth. The two lines within the graph correspond to different values of the coefficient of relative risk aversion (γ): 3 (solid line) and 8 (dashed line). Portfolio shares are given in percentage points and they are based on mean-preserving increases in the standard deviation of log labor income growth (σ_ξ). This parameter takes values in the interval $[0, 0.20]$. Expected log labor income growth (g) takes values so that expected labor income growth stays constant at 3 percent (i.e., $\mathrm{E}_t[Y_{t+1}/Y_t] = \exp\{g + \sigma_\xi^2/2\} = 1.03$). The rest of the parameters in the model take the following baseline values: $R_f = 1.02$, $\mathrm{E}_t[R_{1,t+1}/R_f] = \exp\{\mu + \sigma_u^2/2\} = 1.06$, $\sigma_u = 0.18$. The expected lifetime after retirement is 10 years. All covariances are set to zero.

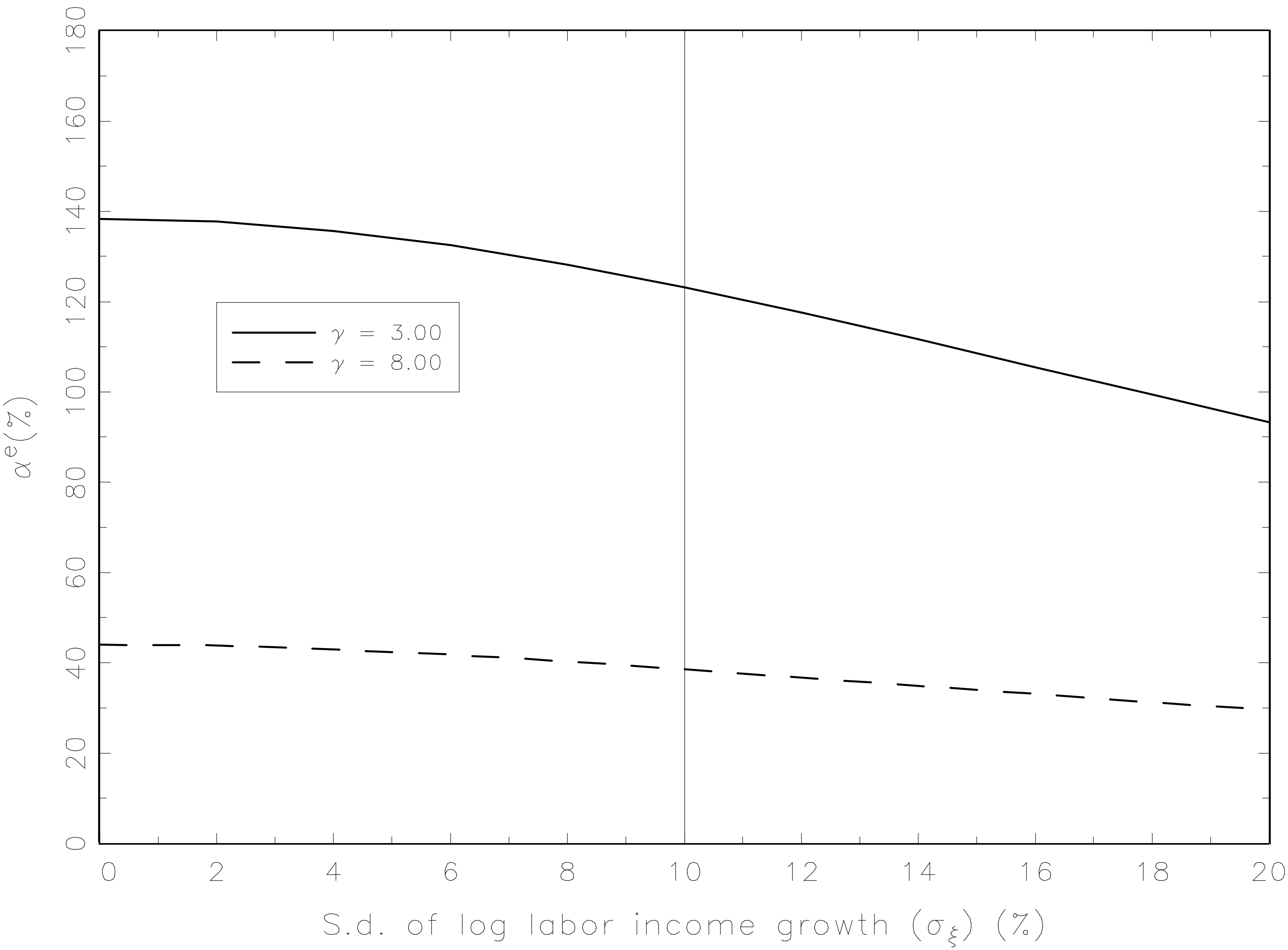
Figure 3. Mean asset holdings in the employment state plotted against the standard deviation of log labor income growth. The two lines within the graph correspond to different values of the coefficient of relative risk aversion (γ): 3 (solid line) and 8 (dashed line). Mean asset holdings are given in natural units and they are based on mean-preserving increases in the standard deviation of log labor income growth, σ_ξ. This parameter takes values in the interval $[0, 0.20]$. Expected log labor income growth (g) takes values so that expected labor income growth stays constant at 3 percent (i.e., $\mathrm{E}_t[Y_{t+1}/Y_t] = \exp\{g + \sigma_\xi^2/2\} = 1.03$). The rest of the parameters in the model take the

following baseline values: $R_f = 1.02$, $\mathrm{E}_t[R_{1,t+1}/R_f] = \exp\{\mu + \sigma_u^2/2\} = 1.06$, $\sigma_u = 0.18$. The expected lifetime after retirement is 10 years. All covariances are set to zero.

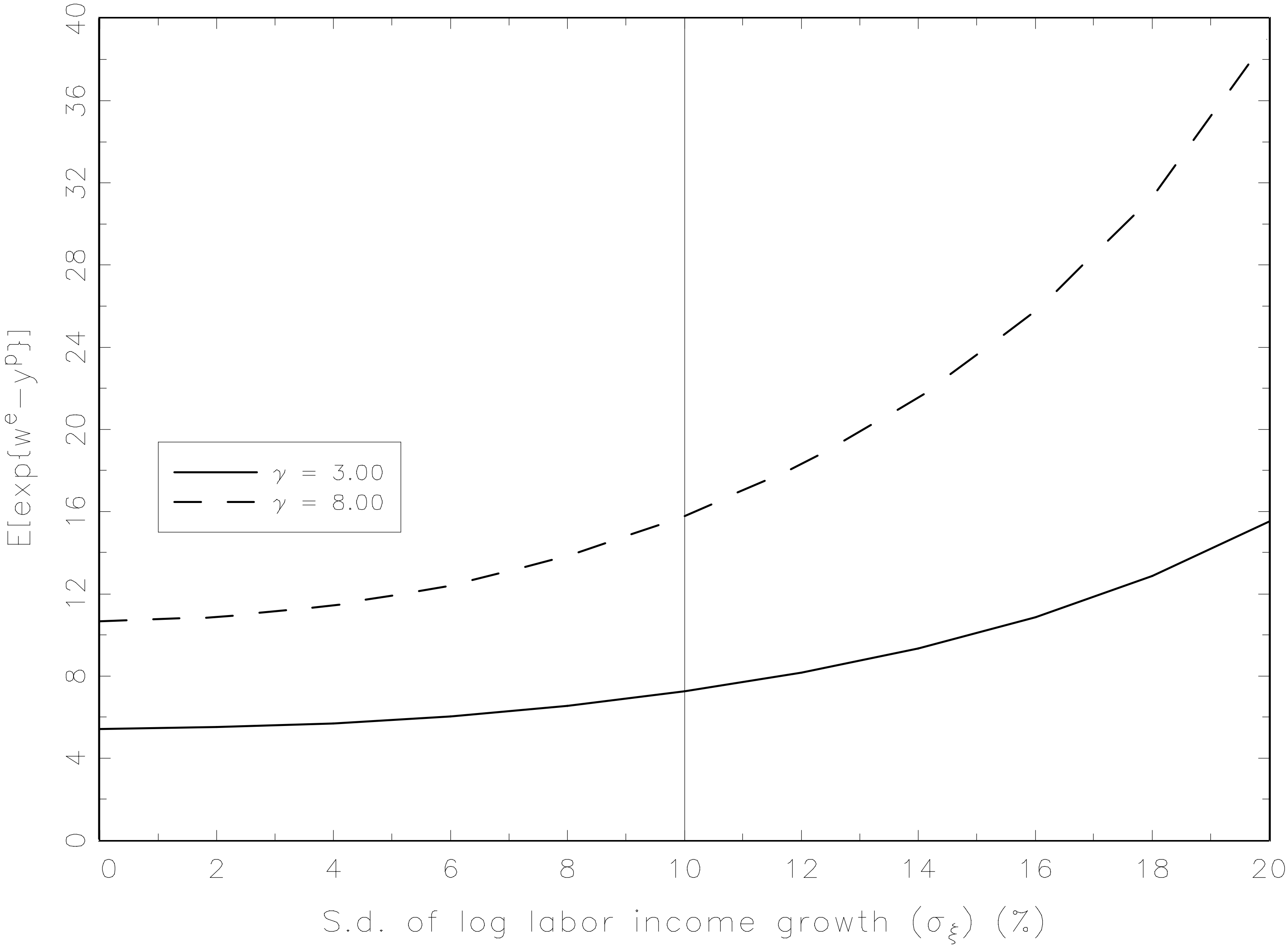
Figure 4. Optimal portfolio allocation to stocks in the employment state: Numerical and approximate analytical solutions. The graph plots the numerical and the approximate analytical solution for the optimal portfolio allocation to stocks against the financial wealth-labor income ratio. The dashed horizontal lines within the graph correspond to the approximate analytical solution, and the solid lines to the numerical solution. The graph plots the solutions for three expected retirement horizons: 30 years (top lines), 20 years (middle lines) and 10 years (bottom lines). These solutions assume the following parameter values: $\gamma = 3$, $\delta = 1/(1.10)$, $\mathrm{E}_t[Y_{t+1}/Y_t] = \exp\{g + \sigma_\xi^2/2\} = 1.03$, $\sigma_\xi = 0.10$, $R_f = 1.02$, $\mathrm{E}_t[R_{1,t+1}/R_f] = \exp\{\mu + \sigma_u^2/2\} = 1.06$, $\sigma_u = 0.18$. The expected lifetime after retirement is 10 years. All covariances are set to zero.

Figure 5. Optimal consumption-labor income ratio in the employment state: Numerical and approximate analytical solution. Each graph within the figure plots the numerical and the approximate analytical solution for the optimal consumption-labor income ratio against the financial wealth-labor income ratio. The dashed horizontal line within each graph corresponds to the approximate analytical solution, and the solid line to the numerical solution. The figure plots the solutions for three expected retirement horizons: 30 years (top graph), 20 years (middle graph) and 10 years (bottom graph). These solutions assume the following parameter values: $\gamma = 3$, $\delta = 1/(1.10)$, $\mathrm{E}_t[Y_{t+1}/Y_t] = \exp\{g + \sigma_\xi^2/2\} = 1.03$, $\sigma_\xi = 0.10$, $R_f = 1.02$, $\mathrm{E}_t[R_{1,t+1}/R_f] = \exp\{\mu + \sigma_u^2/2\} = 1.06$, $\sigma_u = 0.18$. The expected lifetime after retirement is 10 years. All covariances are set to zero.

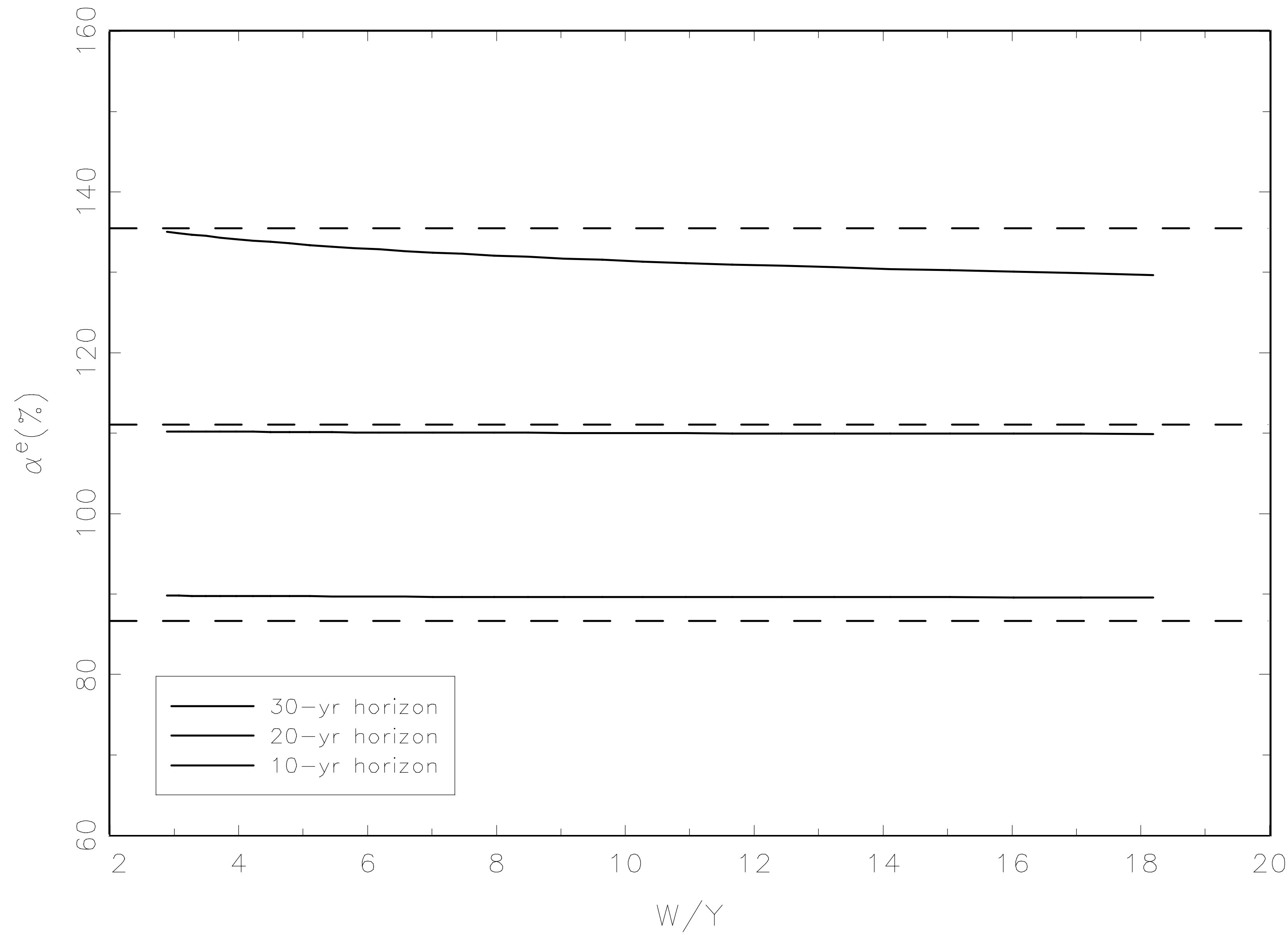

30−yr horizon
20−yr horizon
10−yr horizon
α^e(%)
Time preference rate (−log(δ)) (%)
140
130
120
110
100
90
80
70
0
2
4
6
8
10
12



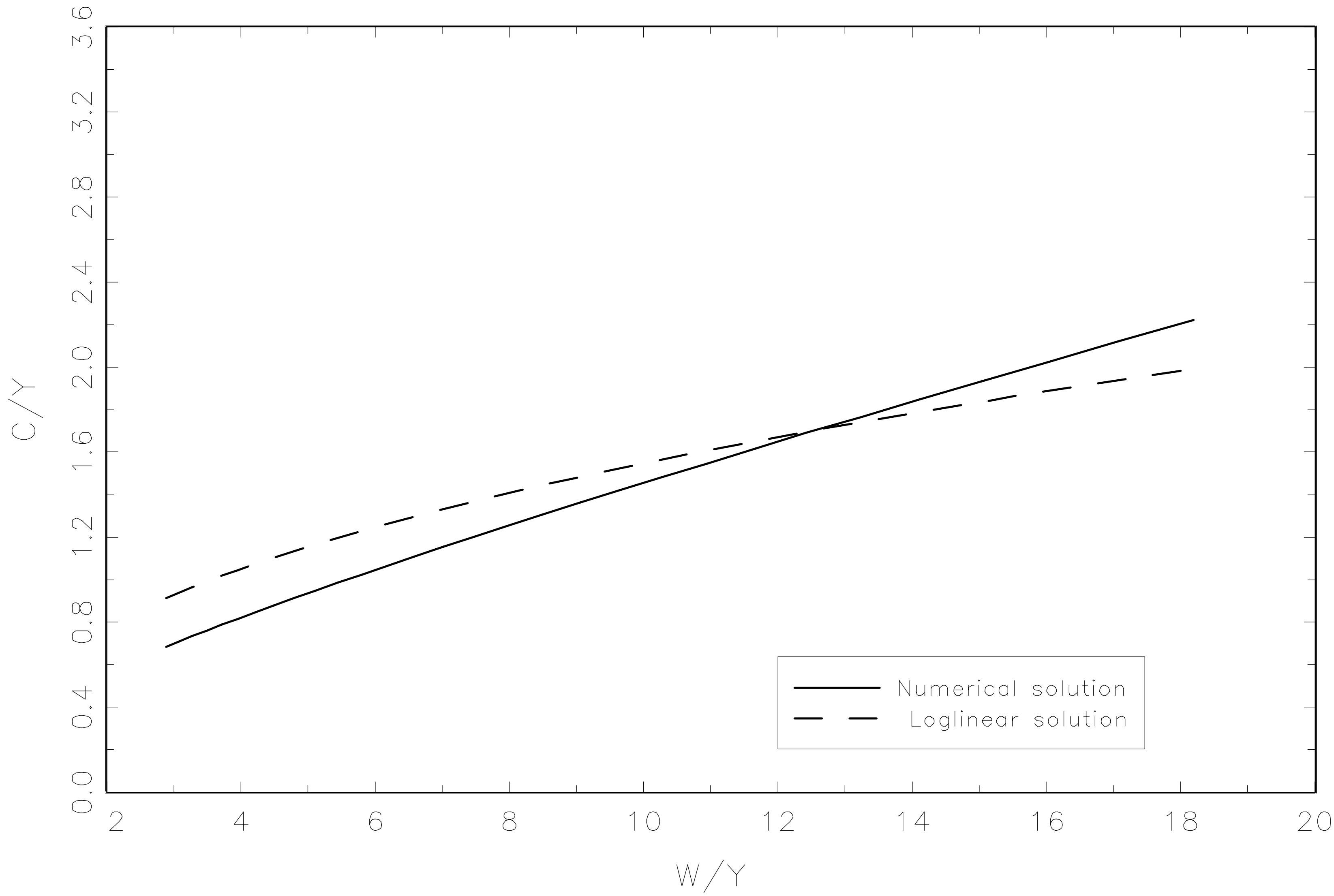
α^e(%)
γ = 3.00
γ = 8.00
S.d. of log labor income growth (σ_ξ) (%)



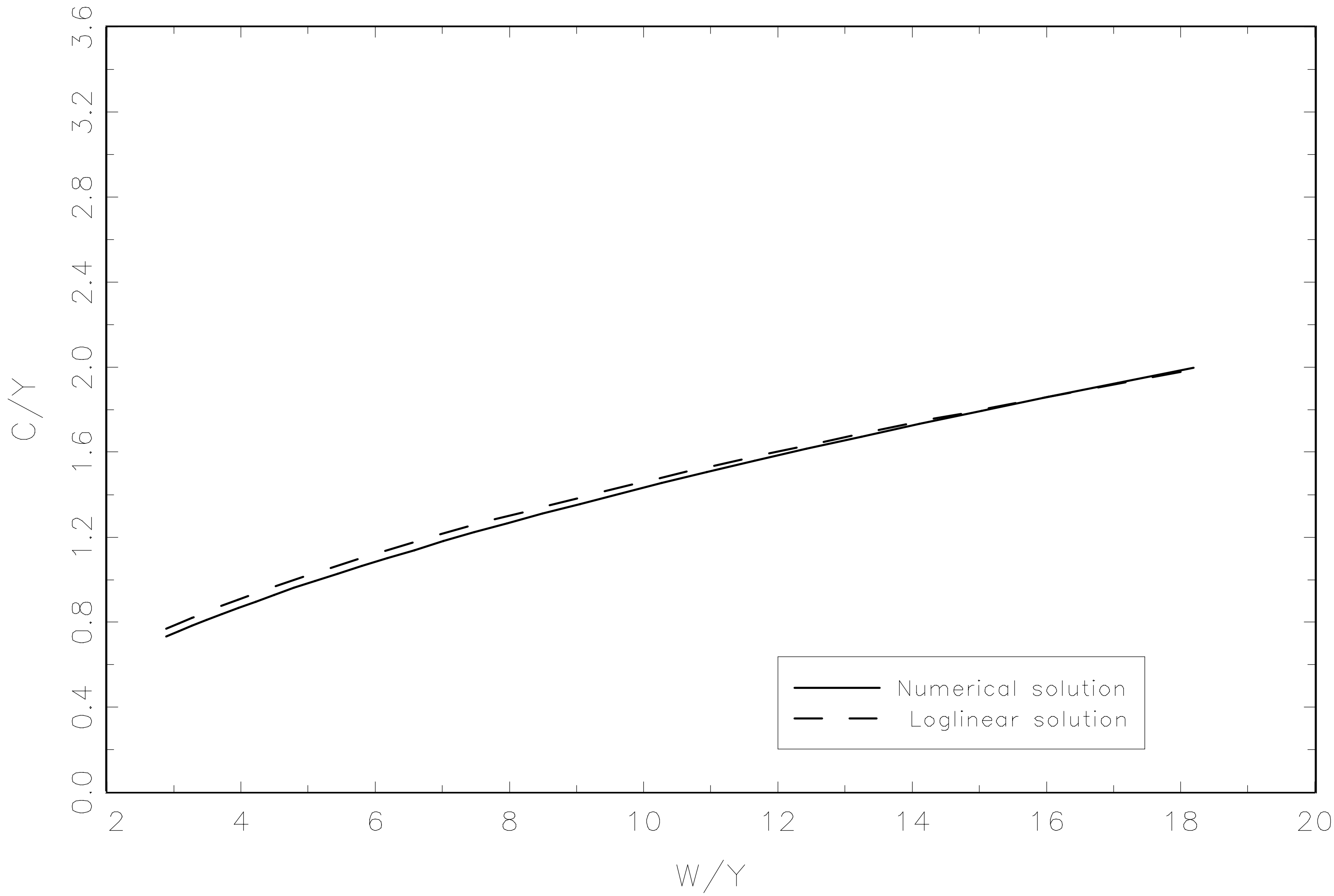
E[exp{w^e − y^P}]
40
36
32
28
24
20
16
12
8
4
0
γ = 3.00
γ = 8.00
0
2
4
6
8
10
12
14
16
18
20
S.d. of log labor income growth (σ_ξ) (%)

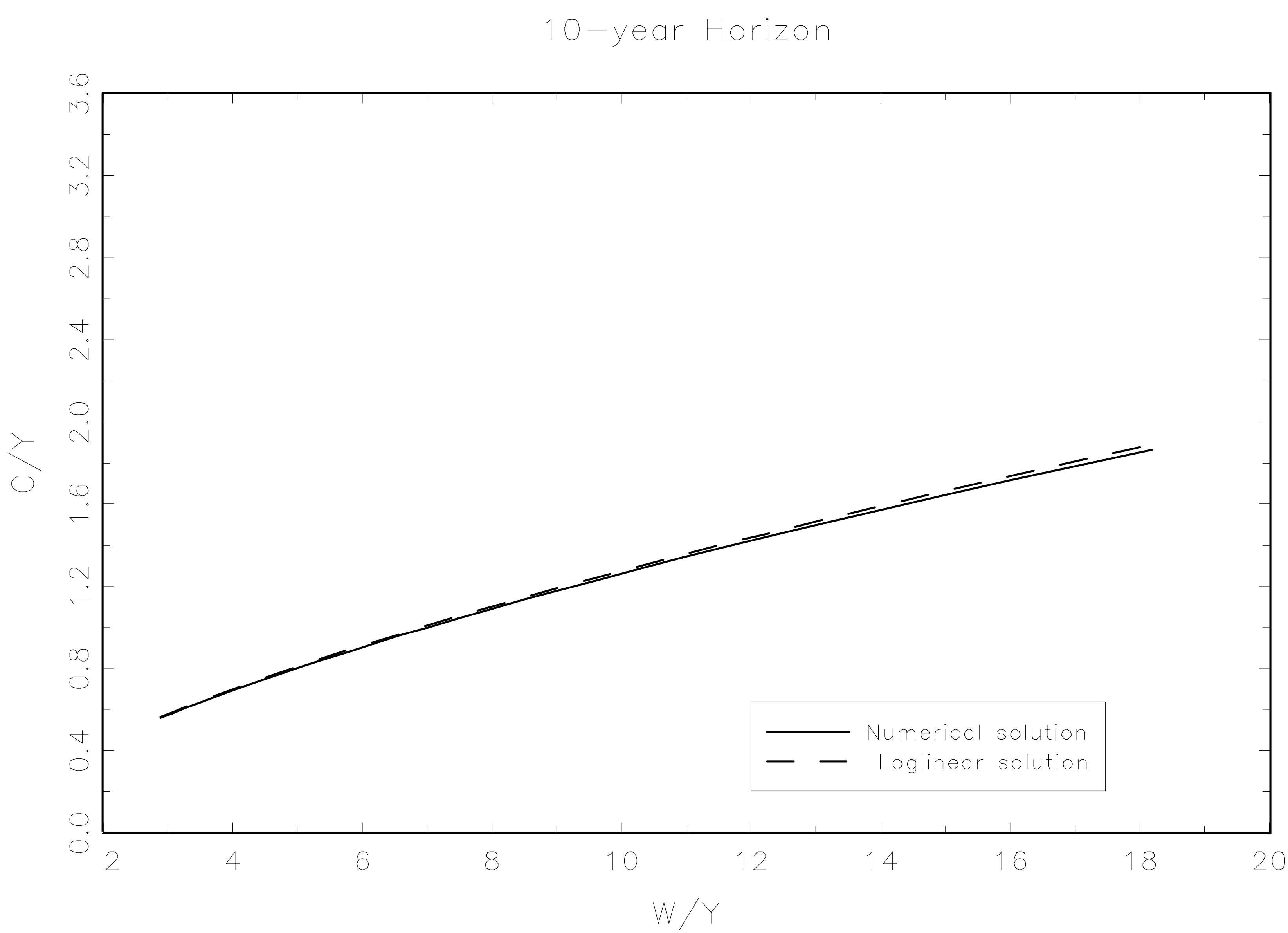

30−yr horizon
20−yr horizon
10−yr horizon
W/Y
$\alpha^e(\%)$

30-year Horizon



20-year Horizon





10-year Horizon
C/Y
W/Y
0.0
0.4
0.8
1.2
1.6
2.0
2.4
2.8
3.2
3.6
2
4
6
8
10
12
14
16
18
20
Numerical solution
Loglinear solution